Matthew Gentry, Tong Li, Jingfeng Lu

Existence of monotone equilibrium in first price auctions with private risk aversion and private initial wealth

Article (Accepted version) (Refereed)

Original citation:
Gentry, Matthew, Li, Tong and Lu, Jingfeng (2015) *Existence of monotone equilibrium in first price auctions with private risk aversion and private initial wealth.* Games and Economic Behavior, 94. pp. 214-221. ISSN 0899-8256

DOI: 10.1016/j.geb.2015.07.003

Existence of Monotone Equilibrium in First Price Auctions with Private Risk Aversion and Private Initial Wealth*

Matthew Gentry[†] Tong Li[‡] Jingfeng Lu[§]

July 2015

Abstract

In this paper, we study the existence of monotone equilibrium in first price auctions where bidders have a three-dimensional private type, i.e. their private values, degrees of risk aversion and initial wealth. Bidders' utility functions belong to the class of constant relative risk aversion (CRRA) or constant absolute risk aversion (CARA). The bidders' types are independent across bidders, while a bidder's private value, initial wealth and degree of risk aversion are allowed to be correlated. We show that a monotone equilibrium always exists in a general setting allowing for asymmetric bidders. Moreover, with symmetric bidders, a symmetric monotone equilibrium strategy must exist. A bidder's equilibrium strategy increases with bidders' private values and degrees of risk aversion. When bidders have CRRA utility, equilibrium bids decrease with initial wealth; when bidders have CARA utility, equilibrium bids are invariant to initial wealth.

JEL Nos: C7, D7

Key Words: Constant absolute risk aversion (CARA), Constant relative risk aversion (CRRA), First price auction, Initial wealth, Monotone equilibrium.

1 Introduction

Risk aversion is a core notion for analyzing economic agents' decisions under uncertainty. Since Pratt (1964)'s formalization of risk aversion, a rich literature has been devoted to analyzing behavior of risk averse agents in a variety of situations. In particular, a large body of theoretical, experimental and empirical research has demonstrated that bidders' risk aversion is an important determinant of their bidding behavior in auctions. As is well known, risk aversion leads bidders to bid more aggressively in first-price auctions relative to other standard formats, helping to rationalize the extensively observed "overbidding" relative to the risk-neutral Bayesian Nash equilibrium in experiments (e.g. Cox et al. (1988) and Goeree, Holt and Palfrey (2002) among others). Adopting a structural approach, Bajari and Hortaçsu (2005) show that the risk aversion model provides the best fit to experimental data among several competing models. Abundant

*We are grateful to the editors and two anonymous reviewers for very insightful comments and suggestions that significantly improved the quality of the paper. We thank Ali Khan, Bin Liu and Yeneng Sun for their very helpful discussions. All remaining errors are our own.

[†]Matthew Gentry: Department of Economics, London School of Economics, Houghton Street, London, UK WC2A 2AE. Email: m.l.gentry@lse.ac.edu.

[‡]Tong Li: Department of Economics, Vanderbilt University, Nashville, TN 37235-1819. Email: tong.li@vanderbilt.edu.

[§]Jingfeng Lu: Department of Economics, National University of Singapore, Singapore 117570. Email: ecsljf@nus.edu.sg.

evidence for bidders' risk aversion has also been identified in field auction data, including Baldwin (1995), Athey and Levin (2001), Perrigne (2003), Lu and Perrigne (2008), Campo, Guerre, Perrigne and Vuong (2011), Campo (2012), Fang and Tang (2014), and Li, Lu and Zhao (2015) among others.

Two commonly adopted measures of risk aversion are the Arrow-Pratt coefficient of absolute risk aversion and the Arrow-Pratt-De Finetti coefficient of relative risk aversion. A Bernoulli utility function that generates a constant coefficient of absolute risk aversion is called a constant absolute risk aversion (CARA) utility function, while a utility function that generates a constant coefficient of relative risk aversion is called a constant relative risk aversion (CRRA) utility function. These two classes of utility functions are the preeminent tools by which to model risk aversion in a wide range of settings, including auctions in particular.

It has long been recognized in the auction literature that bidders' risk preferences can be heterogeneous and that such heterogeneous risk preferences can be bidders' private information. Cox, Roberson, and Smith (1982) and Cox, Smith and Walker (1982) provide the first characterizations of closed form pure strategy equilibria in first-price auctions when bidders' degrees of risk aversion are modeled as part of their private types. Specifically, they adopt a constant relative risk aversion (CRRA) framework and assume the bidders' private values follow a uniform distribution. Cox, Smith and Walker (1988) further find that a CRRA model with private degrees of risk aversion fits well the "overbidding" data generated by their first price auction experiments. In these studies, however, the existence of a pure strategy equilibrium is not fully established, since the hypothetical bidding strategy is only partially identified and it is thus difficult to verify its monotonicity over the whole ranges of degrees of risk aversion and values. Van Boening, Rassenti and Smith (1998) further explore numerically solving the differential equations that a hypothetical equilibrium bidding strategy must satisfy. Their simulation results verify the existence of isotone equilibrium for a specific CRRA model with four bidders whose values follow a uniform distribution and their degrees of risk aversion follow a beta distribution on $[0, 1]$. The bidding strategies obtained increase in both bidders' degrees of risk aversion and their values. In all these studies, the initial wealth of bidders is fixed at zero.

Assuming the existence of pure strategy equilibrium in the model of Cox, Smith and Walker (1988) and Van Boening, Rassenti and Smith (1998), Pevnitskaya (2001) extends Levin and Smith (1994) by allowing for bidders who hold private information on their degrees of risk aversion before making their entry decisions. Potential bidders who observe their private degrees of risk aversion must decide simultaneously whether to incur an entry cost to discover their values before bidding in a first price auction. Pevnitskaya (2001) finds a self-selection effect that relatively more risk tolerant bidders choose to enter in a symmetric entry equilibrium, and this effect increases with the entry cost. This self-selection effect leads to less aggressive bids than those from an exogenous pool of bidders. The experimental study conducted by Palfrey and Pevnitskaya (2008) confirms these theoretical predictions.

Nevertheless, even with fixed initial wealth for bidders, the existence of monotone equilibrium in first price auction remains an open question in a general CRRA model with an arbitrary set of bidders whose joint distributions of values and degrees of risk aversion are left unrestricted. Verifying the existence of monotone equilibrium for a general model by checking the properties of the solution of the differential equations that Cox, Smith and Walker (1988) and Van Boening, Rassenti and Smith (1998) identify is not easy to implement. It is in general infeasible to study the properties of the hypothetical strategy for a general model without solving these equations analytically. If instead the bidders' utility functions are assumed to have a CARA form, this procedure would face new challenge since the linear segment of the bidding strategy identified in the existing literature on the CRRA case no longer holds. More technical complications could arise if correlation between bidders' value and degree of risk aversion is introduced into the CRRA or CARA model.

While values and private risk preferences have received relatively more attention in the literature, in settings with risk aversion it is also natural to interpret bidders' initial wealth levels as being private information. With risk averse bidders, variation in initial wealth would in general affect bidding behavior, but little is known about properties of equilibrium when wealth is private. In this study, we establish the existence of isotone equilibrium in first price auctions when bidders have three dimensional private types, i.e. their initial wealth levels, private values and degrees of risk aversion. To our knowledge, existence in this setting has not been previously studied in general.

To avoid the difficulties of solving differential equations that a hypothetical equilibrium strategy must satisfy, we follow an alternative approach along the lines of Reny (1999, 2011), Athey (2001) and McAdams (2003) to address the existence of monotone equilibrium in first price auctions where bidders have three-dimensional private information about their values, initial wealth and degrees of risk aversion. Our approach accommodates both CRRA and CARA specifications with an arbitrary number of potentially asymmetric bidders. We assume that private information – i.e. bidders' values, degree of risk aversion, and levels of initial wealth – is drawn independently across bidders, but leaves the joint distribution of values, initial wealth, and risk aversion for each bidder unrestricted. We find that a pure strategy equilibrium that is monotone in all three dimensions of private information must exist. Moreover, a symmetric pure strategy monotone equilibrium must exist for symmetric bidders. Our results thus complement existing studies on private risk aversion and strengthen the foundation for future theoretical, experimental and empirical studies on first price auctions when bidders are endowed with private information on their risk attitudes and/or initial wealth.

The rest of the paper is organized as follows. Section 2 describes the model and outlines notation used to unify discussion of the CARA and CRRA cases. Section 3 establishes existence of monotone equilibria in the general model with asymmetric bidders. Section 4 specializes this to existence of symmetric monotone equilibria in settings with symmetric bidders. Section 5 concludes. Some technical proofs are relegated to the online appendix.

2 First Price Auctions with Private Values, Wealth and Risk Aversion

In this paper, we consider a sealed-bid first price auction of a single item. A reservation price r is imposed. Ties in winning bids are broken randomly. There are N potentially asymmetric bidders who are risk averse. Each bidder is endowed with three-dimensional private information about his value, initial wealth and degree of risk aversion.

Each bidder i enters the auction with private initial wealth w_i, assigns private value v_i to the object being sold, and has preferences over final wealth realizations x described by a concave Bernoulli utility function $u(x; a_i)$, where a_i is a scalar parameter indexing bidder i's private risk preferences. While the notation $u(x; a_i)$ is adopted to unify discussion, in practice we focus on cases where the utility function $u(x; a_i)$ takes either CARA or CRRA form. This leads to two distinct specifications of the model, which we refer to as *private CRRA preferences* and *private CARA preferences*, respectively.

