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ALTERNATIVE PROOFS FOR KOCIK'S GEOMETRIC DIAGRAM FOR RELATIVISTIC VELOCITY ADDITION

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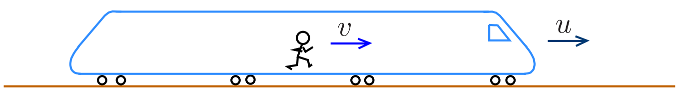
Abstract. A geometric construction for the Poincaré formula for relativistic addition of velocities in one dimension was given by Jerzy Kocik in *Geometric Diagram for Relativistic Addition of Velocities*, American Journal of Physics, volume 80, page 737, 2012. While the proof given there used Cartesian coordinate geometry, three alternative approaches are given in this article: a trigonometric one, one via Euclidean geometry, and one using projective geometry.

1. Introduction

Imagine a train moving at speed u with respect to the ground (as reckoned by someone sitting on the ground), and further that a person P is running with a speed v on the train (as reckoned by somebody sitting in the train). Before 1905, Newtonian physics dictated that the speed of the person P as observed by someone on the ground is $u+v$, while we now know better; the relativistic formula for velocity addition says that the speed should be $(u \oplus v) := (u+v)/(1+uv)$, in units in which the speed of light is 1.



In [1], a geometric diagram for the construction of $u \oplus v$ from u and v was given. We recall it below.

Theorem 1.1 ([1]). *Draw a circle with center O and radius 1. Mark points U, V at distances u, v from O along the radius OC perpendicular to a diameter AB. Let the line joining B to V meet the circle at V', and let the line joining A to U meet the circle at U'. Then $u \oplus v = OW$, where W the point of intersection of $U'V'$ with the radius OC.*

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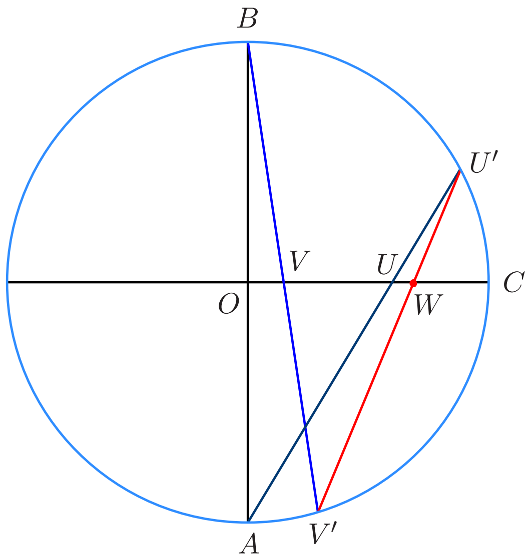
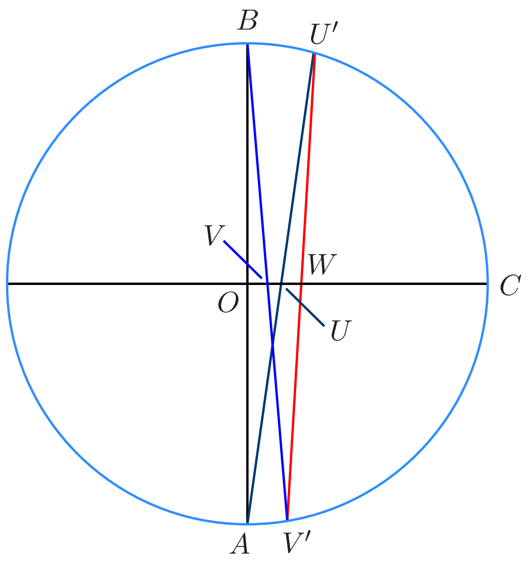


This construction allows visual justification of the following properties of $\oplus$. For all $u, v \in [0, 1]$, $u \oplus v \in [0, 1]$, $v \oplus 1 = 1$, $v \oplus 0 = v$, and when $0 \leq u, v \ll 1$, then $u \oplus v \approx u + v$. For example, let us justify this last fact geometrically. If $u, v \ll 1$, then $\angle OBV \approx 0$, and AV' is almost parallel to OV.



