Phoebe Koundouri, Nikolaos Kourogenis, Nikitas Pittis

Statistical modeling of stock returns: explanatory or descriptive? A historical survey with some methodological reflections

Article (Accepted version) (Refereed)

Original citation:
Koundouri, Phoebe, Kourogenis, Nikolaos and Pittis, Nikitas (2016) *Statistical modeling of stock returns: explanatory or descriptive? A historical survey with some methodological reflections.* Journal of Economic Surveys, 30 (1). pp. 149-164. ISSN 0950-0804

DOI: 10.1111/joes.12093

Statistical Modeling of Stock Returns: Explanatory or Descriptive? A Historical Survey with Some Methodological Reflections.

Phoebe Koundouri
Department of International and European Economic Studies,
Athens University of Economics and Business
and
Grantham Research Institute on Climate Change and the Environment,
London School of Economics*

Nikolaos Kourogenis
Department of Banking and Financial Management, University of Piraeus

Nikitas Pittis
Department of Banking and Financial Management, University of Piraeus

Abstract. The purpose of this paper is twofold: first, to survey the statistical models of stock returns that have been suggested in the finance literature since the middle of the twentieth century; second, to examine under the prism of the contemporary philosophy of science, which of the aforementioned models can be classified as explanatory and which as descriptive. Special emphasis is paid on tracing the interactions between the motivation for the birth of statistical models of stock returns in any given historical period and the concurrent changes of the theoretical paradigm in financial economics, as well as those of probability theory.

Keywords: Stock Returns, Statistical Model, Explanatory Model, Scientific Explanation, Market Efficiency, Brownian Motion.

JEL classification: C58, C51, G17.

*Corresponding author: Phoebe Koundouri, Department of International and European Economic Studies, Athens University of Economics and Business, 76 Patission str., GR-104 34 Athens, Greece. email: pkoundouri@aueb.gr Tel: +302108203455.

1 Introduction

The statistical modeling of returns of financial assets has attracted a great deal of research interest for more than a century. All the statistical models of asset returns that have been advanced so far, aim at either describing or explaining the empirical regularities exhibited by asset returns. The method by which a particular model is derived is heavily affected by whether *description* or *explanation* is the aim of the model. However, before the statistician develops a statistical model to either describe or explain an empirical regularity, she must first define clearly what this regularity is. This is not an easy task, especially in view of the fact that defining a regularity contains a great deal of subjective judgement. This judgement depends on the statistician's theoretical perceptions, as they are formed by the probabilistic concepts available while the regularity is being examined. In other words, an empirical regularity is not a purely objective property of the data, but it is partly defined by the manner in which the observer interprets the patterns in the data that she was able to discern. This subjectivity results in the empirical regularities being contingent, that is, the same regularity or the same pattern of behavior may be interpreted in radically different ways at different points in time.

The preceding discussion has identified two factors affecting the emergence of statistical models of asset returns: whether the model aspires to be explanatory or descriptive and how the empirical regularities that motivate the model have been identified and interpreted. The effects of these two factors on the statistical modeling of asset returns over time are the central theme of the present paper. Specifically, this paper aims at answering the following questions: Which factors motivated the birth of the various statistical models of asset returns over time? Or more simply, where have these models come from? What do these models do? Do they explain or simply describe the identified empirical regularities of asset returns at each particular historical period? If explanation provides 'something', *over and above* description, then how can this be defined?

The analysis of the origins and the nature of statistical models of stock returns can be conveniently organized around two basic hypotheses, namely the Independence hypothesis (IN) and the Normality hypothesis (N). In a nutshell, the Independence/Normality hypothesis states that the returns of any asset across time can be thought of as realizations of time-independent/normal random variables. It is no exaggeration to say that the question of whether or not asset returns exhibit Independence and/or Normality has motivated the production of almost the totality of asset returns models that have been suggested in the literature since the middle of the twentieth century. The questions revolving around the Independence and Normality hypotheses, that will be dealt with in the sequel, include the following: To begin with, why IN and N were deemed to be interesting hypotheses to test? Was IN consistent with the view that prices are determined by means of economic laws? Related to this, what kind of economic theory was consistent with IN? Did the acceptance of IN result in a radical change in the theoretical paradigm? How was IN thought to be related to the so-called Efficient Market Hypothesis (EMH)? Were IN and N related in some way? More specifically, did IN imply to some extent N? What was the role of the central limit theorem (CLT) for the emergence of N as a property deducible from IN? In view of the empirical evidence against N, and in favor of leptokurtic distributions that has been accumulating since the early 1960s, what were the potential explanations for leptokurtosis? Related to this, what kind of limit theorems predicted non-normal limiting distributions? What were the implications of the alternative explanations for leptokurtosis for the theoretical paradigm existed at the time, and especially for the definition and measurement of financial risk? Put differently, was leptokurtosis interpreted as evidence of excess (compared to normal) instability of the economic system itself or was it simply viewed as a manifestation of our inability to conduct controlled experiments, namely to keep the experimental condition constant in repeated trials? Within the time span under study, can we identify a specific period in which the search for explanatory models was succeeded by that for descriptive ones, whose sole task was to capture the empirical properties of stock returns as perceived and interpreted at that time? Did the weaning of EMH from IN contribute to the switch from explanatory to descriptive statistical modeling?

This paper is organized as follows: Sections 2 and 3 analyze the origins and implications of IN and N, respectively. Section 3 also surveys the empirical evidence against the normality hypothesis that started to accumulate in the early 1960s. Then, it compares the two main competing "explanations" for the presence of non-normal leptokurtic return distributions, namely the "infinite-variance" and "finite-variance" ones that were put forward in the early 1960s and early 1970s, respectively. Section 4 discusses the theoretical and empirical motivation that gave birth to a different class of models than those discussed in the previous sections, the so-called factor models. These models are multivariate in nature, aiming at explaining the behaviour of stock returns in terms of the behaviour of a set of risk factors. Section 5 focuses on a new class of models, put forward in the beginning ot the 1980s, which are referred to as GARCH models. These models interpret the empirical regularities of stock returns in a fundamentally different way, namely in terms of second-order temporal dependence or dynamic conditional heteorskedasticity. The main argument of this section is that the explanatory power of the GARCH models is substantially lower than that of the aforementioned models of the previous decades due to their inability to connect the observed regularities to the chance mechanism at work. Section 6 concludes the paper.

2 Independence

In this section, we examine the origins and implications of the assumption of Independence for the returns process. As will be discussed below, the introduction of IN to financial economics was made by a non economist and initially caused embarassement and confusion to the profession. Soon afterwards, however, IN was interpreted in a radically different way and eventually became the flagship of the new theoretical paradigm in financial economics.

2.1 Empirical Motivation and Theoretical Justification

The period during which stock returns data was mainly interpreted as realizations from an Independent and Identically Distributed (IID) process covered approximately the period 1953-1982. This interpretation stemmed from the following two main sources. First, the publication of the first systematic study on the statistical properties of stock returns data by the eminent statistician Maurice Kendall, in 1953. The second source was of theoretical nature and had to do with the increasing awareness of the economic profession on the anticipatory nature of asset prices and the proper definition of economic rationality associated with it. Let us analyze the above in some detail.