Private CRRA preferences When bidders have *private CRRA preferences*, we take a_i to be bidder i's level of constant relative risk aversion, assume that a_i is supported on a closed interval $\mathcal{A}_i = [\underline{a}_i, \bar{a}_i] \subset [0, \bar{a}]$ with $\bar{a} < 1$, and define $u(\cdot; a_i)$ as follows:

$$u(x; a_i) \equiv \begin{cases} \frac{x^{1-a_i}}{1-a_i} & \text{if } x \geq 0, \\ x & \text{if } x < 0. \end{cases}$$

The condition $a_i \leq \bar{a} < 1$ guarantees that $u(\cdot; a_i)$ exists and is continuous on $\mathbb{R}^+$. The extension of $u(\cdot; a_i)$ to negative wealth values is made for technical convenience but is otherwise irrelevant for our results. In particular, since in equilibrium bidders will never bid above their values, any equilibrium arising when $u(\cdot; a_i)$ is defined as above is also an equilibrium when $u(\cdot; a_i)$ is restricted to $\mathbb{R}^+$. Furthermore, the form of the extension is arbitrary; so long as $u(\cdot; a_i)$ is continuous and increasing for all $a_i \in \mathcal{A}_i$, any other extension to $x < 0$ would work equally well.

Private CARA preferences When bidders have *private CARA preferences*, we take a_i to be bidder i's level of constant risk aversion, assume that a_i is supported on a closed interval $\mathcal{A}_i = [\underline{a}_i, \bar{a}_i] \subset \mathbb{R}^+$, and define $u(\cdot; a_i)$ as follows:

$$u(x; a_i) = \begin{cases} \frac{1-\exp(-a_i x)}{a_i}, & a_i > 0 \\ x, & a_i = 0. \end{cases}$$

In this case continuity of $u(\cdot; a_i)$ on $\mathbb{R}$ follows by construction, therefore no extension is required.

Distribution of private information Bidder i's private type includes his value v_i, initial wealth w_i and degree of constant risk aversion a_i. Each bidder i draws his type vector (v_i, a_i, w_i) from a joint distribution described by cumulative distribution function $F_i(v, a, w)$, with types drawn independently across bidders. We assume the distribution F_i is supported on $\mathcal{V}_i \times \mathcal{A}_i \times \mathcal{W}_i$ and admits a bounded density $f_i(v, a, w)$ everywhere positive on this support, where $\mathcal{V}_i = [\underline{v}_i, \bar{v}_i]$ with $\underline{v}_i \geq 0$, $\mathcal{W}_i = [0, \bar{w}_i]$, and $\mathcal{A}_i$ is defined as above depending on the class of risk preferences considered. Type draws (v_i, a_i, w_i) are private information of each bidder, but distributions $F_1, ..., F_N$ are common knowledge to all players. The reservation price r falls in $[\underline{v}, \overline{v}]$ where $\underline{v} = \min_i \underline{v}_i$ and $\overline{v} = \max_i \overline{v}_i$, with ties broken randomly across relevant players.

Actions, strategies, payoffs, equilibria In developing our results, we employ the following (standard) definitions:

- An *action* for player i is a bid $b_i \in \mathbb{R}$. Let $\mathcal{B}_i$ denote the set of feasible equilibrium bids for bidder i. Without loss of generality, we can take $\mathcal{B}_i = \mathcal{B} = [0, \bar{v}]$. Let $\mathcal{B}^N$=$[0, \bar{v}]^N$.
- We denote the *type space for* i by $\mathcal{T}_i \equiv \mathcal{V}_i \times \mathcal{A}_i \times \mathcal{W}_i$, the *type space* for the game by $\mathcal{T} \equiv \times_i \mathcal{T}_i$. Let $\mathbf{F} = \times_i F_i$ be the measure over $\mathcal{T}$ induced by primitive distributions $F_1, ..., F_N$.

- A *pure strategy* for player i is a map $s_i : \mathcal{T}_i \to \mathcal{B}_i$, and a *pure strategy profile* is a vector $\mathbf{s} = (s_1, ..., s_N)$ which specifies a strategy for each player. Let $\mathcal{S}_i$ denote the set of pure strategies for player i and $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_N$ denote the set of pure strategy profiles.

- *Payoffs* for player i are described by an ex ante expected utility function $\pi_i : \mathcal{S} \to \mathbb{R}$ defined as follows:

$$\begin{aligned}\pi_i(\mathbf{s}) \quad = \quad & E_{(V_i,A_i,W_i)}\big\{u(W_i; A_i) \cdot [1 - p_i(s_i(V_i, A_i, W_i); \mathbf{s}_{-i})] \\ & + u(W_i + V_i - s_i(V_i, A_i, W_i); A_i) \cdot p_i(s_i(V_i, A_i, W_i); \mathbf{s}_{-i})\big\},\end{aligned}$$

  where $p_i(b; \mathbf{s}_{-i})$ is the probability that bid b leads player i to win the auction when rivals draw from distribution profile $\mathbf{F}_{-i}$ and play according to strategy profile $\mathbf{s}_{-i}$. Let $\pi(\mathbf{s}) = (\pi_1(\mathbf{s}), ..., \pi_N(\mathbf{s}))$ be the ex ante *vector payoff function* for all players in the auction game. Note that with private CRRA preferences the condition $\bar{a} < 1$ ensures that $u(\cdot; \cdot)$ is bounded.

- A *pure strategy equilibrium* is a strategy profile $\mathbf{s}^* \in \mathcal{S}$ such that s_i^* is a best response to $\mathbf{s}^*_{-i}$ for all i. A *restricted pure strategy equilibrium* is a class of pure strategies $\mathcal{S}^C \subset \mathcal{S}$ plus a strategy profile $\mathbf{s}^* \in \mathcal{S}^C$ such that for each i,

$$\pi_i(s_i^*; \mathbf{s}^*_{-i}) = \sup_{s_i \in \mathcal{S}_i^C} \pi_i(s_i; \mathbf{s}^*_{-i}).$$

With these definitions established, we now turn to our main focus in this paper: existence of monotone equilibria when bidders have private values, wealth, and risk preferences.

3 Existence of Monotone Equilibrium under Private CRRA Preferences

We first establish that a monotone pure strategy equilibrium exists when bidders have private CRRA preferences. In developing the argument, we rely heavily on the following monotonicity result: no matter what strategies i's rivals play, i's best response bid must be increasing in v_i, increasing in a_i, and decreasing in w_i. We formalize this observation as follows:

Lemma 1. *Suppose that bidders have private CRRA preferences. Let $\succeq_R$ be the partial order on $\mathcal{T}_i, \forall i$ defined as follows: for all (v_i', a_i', w_i'), $(v_i'', a_i'', w_i'') \in \mathcal{T}_i$,*

$$(v_i'', a_i'', w_i'') \succeq_R (v_i', a_i', w_i') \quad \textit{iff} \quad v_i'' \geq v_i', a_i'' \geq a_i', w_i'' \leq w_i'.$$

Consider any bidder i, any action space $\mathcal{B}_i \subset [0, \bar{v}]$ and any rival strategy profile $\mathbf{s}_{-i}$ defined on any action space $\mathcal{B}_{-i} \subset [0, \bar{v}]^{N-1}$. Let (v_i', a_i', w_i'), $(v_i'', a_i'', w_i'') \in \mathcal{T}_i$ be any type realizations such that $(v_i'', a_i'', w_i'') \succeq_R (v_i', a_i', w_i')$, and let b_i', b_i'' be any best responses by i to $\mathbf{s}_{-i}$ at types (v_i', a_i', w_i'), (v_i'', a_i'', w_i'') respectively. Then $b_i'' \geq b_i'$.

Proof. See online Appendix. ∎

The implications of Lemma 1 are three-fold: Firstly, it implies that any equilibrium will be in monotone strategies, hence will involve pure strategies almost everywhere. Secondly, there is no non-monotone pure strategy equilibrium. Thirdly, in any monotone pure strategy equilibrium, the bid must increase with the value and degree of risk aversion, and decrease with the wealth.

It remains to show, however, that such an equilibrium exists. Our basic proof strategy proceeds in two steps. We first restrict the space of feasible bids to be discrete, noting that in this case existence follows immediately from Reny (2011). Alternatively, one can invoke Theorem 4 of Milgrom and Weber (1985) to obtain existence of a pure strategy equilibrium, then note that any such equilibrium is necessarily monotone by Lemma 1. In the second step, building on Reny (1999, 2011),[1] we then use this result on existence in discrete spaces to construct a sequence of monotone ε-equilibria in the unrestricted bid space $\mathcal{B}$, which converges to a monotone equilibrium in the unrestricted game.

3.1 Existence of Monotone Equilibrium on a Finite Grid

We first establish that a monotone pure strategy equilibrium exists when the bid space is restricted to be discrete. In particular, given any positive integer K, let $\mathcal{B}_K = \{b_1, ..., b_K\}$ be any size-K grid on $\mathcal{B} = [0, \bar{v}]$, where $b_{k+1} \geq b_k$ for all $k \leq K - 1$. We establish the existence of a monotone equilibrium on $\mathcal{B}_K$.