So ΔBOV is almost similar to $\Delta BAV'$, giving

$$AV' \approx \frac{AB}{OB} \cdot OV = \frac{2}{1} \cdot OV = 2v.$$

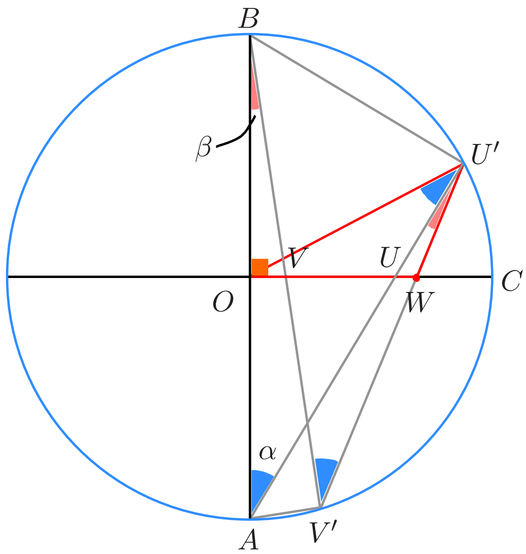
Since AV' is almost parallel to OC, $\Delta U'UW$ is almost similar to $\Delta U'AV'$. Moreover, as $u, v \ll 1$, $U'V' \approx AB = 2$, and $U'W \approx OB = 1$. Hence

$$UW \approx \frac{U'W}{U'V'} \cdot AV' \approx \frac{1}{2} \cdot 2v = v.$$

Thus if $w := OW$, then $w - u = UW \approx v$, that is, $w \approx u + v$.

In [1], Theorem 1.1 was proved using Cartesian coordinate geometry. In the next three sections, we give three alternative proofs of this result. (The more proofs, the merrier!)

2. A trigonometric proof



We refer to the picture above, calling

$$\angle BAU' = \angle OAU =: \alpha \quad \text{and} \quad \angle ABV' = \angle OBV =: \beta.$$

Let W be the point of intersection of $U'V'$ and OC, and set $OW =: w$. Then by looking at the right triangles ΔBOV and ΔAOU, we see that

$$\tan\beta = v \text{ and } \tan\alpha = u.$$

Using the Sine Rule in $\Delta OWU'$, we have

$$\frac{1}{\sin\angle OWU'} = \frac{OU'}{\sin\angle OWU'} = \frac{OW}{\sin\angle OU'W} = \frac{w}{\sin\angle OU'W},$$

giving

$$w = \frac{\sin\angle OU'W}{\sin\angle OWU'}. \tag{1}$$

The proof will be completed by showing (below) that $\angle OU'W = \alpha + \beta$ and $\angle OWU' = 90^\circ + (\alpha - \beta)$, so that (1) yields

$$\begin{aligned} w &= \frac{\sin(\alpha+\beta)}{\sin(90^\circ + (\alpha-\beta))} = \frac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{(\sin\alpha)(\cos\beta)+(\cos\alpha)(\sin\beta)}{(\cos\alpha)(\cos\beta)+(\sin\alpha)(\sin\beta)} \\ &= \frac{\tan\alpha+\tan\beta}{1+(\tan\alpha)(\tan\beta)} = \frac{u+v}{1+uv}, \end{aligned}$$

as desired.

First we will show $\angle OU'W = \alpha+\beta$. Note that $\Delta OAU'$ is isosceles with $OA = OU' = 1$ and so $\angle OU'U = \angle OAU = \alpha$. The chord AV' subtends equal angles at B and U', and so $\angle UU'W = \angle ABV = \beta$. Hence

$$\angle OU'W = \angle OU'U + \angle UU'W = \alpha + \beta.$$

Next, let us show that $\angle OWU' = 90^\circ + (\alpha - \beta)$. To this end, note that $\angle OUU'$ is the common exterior angle for ΔAOU and $\Delta OU'U$, and using the fact that this equals the sum of the opposite interior angles in each triangle, we obtain

$$90^\circ + \alpha = \angle OUU' = \beta + \angle UWU',$$

so that $\angle OWU' = \angle UWU' = 90^\circ + (\alpha-\beta)$, completing the proof.