Kendall (1953) analyzed a set of 22 asset price series, observed at a weekly frequency, which included both commodity and stock prices. One of his major objectives was to determine whether these prices exhibited systematic versus random behavior over time. It must be noted that in Kendall's era, the concept of probabilistic independence was implicitly equated to that of serial non-correlation. By failing to produce evidence against non-zero serial correlation coefficients, Kendall concludes that "the price series was much less systematic than is generally believed". In fact he went so far as to declare that "The series looks like a "wandering" one, almost as if once a week the Demon of Chance drew a random number from a symmetrical population of fixed dispersion and added it to the current price to determine next week's price" (1953, p. 13).

Kendall's results appeared to be rather surprising at the time. The independence of successive commodity price changes was rather difficult to be meaningfully interpreted within the existing theoretical paradigm, since it seemed to defy fundamental economic "laws". Samuelson (1973) describes the situation as follows: "...there are the "fundamentalists" and economists who think that the future algebraic rise in the price of wheat will have something to do with possibly discernible patterns of what is going to happen to the weather in the plains states, the price of nitrogen fertilizer, the plantings of corn, and the fad for reducing diets." (1973, p. 5). And also: "The economists who served as discussants for Kendall's 1953 paper were outraged, as he expected them to be, at the notion that there is no economic law governing the wanderings of price, but rather only blind chance. Such nihilism seemed to strike at the very heart of economic science." (1973, p. 18). Kendall's results seemed to echo in the field of economic theory Bertrand's (1889) famous question "is not chance the antithesis of all laws?"

However, soon after the publication of Kendall's paper, the attitude of the economics profession towards the concept of independence of successive price changes changed dramatically. This was mainly due to the theoretical developments that took place in economic theory. In particular, the economic science experienced a radical switch from a state in which "independence" was inconsistent with the existing background theory, to a new state in which "independence" was dictated by the new theory. Samuelson (1965) offered a formal proof of the statement that "properly anticipated prices fluctuate randomly". These new theoretical developments, under the guidance and influence of economists such as Samuelson and Fama, were eventually developed to a brand new theoretical paradigm which became known under the rubric of Efficient Market Hypothesis (EMH). At the heart of EMH was the "anticipatory" nature of speculative prices and the assumption that investors process available information, Φ_t, rapidly and accurately. Under EMH, investors' subjective expectations $\mathcal{E}(P_{t+1} \mid \Phi_t)$ are rational, in the Muthian sense, that is, they coincide with the true objective expectation $E(P_{t+1} \mid \Phi_t)$ (see Muth 1961). The instantaneous and accurate processing of Φ_t at t implies that the totality of available information at t is reflected on the current price P_t. In other words, there is no subset of Φ_t that will affect the (logarithmic) price change $p_{t+1} - p_t$. In a similar fashion, P_{t+1} will be determined entirely by Φ_{t+1}, and so on, while the successive (logarithmic) price changes $p_{t+1} - p_t$, $p_{t+1} - p_t$, ... will be independent. The initial argument in favour of rationality had the structure of a "reductio ad absurdum" argument. Specifically, it shows that non-rationality cannot last for long since this would imply unexploited profit opportunities. Roberts (1959, p. 7) justifies rationality through the independence of price changes as follows: "If the stock market behaved like a mechanically imperfect roulette wheel, people would notice the imperfections and, by acting on them, remove them". The assumption of rationality is crucial for the independence property of stock returns. In fact it is easy to show that non-rationality is sufficient for non-independence.

The radical change of the theoretical attitude towards IN, analyzed above, was accompanied by a similar change to the frequency and clarity of observational statements concerning the detection of independence in empirical data. Indeed, under the guidance of the new theory of efficient markets, economists started a battery of statistical tests for independence which were not limited to estimating serial correlation coefficients (see, for example, Alexander 1961, Moore 1962). Other studies tested the independence hypothesis by equating the concept of independence with that of unpredictability, and examining the extent to which professional fund managers suceed in generating systematically abnormal returns (see Jensen 1968). Relatively soon a solid body of evidence in favor of the independence hypothesis was accumulated. Fama (1965b) went so far as to declare "I know of no study in which standard statistical tools have produced evidence of important dependence in series of successive price changes" (1965b, p. 57). This statement, however, seems to exaggerate on the level of aggreement that was achieved in the empirical literature with respect to the empirical validity of the independence hypothesis, since there were quite a few studies in which the evidence for independence was at best mixed (see, for example, Houthakker 1961, Cootner 1962, and Steiger 1964).

In concluding this section, it is of some historical interest to note that some studies suggesting that the independence hypothesis was consistent with rationality, existed in the literature even before 1953. Had the theoretical economists paid more attention to the studies by Working (1934), Taussig (1921), and especially Bachelier (1900), they would have been protected from the embarassement caused by Kendall's results. To this end, an important study on the anticipatory nature of asset prices, published in 1958 by Working, seems to have paved up the way for the emergence of the efficient market hypothesis.

3 Normality

In this section, we examine the origins and implications of the assumption of Normality for the returns generating process.

3.1 Empirical Motivation and Theoretical Justification

Once the independence of asset returns was recognized to be a consequence of economic rationality, the property of Gaussianity of the distribution of returns immediately followed. This in turn was due to the probabilistic background theory available at that period, which was dominated by the Central Limit Theorem (CLT). More specifically, a consequence of the idea that an asset price is continuously bombarded by independent news is that (logarithmic) price change within a given interval, say a day, is the sum of the elementary returns from transaction to transaction occured within this interval. To this end Osborne (1959) argues as follows: "This nearly normal distribution in the changes of logarithm of price changes suggests that it may be a consequence of many independent random variables contributing to the changes in values. The normal distribution arises in many stochastic processes involving large numbers of independent variables, and certainly the market place should fulfill this condition, at least" (Osborne 1959, p. 151). Formally, the random variable R_t, denoting the returns of day t, may be thought of as the sum of the elementary rates of return ξ_{tj} in that day, $R_t = \sum_{j=1}^{n} \xi_{tj}$, where n denotes the number of transactions in day t. The assumption of independence of the random variables ξ_{tj} together with some additional moment conditions, such as those in Feller (1935), allowed the application of the Central Limit Theorem (CLT), according to which the limiting distribution of R_t is the normal. Fama (1963) states this reasoning as follows: "If the price changes from transaction to transaction are independent, identically distributed random variables with finite variance and if transactions are fairly uniformly spaced through time, the central-limit theorem leads us to believe that price changes across differencing intervals such as a day, a week, or a month will be normally distributed since they are simple sums of the changes from transaction to transaction" (1963, p. 420). Normality was formally proved (in continuous time) by Osborne (1959) and Bachellier (1900).

