Matthew Levy, Balázs Szentes

An alternative to signaling: directed search and substitution

Article (Published version)
(Refereed)

Original citation:
Levy, Matthew and Szentes, Balázs (2016) *An alternative to signaling: directed search and substitution.* American Economic Journal: Microeconomics, 8 (4). pp. 1-15. ISSN 1945-7669

DOI: 10.1257/mic.20150116

American Economic Journal: Microeconomics 2016, 8(4): 1–15
http://dx.doi.org/10.1257/mic.20150116

An Alternative to Signaling: Directed Search and Substitution[†]

By Matthew Levy and Balázs Szentes*

This paper analyzes a labor market, where: workers can acquire an observable skill at no cost, firms differ in unobserved productivity, workers' skill and firms' productivity are substitutes, and firms' search is directed. The main result is that, if the entry cost of firms is small, no worker acquires the skill in the unique equilibrium. For intermediate entry costs, a positive measure of workers obtain the skill, and the number of skilled workers goes to one as entry costs become large. Welfare is highest when the entry cost is high. (*JEL* D21, D24, D82, D83, J24)

Signaling theory is often used to explain seemingly inefficient investments. The peacock's large and colorful tail is often explained as a costly signal from males with high but unobservable reproductive value (Zahavi 1975). In the context of economics, individuals might invest in education in order to signal their high ability to the labor market (Spence 1973). This paper puts forward a model that predicts inefficient investments of a different kind. Our central departure is that workers' ability is a substitute for, rather than a complement of, firms' productivity. Workers may costlessly acquire (or, more provocatively, destroy) a skill which is perfectly observed by potential firm matches. Rather than acquire the productive skill, workers may strategically decide to remain low-ability types in order to avoid a poor match.

We analyze a model with many workers and many firms. Workers must decide whether to acquire a productive skill at no cost. At the same time, firms must decide whether to enter the market at some positive cost. Entering firms then draw a stochastic productivity. The skill of a worker is observable, and an entrant firm can direct its search toward a particular type of worker. We consider a stark version of a directed search model, where there are two markets, one for skilled workers and one for unskilled ones. In each market, the maximum number of matches are then created.[1] The surplus created by a match is equally shared by the worker and the firm.

* Levy: Department of Economics, London School of Economics, Houghton Street, London WC2A 2AE, UK (e-mail: m.r.levy@lse.ac.uk); Szentes: Department of Economics, London School of Economics, Houghton Street, London WC2A 2AE, UK (e-mail: b.szentes@lse.ac.uk). We have benefited from comments by John Geanakoplos, Stephen Morris, Phil Reny, Joel Sobel, and especially Jeff Ely.

[†] Go to http://dx.doi.org/10.1257/mic.20150116 to visit the article page for additional materials and author disclosure statement(s) or to comment in the online discussion forum.

[1] This assumption captures the idea that it is more costly to search for a type which is more demanded. Search in our model is more strongly directed than in, for example, Shi (2002) where a "high-tech" firm's strategy may be to match with both skilled and unskilled workers with positive probability.

Our key assumption is that the productivity of the firm and the skill of the worker are substitutes.

Our main result is that if the firms' entry cost is low, no worker acquires the skill. The intuition behind this result can be explained as follows. Suppose that some workers acquire the skill. Since skilled workers generate larger surplus than unskilled workers, there will be more excess demand for skilled workers. Therefore, firms entering the market for skilled workers face a more severe search friction than those firms who search for unskilled workers. Since a worker's skill and a firm's productivity are substitutes, more productive firms are less willing to endure this search friction. There will be a productivity cutoff below which firms search for skilled workers and above which firms search for unskilled workers. Conditional on being matched, a worker is better off in the low-skill market.[2] To attract any skilled workers, the match probability must be sufficiently higher in the high-skill market than in the low-skill market. When the entry cost is low, however, there will be enough firms entering to guarantee a match for even low-skilled workers. As a consequence, workers have no incentive to obtain the skill.

Our second result is that for higher firm entry costs, there exists a unique equilibrium, in which some workers acquire the skill. All skilled workers are matched with a firm, while some unskilled workers remain unmatched. In equilibrium, the highest productivity firms go to the low-skill market where they are matched with probability one. This sorting explains the inefficient investment in skills by workers, as they are willing to remain unskilled (and potentially unmatched) in order to avoid the low-productivity firms in the high-skill market. We then show that when entry costs are sufficiently high, all workers obtain the skill.

That we obtain inefficient investment in the skill may superficially resemble the typical (inefficient) separating equilibrium in signaling models of education. However, our model is in many ways opposite. For example, Spence (1973) and related models generate inefficiency through overinvestment in costly but nonproductive education because of its signaling value. Instead, here inefficiency comes through the underinvestment in costless (or even negative cost) but productive education because of the substitutability of worker and firm productivity. Furthermore, unlike in signaling models, inefficient investments are made by the side of the market whose type is observable.

Because we consider nontransferable utility, our result does not follow from the previous literature, which has focused on the link between negative assortative matching and submodularity of the production function in settings with transferable utility (e.g., Atakan 2006; Shimer and Smith 2000; Becker 1973, 1974). In such settings, submodular production technology leads to negative assortativity because high-ability workers' marginal product—and hence their share of the surplus—is higher in a low-ability match. In fact, we shall argue that workers would acquire the skill if utilities were perfectly transferable in our model. Instead, we assume that the surplus from any match is shared equally, which means that workers focus on their total rather than marginal product. In models with nontransferable utility,

[2] We will assume that the productivity of a skilled worker is not too high relative to that of an unskilled worker.

such as Smith (2006), monotonicity of preferences—i.e., all types prefer matching with higher types than lower types—guarantees that equilibrium matches are never negatively assortative.[3] We show that frictions from directed search can nevertheless result in high-quality firms matching with low-ability workers.