Athey (2001) first establishes two sufficient non-primitive conditions for the existence of monotone pure strategy equilibrium with finite-action space: (a) (non-empty monotone best replies) each player has a monotone best response when others adopt monotone strategies; (b) (convex monotone best replies) the set of monotone best replies is convex. McAdams (2003) generalizes Athey's approach to settings with multidimensional actions and multidimensional types, with sufficient non-primitive conditions still being that there are non-empty and convex monotone best replies.[2] Reny (2011) insightfully points out that convexity of the monotone best-reply set is not indeed needed and contractibility is sufficient. Moreover, Reny (2011) showed by an ingenious argument that contractibility is automatically satisfied given any non-empty monotone best-replies. Therefore, to establish existence of a monotone pure strategy equilibrium with finite-action space, it suffices to show that each player has a monotone best reply whenever others adopt monotone strategies.[3]

In this paper's setting with a finite bid-grid, the existence of a monotone best reply comes essentially automatically in view of Lemma 1. Note that finiteness of $\mathcal{B}_K$ implies player i's best response set will be nonempty for all type realizations $t_i \in \mathcal{T}_i$ and all rival strategy profiles $\mathbf{s}_{-i}$. Furthermore, by Lemma 1 above, player i's interim best response to any rival strategy profile $\mathbf{s}_{-i}$ is monotone in his type realization t_i, where monotonicity is defined relative to the relevant partial order defined in Lemma 1. By Proposition 4.4 in Reny (2011), it follows that player i's set of monotone pure strategy best replies is nonempty and join-closed.[4]

We then can establish the existence of a monotone equilibrium on $\mathcal{B}_K$ by invoking Theorem 4.1 of Reny (2011). For this purpose, it suffices to verify Conditions G.1-G.6 of Reny (2011). Those conditions are defined to apply to settings where action spaces may not even be a lattice (only semi-lattice struc-

[1] Specifically, we apply the standard methods as in Reny (2011) for establishing his Corollary 5.2.

[2] Much as in Athey (2001), the key insight of McAdams (2003) is in how to establish convexity of the monotone best-reply set.

[3] We thank an anonymous referee for suggesting us to provide the explanations in this paragraph.

[4] Condition (ii) of Proposition 4.4 in Reny (2011) holds.

ture is applied). Among other things, this paper's model lies squarely in the more "traditional" sort of multidimensional example as in McAdams (2003), with the extra simplification that the action space is one-dimensional. To avoid cluttering the text and adding unnecessary burden on the reader, we can rewrite Reny's Conditions G.1-G.6 as the following three conditions using relatively familiar terms for our first price auction setting.

Recall player i's type space is $\mathcal{T}_i = \mathcal{V}_i \times \mathcal{A}_i \times \mathcal{W}_i = [\underline{v}_i, \overline{v}_i] \times [\underline{a}_i, \overline{a}_i] \times [0, \overline{w}_i] \in \mathbb{R}^3$, and $\mathcal{T} = \times_i \ \mathcal{T}_i$. Let $\succeq_R$ be the partial order as defined in Lemma 1.

- H.1: The probability measure on $\mathcal{T}_i$ is atomless.
- H.2: $\mathcal{B}_K$ is a compact subset of Euclidean space $\mathbb{R}$, and the maximum of two bids is itself a feasible bid.
- H.3: Player i's Bernoulli utility function $u(\cdot; a_i)$ is bounded, jointly measurable, and continuous in bids $\mathbf{b} = (b_i) \in (\mathcal{B}_K)^N$ for every type $\mathbf{t} = (t_i) \in \mathcal{T}$.

H.1, which is G.2 in Reny (2011), is satisfied since density f_i is everywhere positive over $\mathcal{T}_i$. Given H.1 is satisfied and that $\mathcal{T}_i$ is a rectangle in $\mathbb{R}^3$, G.1 and G.3 in Reny (2011) automatically hold.[5] Thus given our specification of $\mathcal{T}_i$, there is no need to write conditions to check G.1 and G.3 as they are implied by H.1. By Proposition 3.1 in Reny (2011), H.2 implies G.4 and G.5 in Reny (2011). H.2 is trivially satisfied since $\mathcal{B}_K$ is finite. H.3 is identical to G.6 in Reny (2011). H.3 trivially holds as the action space $\mathcal{B}_K$ is a totally ordered finite set. Because of one-dimensionality, the quasisupermodularity assumption in McAdams (2003) is automatically satisfied.

Hence by Theorem 4.1 of Reny (2011), there exists a monotone pure strategy equilibrium on the restricted action space $\mathcal{B}_K$.

Proposition 1. *Suppose that bidders have private CRRA preferences. Then a monotone pure strategy equilibrium exists for any discrete bid space $\mathcal{B}_K$, with monotonicity interpreted relative to the partial order $\succeq_R$ on $\mathcal{T}_i, \forall i$ defined in Lemma 1.*

3.2 Extension to Existence on Continuous $\mathcal{B}$

Following Athey (2001), McAdams (2003), and Reny (2011), we next extend existence of monotone equilibrium on the restricted bid spaces $\mathcal{B}_K$ to existence of monotone equilibrium on the continuous bid space $\mathcal{B}$.[6] Feasibility of this extension follows from two auxiliary results due to Reny (1999, 2011):

- Suppose the unrestricted game is *better-reply secure* in the sense of Reny (1999). By Remark 3.1 on page 1038 in Reny (1999), better-reply security implies that the limits of a convergent sequence of pure-strategy ε-equilibria, as ε tends to zero, are pure strategy equilibria.
- Let $\mathcal{M}$ denote the set of monotone functions from $\mathcal{T} \to \mathcal{B}^N$. Under Conditions G.1, G.3, and G.4, Appendix A.3 in Reny (2011) shows how to construct a metric δ on $\mathcal{M}$ such that the resulting metric

[5] G.3 holds since H.1 holds and since we can pick the subset $\{t_i = (v_i, a_i, w_i) \in \mathcal{T}_i : \ v_i, a_i, w_i \in \mathbb{Q}\}$, which is countable and dense in $\mathcal{T}_i$ when the metric is the Euclidean metric.

[6] See, for instance, the proof of Corollary 5.2 in Reny (2011).

space $(\mathcal{M}, \delta)$ is the space of equivalence classes of monotone functions that are equal μ-a.e., where μ is the measure on $\mathcal{T}$ induced by $\mathbf{F}$. By Lemma A.13 in Reny (2011), the metric space $(\mathcal{M}, \delta)$ is compact.

Existence on $\mathcal{B}_K$ can then be extended to existence on $\mathcal{B}$ in four steps. First, building on Reny (1999), we establish that the auction game is better-reply secure. Second, as in Reny (2011), we use existence of equilibrium for each $\mathcal{B}_K$ to construct a sequence of ε-equilibria on $\mathcal{B}$ for which ε tends to zero. Third, by compactness of $(\mathcal{M}, \delta)$, this sequence of ε-equilibria has a convergent subsequence. Finally, by better-reply security, the limit of this subsequence is a monotone pure strategy equilibrium.

We first verify that the auction game is *better-reply secure* in the sense of Reny (1999). Reny (1999) defines the concept of better-reply security formally as follows:

Definition 1 (Secure a Payoff)**.** *Player i can* secure a payoff *of $\alpha \in \mathbb{R}$ at $\mathbf{s} \in \mathcal{S}$ if there exists $\bar{s}_i \in \mathcal{S}_i$ such that $\pi_i(\bar{s}_i, \mathbf{s}_{-i}) \geq \alpha$ for all $\mathbf{s}'_{-i}$ in some open neighborhood of $\mathbf{s}_{-i}$.*

Definition 2 (Better-Reply Secure)**.** *A game $G = (\mathcal{S}_i, \pi_i)_{i=1}^N$ is* better-reply secure *if whenever $(\tilde{\mathbf{s}}, \tilde{\boldsymbol{\pi}})$ is in the closure of the graph of the vector payoff function $\boldsymbol{\pi}(\cdot)$ and $\tilde{\mathbf{s}}$ is not an equilibrium, some player i can secure a payoff strictly above $\tilde{\pi}_i$ at $\tilde{\mathbf{s}}$.*

Building on the arguments in Reny (1999), it is straightforward to show that our auction game is better-reply secure. For completeness, we state this fact formally as a lemma.

Lemma 2. *The first price auction of Section 2 is better-reply secure.*

Proof. See online Appendix. ∎

Following Reny (2011), we next apply Proposition 1 (existence of monotone equilibria in discrete bid spaces) to establish a second intermediate result: for each $\varepsilon > 0$, there exists a monotone ε-equilibria in the continuous bid space $\mathcal{B}$.

Lemma 3. *For each $\varepsilon > 0$, there exists an ε-equilibrium $\mathbf{s}^\varepsilon = (s_1^\varepsilon, ..., s_N^\varepsilon)$ of the auction with private CRRA preferences, where each $s_i^\varepsilon : \mathcal{V}_i \times \mathcal{A}_i \times \mathcal{W}_i \to \mathcal{B} = [0, \overline{v}]$ is monotone in the sense of Lemma 1.*

Proof. See online Appendix. ∎

Having established Lemmas 2 and 3, we are now in position to complete the proof. Recall that by Remark 3.1 of Reny (1999), if a game satisfies better-reply security then the limit of any convergent sequence of ε-equilibria for which $\varepsilon \to 0$ will be an equilibrium. Lemma 2 establishes that our auction game is better-reply secure, and Lemma 3 shows existence of a sequence of monotone ε-equilibria for which $\varepsilon \to 0$. Hence to complete the proof it only remains to establish that the sequence of equilibria in Lemma 3 has a convergent subsequence. But in light of Appendix A.3 in Reny (2011) this follows automatically from monotonicity: there exists a metric δ under which the space of monotone functions from $\mathcal{T}$ to $\mathcal{B}^N$ is compact, with equality under δ implying equality almost everywhere with respect to the measure on $\mathcal{T}$ induced by $\mathcal{F}$. Hence for any sequence of monotone ε-equilibria $\{\mathbf{s}_k\}_{k=1}^\infty$ constructed above there will exist a subsequence $\{\mathbf{s}_{k_j}\}_{j=1}^\infty$ such that $\mathbf{s}_{k_j}$ converges (with respect to δ) to a limit $\mathbf{s}^* \in \mathcal{M}$, and by Remark 1 in Reny (1999) this limit $\mathbf{s}^*$ will be an equilibrium of the underlying auction game. We therefore conclude:

Theorem 1. *In the first-price auction game with private CRRA preferences, there exists a monotone pure strategy equilibrium. Furthermore, in any such equilibrium, bids are increasing in values, increasing in private risk aversion, and decreasing in private wealth.*

4 Existence of Monotone Equilibrium under Private CARA Preferences

We now turn to consider existence of equilibrium under private CARA preferences. The proof strategy in this case is essentially identical; one first establishes existence of a monotone equilibrium on discrete bid spaces, then extends this to the continuous case via the limiting arguments above. However, the proof involves two differences in detail, which we outline briefly here.