Studies supporting the normality (or approximate normality) hypothesis for stock returns include Osborne (1959) and Larson (1960). Kendall's (1953) results were also supportive for the normality hypothesis in the price, rather than logarithmic price, changes. However, all these authors expressed with one way or another some reservations concerning the extent to which the normal distribution does in fact adequately fit the data. Osborne refers to the empirical distributions as "nearly normal" (1959, p. 129). Larson expressed more serious doubts by arguing: "The distribution ...has mean near zero, and is symmetrical and very nearly normally distributed for the central 80 per cent of the data, but there is an excessive number of extreme values. Also, some of these are quite extreme, being 8 or 9 standard deviations from the mean" (1960, p. 318). Although Kendall himself stated clearly that distributions look "very much like a normal form" (1953, p. 23), he nevertheless identified cases in which "The distributions are accordingly rather leptokurtic" (1953, p. 13).

3.2 Empirical Puzzles, Evidence of Non-Normality and the Role of Central Limit Theorem

Apart from the aforementioned comments on the approximate character of the normal distribution as a model for asset returns, there were studies whose results were substantially more negative for the normality hypothesis. For example, in commenting on Osborne's (1959) results, Alexander (1961) argues as follows: "But Osborne did not rigorously test the normality of the distribution. A rigorous test, for example the application of the chi-square test to some of the data used by Osborne, would lead us strongly to dismiss the hypothesis of normality" (1961, p. 16). This type of non-favorable results for the "'normality hypothesis" caused some confusion to the newly established efficient market paradigm.

The preceding discussion suggests that if the much desired reconcilation of independence with non-normality of asset returns were to be achieved, the reasons that caused failure of CLT (in the presence of independence) had to be identified. This identification depended on the following two questions: (i) Did the state of the art in probability theory at the beginning of the 1960s provide results showing the conditions under which a sum of independent random variables converges to a non-normal distribution? (ii) Were the econometricians at the time aware of such results? As will be shown below, the answer to the first question is affirmative whereas that of the second question is negative. In fact, probability theory had already identified at least two cases in which the limiting distribution of a sum of independent random variables is a non-normal distribution. The first of these cases is the one developed by Lévy (1925) and is the main reason why Bachelier's early derivation of Brownian motion was incomplete. This case was adopted in 1963 by Benoit Mandelbrot. However, the reconcilation between independence and non-normality, put forward by Mandelbrot, did not come without a price for the existing theoretical paradigm.

3.3 Leptokurtosis I: Infinite Variance

In 1963, Benoit Mandelbrot expressed forcefully and without any reservations the argument that the distribution of stock returns was not Gaussian. Moreover, Mandelbrot offered an elegant explanation of the observed leptokurtosis of R_t, which - importantly-was consistent with the theoretically desirable independence hypothesis. However, as will be discussed below, Mandelbrot's interpretation did not come without any cost for the existing theoretical paradigm. Specifically, Mandelbrot argued that the only assumption that had to be made in order to obtain leptokurtosis is that (the independent) ξ_{tj}s have infinite variance. Why did he make such an assumption? We may identify the following two reasons that motivated Mandelbrot's choice: (i) Mandelbrot was aware of the probability theory results of Lévy (1925) according to which the family of limiting distributions for sequences of sums of independent and identically distributed random variables is the so-called Stable family, $\mathcal{L}$, with the Gaussian distribution being just a member (though the most important one) of this family. Quite importantly, the normal is the only distribution in this family with a finite variance (see also Khintchine, 1933). The extent to which a particular sequence converges to the Gaussian or some other member of $\mathcal{L}$ depends on whether the random variables of this sequence possess finite second moments. Consequently, in order for Mandelbrot to derive the desired result (convergence to a non-Gaussian leptokurtic Stable distribution) he had to abandon the assumption that the ξ_{tj}s have finite variances. (ii) Mandelbrot was willing to accept the empirical implications of the infinite-variance assumptions, namely that elementary stock returns can be arbitrarily large. Equivalently - in a continuous time framework - Mandelbrot was at peace with the assumption that stock price paths are discontinuous.

The preceding discussion implies that the probability theory available at the time had already produced the necessary theoretical results for a potential explanation of the observed leptokurtosis. More importantly, this explanation did not have to sacrifice the independence hypothesis. The only change in the set of the existing assumptions that had to be made is to replace the finite variance assumption of the elementary stock returns with the infinite-variance one. To this end, Cootner (1964) speaks apologetically on behalf of the financial economists of the time about "... our guilt at our failure to appreciate the possibility of non-Gaussian central limit theorems ..." (p. 413).

How inoccuous was the adoption of the infinite-variance hypothesis for the theoretical paradigm existed at the time? To answer this question, we must analyze the implications of this assumption for the concept and existing measures of financial risk? These implications will be dealt with in detail in subsequent sections. For the moment, suffices to say that economists did not rush to embrace Mandelbrot's interpretation, despite the fact that this interpetation left the independence assumption intact. Cootner (1964) summarizes the discomfort that the infinite-variance assumption caused to the academic community as follows: "Mandelbrot, like Prime Minister Churchill before him, promises us not utopia but blood, sweat, toil and tears. If he is right, almost all of our statistical tools are obsolete - least squares, spectral analysis, workable maximum likelihood solutions, all our established sample theory, closed distribution functions. Almost without exceptions, past econometric work is meaningless" (Cootner, 1964, p. 337). Although Mandelbrot had succeeded in producing an empirically adequate model for stock returns, his model was not fully compatible with the emerging paradigm of "efficient markets with controllable risk" and its adoption would have meant the collapse of a substantial part of the paradigm itself. This was a rather unwelcome outcome as "surely, before consigning centuries of work to the ash pile, we should like to have some assurance that all our work is truly useless. If we have permitted ourselves to be fooled as long as this into believing that the Gaussian assumption is a workable one, is it not possible that the Paretian revolution is similarly illusory?" (Cootner 1964, p. 337).

It is worth noting that apart from leptokurtosis Mandelbrot was the first to detect, another empirical regularity of asset returns, namely that "large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes." (Mandelbrot, 1963, p. 418). This regularity, usually referred to as "volatility clustering" went largely unnoticed for almost two decades, that is, between the early 1960s and early 1980s. This effect was usually referred to as "curious behavior of volatility" and was thought of as another manifestation of the "infinite variance" effect or, alternatively, as a symptom of the non-uniformity of transactions over time (see below). As will be discussed in the last section of the paper, a re-interpretation of this regularity formed the basis for the emergence of a new statistical paradigm for describing asset returns, at the heart of which was the concept of "conditional heteroskedasticity".