It is an open question whether firms' and workers' productivity are complements or substitutes in production. Most empirical research takes complementarity as given (Postel-Vinay and Robin 2002; Krusell et al. 2000; Goldin and Katz 1998). Our results suggest that this assumption rests on firm theoretical ground, since in markets with cheap entry, workers will never invest in skills which are substitutes for firm productivity.

Finally, we note that our paper was motivated by the work of Mailath and Postlewaite (2006). They consider a population of men and women who, each period, are matched and produce offspring. Agents differ in their non-storable endowments, and care about the consumption of their descendants. In addition, some agents have a particular physical attribute, such as blue eyes, which is inherited by offspring. There exist equilibria in which the attribute has a value—that is, agents with the attribute are better off than agents without it. In this type of equilibrium, high-endowment agents without the attribute prefer to match with low-endowment agents with the attribute rather than with high-endowment agents without it. Such preferences arise from risk-aversion among agents; high-endowment individuals are willing to forgo present consumption in order to increase the expected consumption of their offspring by equipping them with the attribute. In other words, the biological attribute is used to transfer wealth to future generations. In our setup agents are risk-neutral, so they have no incentive to transfer wealth across periods. In Mailath and Postlewaite (2006), individuals who are not endowed with the attribute consume less than others. Since reproduction is unaffected by consumption, the frequency of the attributes are constant over time. We observed that if reproductive value would be determined by consumption, the valued attribute would be more and more frequent in the population and could not be used to transfer wealth across generations. In other words, the frequency of attributes can only be stable in the population if the valued attribute is biologically disadvantageous. Indeed, our original motivation was to try to develop a theory whereby a disadvantageous attribute could survive evolution. In a biological version of our model, it can be shown that a disadvantageous male trait, e.g., the peacock's tail, can survive evolution if it is substitutable with female fitness.

The paper proceeds as follows. Section I presents the model. Section II derives preliminary results. Section III characterizes the equilibrium for low, intermediate, and high entry costs, and presents an example for exposition. Section IV considers the welfare implications of the model, and Section V concludes.

I. Model

We consider a labor market setting with a unit mass of workers and an unlimited number (continuum) of firms. There are two time periods. In the first period,

[3] In addition, log-supermodular production guarantees strict positive assortativity. While we will assume log-submodular production, preferences are clearly monotonic.

workers decide whether to acquire a skill and simultaneously firms decide whether or not to enter. The type of a skilled worker is denoted by H and the type of an unskilled worker is denoted by L. Acquiring the skill is free, but the entry cost of a firm, c, is strictly positive. Upon paying the entry fee, a firm draws a productivity π which is uniformly distributed on $[0, 1]$.[4] If a firm does not enter, its payoff is zero. We assume that a worker's decision whether to acquire the skill is publicly observable but that a firm's productivity is its private information.

In the second period, after observing the measure of skilled and unskilled workers, firms search for workers, and produce if matched with a worker. We assume that search is directed. To be more specific, there are two markets: one for H workers and one for L workers. If there are f firms and l workers in a market, then $\min\{f, l\}$ firms and workers are matched and produce in that market. The remaining unmatched workers or firms do not produce, and receive a payoff of zero. If a worker of type T is matched with a firm of productivity π, they create a positive surplus of $2S(T, \pi)$ and share it equally.[5] In order to guarantee that firms might enter in this market, we assume that $c < E[S(H, \pi)]$. The timing of the model is shown in Figure 1.

We assume that the function S is continuous and strictly increasing in π for all $T \in \{L, H\}$. Skilled workers are strictly more productive than unskilled ones; that is, $S(L, \pi) < S(H, \pi)$ for all $\pi \in [0, 1)$. Furthermore, we make the following assumptions on the surplus function S :

ASSUMPTION 1: $S(L, \pi)/S(H, \pi)$ *is strictly increasing in* π.

ASSUMPTION 2: $E[S(L, \pi)|\pi \geq \overline{\pi}] > E[S(H, \pi)|\pi \leq \overline{\pi}]$ *for all* $\overline{\pi} \in [0, 1]$.

Assumption 1 means that production is log-submodular,[6] and thus the productivity of a firm substitutes for the skill of the worker. Indeed, this assumption requires that the surplus of an L-worker grows faster in π than does the surplus of an H-worker. Assumption 2 implies that the productivity of a skilled worker is not too high relative to that of an unskilled worker. Specifically, this assumption means that the expected surplus generated by an unskilled worker conditional on being matched with a firm with productivity larger than $\overline{\pi}$ is greater than the expected surplus generated by a skilled worker conditional on being matched with a firm with productivity less than $\overline{\pi}$.

For concreteness, consider the following production technology:

Example: Let $L, H \in (0, 1)$, $L < H$, and $S(T, \pi) = T + (\alpha - T)\pi$. Assumption 1 is satisfied whenever α is positive. Assumption 2 is satisfied whenever $\alpha > 2H - L$.

Our objective is to characterize the set of Weak Perfect Bayesian equilibria in this economy. In equilibrium, the strategies of the firms and workers must satisfy three

[4] This uniformity assumption is without loss because π can be always thought of as percentiles of a general distribution.

[5] This may be thought of as the result of Nash bargaining with equal powers, such as in Mortensen and Pissarides (1994), though any fixed division of the surplus would suffice in our static model.