First, as is well known, when bidders have CARA utility initial wealth w_i is irrelevant for bidding in the sense that realizations of w_i do not affect bidder i's set of best responses. We therefore restrict attention to "natural" equilibria in which equilibrium bidding strategies are constant over w_i. Note that any such equilibria will be trivially monotone in w_i, thus will fulfill the objectives of this study.

Second, since the Bernoulli utility function $u(x; a_i)$ is now CARA rather than CRRA, the arguments used to establish existence of ε-equilibrium differ slightly from those underlying Lemma 3. We thus provide a separate proof of existence of ε-equilibrium for the CARA case. We refer interested readers to the online appendix for details.

Monotonicity of best responses Under private CARA preferences, best responses for bidder i are again increasing in both v_i and a_i for any initial wealth w_i and rival strategies $\mathbf{s}_{-i}$. In contrast to the CRRA case above, however, i's best-response set is now invariant to w_i, so that a best response at wealth $w_i = w_0$ is also a best response for any other initial wealth. Formally:

Lemma 4. *Suppose that bidders have private CARA preferences. Consider any bidder i, any bid space $\tilde{\mathcal{B}} \subset \mathbb{R}^+$ and any rival strategy profile $\mathbf{s}_{-i} : \mathcal{V}_{-i} \times \mathcal{A}_{-i} \times \mathcal{W}_{-i} \to \tilde{\mathcal{B}}^{N-1}$. Fixing $w_i = w_0$, let (v_i', a_i') and (v_i'', a_i'') be any points in $\mathcal{V}_i \times \mathcal{A}_i$ such that $(v_i', a_i') \leq (v_i'', a_i'')$, and let b_i', b_i'' be any interim best responses by i to $\mathbf{s}_{-i}$ at type realizations (v_i', a_i', w_0), (v_i'', a_i'', w_0) respectively. Then $b_i' \leq b_i''$. Furthermore, for all $(v_i, a_i) \in \mathcal{V}_i \times \mathcal{A}_i$, if b_i is an interim best response to $\mathbf{s}_{-i}$ at (v_i, a_i, w_0) then b_i is an interim best response to $\mathbf{s}_{-i}$ at (v_i, a_i, w_i) for any $w_i \in \mathcal{W}_i$.*

Proof. See online Appendix. ∎

Existence of monotone equilibrium The second half of Lemma 4 implies that if there exists a best response strategy s_i^* to s_{-i}, then there exists a best response strategy $s_i^{*\prime}$ which is invariant (hence trivially monotone) in w_i. With this modification, the argument in Section 3.1 can be applied to establish existence of a monotone equilibrium on any discrete bid space $\mathcal{B}_K$, with the additional property that bidding strategies in this equilibrium are invariant to w_i for all i. The argument in Lemma 2 then establishes the game is better-reply secure, so to complete the proof we need only construct a sequence of ε-equilibria for which $\varepsilon \to 0$. The technical argument underlying this construction is slightly different than that used to establish

Lemma 3, so we provide a separate proof in the online appendix. Again, however, it is straightforward to show that for any $\varepsilon > 0$ there exists an ε-equilibrium of the continuous auction game.

After accounting for these minor technical differences, the existence argument in Section 3 applies directly to the case of private CARA preferences. We therefore conclude:

Theorem 2. *In the first-price auction game with private CARA preferences, there exists a monotone pure strategy equilibrium such that bids are increasing in values, increasing in private risk aversion, and constant in initial wealth.*

5 Existence of Symmetric Monotone Equilibrium with Symmetric Bidders

Finally, we consider the special case in which bidders are symmetric: $\mathcal{V}_i = \mathcal{V}$, $\mathcal{A}_i = \mathcal{A}$, $\mathcal{W}_i = \mathcal{W}$, and $F_i = F$ for all $i = 1, ..., N$. We seek to show that in environments with symmetric bidders there exists a *symmetric* monotone equilibrium. Toward this end, we refine the arguments above as follows.

By hypothesis, neither the auctioneer's allocation rule nor the bid spaces $\mathcal{B}$ and $\mathcal{B}_K$ defined above depends on bidder identities, and we are now assuming type spaces and distributions over types to be symmetric. Hence the ex ante payoff vector π is exchangeable in type-action pairs, and the auction game as a whole is therefore symmetric as defined in Section 4.2 of Reny (2011). Furthermore, for any discrete bid space $\mathcal{B}_K$, the argument in Section 3.1 established that bidder i's set of monotone best replies is nonempty and join-closed for arbitrary rival strategies, hence in particular this set is nonempty and join-closed when rivals employ the *same* pure strategy. Theorem 4.5 of Reny (2011) then implies existence of a *symmetric* monotone pure strategy equilibrium on any discrete bid space $\mathcal{B}_K$:

Proposition 2. *Suppose that potential bidders are symmetric and have either private CRRA or private CARA preferences. Then there exists a symmetric monotone pure strategy equilibrium on any discrete bid space $\mathcal{B}_K$.*

In turn, application of the arguments in Section 3.2 to symmetric equilibria of the form in Proposition 2 yields existence of a sequence of *symmetric* monotone ε-equilibria converging to an equilibrium of the symmetric auction game. Noting that the limit of any sequence of symmetric strategy profiles will be symmetric, we therefore conclude:

Theorem 3. *Suppose that potential bidders are symmetric and have either private CRRA or private CARA preferences. Then there exists a symmetric monotone pure strategy equilibrium of the first-price auction game.*

The nature of monotonicity in w_i will of course be determined by the specification of either CRRA or CARA preferences, but in either case there will exist an equilibrium in which bids are increasing in values, increasing in private risk aversion, and at least weakly decreasing in private wealth.

6 Concluding Remarks

In this paper, we establish the existence of monotone equilibrium in first price auctions when bidders' values, degrees of risk aversion, and levels of initial wealth are all private information. Furthermore, when bidders are symmetric, a symmetric monotone equilibrium must exist. Both CRRA and CARA specifications are accommodated in our study. For both specifications, the equilibrium bidding strategies increase with values and degrees of risk aversion. For CRRA, the equilibrium bidding strategies decrease with initial wealth, while for CARA the equilibrium bids do not depend on initial wealth.

We followed a procedure along the lines of Reny (1999, 2011), Athey (2001) and McAdams (2003) to establish the equilibria. We considered an environment where no mass points exist in either dimension of the type space, i.e. value, degree of risk aversion and wealth. In principle, the approach applies in general when any one or two of these three variables are discretely distributed. To apply Reny (2011), the joint distribution needs only be atomless and satisfies Reny's condition G.3.[7] In particular, we can allow one or two variable to be public information. Nevertheless, one has to be cautious about some technical caveats that might have to be observed when applying the procedure to extend the existence of monotone equilibrium to continuous bids. For example, when initial wealth has a mass point at zero and the utility function takes a CRRA form, then a technical issue arises when establishing the ε-equilibria of the game since the first order derivative of the utility function is not bounded. This is the case considered by Cox, Smith and Walker (1988) and Van Boening, Rassenti and Smith (1998). More work needs to be done to fully resolve this case.

As evidenced in experimental and empirical studies on auctions, it is important to take risk aversion into account and also consider a general setting with multi-dimensional private information as considered in this paper. Our theoretical results on existence of monotone equilibrium when values, degree of risk aversion, and initial wealth are all bidder's private information make progress in providing the theoretical foundation for further facilitating experimental and empirical research on auctions in environments with private risk aversion and/or private initial wealth.

References

[1] Athey, S., and J. Levin (2001): Information and Competition in U.S. Forest Service Timber Auctions, *Journal of Political Economy*, 109, 375-413.

[2] Athey, S., (2001): Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information, *Econometrica*, 69, 861–889.

[3] Bajari, P., and A. Hortaçsu (2005): Are Structural Estimates of Auction Models Reasonable? Evidence from Experimental Data, *Journal of Political Economy*, 113, 2005, 703-741.

[4] Baldwin, L., (1995): Risk Averse Bidders at Oral Ascending-Bid Forest Service Timber Sales, Unpublished Manuscript, RAND Corporation.

[7] These requirements are satisfied when one or two variables are discrete. The other conditions for applying Reny's Theorem 4.1 are also satisfied for establishing the existence of monotone equilibrium with finite bids. We thank an anonymous referee for suggesting us to discuss the cases where one or two variables are discrete.

[5] Campo, S., (2012): Risk Aversion and Asymmetry in Procurement Auctions: Identification, Estimation and Application to Construction Procurements, *Journal of Econometrics*, 168, 96–107.