Yet another trigonometric proof can be obtained by focussing on $\Delta U'CW$, determining all its angles, and the side length $U'C$ (using the isosceles triangle $\Delta OU'C$), enabling the determination of WC $(= 1 - w)$. The details are as follows. In the isosceles triangle $\Delta OU'C$, we have

$$\angle U'OC = 90^\circ - \angle BOU' = 90^\circ - 2\angle BAU' = 90^\circ - 2\alpha.$$

As $OU' = OC = 1$, we obtain $\angle OCU' = 45^\circ + \alpha$ and $U'C = 2\cos(45^\circ + \alpha)$. Also $\angle WU'C = \angle V'U'C = \angle V'BC = \angle ABC - \angle ABV' = 45^\circ - \beta$. This yields $\angle U'WC = 180^\circ - (\angle WU'C + \angle U'CW) = 90^\circ + (\beta - \alpha)$. Again, by the Sine Rule, this time in $\Delta U'WC$, we have

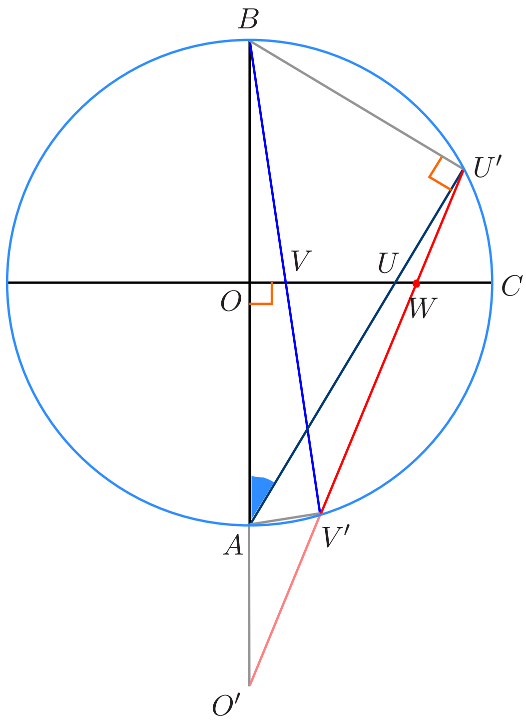
$$\frac{1-w}{\sin\angle WU'C} = \frac{WC}{\sin(45^\circ-\beta)} = \frac{U'C}{\sin\angle U'WC} = \frac{2\cos(45^\circ+\alpha)}{\sin(90^\circ+(\beta-\alpha))},$$

that is,

$$\begin{aligned} 1-w &= \frac{2\cos(45^\circ+\alpha)\sin(45^\circ-\beta)}{\sin(90^\circ+(\beta-\alpha))} = \frac{(\cos\alpha-\sin\alpha)(\cos\beta-\sin\beta)}{(\cos\beta)(\cos\alpha)+\sin\alpha)(\sin\beta)} \\ &= \frac{(1-\tan\alpha)(1-\tan\beta)}{1+(\tan\alpha)(\tan\beta)} = \frac{(1-u)(1-v)}{1+uv}, \end{aligned}$$

which, upon solving for w, gives $w = \dfrac{u+v}{1+uv}$.

3. A Euclidean geometric proof



As $\angle AOU = 90^\circ = \angle AU'B$ and $\angle OAU = \angle U'AB$ (common), by the AA Similarity Rule, $\Delta AOU \sim \Delta AU'B$. So

$$\frac{AU'}{2} = \frac{AU'}{AB} = \frac{AO}{AU} = \frac{1}{\sqrt{1+u^2}},$$

giving $AU' = 2/\sqrt{1+u^2}$. Hence

$$UU' = AU' - AU = \frac{2}{\sqrt{1+u^2}} - \sqrt{1+u^2} = \frac{1-u^2}{\sqrt{1+u^2}}.$$

Proceeding similarly, $BV' = 2/\sqrt{1+v^2}$ and $VV' = (1-v^2)/\sqrt{1+v^2}$. Let W be the point of intersection of $U'V'$ and OC, and set $OW =: w$. Let the extension of $U'V'$ meet the extension of AB at O'. Menelaus's Theorem applied to ΔAOU with the line $O'U'$ gives

$$\frac{w-u}{w} \cdot \frac{OO'}{OO'-1} \cdot \frac{2/\sqrt{1+u^2}}{(1-u^2)/\sqrt{1+u^2}} = \frac{UW}{OW} \cdot \frac{OO'}{AO'} \cdot \frac{AU'}{UU'} = 1.$$

This yields

$$\frac{1}{OO'} = 1 - \frac{2}{1-u^2} \cdot \frac{w-u}{w}. \tag{2}$$

Similarly, Menelaus's Theorem applied to ΔBOV with the line $O'U'$ gives

$$\frac{w-v}{w} \cdot \frac{OO'}{OO'+1} \cdot \frac{2/\sqrt{1+v^2}}{(1-v^2)/\sqrt{1+v^2}} = \frac{VW}{OW} \cdot \frac{OO'}{BO'} \cdot \frac{BV'}{VV'} = 1.$$

This yields

$$\frac{1}{OO'} = \frac{2}{1-v^2} \cdot \frac{w-v}{w} - 1. \tag{3}$$

Equating the right-hand sides of (2) and (3) gives $w = \dfrac{u+v}{1+uv}$.