3.3.1 The "Silent" Decade: 1963-1973

Despite Cootner's objections mentioned above, Mandelbrot's clear statement on the presence of leptokurtosis in the empirical distributions of stock returns, along with his interpretation of this leptokurtosis delivered a severe blow against the "approximate normality" view, which prevailed in the literature up to 1963. His analysis forced the econometricians to look the naked truth about leptokurtosis and recognize it as a clear and distinct empirical regularity of stock returns. In his comment on Mandelbrot's paper, Cootner (1964) describes the situation as follows: "Dr Mandelbrot's series of papers on the application of Paretian distributions to economic phenomena has forced us to face up in a substantive way to those *uncomfortable* empirical observations that there is little doubt most of us have had to sweep under the carpet up to now." (1964, p. 333). As a result, econometricians chose to remain silent on this issue until they were able to come up with an explanation of this leptokurtosis that does not have to abolish the finite-variance hypothesis. In fact, between 1963 and 1973, the only published papers on the type of stock returns distribution were by Fama (1965c), Fama and Blume (1966), Teichmoeller (1971) and Officer (1972), all of which advancing further either theoretically or empirically the Stable Paretian hypothesis.

3.4 A "Finite-Variance" Explanation of Leptokurtosis

The long awaited "finite-variance" explanation of leptokurtosis finally came in the beginning of 1970s with the works of Praetz (1972) and Clark (1973) although the roots of the basic idea can be traced in Press (1968). These authors followed a line of reasoning similar to that of Mandelbrot. In particular, they also recognized the fact that since R_t is the sum of independent random variables ξ_{tj}, the non-normality of the distribution of R_t implies failure of CLT. But instead of assuming that the CLT failure was due to the non-finiteness of the variance of ξ_{tj}, they put forward the hypothesis that CLT failure was due to the *randomness* of the number of summands, n. Let us analyze Clark's ideas in slightly more detail: As Mandelbrot employed non-standard limit theorem results to interpret the observed leptokurtosis, so did Clark. However, instead of employing Paretian limit theorems, Clark employed limit theorems for *random* sums of random variables, that had appeared in the probability literature since 1948. More specifically, for each t, let $\{\xi_{t,j}\}_{j\geq 1}$ be an iid sequence of random variables with finite $E(\xi_{t,j}) = \mu_\xi$ and $Var(\xi_{t,j}) = \sigma_\xi^2 > 0$. Moreover, let $\{N_n\}_{n\geq 1}$ be a sequence of non-negative, integer-valued random variables. The random sum process is defined as

$$R_{t,N_n} = \sum_{j=1}^{N_n} \xi_{t,j}. \tag{1}$$

The question that was raised in the relevant probability theory was the following: Under what conditions does the properly normed and centered random sum, R_{t,N_n}, converge in law to some random variable, Z, and, further, under what additional conditions is Z distributed as $N(0,1)$. Robbins (1948) obtained sufficient conditions for the convergence in law of the properly centered and normed sequence, R_{t,N_n}, to the normal distribution, under the assumption that N_n is independent of the summands, $\xi_{t,1}, \xi_{t,2}, \ldots$. Renyi (1960) and Blum, Hanson and Rosenblatt (1963) derived sufficient conditions that are similar to those of Robbins (1948) without the assumption of independence between N_n and the summands. Whether the properly centered and normed sequence, R_{t,N_n} converges to the $N(0,1)$ or not depends on the variability of N_n around n as n increases. Specifically, if N_n exhibits a "substantial rate of variation" around n, in the sense that the condition

$$p \lim_{n \longrightarrow \infty} \frac{N_n}{n} = 1 \tag{2}$$

is violated, then the limiting distribution of the properly centered and normed sequence, R_{t,N_n} is not the standard normal and depends on the distribution of N_n. Clark assumes that $N_n = [Zn]$ where Z is a random variable with mean 1 and variance $\Gamma > 0$ and [] denoting "the largest integer less than". It is easy to show that under the aforementioned assumptions on the random variables $\xi_{t,j}$, and for any given realization of Z, we have that

$$\frac{1}{\sigma_\xi \sqrt{n}} R_{t,N_n} = \sqrt{\frac{[nZ]}{n}} \frac{1}{\sigma_\xi \sqrt{[nZ]}} \sum_{j=1}^{[nZ]} \xi_{t,j} \xrightarrow{L} N(0, Z).$$

We observe that the variance of the limit distribution is the random variable Z, hence the unconditional limiting distribution is a mixture of normals. Such a distribution may well be leptokurtic. Clark (1973) justified the appeal to random limit theory on the grounds that transactions are not spread uniformly across time but instead display substantial variation.

The concept of "substantial variation" is central in delivering non-normal limiting distributions. Put differently, randomness of N_n *per se* does not ensure convergence to a non-normal distribution. As mentioned above, if (2) is satisfied then the limiting distribution of the properly centered and normed sequence R_{t,N_n} is $N(0, 1)$. The importance of (2) calls for a further ellaboration of the concept of substantial variation. More specifically, condition (2) imposes restrictions on the variability of N_n around n, as n increases. This property, referred to as "theoretical variation" refers to the relationship, $g(n)$, between the variance of N_n and n, that is $Var(N_n) = g(n)$. It is the functional form of $g(n)$ that affects the convergence properties of the random sum sequence. Koundouri and Kourogenis (2011) assume that N_n is defined by

$$N_n = n^r U + n, \text{ for some } 0 \leq r \leq \frac{1}{2} \tag{3}$$

where U is a random variable with $E(U) = 0$ and $Var(U) = c > 0$. This assumption implies the following function $g()$:

$$g(n) = Var(N_n) = cn^{2r} \ . \tag{4}$$

It can be shown that for $0 \leq r < \frac{1}{2}$, that is, when theoretical variation is moderate, the properly centered and normed sequence R_{N_n} converges to $N(0,1)$. On the other hand, for $r = \frac{1}{2}$, that is, when theoretical variation is substantial, the limiting distribution is a mixture of normals.

Once a "finite-variance" explanation of leptokurtosis was available in terms of substantial variation in the number of elementary transactions across days, the empirical literature on the type of stock returns distribution took off. Blattberg and Gonedes (1974) assume that Z^{-1} follows a gamma-2 distribution which implies that the resulting limiting distribution of $\frac{1}{\sigma_\xi \sqrt{n}} R_{t,N_n}$ is the student. Kon (1984) offers evidence in favor of the assumption that the stock returns distribution is a discrete mixture of normals.

4 Factor Models

Apart from the univariate statistical models of stock returns analyzed in the previous sections, a class of multivariate models, usually referred to as "factor models" , was introduced in the begining of the 1970's. The multiple factor model (MFM) is defined as follows:

$$R_{i,t} = a_i + \beta_{1,i} M_{1,t} + \beta_{2,i} M_{2,t} + ... + \beta_{k,i} M_{k,t} + u_{i,t}, \ i = 1, 2, ..., n, \tag{5}$$

where $M_{l,t}$, $l = 1, 2, ..., k$ is the unanticipated change from time $t-1$ to time t of the risk factor, $X_{l,t}$, that is $M_{l,t} = X_{l,t} - \mathcal{E}(X_{l,t} \mid \Phi_{t-1})$ and $\mathcal{E}(X_{l,t} \mid \Phi_{t-1})$ are the agents' subjective expectations formed at time t-1. Concerning, the non-systematic term, $u_{i,t}$, a minimal assumption is that it is a martingale-difference with respect to the publicly available information set Φ_{t-1}. MFMs can be thought of as multi-period extentions of the one-period factor model originated in the context of the Arbitrage Pricing Theory (APT) developed by Ross (1976). In fact, APT is built on the hypothesis that returns for all assets i, $i = 1, 2, ..., m$, are linearly related to a common set of k risk factors along with the hypothesis that there are no arbitrage opportunities in the market. The main result of APT is that expected returns are linearly related to $\beta_{1,i}, \beta_{2,i}, ..., \beta_{k,i}$. To keep the analysis as simple as possible, for the rest of this section we assume the presence of a single risk factor, that is $l = 1$.