[6] Cox, J., Roberson, B., and V. Smith (1982): Theory and Behavior of Single Object Auctions, in Vernon L.Smith (ed.), *Research in Experimental Economics*, vol. 2, Greenwich, CT: JAI Press.

[7] Cox, J., V. Smith and J. Walker (1982): Auction Market Theory of Heterogenous Bidders, *Economic Letters*, 9, 319-325.

[8] Cox, J., V. Smith and J. Walker (1988): Theory and Individual Behavior of First-Price Auctions, *Journal of Risk and Uncertainty*, 1, 61-99.

[9] Campo, S., E. Guerre, I. Perrigne, and Q. Vuong (2011): Semiparametric Estimation of First-Price Auctions with Risk-Averse Bidders, *Review of Economic Studies*, 78(1), 112-147.

[10] Fang, H., and X. Tang (2014): Inference of Bidders' Risk Attitudes in Ascending Auctions with Endogenous Entry, *Journal of Econometrics*, 180, 198–216.

[11] Levin, Dan and James L. Smith (1994): Equilibrium in Auctions with Entry. *American Economic Review*, Volume 84, Issue 3, June: 585-599.

[12] Li, T., J. Lu, and L. Zhao (2015): Auctions with Selective Entry and Risk Averse Bidders: Theory and Evidence, *RAND Journal of Economics*, 46(3), 524-545.

[13] Lu, J., and I. Perrigne (2008): Estimating Risk Aversion from Ascending and Sealed-Bid Auctions: The Case of Timber Auction Data, *Journal of Applied Econometrics*, 23, 871-896.

[14] McAdams, D., (2003): Isotone Equilibrium in Games of Incomplete Information, *Econometrica*, 71, 1191–1214.

[15] Milgrom, P., and R. Weber (1985): Distributional Strategies for Games with Incomplete Information, *Mathematics of Operations Research*, 10, 619-632.

[16] Palfrey, T. R. and Pevnitskaya, S., (2008): Endogenous entry and self-selection in private value auctions: An experimental study, *Journal of Economic Behavior & Organization*, 66(3-4), 731-747.

[17] Perrigne, I., (2003): Random Reserve Prices and Risk Aversion in Timber Auctions, Unpublished manuscript, Pennsylvania State University.

[18] Pevnitskaya, S. (2001): First-Price Private Value Auctions with an Endogenous Number of Heterogenous Bidders, Working paper, University of Southern California.

[19] Pratt, J.W., (1964): Risk Aversion in the Small and in the Large, *Econometrica*, 32, 122-136.

[20] Reny, P. (1999): On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games, *Econometrica*, 67, 1029-1056.

[21] Reny, P. (2011): On the Existence of Monotone Pure Strategy Equilibria in Bayesian Games, *Econometrica,* 79, 499-553.

[22] Van Boening, M., Rassenti, S., and V. Smith (1998): Numerical Computation of Equilibrium Bid Functions in a First-Price Auction with Heterogeneous Risk Attitudes, *Experimental Economics*, 1:147-159.

Online Appendix for "Existence of Monotone Equilibrium in First Price Auctions with Private Risk Aversion and Private Initial Wealth" by Matthew Gentry, Tong Li and Jingfeng Lu

Proof of Lemma 1

Proof. Recall $p_i(b;\mathbf{s}_{-i})$ be the winning probability of bidder i if he bids b given rival strategy profile $\mathbf{s}_{-i}$. Given the first price auction rule, $p_i(b;\mathbf{s}_{-i})$ is increasing in b.
[**Monotonicity in value**]: Take fixed a_i and w_i and consider two values $v_i' < v_i''$. We want to show $b_i'' \geq b_i'$. Clearly we should have $b_i' \leq v_i'$, $b_i'' \leq v_i''$. If $b_i'' \geq v_i'$, then clearly $b_i'' \geq b_i'$. We now assume $b_i'' \leq v_i'$. Incentive compatibility conditions give

$$(v_i'-b_i'+w_i)^{1-a_i}p_i(b_i';\mathbf{s}_{-i})+{w_i}^{1-a_i}[1-p_i(b_i';\mathbf{s}_{-i})] \geq (v_i'-b_i''+w_i)^{1-a_i}p_i(b_i'';\mathbf{s}_{-i})+{w_i}^{1-a_i}[1-p_i(b_i'';\mathbf{s}_{-i})]$$

and

$$(v_i''-b_i''+w_i)^{1-a_i}p_i(b_i'';\mathbf{s}_{-i})+{w_i}^{1-a_i}[1-p_i(b_i'';\mathbf{s}_{-i})] \geq (v_i''-b_i'+w_i)^{1-a_i}p_i(b_i';\mathbf{s}_{-i})+{w_i}^{1-a_i}[1-p_i(b_i';\mathbf{s}_{-i})].$$

We thus have

$$\frac{(v_i'-b_i'+w_i)^{1-a_i}-{w_i}^{1-a_i}}{(v_i'-b_i''+w_i)^{1-a_i}-{w_i}^{1-a_i}} \geq \frac{p_i(b_i'';\mathbf{s}_{-i})}{p_i(b';\mathbf{s}_{-i})} \geq \frac{(v_i''-b_i'+w_i)^{1-a_i}-{w_i}^{1-a_i}}{(v_i''-b_i''+w_i)^{1-a_i}-{w_i}^{1-a_i}},$$

i.e.

$$\frac{(v_i'-b_i'+w_i)^{1-a_i}-{w_i}^{1-a_i}}{(v_i''-b_i'+w_i)^{1-a_i}-{w_i}^{1-a_i}} \geq \frac{(v_i'-b_i''+w_i)^{1-a_i}-{w_i}^{1-a_i}}{(v_i''-b_i''+w_i)^{1-a_i}-{w_i}^{1-a_i}}.$$

We only need to show that

$$\Psi(b)=\frac{(v_i'-b+w_i)^{1-a_i}-{w_i}^{1-a_i}}{(v_i''-b+w_i)^{1-a_i}-{w_i}^{1-a_i}}$$

decreases with $b \leq v_i' < v_i''$.

Note that

$$\ln\Psi(b)=\ln[(v_i'-b+w_i)^{1-a_i}-{w_i}^{1-a_i}]-\ln[(v_i''-b+w_i)^{1-a_i}-{w_i}^{1-a_i}].$$

Thus

$$\begin{aligned}
\frac{d\ln\Psi(b)}{db} &= \frac{-(1-a_i)(v_i'-b+w_i)^{-a}}{(v_i'-b+w_i)^{1-a_i}-{w_i}^{1-a_i}}-\frac{-(1-a_i)(v_i''-b+w_i)^{-a}}{(v_i''-b+w_i)^{1-a_i}-{w_i}^{1-a_i}} \\
&= \frac{(1-a_i)}{(v''-b+w_i)^a[(v_i''-b+w_i)^{1-a_i}-{w_i}^{1-a_i}]}-\frac{(1-a_i)}{(v_i'-b+w_i)^a[(v_i'-b+w_i)^{1-a_i}-{w_i}^{1-a_i}]} \\
&< 0,
\end{aligned}$$

as required.

[**Monotonicity in risk aversion**]: Take fixed v_i and w_i and consider two values $a_i' < a_i''$. We want to show $b_i'' \geq b_i'$. Clearly we should have $b_i', b_i'' \leq v_i$. Incentive compatibility conditions gives

$$(v_i - b_i' + w_i)^{1-a_i'} p_i(b_i'; \mathbf{s}_{-i}) + {w_i}^{1-a_i'}[1 - p_i(b_i'; \mathbf{s}_{-i})] \geq (v_i - b_i'' + w_i)^{1-a_i'} p_i(b_i''; \mathbf{s}_{-i}) + {w_i}^{1-a_i'}[1 - p_i(b_i''; \mathbf{s}_{-i})]$$

and

$$(v_i - b_i'' + w_i)^{1-a_i''} p_i(b_i''; \mathbf{s}_{-i}) + {w_i}^{1-a_i''}[1 - p_i(b_i''; \mathbf{s}_{-i})] \geq (v_i - b_i' + w_i)^{1-a_i''} p_i(b_i'; \mathbf{s}_{-i}) + {w_i}^{1-a_i''}[1 - p_i(b_i'; \mathbf{s}_{-i})].$$

We thus have

$$\frac{(v_i - b_i' + w_i)^{1-a_i'} - {w_i}^{1-a_i'}}{(v_i - b_i'' + w_i)^{1-a_i'} - {w_i}^{1-a_i'}} \geq \frac{p_i(b_i''; \mathbf{s}_{-i})}{p_i(b_i'; \mathbf{s}_{-i})} \geq \frac{(v_i - b_i' + w_i)^{1-a_i''} - {w_i}^{1-a_i''}}{(v_i - b_i'' + w_i)^{1-a_i''} - {w_i}^{1-a_i''}},$$

which is

$$\frac{(v_i - b_i' + w_i)^{1-a_i'} - {w_i}^{1-a_i'}}{(v_i - b_i' + w_i)^{1-a_i''} - {w_i}^{1-a_i''}} \geq \frac{(v_i - b_i'' + w_i)^{1-a_i'} - {w_i}^{1-a_i'}}{(v_i - b_i'' + w_i)^{1-a_i''} - {w_i}^{1-a_i''}}.$$

We only need to show that

$$\Psi(b) = \frac{(v_i - b + w_i)^{1-a_i'} - {w_i}^{1-a_i'}}{(v_i - b + w_i)^{1-a_i''} - {w_i}^{1-a_i''}}$$

decreases with $b \leq v_i$.