[6] Note that Assumption 1 can be equivalently stated as $\partial \log S(L, \pi)/\partial \pi > \partial \log S(H, \pi)/\partial \pi$.

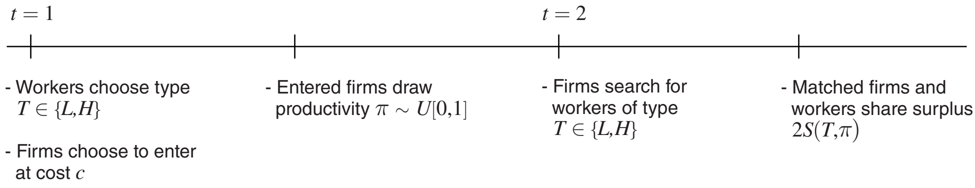

Figure 1. Timing of Model

sets of criteria. First, a firm optimally chooses a market in the second-stage conditional on its productivity. Second, each worker optimally chooses whether or not to acquire a skill. Finally, each firm makes the entry decision optimally.

Note that if each worker is of the same type, say L, then no worker will be in the H-market in the second period. As a consequence, when a single worker decides to acquire the skill, the measure of workers in the H-market is still zero. Therefore, a firm's choice to go to the H-market can have a large effect on the search friction. Indeed, if only one firm enters the H-market it will be surely matched with a worker, but if a second firm enters the probability of being matched is halved. In order to avoid this problem, we assume that there is an $\varepsilon(> 0)$ measure of workers in each market in addition to those who strategically decide to be there. We characterize equilibria in the limit where ε tends to zero.

II. Preliminaries

We first establish some preliminary results which will be useful in characterizing the equilibria.

LEMMA 1: *The expected payoff of each firm is zero in each equilibrium.*

The statement of Lemma 1 follows trivially from the unlimited number of firms. Since entry is costly and the total surplus in the labor market is bounded, some firms do not enter and earn a payoff of zero. Since firms must be indifferent between entering the market and staying out, the payoff of the entrants are also zero in every equilibrium.

Next, we show that a consequence of the substitution assumption (Assumption 1) is that there is negative assortative matching in equilibrium: more productive firms are matched with unskilled workers and less productive firms are matched with skilled workers.

LEMMA 2: *Suppose that a firm with productivity π goes to the L-market in equilibrium. Then, if $\pi' > \pi$, a firm with productivity π' also enters the L-market in that equilibrium.*

The intuition behind the statement of this lemma is central for our theory and can be explained as follows. Since skilled workers generate larger surplus than unskilled ones, there will always be higher demand for skilled workers. In other words, firms face a more severe search friction in the H-market than in the L-market. Since productivity and skill are substitutes, the value added of a skilled worker is higher to a low-productivity firm than to a high-productivity firm. As a consequence, low-productivity firms are more willing to put up with the search friction in the H-market while more productive firms are willing to settle for an unskilled worker but guarantee that they are matched with high probability. It is worth noting the importance of log-submodular production for this result. If production were log-supermodular, firm sorting would be reversed and workers would always prefer to obtain the skill.

PROOF:

Let p_T denote the probability that a firm is matched in the T-market. Then a firm with productivity π is better off going to the L market if and only if $p_L S(L, \pi) \geq p_H S(H, \pi)$. By Assumption 1, this inequality implies that $p_L S(L, \pi')$ $> p_H S(H, \pi')$ whenever $\pi' > \pi$. Therefore, a firm with productivity π' has a higher payoff in the L-market. ∎

An implication of this lemma is that the equilibrium strategy of the firms in the second period can be described by a threshold, π^*. Firms with productivity above π^* enter the L-market and firms with productivity below π^* enter the H-market.

We next establish whether firms or workers will be the short side of each market. In what follows, let μ^* denote the equilibrium measure of type-H workers.

LEMMA 3: *In every equilibrium,*

(*i*) *there are more firms than workers in the H-market, and*

(*ii*) *if* $\mu^* > 0$, *then there are strictly more workers than firms in the L-market.*

PROOF:

To prove (i), recall that there is at least an ε measure of workers in the H-market. If there were fewer firms than workers in the H market, then a firm could enter and achieve a payoff of $E[S(H, \pi)]$. Since $c < E[S(H, \pi)]$, this violates the zero-profit condition (see Lemma 1).

Conditional on being matched, the payoff of an L-worker is $E[S(L, \pi) | \pi \geq \pi^*]$. Similarly, the payoff of an H-worker is $E[S(H, \pi) | \pi \leq \pi^*]$ if she is matched. By Assumption 2, $E[S(L, \pi) | \pi \geq \pi^*] > E[S(H, \pi) | \pi \leq \pi^*]$, so an L-worker is strictly better off conditional on being matched. Note that $\mu^* > 0$ implies that a worker is weakly better off acquiring the skill than remaining unskilled, so it must be the case that an L-worker is matched with a strictly lower probability than an H-worker. An L-worker's probability of being matched must therefore be strictly smaller than one. That is, there are more workers in the L-market than firms. ∎

Assumption 2, which limits how much more productive an H-worker is than an L-worker, is critical to this result. If instead the skill granted a sufficient increase in expected surplus, it would be the H-workers who are better off conditional on being matched, and part (ii) of the lemma would not hold.

Finally we note that the ratio of the unconditional expected surplus for a low-type relative to the unconditional expected surplus for a high-type worker is smaller than the ratio conditional on being matched with the most productive firm.