4. A projective geometric proof

We recall the notion of the cross ratio in projective geometry. If A, B, C, D are collinear points that are projected along four concurrent lines meeting at P, to the collinear points A', B', C', D', respectively, then we know that the cross ratio is preserved, that is,

$$(A, B; C, D) := \frac{AC}{AD} \Big/ \frac{BC}{BD} = \frac{A'C'}{A'D'} \Big/ \frac{B'C'}{B'D'} =: (A', B'; C', D').$$

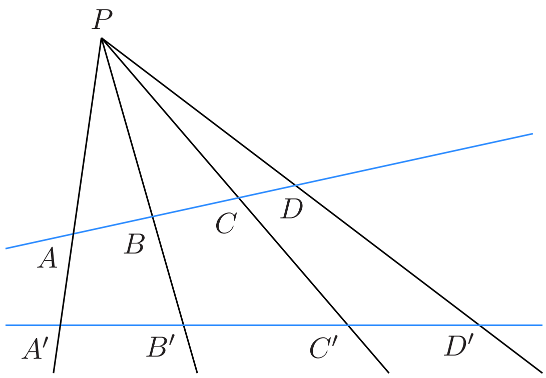
Recall that this is an immediate consequence of the Sine Rule for triangles, using which one can see that

$$\frac{AC}{AP} = \frac{\sin \angle APC}{\sin \angle PCA}, \qquad \frac{AD}{AP} = \frac{\sin \angle APD}{\sin \angle PDA},$$
$$\frac{BD}{BP} = \frac{\sin \angle BPD}{\sin \angle PDB}, \qquad \frac{BC}{BP} = \frac{\sin \angle BPC}{\sin \angle PCB},$$

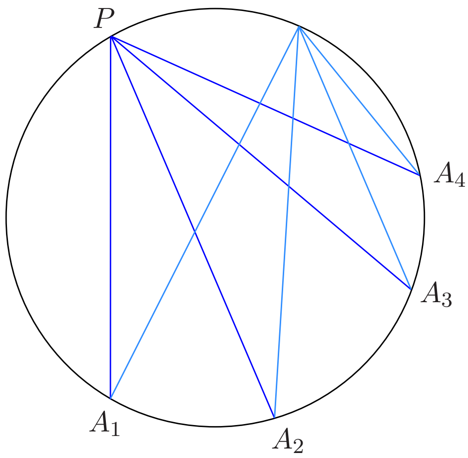
and so

$$(A, B; C, D) = \frac{\sin \angle APC}{\sin \angle APD} \Big/ \frac{\sin \angle BPC}{\sin \angle BPD}.$$

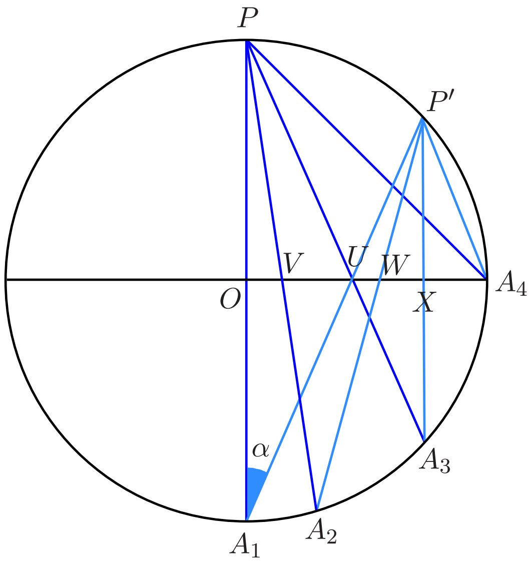
In light of this invariance, we refer to the cross ratio of the four concurrent lines instead of particular collinear points on the lines.