Note that

$$\ln \Psi(b) = \ln[(v_i - b + w_i)^{1-a_i'} - {w_i}^{1-a_i'}] - \ln[(v_i - b + w_i)^{1-a_i''} - {w_i}^{1-a_i''}].$$

Thus

$$\begin{aligned}
\frac{d \ln \Psi(b)}{db} &= \frac{-(1 - a_i')(v_i - b + w_i)^{-a_i'}}{(v_i - b + w_i)^{1-a_i'} - {w_i}^{1-a_i'}} - \frac{-(1 - a_i'')(v_i - b + w_i)^{-a_i''}}{(v_i - b + w_i)^{1-a_i''} - {w_i}^{1-a_i''}} \\
&= \frac{(1 - a_i'')(v_i - b + w_i)^{-a_i''}}{(v_i - b + w_i)^{1-a_i''} - {w_i}^{1-a_i''}} - \frac{(1 - a_i')(v_i - b + w_i)^{-a_i'}}{(v_i - b + w_i)^{1-a_i'} - {w_i}^{1-a_i'}} \\
&= \frac{1}{w_i}\left[\frac{(1 - a_i'')}{\frac{v_i - b + w_i}{w_i} - (\frac{v_i - b + w_i}{w_i})^{a_i''}} - \frac{(1 - a_i')}{\frac{v_i - b + w_i}{w_i} - (\frac{v_i - b + w_i}{w_i})^{a_i'}}\right] \\
&= \frac{1}{v_i - b + w_i}\left[\frac{(1 - a_i'')}{1 - (\frac{w_i}{v_i - b + w_i})^{1-a_i''}} - \frac{(1 - a_i')}{1 - (\frac{w_i}{v_i - b + w_i})^{1-a_i'}}\right].
\end{aligned}$$

Define $\varphi(x) = \frac{x}{1-\alpha^x}$ where $\alpha \in [0, 1], x \in (0, 1]$. We need to show that $\varphi'(x) > 0$.

Note that

$$\begin{aligned}
\varphi'(x) &= \frac{1}{1-\alpha^x} + \frac{x(\ln\alpha)\alpha^x}{(1-\alpha^x)^2} \\
&= \frac{1}{(1-\alpha^x)^2}[(1-\alpha^x) + x(\ln\alpha)\alpha^x].
\end{aligned}$$

Let $y = \alpha^x \in [\alpha, 1)$, i.e. $x = \frac{\ln y}{\ln \alpha}$. Term $[(1-\alpha^x) + x(\ln\alpha)\alpha^x]$ can be written as

$$\begin{aligned}
\eta(y) &= (1-y) + \frac{\ln y}{\ln\alpha}(\ln\alpha)y \\
&= (1-y) + y\ln y; \\
\eta'(y) &= \ln y < 0, \text{ where } y \in [\alpha, 1), \\
\eta(1) &= 0.
\end{aligned}$$

Thus $\eta(y) \geq 0$. We thus have $\frac{d\ln\Psi(b)}{db} \leq 0$ as required.

[**Monotonicity in initial wealth**]: Take fixed v_i and a_i and consider two values $w_i' > w_i''$. We want to show $b_i'' \geq b_i'$. Clearly we should have b_i', $b_i'' \leq v_i$. Incentive compatibility conditions gives

$$(v-b'+w_i')^{1-a_i}p_i(b';\mathbf{s}_{-i}) + w_i'^{1-a_i}[1-p_i(b';\mathbf{s}_{-i})] \geq (v-b''+w_i')^{1-a_i}p_i(b'';\mathbf{s}_{-i}) + w_i'^{1-a_i}[1-p_i(b'';\mathbf{s}_{-i})]$$

and

$$(v-b''+w_i'')^{1-a_i}p_i(b'';\mathbf{s}_{-i}) + w_i''^{1-a_i}[1-p_i(b'';\mathbf{s}_{-i})] \geq (v-b'+w_i'')^{1-a_i}p_i(b';\mathbf{s}_{-i}) + w_i''^{1-a_i}[1-p_i(b';\mathbf{s}_{-i})].$$

We thus have

$$\frac{(v-b'+w_i')^{1-a_i} - w_i'^{1-a_i}}{(v-b''+w_i')^{1-a_i} - w_i'^{1-a_i}} \geq \frac{p_i(b'';\mathbf{s}_{-i})}{p_i(b';\mathbf{s}_{-i})} \geq \frac{(v-b'+w_i'')^{1-a_i} - w_i''^{1-a_i}}{(v-b''+w_i'')^{1-a_i} - w_i''^{1-a_i}},$$

which is

$$\frac{(v-b'+w_i')^{1-a_i} - w_i'^{1-a_i}}{(v-b'+w_i'')^{1-a_i} - w_i''^{1-a_i}} \geq \frac{(v-b''+w_i')^{1-a_i} - w_i'^{1-a_i}}{(v-b''+w_i'')^{1-a_i} - w_i''^{1-a_i}},$$

We only need to show that

$$\Psi(x) = \frac{(x+w_i')^{1-a_i} - w_i'^{1-a_i}}{(x+w_i'')^{1-a_i} - w_i''^{1-a_i}}$$

increases with $x \geq 0$.

Note that

$$\ln\Psi(x) = \ln[(x+w_i')^{1-a_i} - w_i'^{1-a_i}] - \ln[(x+w_i'')^{1-a_i} - w_i''^{1-a_i}].$$

Thus

$$\begin{aligned}\frac{d\ln\Psi(x)}{dx} &= \frac{(1-a_i)(x+w_i')^{-a_i}}{(x+w_i')^{1-a_i}-w_i'^{1-a_i}} - \frac{(1-a_i)(x+w_i'')^{-a_i}}{(x+w_i'')^{1-a_i}-w_i''^{1-a_i}} \\ &= (1-a_i)[\frac{1}{(x+w_i')-(x+w_i')^{a_i}w_i'^{1-a_i}} - \frac{1}{(x+w_i'')-(x+w_i'')^{a_i}w_i''^{1-a_i}}].\end{aligned}$$

Define $\varphi(z) = (x+z)-(x+z)^{a_i}z^{1-a_i}$ where $x \geq 0, z \geq 0$. We need to show that $\varphi'(z) \leq 0$.

We have

$$\varphi'(z) = 1 - a_i(x+z)^{a_i-1}z^{1-a_i} - (1-a_i)(x+z)^{a_i}z^{-a_i}.$$

Let $\lambda = (x+z)z^{-1} \geq 1$. Then

$$\begin{aligned}& a_i(x+z)^{a_i-1}z^{1-a_i} + (1-a_i)(x+z)^{a_i}z^{-a_i} \\ = \;& a_i\lambda^{a_i-1} + (1-a_i)\lambda^{a_i} = \xi(\lambda).\end{aligned}$$

We next show $\xi(\lambda)$ is minimized at $\lambda = 1$. This actually is clear as $\xi'(\lambda) = a_i(1-a_i)\lambda^{a_i-2}[\lambda-1] \geq 0$ when $\lambda \geq 1$. We thus have $\varphi'(z) \leq 0$, which leads to $\frac{d\ln\Psi(x)}{dx} \geq 0$ and $\Psi(x)$ increases with $x \geq 0$. Therefore, $b' \leq b''$. □

Proof of Lemma 2

Proof. Let $(\tilde{\mathbf{s}}, \tilde{\boldsymbol{\pi}})$ be any point in the closure of the graph of $\boldsymbol{\pi}(\cdot)$. By definition, there exists a sequence of strategies $\{\mathbf{s}^m\}_{m=1}^{\infty}$ converging to $\tilde{\mathbf{s}}$ such that $\tilde{\boldsymbol{\pi}} = \lim_{m\to\infty}\boldsymbol{\pi}(\mathbf{s}^m)$. Now focus on the limit strategy $\tilde{\mathbf{s}}$, and consider possible cases.

First suppose relevant ties occur with probability zero at $\tilde{\mathbf{s}}$. Then $\boldsymbol{\pi}(\cdot)$ is also continuous at $\tilde{\mathbf{s}}$, which in turn implies $\boldsymbol{\pi}(\tilde{\mathbf{s}}) = \tilde{\boldsymbol{\pi}}$. Recall that $\bar{\pi}_i(\tilde{\mathbf{s}}_{-i})$ is player i's supremum payoff at $\tilde{\mathbf{s}}_{-i}$. By hypothesis, $\tilde{\mathbf{s}}$ is not an equilibrium, which implies $\bar{\pi}_i(\tilde{\mathbf{s}}_{-i}) > \pi_i(\tilde{\mathbf{s}}) = \tilde{\pi}_i$ for at least one bidder i. From above, for any $\varepsilon > 0$ there exists a strictly monotone strategy s_i^{ε} yielding a payoff within ε of $\bar{\pi}_i(\tilde{\mathbf{s}}_{-i})$, with $\pi_i(s_i^{\varepsilon}, \cdot)$ continuous in $\mathbf{s}_{-i}$ at $(s_i^{\varepsilon}, \tilde{\mathbf{s}}_{-i})$. In particular, for small enough ε, we will have $\pi_i(s_i^{\varepsilon}, \tilde{\mathbf{s}}_{-i}) > \tilde{\pi}_i$. By definition of continuity, it follows that there exists an open neighborhood of $\tilde{\mathbf{s}}_{-i}$ in $\mathcal{S}_{-i}$ such that $\pi_i(s_i^{\varepsilon}, \mathbf{s}'_{-i}) > \tilde{\pi}_i$ for all $\mathbf{s}'_{-i}$ in this neighborhood.