LEMMA 4: *Assumption* 1 *implies that*

$$\frac{E\left[S(L,\pi)\right]}{E\left[S(H,\pi)\right]} < \frac{S(L,1)}{S(H,1)}.$$

PROOF:

By Assumption 1 it follows that for all $\pi \in [0,1)$:

$$\frac{S(L,\pi)}{S(L,1)} < \frac{S(H,\pi)}{S(H,1)}.$$

Taking expectations of both sides with respect to π yields

$$\frac{E\left[S(L,\pi)\right]}{S(L,1)} < \frac{E\left[S(H,\pi)\right]}{S(H,1)},$$

which is equivalent to the lemma's statement. ∎

III. Results

We now turn to the main results. First, we characterize the unique equilibrium for low entry cost. We show that no worker acquires the skill. Then we turn our attention to higher entry cost. We show that a positive measure of workers acquire the skill, and that both this measure and the fraction of firms in the H-market go to one as entry costs increase, and may remain at one for an interval of high entry costs.

For expositional ease, we define three cost thresholds: $c_L = E[S(L,\pi)]$, $c_M = [S(L,1)/S(H,1)]E[S(H,\pi)]$, and $c_H = E[S(H,\pi)]$. Note that $c_L < c_M \leq c_H$, where the first inequality follows from Lemma 4 and the second one from $S(L,1) \leq S(H,1)$.

A. *Low Entry Cost*

Our main result establishes that, for low entry costs, no worker becomes skilled in equilibrium.

In what follows, let λ^* denote the equilibrium measure of entrant firms and recall that μ^* denotes the equilibrium measure of type-H workers.

THEOREM 1: *If* $c < c_L$, *then* $\mu^* = 0$ *and* $\pi^* = 0$.

This theorem states that when entry costs are so low that a firm would still enter if it knew it would be matched with an L-type worker with probability one, then the unique equilibrium is for all workers and all entrant firms to go to the L-market. If any workers decided to acquire the skill, then the H-market would be overrun with low-quality firms hoping to be matched. To avoid this severe selection problem, workers remain unskilled, despite the fact that they obtain an equal share of the surplus they generate.

We note that workers' incentives are strict in this equilibrium, that is, a worker strictly prefers to remain unskilled. This immediately yields the striking result that workers would be willing to pay a strictly positive amount in order to avoid becoming skilled. That is, workers endowed with the skill would be willing to pay to actively destroy their human capital in order to achieve a match with a higher quality firm.

PROOF:

First, we show that μ^* cannot be strictly positive in an equilibrium. If $\mu^* > 0$, then Lemma 3 implies that there must be strictly more workers than firms in the L-market. A firm could therefore match with an L-worker with probability one, which implies that expected post-entry profits must be at least $E[S(L, \pi)]$. This leads to positive expected profits for $c < c_L$, which contradicts firms' zero-profit condition.

It remains to show that $\mu^* = 0$ and $\pi^* = 0$ is indeed an equilibrium. If a worker deviates and acquires a skill, then she will be matched with the least productive firm. By Assumption 2, $S(H, 0) < E\big[S(L, \pi)\big]$, so this deviation is not profitable. Since there are no workers in the H-market, $\pi^* = 0$ is a best response of the firms since they can only be matched in the L-market. Thus $\mu^* = \pi^* = 0$ is an equilibrium. ∎

Our directed search assumption and nontransferable utility are central to this result. To understand their importance, suppose instead that the labor market is perfectly competitive and workers are compensated through marginal product pricing. Assumption 1 would still imply negative assortative matching and the existence of a cutoff productivity, π^c, above which a firm would hire an unskilled worker. The wages of the skilled and unskilled workers, w_L and w_H, would be determined by the indifference condition of this firm:

$$S(\pi^c, L) - w_L = S(\pi^c, H) - w_H.$$

This condition implies that $w_H > w_L$, and, hence, a worker always prefers to be skilled. In fact, the same conclusion can be drawn about the stable outcome in matching models where utilities are perfectly transferable.

Now suppose that surplus is shared equally, but search is not directed. For example, there is just one market for the workers and the maximum number of matches are created. Then the surplus of a matched worker of type T is $E\big[S(T, \pi)\big]$. Again, workers would strictly prefer to become skilled even at a positive cost. The same argument implies that workers would obtain the skill if they could direct their search toward particular firm types.

B. *Intermediate Entry Cost*

We now turn our attention to the case where the cost of entry is larger than $c_L = E\big[S(L,\pi)\big]$, that is, firms would not enter if all workers were unskilled. The next theorem characterizes the unique equilibrium where the entry cost is larger than c_L but smaller than $c_M = \big[S(L,1)/S(H,1)\big]\, E\big[S(H,\pi)\big]$. For this range of entry costs, there will be a unique equilibrium featuring a positive measure of both worker types. An interior equilibrium (where $\mu^*, \pi^* \in (0,1)$) is defined by the following three constraints:

Worker Indifference.— By Lemma 3, a skilled worker is surely matched and, if $\mu^* < 1$, there are more workers than firms in the L-market. Note that if λ^* is the measure of entering firms, and π^* is the productivity cutoff above which a firm enters the L-market, then $\lambda^*(1-\pi^*)$ is the measure of firms in the L-market. Therefore, the probability that an unskilled worker is matched is $\lambda^*(1-\pi^*)/(1-\mu^*)$ and the indifference condition of a worker is

$$(1) \qquad E\big[S(H,\pi)|\pi \le \pi^*\big] = \frac{\lambda^*(1-\pi^*)}{1-\mu^*} E\big[S(L,\pi)|\pi \ge \pi^*\big].$$