It is important to note that MFM was born out of empirical rather than theoretical considerations. Roll and Ross (1980) state explicitly that MFM is motivated by "the single most widely-acknowledged empirical regularity of asset returns, their common variability" (1980, p. 1073). However, this statement does not do justice to the theoretical origins of MFM. Even if all that motivated MFM was to account for the common variability of returns, the specific way by which MFM does so is causal. Specifically, MFM accounts for the common variabilty of stock returns by adopting the "common cause principle", (CCP). This principle states that if two random variables, say R_{1t} and R_{2t} are found to be correlated then either R_{1t} causes R_{2t} or R_{2t} causes R_{1t} or there is a third variable say $M_{1,t}$ which causes both R_{1t} and R_{2t}. In the third case, $M_{1,t}$ "screens-off"the correlation between R_{1t} and R_{2t}. This means that conditioning on the common cause renders R_{1t} and R_{2t} independent.

It is important to emphasize that the CCP interpretation of MFM assumes that $M_{1,t}$ is the *only* common cause of $R_{i,t}$, $i = 1, 2, ..., n$. This in turn implies that the observed correlations among the $R_{i,t}$'s stem solely from their common causal relationship to $M_{1,t}$. This causal interpretation of MFM imposes a restriction on the error terms $u_{i,t}$, $i = 1, 2, ..., n$ (which is sometimes ignored in the literature), namely that their correlation matrix, Σ_u, is diagonal. The diagonality of Σ_u may be thought of as a "theoretical" restriction reflecting the view that the common risk factor $M_{1,t}$ "screens off" the correlations among the $R_{i,t}$'s.

Concerning the issue of how the agents' subjective expectations are formed, MFM assumes rational expectations (REH). REH asserts that investors' subjective expectations, $\mathcal{E}(X_t \mid \Phi_{t-1})$, coincide with the objective mathematical expectations, that is $\mathcal{E}(X_t \mid \Phi_{t-1}) = E(X_t \mid \Phi_{t-1})$. MFM in conjuction with REH implies that $\{R_{i,t}\}$ (or more accurately $\{R_{i,t} - a_i\}$) is a martingale difference sequence (MD). This can be easily seen by operating on both sides of (5) with the objective operator $E(\cdot \mid \Phi_{t-1})$. On the contrary, if $\mathcal{E}(X_t \mid \Phi_{t-1}) \neq E(X_t \mid \Phi_{t-1})$, then it can be shown that

$$E(R_{i,t} \mid \Phi_{t-1}) = a_i + \beta_{1,i}\mathcal{E}(X_{1,t} \mid \Phi_{t-1}).$$

In such a case, $\{R_{i,t} - a_i\}$ is not MD with respect to Φ_{t-1}. It is important to note that in the context of MFM, the MD property of asset returns (which is a weak form of independence) is "derived", rather than imposed. Indeed, MD is the result of REH with the latter being directly related to EMH.

One important caveat of MFM that has serious implications for its causal interpretation concerns the identity of the risk factors. Roll and Ross themselves raise the question "What are the common or systematic factors?" (1980, p. 1077). The finance literature has approached the problem of the specification of risk factors in mainly three alternative ways:

(i) The first approach does not aim at identifying the true set of risk factors, or even a subset of it, but rather to "approximate" them with another set of approximating variables. In the context of this approach one can distinguish solutions of purely statistical nature such as principal component analysis (see Chamberlain and Rothschild 1983, Connor and Korajczyk 1985, 1986). Another popular solution within the first approach amounts to approximating the risk factors by the so-called "mimicking portfolios." A well-known example in this direction is the approach of Fama and French (1993) in which characteristics such as the firm size or the book-to-market ratio are supposed to capture the effects of some unobservable risk factors (see also, Rosenberg, Reid and Lanstein 1984, Chan, Hamao and Lakonishok 1991). An immediate implication of such an approach is that the mimicking portfolios are by definition symptomatic factors in the sense that if the true causal factors had been included, the mimicking portfolios would become causally irrelevant. In other words, the true factors would screen off the mimicking portfolios. This in turn raises questions on the explanatory power, in terms of causal relevance, of the factor models that employ mimicking portfolios as factors.

(ii) The second approach assumes that the set $\mathcal{X}$ of the true causal factors is a subset of the set $\mathcal{D}$ of all possible observable macroeconomic variables. Potential identification of $\mathcal{X}$ takes place in two steps. First, linear time series regressions are run in which the dependent variable is $R_{i,t}$ and the independent variables are those in $\mathcal{D}$. In this step the set $\mathcal{D}^s$ of statistically relevant variables is determined. The second step aims at identifying the subset $\mathcal{D}^c$ of $\mathcal{D}^s$ which includes only the variables that account for the cross sectional variation of stock returns, that is the set of variables that are actually priced by the market. The final set $\mathcal{D}^c$ of the surviving variables in both steps of the screening-off type procedure outlined above are assumed to be the causally relevant factors. Chen, Roll and Ross (1986), follow this approach and identify the following risk factors for stock returns: unanticipated changes in inflation, unanticipated changes in GDP, unanticipated changes in the default premium of corporate bonds, and unanticipated shifts in the yield curve. A rather surprising result emerging from the study of Chen, Roll and Ross (1986) is that "well established " factors such as the value-weighted New York Stock Exchange Index turn out to be symptomatic since they are screened off in the second step by apparently, more relevant macroeconomic factors.

(iii) The third approach aims at identifying $\mathcal{X}$ by means of theoretical considerations. More specifically, in the context of this approach the set $\mathcal{X}$ is derived from a relevant theory. A prominent example of such an approach is the celebrated capital asset pricing model (CAPM) in which there is a unique risk factor, namely the returns of the market portfolio.