Next suppose that ties occur with positive probability at $\tilde{\mathbf{s}}$. Then $\pi_i(\cdot)$ may be discontinuous in $\mathbf{s}$, so in general $\boldsymbol{\pi}(\tilde{\mathbf{s}}) \neq \tilde{\boldsymbol{\pi}} = \lim_{m\to\infty}\boldsymbol{\pi}(\mathbf{s}^m)$. Now consider the set of bidders who tie with positive probability at $\tilde{\mathbf{s}}$. By construction, at each point $\mathbf{s}^m$ along the sequence $\{\mathbf{s}^m\}_{m=1}^{\infty}$, at least one bidder tying at $\tilde{\mathbf{s}}$ must lose with probability bounded away from 0. The identity of this bidder could conceivably change across points in $\{\mathbf{s}^m\}_{m=1}^{\infty}$. However, we can always select a *subsequence* $\{\mathbf{s}^{m_k}\}_{k=1}^{\infty}$ such that a *fixed* (set of) bidders who tie at $\tilde{\mathbf{s}}$ lose with probability bounded away from 0 on $\{\mathbf{s}^{m_k}\}_{k=1}^{\infty}$. Let i denote any such bidder, and $\tilde{\pi}_i = \lim_{k\to\infty}\pi_i(\mathbf{s}^{m_k})$ be the limit of this bidder's payoffs along the sequence $\{\mathbf{s}^{m_k}\}_{k=1}^{\infty}$. By construction, $\mathbf{s}^{m_k} \to \tilde{\mathbf{s}}$. Hence for each $\varepsilon > 0$, there will be a $K < \infty$ and a strategy s_i' such that for each $k > K$, s_i' is within ε of $s_i^{m_k}$ (i.e. satisfying

$\delta(s_i', s_i^{m_k}) < \varepsilon$) such that every *tie at* $(\tilde{s}_i, \tilde{\mathbf{s}}_{-i})$ becomes a *win at* $(s_i', \mathbf{s}_{-i}^{m_k})$. For a tie to occur with positive probability, bidder i must submit the same bid on a set of types of positive measure. Since bidders never bid above their values, bidder i must be bidding below value on almost every element of this set. The strategy s_i' described above yields a discrete increase in winning probability at every such point, which for large enough k (and small enough ε) must yield a discrete increase in expected payoffs. Hence for large enough k player i's supremum payoff $\bar{\pi}_i^k = \sup_{s_i \in \mathcal{S}_i} \pi_i(s_i, \mathbf{s}_{-i}^{m_k})$ must be *bounded away from* $\pi_i(s_i^{m_k}, \mathbf{s}_{-i}^{m_k})$. Since this is true at each point in $\{\mathbf{s}^{m_k}\}_{k=1}^{\infty}$, it is also true at the limit $\tilde{\pi}_i = \lim_{k\to\infty} \pi_i(\mathbf{s}^{m_k})$. From above, player i can attain a payoff within ε of his supremum at $\tilde{\mathbf{s}}_{-i}$ by employing some strictly increasing strategy s_i^{ε}, and for this strategy $\pi_i(s_i^{\varepsilon}, \cdot)$ will be continuous in $\mathbf{s}_{-i}$ at $\tilde{\mathbf{s}}_{-i}$. It follows that player i can secure a payoff strictly above $\tilde{\pi}_i$ at $\tilde{\mathbf{s}}_{-i}$.

Combining these cases, we conclude that our auction game is better-reply secure. □

Proof of Lemma 3

Proof. The proof builds on arguments in Reny (2011), but involves substantial differences in detail. For each integer $k \in \{1, ..., \infty\}$, define the $k+1$ element grid $\mathcal{B}_k = \{0, \frac{1}{k}\overline{v}, \frac{2}{k}\overline{v}, ..., \frac{k}{k}\overline{v}\}$. From Proposition 1, we know there exists a monotone pure strategy equilibrium $\mathbf{s}_k$ corresponding to each $\mathcal{B}_k$ thus defined. We wish to show that for k sufficiently large these $\mathbf{s}_k$ are ε-equilibria on the unrestricted bid space $\mathcal{B} = [0, \overline{v}]$.

Fix player i. It suffices to show that for all k sufficiently large and all monotone pure strategies $s_i : \mathcal{T}_i \to \mathcal{B}$ for player i, there is a monotone pure strategy $s_{i,k} : \mathcal{T}_i \to \mathcal{B}_k$ such that player i's utility loss from using $s_{i,k}$ instead of s_i is no greater than ε uniformly in rival strategies.

Let $s_{i,k}^*$ be the unrestricted best response of bidder i given strategies $\mathbf{s}_{-i,k}$ played by rivals. By Lemma 1, we have that $s_{i,k}^*$ is monotone. Define an approximation $\tilde{s}_{i,k}$ (on $\mathcal{B}_k$) of $s_{i,k}^*$ as follows: set $\tilde{s}_{i,k}(v_i, a_i, w_i)$ equal to the lowest $\frac{l}{k}\overline{v}$ that is weakly higher than $s_{i,k}^*(v_i, a_i, w_i)$. Thus $\tilde{s}_{i,k}(v_i, a_i, w_i)$ is weakly monotone. Furthermore, since $\mathbf{s}_k$ is an equilibrium on $\mathcal{B}_k$, we must have $\pi_i(s_i = s_{i,k}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) \geq \pi_i(s_i = \tilde{s}_{i,k}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k})$.

We next show that $\forall \varepsilon > 0$, there exists a $K > 0$ such that when $k \geq K$ we have $\pi_i(s_i = s_{i,k}^*; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) - \pi_i(s_i = \tilde{s}_{i,k}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) \leq \varepsilon$. We consider three events which could arise for a given type profile realization $(\mathbf{v}, \mathbf{a}, \mathbf{w})$.

First, consider the event that bidder i does not win under strategy $\tilde{s}_{i,k}$. In this case, he also does not win under strategy s_{ik}^* as $\tilde{s}_{i,k}(v_i, a_i, w_i) \geq s_{ik}^*(v_i, a_i, w_i)$. Hence in this event bidder i's payoff does not change.

Second, consider the event that bidder i wins under strategy $s_{i,k}^*$. In this case, he must also win under strategy $\tilde{s}_{i,k}$ as $\tilde{s}_{i,k}(v_i, a_i, w_i) \geq s_{i,k}^*(v_i, a_i, w_i)$. Note that $0 \leq \tilde{s}_{i,k}(v_i, a_i, w_i) - s_{ik}^*(v_i, a_i, w_i) \leq 1/k$ by construction. Therefore, for these events, bidder i's payoff decreases by at most $1/k$.

Third, consider the event that bidder i wins under strategy $\tilde{s}_{i,k}$ but not under strategy $s_{i,k}^*$. In this case, bidder i's payoff is 0 under strategy $s_{i,k}^*$. If bidder i wins under strategy $\tilde{s}_{i,k}$ with a bid lower than v_i, then his gain is positive. If bidder i wins under strategy $\tilde{s}_{i,k}$ with a bid higher than v_i, then his gain is negative. We next bound this loss. Note $s^*_{i,k}(v_i, a_i, w_i) \leq v_i$, and thus $\tilde{s}_{i,k}(v_i, a_i, w_i) \leq v_i + 1/k$. This fact means that his gain is higher than $(-1/k)$. In summary, for these events, bidder i's gain under strategy $\tilde{s}_{i,k}$ is at most lower by $1/k$ than his gain under strategy $s^*_{i,k}$.

Now consider large k. When $w_i \in [0, \frac{1}{k}]$, the utility loss of bidder i is bounded by $u(\frac{1}{k}; a) - u(0; a) \leq u(\frac{1}{k}; \overline{a})$. When $w_i \in [\frac{1}{k}, 1 + \frac{1}{k}]$, the utility loss of bidder i is bounded by $u(w_i; a) - u(w_i - \frac{1}{k}; a) \leq \frac{1}{k} u'(w_i - \frac{1}{k}; \overline{a}) = \frac{1}{k}(w^i - \frac{1}{k})^{-\overline{a}}$. When $w_i > 1 + \frac{1}{k}$, the utility loss of bidder i is bounded by $u(w_i; a) - u(w_i - \frac{1}{k}; a) \leq \frac{1}{k} u'(w_i - \frac{1}{k}; 0) = \frac{1}{k}$.

Let Φ_i denote the marginal cumulative distribution function of w_i. Based on the above arguments, when k is large the expected utility of bidder i under strategy $\tilde{s}_{i,k}$ is at most lower by $u(\frac{1}{k}; \overline{a})\Phi_i(\frac{1}{k}) + \int_{\frac{1}{k}}^{1+\frac{1}{k}} \frac{1}{k}(w_i - \frac{1}{k})^{-\overline{a}} d\Phi_i(w_i) + \int_{1+\frac{1}{k}}^{\bar{w}_i} \frac{1}{k} d\Phi_i(w_i)$ than his expected utility under strategy $s^*_{i,k}$, i.e. $0 \leq \pi_i(s_i = s^*_{ik}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) - \pi_i(s_i = \tilde{s}_{i,k}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) \leq u(\frac{1}{k}; \overline{a})\Phi_i(\frac{1}{k}) + \frac{1}{k}[\int_{\frac{1}{k}}^{1+\frac{1}{k}} (w_i - \frac{1}{k})^{-\overline{a}} d\Phi_i(w_i) + \int_{1+\frac{1}{k}}^{\bar{w}_i} d\Phi_i(w_i)]$. As density $f_i(\cdot, \cdot, \cdot)$ is bounded, we have $\Phi'_i(w_i)$ is bounded. Therefore, $\int_{\frac{1}{k}}^{1+\frac{1}{k}} (w_i - \frac{1}{k})^{-\overline{a}} d\Phi_i(w_i) + \int_{1+\frac{1}{k}}^{\bar{w}_i} d\Phi_i(w_i)$ is bounded for large k. Note $\int_0^1 (w_i)^{-\overline{a}} dw_i < +\infty$, and $u(\frac{1}{k}; \overline{a})\Phi_i(\frac{1}{k})$ converges to zero when k goes to infinity. Thus for all $\varepsilon > 0$, there exists $K < \infty$ such that $0 \leq \pi_i(s_i = s^*_{ik}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) - \pi_i(s_i = \tilde{s}_{i,k}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) \leq \varepsilon$ for all $k > K$.