Firm Indifference.— Again, by Lemma 3, a firm is matched for sure in the L-market. The probability that a firm is matched in the H-market is $\mu^*/(\lambda^*\pi^*)$. Therefore, a firm with cutoff productivity π^* is indifferent between the two markets if

$$(2) \qquad \frac{\mu^*}{\lambda^*\pi^*} S(H,\pi^*) = S(L,\pi^*).$$

Zero-Profit Condition.— By Lemma 1, the payoff of the entering firm is zero. This constraint is captured by the following condition:

$$(3) \qquad c = \left(\frac{\mu^*}{\lambda^*\pi^*}\right)\pi^* E\big[S(H,\pi)|\pi \le \pi^*\big] + (1-\pi^*)\, E\big[S(L,\pi)|\pi \ge \pi^*\big].$$

The left-hand side is the cost of entry. The right-hand side decomposes the post-entry payoff of the firm depending on whether its productivity is smaller or larger than π^*. If $\pi \le \pi^*$, which happens with probability π^*, the firm enters the H-market and is matched with probability $\mu^*/(\lambda^*\pi^*)$. This explains the first term on the right-hand side. If $\pi > \pi^*$, which happens with probability $(1-\pi^*)$, the firm enters the L-market and is surely matched. This explains the second term.

We now characterize the unique equilibrium for large entry costs. In what follows, $\mu^*(c)$, $\lambda^*(c)$, and $\pi^*(c)$ denote the equilibrium values of the fraction of skilled workers, the measure entering firms, and the cutoff productivity, respectively, if the entry cost is c.

THEOREM 2: *Suppose that* $c \in (c_L, c_M)$. *Then, there is a unique equilibrium, where* $\mu^*(c), \pi^*(c) \in (0,1)$. *In addition,* $\mu^*(c)$ *and* $\lambda^*(c)$ *are continuous in c and*

(*i*) $\mu^*(c), \pi^*(c) \to 0$, $\lambda^*(c) \to S(H,0)/E[S(L,\pi)]$ as $c \to c_L$.

(*ii*) $\mu^*(c), \pi^*(c) \to 1$, $\lambda^*(c) \to S(H,1)/S(L,1)$ as $c \to c_M$.

This theorem states that both the measure of skilled workers, $\mu^*(c)$, and the firms' cutoff, $\pi^*(c)$, converge to zero at c_L and to one at c_M, and are continuous in between. In other words, as the entry cost becomes larger, more and more workers acquire the skill and more and more firms search for them. In fact, there are also more entrant firms at $c = c_M$ than at $c = c_L$. The theorem does not claim that these functions are pointwise monotonic, although one could provide additional technical assumptions to guarantee monotonicity, for example as in Section IIID.

It is worth pointing out that there is a discontinuity in λ^* at $c = c_L = E[S(L,\pi)]$. Recall that Theorem 1 implies that λ^* is c_L/c as long as $c < c_L$. As c converges to c_L from below, the measure of entrant firms goes to one. There is an indeterminacy at $c = c_L$. At this cost, the post-entry payoff of each firm is zero if each worker is unskilled and firms do not face search frictions. As a consequence, λ^* can be anything between zero and one. Part (i) of the theorem states that when c becomes a bit higher than c_L, the measure of entrants is again uniquely pinned down and it is $S(H,0)/E[S(L,\pi)]$. By Assumption 2, this is smaller than one, that is, there is a discrete drop in λ^* at $c = c_L$. On the other hand, part (i) also implies that the other variables of our interest, μ^* and π^*, are continuous at c_L.

PROOF:

First, we argue that $\mu^* \in (0,1)$. Since c is larger than $c_L = E[S(L,\pi)]$, the entering firms would make a negative profit if $\mu^* = 0$. If each worker were skilled ($\mu^* = 1$), all entering firms would go to the H-market, and the zero-profit condition of the firms would imply that the measure of entering firms is $c/E[S(H,\pi)]$. The post entry payoff of a firm with $\pi = 1$ would be

$$S(H,1)\,\frac{E[S(H,\pi)]}{c}.$$

If this firm were matched with an L-worker, its payoff would be $S(L,1)$. Since $c < c_M = [S(L,1)/S(H,1)]\,E[S(H,\pi)]$, the firm with $\pi = 1$ would strictly prefer to be matched with an L-worker for sure. Hence, a worker would have incentive to deviate and remain unskilled.

Note that both (2) and (3) depend on μ^* and λ^* only through the ratio μ^*/λ^*. Let x^* denote μ^*/λ^*.

Next, we show that for all $x^* \in [0, S(L,1)/S(H,1)]$ there is a unique $\pi \in [0,1]$ which satisfies (2), that is,

$$x^* = \frac{\pi S(L,\pi)}{S(H,\pi)}. \tag{4}$$

Note that the right-hand side is zero at $\pi = 0$ and $S(L,1)/S(H,1)$ at $\pi = 1$. In addition, the right-hand side is continuous and strictly increasing in π (by Assumption 1). Hence, by the Intermediate Value Theorem, there is indeed a unique π which solves (4). We denote the solution by $\pi(x^*)$. Notice that $\pi(0) = 0$, $\pi\big(S(L,1)/S(H,1)\big) = 1$ and the function $\pi(\cdot)$ is continuous and strictly increasing. Note that $\pi(x^*)$ is the optimal threshold for firm sorting.