5 The New Era: Conditional Heteroskedasticity and Non-Linear Dependence

The 1980s witnessed the generation of a new class of statistical models for asset returns which aim at capturing the non-linear dependence observed in asset return series. More specifically, the major breakthrough of this decade was a probabilistic re-interpretation of the volatility clustering effect, first observed by Mandelbrot as far back as 1963. In this decade, the "curious behavior of volatility" observed by previous authors was re-interpreted in a fundamentally different way. Instead of seeing it as a manifestation of "infinite-variance" or "non uniformity of transactions over time", the new view interpreted the "volatility clustering" effect as temporal non-linear dependence arising from the conditional variance. In other words, the observed data could have been produced by a strictly (or even second-order) stationary process which exhibits conditional heteroskedasticity. The ARCH model of Engle (1982) and its extentions (see the GARCH(p,q) model of Bollerslev 1986, etc.) offered a convenient way for describing such processes. More specifically, the well-known GARCH(1,1) model takes the following form:

$$R_t = c + \varepsilon_t,\ \varepsilon_t = h_t \nu_t,\ h_t^2 = a_0 + a_1 h_{t-1}^2 + a_2 \varepsilon_{t-1}^2\ , \tag{6}$$

where $a_0 > 0$, $a_1 \geq 0$, $a_2 \geq 0$, and $\nu_t \sim IID(0, \sigma_\nu^2)$.

The process $\{R_t - c\}$ where $\{R_t\}$ described by (6) is martingale difference. Early attempts to investigate the probabilistic properties of the process defined by (6) focused mainly on (i) showing that a GARCH process is leptokurtic and (ii) establishing the conditions under which this process is covariance stationary. The first results showed that $\{R_t\}$ is second-order stationary if $a_1 + a_2 < 1$, in which case the unconditional variance of R_t exists and is equal to $a_0/(a_1 + a_2)$. However, the estimates of these parameters were found to be in the viscinity of the unit root area, that is they point towards that $a_1 + a_2 \simeq 1$. These estimates gave rise to the so-called Integrated GARCH process (IGARCH), that is, a process described by (6) with $a_1 + a_2 = 1$. This process is clearly not covariance stationary since the unconditional variance is infinite although it is still strictly stationary and ergodic (see Nelson 1990). The near to unit root estimates of the conditional variance revived the debate on the "infinite-variance" issue. In fact, the "infinite-variance" problem, which came out of the front door with Clark's explanation re-emerged in the context of the IGARCH model from the rear window. Was Mandelbrot right? Should the presence of a unit root in the conditional variance be interpreted as supporting evidence - obtained from a brand new statistical method - for the Mandelbrotian infinite variance hypothesis. The answer to this question is a definitive "no". Kourogenis and Pittis (2008) showed that the unconditional variance of an IGARCH process is "barely infinite", meaning that all the moments with order less than two exist! In the context of (6) with $a_1 + a_2 = 1$ the barely infinite variance hypothesis is stated as $E\left|R_t\right|^{\delta} < \infty$ for every $0 \leq \delta < 2$. The difference between the barely infinite variance IGARCH process defined by (6) with $a_1 + a_2 = 1$ and the independent Stable Paretian process proposed by Mandelbrot is huge as far as their asymptotic properties are concerned. More specifically, in spite of having (barely) infinite variance, an IGARCH process is in the domain of the attraction of the normal law.

It is worth noticing that the "objective" regularities exhibited by stock returns do not seem to have changed in any fundamental way from the beginning of the twentieth century until today. In other words, the fundamental empirical regularities, namely, leptokurtosis of empirical distributions (histograms), very low or zero sample autocorrelation coefficients, and volatility clustering seem to characterize high-frequency asset returns data for any (sufficiently long) sub-sample of this period. What changed drastically was the probabilistic interpretation of these regularities. From the Paretian IID interpetation of Mandelbrot or the Mixed-Normal IID interpretation of Clark the literature took a sharp turn in adopting the GARCH interpretation of Engle and Bollerslev. The implications of this change for the ability of GARCH to explain the observed regularities are analyzed below.

5.1 The Explanatory Status of the Statistical Models of Stock Returns

The main question addressed in this section is the following: how does a statistical model explains an empirical regularity or in what sense a statistical model can be explanatory? More specifically, what is the set of criteria that a statistical model should satisfy in order to be characterized as "explanatory"? Furthermore, is this set unique, or are there many sets of alternative criteria with each one defining a different model of explanation?

In their widely known model of Deductive-Statistical (DS) explanation, Hempel and Oppenheim (1948) argue that an explanation of a statistical regularity is achieved by showing that it can be deduced (or follows with necessity) from a broader regularity or, in other words from one or more statistical laws (and initial conditions in some cases). This means that an explanation of an empirical regularity is an argument to the effect that the regularity to be explained (the explanandum) was to be expected by reason of certain explanatory facts (the explanans) which include at least one more general statistical regularity. Put it slightly differently, the explanation of the regularity of interest amounts to sub-suming it under a broader empirical law (or laws), which is usually referred to as the covering law(s). In the context of the DS model, the characteristic feature of explanation is that the explanadum is deducible from the explanans.

It must be noted that all the statistical models of stock returns summarized in the previous sections - including GARCH - satisfy the explanatory criteria of the D-S model of explanation. Indeed, all these models explain leptokurtosis by showing that this regularity is deduced from one or more broader statistical regularities. More specifically, both Mandelbrot's and Clark's models (Type-I models) explain leptokurtosis of R_t by deducing it from the properties of elementary returns ξ_{tj}, whereas GARCH models (Type-II models) explain leptokurtosis by deducing it from the second-order temporal dependence properties of $\{R_t\}$ itself.

However, the two classes of models mentioned above differ in one important respect. Type-I models investigate the properties of the "chance mechanism at work" by introducing assumptions in terms of "unobservable" entities, namely, the elementary returns ξ_{tj}. On the contrary, Type-II models focus solely on the "outer behavior" of the chance mechanism without exploring its "inner structure". Railton (1978, 1981) argues that the principle aim of explanation is to enhance our understanding of "how the world works." He puts forward the so-called Deductive Nomological Probabilistic (DNP) model of explanation, according to which the mere subsumption of a narrow regularity, say $L1$, under the broader regularity, $L2$, does not constitute an explanation of $L1$ unless $L2$ is backed up with "an account of the mechanism(s) at work." Specifically, $L1$ is explained by placing it within a web of "inter-connected series of law-based accounts of all the nodes and links in the causal network culminating in the explanandum, complete with a fully detailed description of the causal mechanisms involved in the theoretical derivations of all covering laws involved" (1981, p. 174). This means that in the case under study, GARCH in itself cannot form the sole basis for a satisfactory explanation of leptokurtosis unless GARCH is derivable from a theory on the causal mechanism at work.

To sum up: the main explanatory virtue of both Mandelbrot and Clark's models stems from the fact that the empirical regularities of observed asset returns, R_t, were deduced from fundamental laws (assumptions) governing the behavior of the elementary returns ξ_{tj}. In other words, the covering law dictating the behavior of R_t did not emerge inductively, that is, it was not inferred from the observed properties of R_t, but rather deductively from "first principles" concerning the properties of the constituent parts of the chance mechanism at work. On the contrary, GARCH emerged from the probabilistic interpretation of the regularities exhibited by the R_ts themselves, without any attempt to account for the chance mechanism at work. In other words, the birth of GARCH conforms to the "narrowly inductivist view" according to which hypotheses should be inductively inferred from the available evidence (see, Hempel, 1965 for a critique of this view).