Finally, observe that for $k > K$ we have $\pi_i(s_i = s^*_{i,k}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) \geq \pi_i(s_i = s_k; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) \geq \pi_i(s_i = \tilde{s}_{i,k}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k})$. Hence $0 \leq \pi_i(s_i = s^*_{i,k}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) - \pi_i(s_i = s_k; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) \leq \varepsilon$. We conclude that for any $\varepsilon > 0$ there exists $K < \infty$ such that $\mathbf{s}_k$ is an ε-equilibrium for each $k > K$. □

Proof of Lemma 4

Proof. [**Monotonicity in value** v_i]: Take fixed a_i and w_i. Suppose $v'_i < v''_i$. We want to show $b'_i \leq b''_i$. Clearly we should have $b'_i \leq v'_i$, $b''_i \leq v''_i$.

If $b''_i \geq v'_i$, then clearly $b''_i \geq b'_i$. We now assume $b''_i \leq v'_i$. Incentive compatibility conditions gives

$$\begin{aligned} & [1 - \exp(-a_i(v'_i - b'_i + w_i))]p_i(b'_i; s_{-i}) + [1 - \exp(-a_i(w_i))][1 - p_i(b'_i; s_{-i})] \\ \geq \quad & [1 - \exp(-a_i(v'_i - b''_i + w_i))]p_i(b''_i; s_{-i}) + [1 - \exp(-a_i(w_i))][1 - p_i(b''_i; s_{-i})]. \end{aligned}$$

and

$$\begin{aligned} & [1 - \exp(-a_i(v''_i - b''_i + w_i))]p_i(b''_i; s_{-i}) + [1 - \exp(-a_i(w_i))][1 - p_i(b''_i; s_{-i})] \\ \geq \quad & [1 - \exp(-a_i(v''_i - b'_i + w_i))]p_i(b'_i; s_{-i}) + [1 - \exp(-a_i(w_i))][1 - p_i(b'_i; s_{-i})] \end{aligned}$$

We thus have
$$\frac{1 - \exp(-a_i(v'_i - b'_i))}{1 - \exp(-a_i(v'_i - b''_i))} \geq \frac{p_i(b''_i; s_{-i})}{p_i(b'_i; s_{-i})} \geq \frac{1 - \exp(-a_i(v''_i - b'_i))}{1 - \exp(-a_i(v''_i - b''_i))},$$

which leads to

$$[\exp(-a_i v_i') - \exp(-a_i v_i'')][\exp(a_i b_i'') - \exp(a_i b_i')] \geq 0,$$

as required.

[**Monotonicity in risk aversion** a_i]: Fixed v_i and w_i. Suppose $a_i' < a_i''$. We want to show $b_i' < b_i''$. Note $b_i', b_i'' \leq v_i$. Incentive compatibility conditions give

$$\begin{aligned}
& [1 - \exp(-a_i'(v_i - b_i' + w_i))]p_i(b_i'; s_{-i}) + [1 - \exp(-a_i'(w_i))][1 - p_i(b_i'; s_{-i})] \\
\geq \; & [1 - \exp(-a_i'(v_i - b_i'' + w_i))]p_i(b_i''; s_{-i}) + [1 - \exp(-a_i'(w_i))][1 - p_i(b_i''; s_{-i})].
\end{aligned}$$

and

$$\begin{aligned}
& [1 - \exp(-a_i''(v_i - b_i'' + w_i))]p_i(b_i''; s_{-i}) + [1 - \exp(-a_i''(w_i))][1 - p_i(b_i''; s_{-i})] \\
\geq \; & [1 - \exp(-a_i''(v_i - b_i'))]p_i(b_i'; s_{-i}) + [1 - \exp(-a_i''(w_i))][1 - p_i(b_i'; s_{-i})]
\end{aligned}$$

We thus have

$$\frac{1 - \exp(-a_i'(v_i - b_i'))}{1 - \exp(-a_i'(v_i - b_i''))} \geq \frac{p_i(b_i''; s_{-i})}{p_i(b_i'; s_{-i})} \geq \frac{1 - \exp(-a_i''(v_i - b_i'))}{1 - \exp(-a_i''(v_i - b_i''))},$$

which gives

$$\frac{1 - \exp(-a_i''(v_i - b_i'))}{1 - \exp(-a_i'(v_i - b_i'))} \leq \frac{1 - \exp(-a_i''(v_i - b_i''))}{1 - \exp(-a_i'(v_i - b_i''))}.$$

We only need to show that $\gamma(x) = \frac{1-\exp(-a_i''x)}{1-\exp(-a_i'x)}$ decreases with $x > 0$ when $a_i'' > a_i'$, which is verified below.

Note that

$$\begin{aligned}
\gamma'(x) &= \frac{a_i'' \exp(-a_i''x)}{1 - \exp(-a_i'x)} - \frac{[1 - \exp(-a_i''x)]a_i' \exp(-a_i'x)}{[1 - \exp(-a_i'x)]^2} \\
&= \frac{[1 - \exp(-a_i'x)]a_i'' \exp(-a_i''x) - [1 - \exp(-a_i''x)]a_i' \exp(-a_i'x)}{[1 - \exp(-a_i'x)]^2} \\
&= \frac{[a_i'' \exp(a_i'x) - a_i' \exp(a_i''x)] - (a_i'' - a_i')}{[1 - \exp(-a_i'x)]^2 \exp(a_i''x) \exp(a_i'x)} \\
&= \frac{a_i''[\exp(a_i'x) - 1] - a_i'[\exp(a_i''x) - 1]}{[1 - \exp(-a_i'x)]^2 \exp(a_i''x) \exp(a_i'x)}.
\end{aligned}$$

We next show that

$$\frac{\exp(a_i'x) - 1}{a_i'x} < \frac{\exp(a_i''x) - 1}{a_i''x},$$

i.e. function $\lambda(z) = \frac{\exp(z)-1}{z}$increases with $z > 0$, which clearly holds as $\lambda'(z) = \frac{\exp(z)}{z} - \frac{\exp(z)-1}{z^2} = \frac{(z-1)\exp(z)+1}{z^2} > 0$. $\beta(z) = (z-1)\exp(z) + 1$. $\beta'(z) = z\exp(z) \geq 0$. $\beta(0) = 0$.

The second half of the Lemma 4 is clear as the solution of a bidder's expected utility maximization problem does not depend on the initial wealth. $\square$

Existence of Monotone ε-Equilibria under Private CARA Preferences

Proof. Define a sequence of bid spaces $\mathcal{B}_k$, a corresponding sequence of equilibrium strategy profiles $\mathbf{s}_k$ on $\mathcal{B}_k$, a best response $s^*_{i,k}$ to $\mathbf{s}_{-i,k}$ and a candidate strategy $\tilde{s}_{i,k}$ for each i, k as in the proof of Lemma 3. By analogous arguments, the final wealth of bidder i under strategy $\tilde{s}_{i,k}$ is at most lower by $1/k$ than his final wealth under strategy $s^*_{i,k}$, provided other bidders play $\mathbf{s}_{-i,k}$. Furthermore, by form of the CARA utility function $u(x; a_i)$, we have for all $a_i \in [0, \bar{a}]$:[1]

$$u_x(x; a_i) = \begin{cases} \exp(-a_i x) \leq 1 & \forall x \geq 0. \\ \exp(-a_i x) \leq \exp(\frac{\bar{a}}{k}) & \forall x \in [-\frac{1}{k}, 0]. \end{cases}$$

Since the final wealth induced by $\tilde{s}_{i,k}$ is at most lower by $1/k$ than that induced by the supremum payoff $s^*_{i,k}$, and final wealth from $s^*_{i,k}$ is non-negative, final wealth from $\tilde{s}_{i,k}$ must be higher than $-1/k$ even when initial wealth $w_i = 0$. Hence from above we have $0 \leq \pi_i(s_i = s^*_{i,k}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) - \pi_i(s_i = \tilde{s}_{i,k}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) \leq (1/k) \exp(\frac{\bar{a}}{k}) < (1/k) \exp(\bar{a})$. Noting that $\pi_i(s_i = s^*_{i,k}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) \geq \pi_i(s_i = s_k; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) \geq \pi_i(s_i = \tilde{s}_{i,k}; \mathbf{s}_{-i} = \mathbf{s}_{-i,k})$, it follows that $0 \leq \pi_i(s_i = s^*_i; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) - \pi_i(s_i = s_k; \mathbf{s}_{-i} = \mathbf{s}_{-i,k}) \leq (1/k) \exp(\bar{a})$. From this, it follows that for each $\varepsilon > 0$ there exists $K < \infty$ such that $\mathbf{s}^*_k$ is a monotone ε-equilibria for each $k > K$. This establishes existence of a sequence of ε-equilibria for which $\varepsilon \to 0$, as was to be shown.

□

[1] Here, we use $\bar{a} \in \mathbb{R}^+$ to denote a uniform upper bound for all a_i. $\bar{a}$ is not restricted to be smaller than 1 for CARA preferences.