Third, we show that for each $c \in (c_L, c_M)$, there is a unique $x^* \in \big[0, S(L,1)/S(H,1)\big]$ such that $\big(x^*, \pi(x^*)\big)$ satisfies the zero-profit condition, (3), that is,

$$c = x^* E\big[S(H,\pi) | \pi \leq \pi(x^*)\big] + \big(1 - \pi(x^*)\big) E\big[S(L,\pi) | \pi \geq \pi(x^*)\big]. \tag{5}$$

We now observe that the right-hand side of (5) is $E\big[S(L,\pi)\big]$ when evaluated at $x^* = 0$, while at $x^* = S(L,1)/S(H,1)$ it is $\big[S(L,1)/S(H,1)\big]\, E\big[S(H,\pi)\big]$. Next, we argue that the right-hand side is strictly increasing in x^*. Suppose that $x_1^* < x_2^*$. Then,

$$\begin{aligned} &x_1^* E\big[S(H,\pi)|\pi \leq \pi(x_1^*)\big] + \big(1 - \pi(x_1^*)\big) E\big[S(L,\pi)|\pi \geq \pi(x_1^*)\big] \\ &\leq x_2^* E\big[S(H,\pi)|\pi \leq \pi(x_1^*)\big] + \big(1 - \pi(x_1^*)\big) E\big[S(L,\pi)|\pi \geq \pi(x_1^*)\big]. \end{aligned}$$

The right-hand side would be the post-entry payoff of a firm who enters the L-market if and only if $\pi \geq \pi(x_1^*)$, but $\mu/\lambda = x_2^*$. Since the optimal threshold is $\pi(x_2^*)$ if $\mu/\lambda = x_2^*$, we conclude that

$$\begin{aligned} &x_2^* E\big[S(H,\pi)|\pi \leq \pi(x_1^*)\big] + \big(1 - \pi(x_1^*)\big) E\big[S(L,\pi)|\pi \geq \pi(x_1^*)\big] \\ &\leq x_2^* E\big[S(H,\pi)|\pi \leq \pi(x_2^*)\big] + \big(1 - \pi(x_2^*)\big) E\big[S(L,\pi)|\pi \geq \pi(x_2^*)\big]. \end{aligned}$$

The previous two inequalities imply that the right-hand side of (5) is increasing in x^*. Of course, the right-hand side of (5) is also continuous in x^*. Therefore, by the Intermediate Value Theorem, for each $c \in (c_L, c_M)$ there is indeed a unique x^* which solves (5).

So far, we proved that for each $c \in (c_L, c_M)$, there is a unique $x^* = \mu^*/\lambda^*$, which satisfies (2) and (3). It remains to pin down μ^* and λ^*. The indifference condition of a worker, (1), can be written as

$$\frac{1}{\lambda^*} = \frac{\big(1 - \pi(x^*)\big) E\big[S(L,\pi)|\pi \geq \pi(x^*)\big]}{E\big[S(H,\pi)|\pi \leq \pi(x^*)\big]} + x^*, \tag{6}$$

which defines λ^* as a function of x^*. Then we can obtain μ^*, since $\mu^* = \lambda^* x^*$. It remains to show that $\mu^* \in [0,1]$, $\mu^* \leq \lambda^* \pi^*$ and $\lambda^*(1 - \pi^*) \leq 1 - \mu^*$. By (4), $\mu^* \leq \lambda^* \pi^*$ is satisfied. By (6), $\lambda^*(1 - \pi^*) \leq 1 - \mu^*$. Finally, $\lambda^*(1 - \pi^*) \leq 1 - \mu^*$ implies that $\mu^* \leq 1$ and $\mu^* = \lambda^* x^*$ implies that $\mu^* \geq 0$.

To prove part (i), suppose that c goes to $E\big[S(L,\pi)\big]$. Then, by (5), x^* converges to zero. This implies that μ^* also converges to zero because $\mu^* = \lambda^* x^*$. As we pointed out above, $\pi(x^*)$ converges to zero as x^* goes to zero. Then, by (6), λ^* converges $S(H,0)/E\big[S(L,\pi)\big]$.

To prove part (ii), suppose that c goes to $\big[S(L,1)/S(H,1)\big]\, E\big[S(H,\pi)\big]$. Then, by (5), x^* converges to $S(L,1)/S(H,1)$. As we pointed out, $\pi^*(x^*)$ converges to one. Then, by (6), μ^* converges to one. Plugging $\pi^*(x^*) = 1$ and $x^* = 1/\lambda^*$ into (5) yields that λ^* converges to $S(H,1)/S(L,1)$. ∎

C. *Large Entry Cost*

Finally, we characterize the unique equilibrium for the case of large entry cost.

THEOREM 3: *Suppose that $c \in (c_M, c_H)$. Then there exists a unique equilibrium in which $\mu^* = \pi^* = 1$ and $\lambda^* = E\big[S(H,\pi)\big]/c$.*

This theorem states that if the entry cost is large enough then each worker becomes skilled and all firms search for these workers. The fraction of entering firms is determined by the zero-profit condition.