The preceding analysis implies that GARCH does not explain leptokurtosis in the stricter D-N-P sense. As already mentioned, D-N-P requires the broader regularity, namely, conditional heteroskedasticity in terms of which the explanation of leptokurtosis is achieved, to be deduced from a theoretical account of the chance mechanism at work. The theoretical origins of GARCH are poor if non-existent. This model was born out of purely empirical considerations of the behavior of stock returns. The success of this model does not stem from its theoretical underpinnings, but rather from its ability to generate forecasts for the volatility of stock returns. It is noteworhty that, despite the widespread adoption of the GARCH models in the empirical finance literature, these models originated in the context of empirical macroeconomics. Engle (2003) describes the genesis of this model as an attempt to "...get variances into macroeconomic models, because some people thought it was actually not the expected value of economic variables but rather their variability that was relevant for business cycle analysis" (2003, p. 1176). Indeed, the first application of this model was made to the UK inflation rate (see Engle 1982). Moreover, the GARCH models were not developed as a direct attempt to capture the volatility patterns that were observed in macroeconomic time series but instead with the aim of obtaining a powerful test for detecting bilinearity. As Engle remarks "...I discover the model from the test, rather than the other way round" (2003, p. 1177).

6 Conclusions

One of the most interesting problems in the philosophy of science is that of finding criteria that define adequate statistical explanations (either of single events or empirical regularities). In particular, the statistical models of stock returns that aspire to be explanatory seem to be motivated by the aim of explanation of empirical regularities rather than that of single events. A critical analysis of the extensive literature on stock returns since the middle of the twentieth century leads to the distinction between explanatory and descriptive stock returns models, and in particular, to the identification of the models that satisfy the criteria of explanatory adequacy, set forth by alternative theories of scientific explanation.

The statistical modeling of asset returns was revolving around three interconnected axes. First, an empirical regularity, $\mathcal{R}$, was detected. Second, $\mathcal{R}$ was given a probabilistic intepretation in terms of a set, $\mathcal{P}$, of properties of a sequence $\{R_t\}_{t\geq 1}$ of random variables. Note that during the aforementioned period, the same empirical regularity was given alternative probabilistic interpretations. Third, a statistical model, $\mathcal{M}$, that accounts for $\mathcal{P}$ was suggested. Whether or not $\mathcal{M}$ explains $\mathcal{R}$ depends on the way by which $\mathcal{M}$ is produced. Specifically, if $\mathcal{M}$ is derived from a theoretical account of the chance mechanism at work, then $\mathcal{M}$ satisfies the conditions for explanatory adequacy imposed by the Deductive-Nomological-Probabilistic model of explanation. In such a case, $\mathcal{M}$ is deemed to be explanatory. On the other hand, if $\mathcal{M}$ is inductively inferred from the available data, without having any theoretical underpinnings, $\mathcal{M}$ is deemed to be descriptive.

Our critical examination of the origins of the statistical models put forward in the period 1950-1980, showed that these models, with their leading examples being Osborne's NIID (1959), Mandelbrot's Stable-Paretian (1963) and Clark's mixture-of-normals (1973), enjoy a sufficiently high D-N-P explanatory status. In addition, the explanatory value of these models is further enhanced by their ability to answer counterfactual questions, that is, questions about the conditions under which the empirical regularities (explananda) generated by these models would have been different.

A common characteristic of all the aforementioned authors - on which the explanatory feature of their models was based - was their insistence on deriving their models for observable returns from alternative sets of primitive assumptions concerning the behaviour of elementary returns ξ_{tj}. In other words, each of these authors derived his corresponding model from his own theoretical account of the returns generating mechanism. The main differences among the aforementioned theoretical accounts center around the properties of elementary transactions ξ_{tj}. Osborne assumed that ξ_{tj} cannot be arbitrarily large and that the number of transactions across time is constant. Mandelbrot retained the constancy of transactions over time but he took the radical view that elementary transactions can be arbitrarily large, a direct implication of which is that the variance of stock returns is infinite. Finally, Clark in his attempt to salvage the finite-variance hypothesis, he elevated the hypothesis of "the substantial temporal variation of the number of transactions" as the most fundamental one concerning the generation of stock returns. Osborne's explanation - being in the spirit of the original explanation of Bachelier - aimed at explaining the empirical regularities of asset returns identified at the time, namely "independence" and "normality." On the other hand, both Mandelbrot and Clark focused on explaining the "new" empirical regularity identified by the beginning of the 1960s, namely, leptokurtosis of the asset return distributions. The deductive power in all the aforementioned explanations stemmed from limit theorems. Osborne employed the classic central limit theorem, Mandelbrot used the limit theorems for sequences of random variables with infinite variance and Clark utilized limit theorems for random sums of random variables.

The realism of the aforementioned assumptions was the subject of heated debate between the few economists who supported Mandelbrot's explanation and those (the majority) who supported Clark's. The reason is that Mandelbrot's interpretation had unpleasant implications for almost all the major theoretical concepts and models of the finance theory that existed at that time. On the other hand, Clark's interpretation identified the origins of leptokurtosis with the inability of the experimenter to conduct controlled experiments, that is, to keep the number of transactions approximately constant over time.

The arrival of the 1980s witnessed new developments in the statistical modeling of asset returns. The recognition that the Efficient Market Hypothesis requires returns to be just a martingale difference process, led to removal of the need for independence in returns. In the early 1980's, the assumption of martingale difference combined with the empirical evidence of non-constant volatility, gave rise to the conditionally heteroskedastic models (GARCH) by Engle and Bollerslev. These models were motivated mostly by an attempt to describe the stylized facts of assets returns, rather than an attempt to explain their generating mechanism. The rise of GARCH models marks the prevalence of the statistical-inductive approach over the explanatory-deductive one.

Of course, the evolution of the modeling of stock returns does not end; the hunt for a model which comes closer to Railton's ideal explanatory text is indeed unended! Our detailed analysis of the history of this evolution seems to uncover at least one source of inspiration for the development of new models for stock returns: the need for the joint exploitation of both substantive and statistical information in the specification of these models. As Wold (1969) had insightfully remarked forty years ago, "the construction process (of models) alternates several times between the empirical and theoretical sides, building up the model by layer after layer of empirical and theoretical knowledge" (1969, p. 437).

References

Alexander, S. 1961. Price Movements in Speculative Markets: Trends or Random Walks. *Industrial Management Review* 2:7–26.

Bachelier, L. 1900. Théorie de la spéculation - Gauthier-Villars (Theory of speculation in: P. Cootner, ed., 1964, The random character of stock market prices, L. Bachelier - 1900 - MIT Press, Cambridge, Mass).

Bertrand, J. 1889. *Calcul des probabilités.* Paris: Gauthier-Villars.

Blattberg, R., and N. Gonedes. 1974. A Comparison of the Stable and Student Distributions as Statistical Models for Asset Prices. *Journal of Business* 47: 244–280.

Blum, J., D. Hanson, and J. Rosenblatt. 1963. On the central limit theorem for the sum of a random number of independent random variables. *Probability Theory and Related Fields* 1: 389–393.