We also recall Chasles's Theorem, which says that if A_1, A_2, A_3, A_4 are four fixed points on a circle, and P is a movable point, then the cross ratio of the lines PA_1, PA_2, PA_3, PA_4 is a constant. This is an immediate consequence of the fact that a chord of a circle subtends equal angles at any point on its major (or minor) arc.



We refer to the geometric diagram for relativistic velocity addition below, with the labelling of points shown. X is the point of intersection of $P'A_3$ with OA_4.



As $\Delta OP'X$ is a right angled triangle, it follows that

$$OX = \cos\angle P'OX = \sin\angle POP' = \sin(2\alpha) = \frac{2\tan\alpha}{1+(\tan\alpha)^2} = \frac{2u}{1+u^2}.$$

Hence

$$\begin{aligned} UX &= OX - OU = \frac{2u}{1+u^2} - u = u\cdot\frac{1-u^2}{1+u^2} \quad \text{and} \\ WX &= OX - OW = \frac{2u}{1+u^2} - w. \end{aligned}$$

By Chasles's Theorem, we have

$$\frac{u}{1}\Big/\frac{u-v}{1-v}=\frac{OU}{OA_4}\Big/\frac{VU}{VA_4}=\frac{UX}{UA_4}\Big/\frac{WX}{WA_4}=\frac{u\cdot\frac{1-u^2}{1+u^2}}{1-u}\Big/\frac{u\cdot\frac{2u}{1+u^2}-w}{1-w}.$$

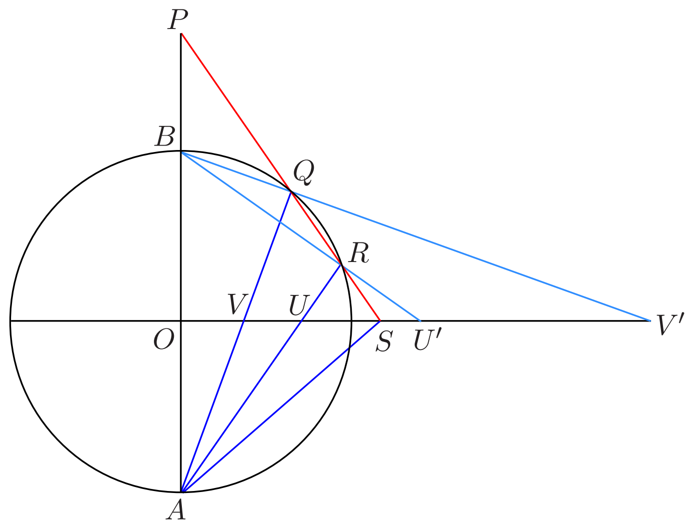
Solving for w, this yields $w=\dfrac{u+v}{1+uv}$.

5. A few remarks

We remark that that the projective perspective also sheds light on the (algebraically easily verified) formula

$$u\oplus v=\frac{1}{u}\oplus\frac{1}{v}.$$

Indeed, let us see the picture below, where U', V' are the images of the points U, V, respectively, under inversion in the circle.



Let $OS =: 1/w$. By the preservation of the cross-ratio for the four collinear lines AP, AQ, AR, AS, we obtain

$$(P,Q;R,S)=(O,V;U,S)=\frac{u}{1/w}\Big/\frac{u-v}{(1/w)-v}.$$

On the other hand, by the preservation of the cross-ratio for the four collinear lines BP, BQ, BR, BS, we obtain

$$(P,Q;R,S)=(O,V';U',S)=\frac{1/u}{1/w}\Big/\frac{(1/v)-(1/u)}{(1/v)-(1/w)}.$$

Thus

$$\frac{u}{1/w}\Big/\frac{u-v}{(1/w)-v}=(P,Q;R,S)=\frac{1/u}{1/w}\Big/\frac{(1/v)-(1/u)}{(1/v)-(1/w)},$$

which gives $w = \dfrac{u+v}{1+uv}$.

We also mention that although we have been considering $u, v \in [0,1]$ for our pictures, one may in fact take $u, v \in [-1,1]$ without any essential change in our derivations. The operation $\oplus$ is associative and the set $(-1,1)$ is a group with the operation $\oplus$.

References

[1] Jerzy Kocik. Geometric diagram for relativistic addition of velocities. *American Journal of Physics*, volume 80, number 8, page 737, 2012.

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