PROOF:

First, we show that the proposed profile $(\mu^*, \pi^*, \lambda^*)$ is indeed an equilibrium. In fact, we show that this is the unique equilibrium in which $\mu^* = 1$. If $\mu^* = 1$, then a firm's expected post-entry payoff is $E\big[S(H,\pi)\big]/\lambda^*$. By Lemma 1, $\lambda^* = E\big[S(H,\pi)\big]/c$. The payoff of a firm with $\pi = 1$ is

$$\frac{S(H,1)}{\lambda^*} = \frac{S(H,1)\,c}{E\big(\big[S(H,\pi)\big]\big)} > S(L,1),$$

where the equality follows from $\lambda^* = E\big[S(H,\pi)\big]/c$ and the inequality follows from $c > c_M = \big[S(L,1)/S(H,1)\big]\, E\big[S(H,\pi)\big]$. Notice that the right-hand side would be payoff of the firm if it enters the L-market and is matched with a worker for sure. So, the firm with $\pi = 1$ strictly prefers to enter the H-market. Then Lemma 2 implies that no firm has an incentive to deviate in the second stage. Since each firm enters the H-market, workers strictly prefer to become skilled. Finally, the entry decisions of the firms are optimal because they make zero profit.

It remains to show that there is no equilibrium where $\mu^* < 1$. If $\mu^* = 0$, then the firm's post-entry payoff is at most $E\big[S(L,\pi)\big] < c$, so the firms would make a negative profit. We argue that there is no interior equilibrium, that is, with $\mu^* \in (0,1)$. In the proof of Theorem 2, we showed that (5) must hold in any interior equilibrium. We have also established that the right-hand side is smaller than $\big[S(L,1)/S(H,1)\big]\, E\big[S(H,\pi)\big]$. As a consequence, (5) cannot hold if $c > \big[S(L,1)/S(H,1)\big]\, E\big[S(H,\pi)\big]$. ∎

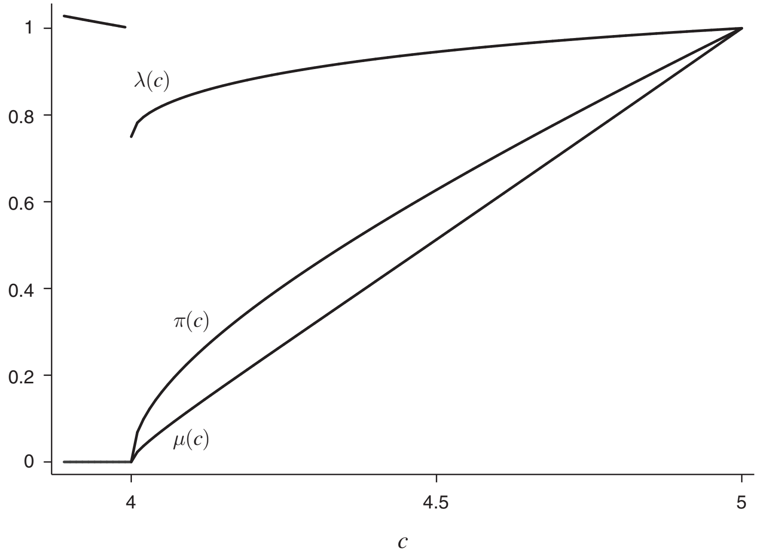

FIGURE 2. EXAMPLE

D. *Example*

To illustrate the intuition behind our results, we briefly discuss an example. Consider $S(T, \pi) = T + (\alpha - T)\pi$, with $L = 1$, $H = 3$, and $\alpha = 7$. Then $c_L = 4$, $c_M = 5$, and $c_H = 5$.[7] The key equilibrium parameters as a function of c are shown in Figure 2.

For entry costs $c > c_H$, it is not profitable for firms to enter. At the opposite extreme, by Theorem 1 the unique equilibrium for entry costs $c < c_L$ is for no worker to obtain the skill and all firms to enter the L-market, while $\lambda^*(c) > 1$ by firms' zero-profit condition.

By Theorem 2, the unique equilibrium for entry costs $c_L < c < c_M$ features a positive measure of workers choosing to obtain the skill, μ^*, and a positive productivity threshold below which firms enter the H-market, π^*. Perhaps counterintuitively, the measure of entrants, λ^*, is increasing in the entry cost c, as the ex ante probability of being matched with an H-type worker increases. In accordance with Lemma 3, however, there are always more workers than firms in the L-market and workers than firms in the H-market. As the cost approaches $c_M = c_H$, all three quantities approach one. Moreover, in this example, $\lambda^*(c)$ and $\mu^*(c)$ are not only continuous in c for intermediate entry costs, but are also monotonically increasing.

IV. Welfare

Although Theorems 1, 2, and 3 establish the relationship between worker skills and firm entry-costs, the impact on overall welfare is potentially more subtle. This

[7] Note that our model only assumes $S(L, \pi) < S(H, \pi)$ on the open interval $[0, 1)$, and $S(L, 1) = S(H, 1) = \alpha$ in this example. Consequently $c_M = c_H$. Whenever $S(L, 1) < S(H, 1)$ is additionally assumed, $c_M < c_H$.

is because, even though more workers obtain the skill in the intermediate-cost case than in the low-cost case, some of them will remain unmatched in equilibrium. In the low-cost case, workers did not obtain the skill, but any search frictions were borne entirely by firms.

THEOREM 4: *Total surplus is strictly higher for* $c \in (c_M, c_H)$ *than for* $c < c_M$*. Furthermore, there exist* $\delta_1, \delta_2 > 0$*, such that*

(*i*) *Total surplus is strictly higher for* $c < c_L$ *than for* $c \in (c_L, c_L + \delta_1)$.

(*ii*) *Total surplus is strictly lower for* $c < c_L$ *than for* $c \in (c_M - \delta_2, c_M)$.

PROOF:

First, note that the firms' zero-profit condition means that it is sufficient to focus only on total worker surplus.