Bollerslev, T. 1986. Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics* 31: 307–327.

Chamberlain, G. and M. Rothschild. 1983. Arbitrage, Factor Structure and Mean-Variance Analysis on Large Asset Markets. *Econometrica* 51: 1281–1304.

Chan, L., Y. Hamao, and J. Lakonishok. 1991. Fundamentals and Stock Returns in Japan. *Journal of Finance* 46, 1739–1764.

Chen, N.F., R. Roll, and S. A. Ross, 1986, Economic Forces and the Stock Market, Journal of Business 59, 383–404.

Clark, P. 1973. A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices. *Econometrica* 41: 135–155.

Connor, G. and R.A. Korajczyk. 1985. Risk and return in an equilibrium APT: theory and tests. *Banking Research Center Working Paper* 129, Northwestern University.

Connor, G. and R.A. Korajczyk. 1986. Performance Measurement with the Arbitrage Pricing Theory: A New Framework for Analysis. *Journal of Financial Economics* 15: 373–394.

Cootner, P. 1962. Stock Prices: Random vs. Systematic Changes. *Industrial Management Review* 3: 24–45.

Cootner, P 1964. *The random character of stock market prices.* Cambridge, MA: MIT Press.

Engle, R. 1982. Autoregressive Conditional Heteroskedasticity With Estimates of the Variance of UK Inflation. *Econometrica* 50: 987–1008.

Engle, R. 2003.The ET Interview: Professor Robert F. Engle. *Econometric Theory* 19: 1159–1193.

Fama, E. 1963. Mandelbrot and the Stable Paretian Hypothesis. *Journal of Business* 36: 420–429.

Fama, E. 1965a. Portfolio Analysis in a Stable Paretian Market. *Management Science* 11: 404–419.

Fama, E. 1965b. Random Walks in Stock Market Prices. *Financial Analysts Journal* 21: 55–59.

Fama, E. 1965c. The Behavior of Stock-Market Prices. *Journal of Business* 38: 34–105.

Fama, E., and M. Blume. 1966. Filter Rules and Stock-Market Trading. *Journal of Business* 39: 226–241.

Fama, E. F. and K. R. French. 1992. The Cross-Section of Expected Stock Returns. *Journal of Finance* 47: 427–465.

Feller, W. 1935. Über den zentralen Genzwertsatz der Wahrscheinlichkeitsrechnung. *Mathematische Zeitschrift* 40: 521–559.

Hempel, C. 1965. *Aspects of Scientific Explanation and Other Essays in the Philosophy of Science*. New York, NY: Free Press.

Hempel, C., and P. Oppenheim. 1948. Studies in the Logic of Explanation. *Philosophy of Science* 15: 135–175.

Houthakker, H. 1961. Systematic and Random Elements in Short-Term Price Movements. *American Economic Review* 51: 164–172.

Jensen, M.1968. The Performance of Mutual Funds in the Period 1945-1964. *Journal of Finance* 23: 389–416.

Kendall, M. 1953. The Analysis of Economic Time-Series-Part I: Prices. .*Journal of the Royal Statistical Society Series A* 116: 11–34.

Khintchine, A. 1933. *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*. Berlin: Julius Springer.

Kon, S. 1984. Models of Stock Returns–A Comparison. *Journal of Finance* 39: 147–165.

Koundouri, P., and N. Kourogenis. 2011. On the Distribution of Crop Yields: Does the Central Limit Theorem Apply? *American Journal of Agricultural Economics* 93: 1341–1357.

Kourogenis, N., and N. Pittis. 2008. Testing for a unit root under errors with just barely infinite variance. *Journal of Time Series Analysis* 29: 1066–1087.

Larson, A. 1960. Measurement of a Random Process in Futures Prices. *Food Research Institute Studies* 1: 313–324.

Lévy, P. 1925. *Calcul des Probabilités*. Paris: Gauthier Villars.

Mandelbrot, B. 1963. The Variation of Certain Speculative Prices. *Journal of Business* 36: 394–419.

Moore, P. 1962. Regression as an Analytical Tool. *Journal of the Royal Statistical Society Series C* 11: 106–119.

Muth, J. 1961. Rational Expectations and the Theory of Price Movements. *Econometrica* 29: 315–335.

Nelson, D. 1990. Stationarity and Persistence in the GARCH(1,1) Model. *Econometric Theory* 6: 318–334.

Officer, R. 1972. The Distribution of Stock Returns. *Journal of the American Statistical Association* 67: 807–812.

Osborne, M. 1959. Brownian Motion in the Stock Market. *Operations Research* 7: 145–173.

Praetz, P. 1972. The Distribution of Share Price Changes. *Journal of Business* 45: 49–55.

Press, S. 1968. A compound events model for security prices. *Journal of Business* 40: 317–335.

Railton, P. 1978. A Deductive-Nomological Model of Probabilistic Explanation. *Philosophy of Science* 45: 206–226.

Railton, P. 1981. Probability, Explanation, and Information. *Synthese* 48: 233–256.

Renyi, A. 1960. On the central limit theorem for the sum of a random number of independent random variables. *Acta Mathematica Hungarica* 11: 97–102.

Roberts, H. 1959. Stock-Market "Patterns" and Financial Analysis: Methodological Suggestions. *Journal of Finance* 14: 1–10.

Robbins, H. 1948. On the Asymptotic Distribution of the Sum of a Random Number of Random Variables. *Proceedings of the National Academy of Sciences* 34: 162–163.

Roll, R. and S.A. Ross. 1980, An Empirical Investigation of the Arbitrage Pricing Theory. *Journal of Finance* 35: 1073–1103.

Rosenberg, B., K. Reid and R. Lanstein. 1984. Persuasive Evidence of Market Inefficiency. *Journal of Portfolio Management* 11: 9–17.

Ross, S.A. 1976. The Arbitrage Theory of Capital Asset Pricing. *Journal of Economic Theory* 13, 341–360.

Samuelson P. 1973. Mathematics of Speculative Price. *SIAM Review* 15: 1–42.

Samuelson, P. 1965. Proof that Properly Anticipated Prices Fluctuate Randomly. *Industrial Management Review* 6: 41–49.

Steiger, W. 1964. A test of nonrandomness in stock price changes. In P. H. Cootner (ed.), *The random character of stock market prices.* Risk Books.

Taussig, F. 1921. Is Market Price Determinate? *Quarterly Journal of Economics* 35: 394–411.

Teichmoeller, J. 1971. A Note on the Distribution of Stock Price Changes. J*ournal of the American Statistical Association* 66: 282–284.

Wold, H. 1969. Mergers of Economics and Philosophy of Science. *Synthese* 20: 427–482.

Working, H. 1934. A Random-Difference Series for Use in the Analysis of Time Series. J*ournal of the American Statistical Association* 29: 11–24.

Working, H. 1958. A Theory of Anticipatory Prices. *American Economic Review* 48: 188–199.