For low costs, i.e., $c < c_L = E[S(L, \pi)]$, by Theorem 1 no worker obtains the skill. Furthermore, firms' entry decision requires that $\lambda = E[S(L, \pi)]/c > 1$, so that workers are matched with certainty. Thus worker surplus, and hence total surplus, is $E[S(L, \pi)]$.

For high costs, i.e., $c \in (c_M, c_H)$, by Theorem 3 all workers obtain the skill. Furthermore, firms' entry decision requires that $\lambda = E[S(H, \pi)]/c > 1$, so that workers are matched with certainty. Thus, worker surplus and, hence, total surplus, is $E[S(H, \pi)]$. Since $S(H, \pi) > S(L, \pi)$ for all $\pi \in [0, 1)$, $E[S(H, \pi)] > E[S(L, \pi)]$.

For intermediate costs, i.e., $c \in (c_L, c_M)$, by Theorem 2 there is an interior equilibrium. The worker indifference condition, (1), implies that worker surplus can be given by the surplus of H-workers, and is thus $E\big[S(H, \pi) | \pi \leq \pi^*\big] < E\big[S(H, \pi)\big]$.

If c goes to $E[S(L, \pi)]$, π^* converges to zero by Theorem 2 and hence total surplus goes to $S(H, 0)$, which is strictly smaller than $E[S(L, \pi) | \pi \geq 0] = E[S(L, \pi)]$ by Assumption 2. Then (i) is implied by continuity of S, $\pi^*(c)$ and the expectations operator.

If c goes to $[S(L, 1)/S(H, 1)E[S(H, \pi)]$, π^* converges to one by Theorem 2 and hence total surplus goes to $E[S(H, \pi)]$. Then (ii) is implied by continuity of S, $\pi^*(c)$ and the expectations operator. ∎

Theorem 4 shows that total surplus is maximized in the high-cost regime. Intuitively, all workers obtain the skill in this case and are matched with probability one, and so worker surplus is maximized. Perhaps surprisingly, total surplus is not minimized in the low-cost case, despite the fact that the measure of skilled workers is also minimized. There is a range of costs in the left tail of the intermediate-cost range where welfare is strictly lower.[8] In this range, the fact that some workers remain unmatched in equilibrium dominates the effect of increased skill acquisition. Finally, we note that in the intermediate-cost case, total surplus is continuous and is monotonically increasing if and only if $\pi^*(c)$ is monotonic.

[8] In the numerical example of Section IIID, welfare is lower for $c \in (4, 4.35)$ than for $c < 4$.

V. Conclusion

We have shown an alternative model of inefficient investment in skills based on an assumption of substitutability of quality rather than signaling. Instead of overinvestment in unproductive skills to signal one's quality to potential matches, we find underinvestment in productive skills in order to avoid low-quality matches. Substitutability of quality between the two halves of a match creates a selection problem, wherein only the lowest-quality firms are willing to enter the congested H-type market in search of a scarce H-type worker and potentially remain unmatched. This congestion is exacerbated as firm entry costs fall—that is, as the market becomes more competitive. Our main result is to show that when entry costs are sufficiently low, the selection problem becomes so severe that it shuts down the H-market entirely and all workers remain unskilled.

While we have focused on the labor market application, the results in this paper are of course applicable to other instances of directed search. Suppose, for example, that the surplus from marriage features substitutability of spousal quality. Someone choosing between pursuing an MBA and an economics PhD (which, given subsequent earnings profiles, may be considered to reduce his human capital) may actually choose the latter option—knowing that only a high-quality mate would consider settling for an economics professor rather than competing over his high-flying financier counterparts.

REFERENCES

Atakan, Alp E. 2006. "Assortative Matching with Explicit Search Costs." *Econometrica* 74 (3): 667–80.

Becker, Gary S. 1973. "A Theory of Marriage: Part I." *Journal of Political Economy* 81 (4): 813–46.

Becker, Gary S. 1974. "A Theory of Marriage: Part II." *Journal of Political Economy* 82 (2): S11–26.

Goldin, Claudia, and Lawrence F. Katz. 1998. "The Origins of Technology-Skill Complementarity." *Quarterly Journal of Economics* 113 (3): 693–732.

Krusell, Per, Lee E. Ohanian, José-Victor Rios-Rull, and Giovanni L. Violante. 2000. "Capital-Skill Complementarity and Inequality: A Macroeconomic Analysis." *Econometrica* 68 (5): 1029–53.

Mailath, George J., and Andrew Postlewaite. 2006. "Social Assets." *International Economic Review* 47 (4): 1057–91.

Mortensen, Dale T., and Christopher A. Pissarides. 1994. "Job Creation and Job Destruction in the Theory of Unemployment." *Review of Economic Studies* 61 (3): 397–415.

Postel-Vinay, Fabien, and Jean-Marc Robin. 2002. "Equilibrium Wage Dispersion with Worker and Employer Heterogeneity." *Econometrica* 70 (6): 2295–2350.

Shi, Shouyong. 2002. "A Directed Search Model of Inequality with Heterogeneous Skills and Skill-Biased Technology." *Review of Economic Studies* 69 (2): 467–91.

Shimer, Robert, and Lones Smith. 2000. "Assortative Matching and Search." *Econometrica* 68 (2): 343–69.

Smith, Lones. 2006. "The Marriage Model with Search Frictions." *Journal of Political Economy* 114 (6): 1124–44.

Spence, Michael. 1973. "Job Market Signaling." *Quarterly Journal of Economics* 87 (3): 355–74.

Zahavi, Amotz. 1975. "Mate Selection—A Selection for a Handicap." *Journal of Theoretical Biology* 53 (1): 205–14.