Xin Guo and Mihael Zervos

Optimal execution with multiplicative price impact

Article (Published version)
(Refereed)

Original citation:
Guo, Xin and Zervos, Mihail (2015) *Optimal execution with multiplicative price impact*. SIAM Journal on Financial Mathematics, 6 (1). pp. 281-306. ISSN 1945-497X
DOI: 10.1137/120894622

SIAM J. FINANCIAL MATH.
Vol. 6, pp. 281–306

Optimal Execution with Multiplicative Price Impact*

Xin Guo† and Mihail Zervos‡

Abstract. We consider the so-called optimal execution problem in algorithmic trading, which is the problem faced by an investor who has a large number of stock shares to sell over a given time horizon and whose actions have an impact on the stock price. In particular, we develop and study a price model that presents the stochastic dynamics of a geometric Brownian motion and incorporates a log-linear effect of the investor's transactions. We then formulate the optimal execution problem as a degenerate singular stochastic control problem. Using both analytic and probabilistic techniques, we establish simple conditions for the market to allow for no arbitrage or price manipulation and develop a detailed characterization of the value function and the optimal strategy. In particular, we derive an explicit solution to the problem if the time horizon is infinite.

Key words. optimal execution problem, multiplicative price impact, singular stochastic control

AMS subject classifications. 93E20, 91G80, 49L20

DOI. 10.1137/120894622

1. Introduction.

We consider an investor who has a large number of stock shares to sell within a given time frame. Rapid selling of the stock may depress the stock price, while slicing the big order into many smaller blocks of orders to be executed sequentially over time may take too long to realize. Such an investor is therefore faced with the problem of how to slice the order, when to trade and at what price, etc. This problem, known as the "optimal execution problem" in algorithmic trading, is concerned with finding a trading strategy that maximizes an appropriate objective function. A key issue of the problem is concerned with modeling the price impact of stock transactions.

The study of the optimal execution problem was initiated by Bertsimas and Lo [8], who analyzed a discrete random walk model and by Almgren and Chriss [5, 6] and Almgren [4], who considered continuous time Bachelier-type models with additive price impact. Since then, the area has attracted considerable interest; an incomplete list of notable contributions in the mathematics literature includes Huberman and Stanzl [21], He and Mamaysky [20], Gatheral, Schied, and Slynko [19], Obizhaeva and Wang [25], Almgren and Lorenz [7], Engle and Ferstenberg [13], Schied and Schöneborn [29], Alfonsi, Fruth, and Schied [1, 2], Schied, Schöneborn, and Tehranchi [30], Predoiu, Shaikhet, and Shreve [27], and Løkka [23].

Modeling stock prices by an arithmetic Brownian motion/random walk with additive impact of large stock sales is a common feature in the references on the optimal execution

*Received by the editors October 10, 2012; accepted for publication (in revised form) February 2, 2015; published electronically April 14, 2015.
http://www.siam.org/journals/sifin/6/89462.html

†Department of Industrial Engineering and Operations Research, UC at Berkeley, Berkeley, CA 94720 (xinguo@ieor.berkeley.edu). This author's research was partially supported by NSF grant DMS 1008759.

‡Department of Mathematics, London School of Economics, London, WC2A 2AE, UK (mihalis.zervos@gmail.com).

problem discussed above. An intriguing consequence of this modeling approach is that optimal strategies turn out to be more or less static or deterministic. Such strategies may lead to predictable trading patterns, which can give rise to market manipulation with techniques such as predatory trading (to this end, see the game formulations studied by Schied and Schöneborn [28] and Moallemi, Park, and Van Roy [24]). Recent work by Schied, Schöneborn, and Tehranchi [30], Gatheral and Schied [17], and Predoiu, Shaikhet, and Shreve [27] has revealed that such deterministic optimal strategies can be recovered by a simple argument involving an integration by parts calculation and an appropriate Euler–Lagrange equation, establishing an effective equivalence between minimizing costs and minimizing the price impact of trading strategies.

Beyond the context of Bachelier-type models, Gatheral and Schied [17] studied a continuous time Black–Scholes-type model with additive price impact. Discrete time models with multiplicative price impact have been considered by Bertsimas and Lo [8] and Bertsimas, Lo, and Hummel [9]. Also, Forsyth et al. [14, 15] proposed a continuous time Black–Scholes-type model with multiplicative price impact and derived its Hamilton–Jacobi–Bellman (HJB) equation using heuristic arguments, which they studied by means of numerical techniques. In these references, it is argued that such models are more natural than those with additive price impact because, e.g., they do not allow for strictly negative prices with nonzero probability.

In this paper, we study the optimal execution problem in the context of a continuous time model with multiplicative price impact. To the best of our knowledge, this model is the very first one in the continuous time optimal execution literature involving singular control rather than absolutely continuous control: this setting does not restrict stock transactions to be realized at a rate over time; instead, it allows for block sales of stock. The objective of the paper is to exhaustively study the model's analytical properties. The development of further realistic and applicable models can be motivated by the one we study here (see Remark 1 for such a generalization).

In particular, we consider an investor who holds $Y_t \geq 0$ shares of stock at time t, not including any transactions made at t. The investor can buy or sell any amount of shares at any time, but short-selling is not allowed. We denote by ξ^{s}_t (resp., ξ^{b}_t) the total amount of shares the investor has sold (resp., bought) up to time t, so that

$$Y_t = y - \xi^{\mathrm{s}}_t + \xi^{\mathrm{b}}_t,$$

where $y \geq 0$ is the number of shares held by the investor at time 0.

We assume that, in the absence of any transactions, stock prices follow a geometric Brownian motion. Also, we assume that (a) the price impact of small transactions is proportional to the stock price at which they are executed as well as proportional to their size, and (b) the price impact of a large transaction is the same as that of any number of smaller transactions of the same total size that are executed at the same time. In section 2, we show that such requirements give rise to the stock price dynamics

$$dX_t = \mu X_t \, dt - \lambda X_t \circ_{\mathrm{s}} d\xi^{\mathrm{s}}_t + \lambda X_t \circ_{\mathrm{b}} d\xi^{\mathrm{b}}_t + \sigma X_t \, dW_t,$$

where $\lambda > 0$ is a constant and the operators $\circ_{\mathrm{s}}$, $\circ_{\mathrm{b}}$ are defined by (2.5)–(2.6) below. Effectively, this is a model with multiplicative price impact: the impact of a transaction is additive to the logarithm of the stock price (see (2.7) below). There are several possible generalizations of these dynamics that exhibit resilience, namely, allowing the effect of transactions on the stock price to fade over time (we briefly discuss one in Remark 1).

The investor has a horizon $\overline{T} \in (0, \infty]$, by which time she exits the market by clearing all her shares. The investor's objective is to maximize the performance criterion

$$\mathbb{E}\left[\int_{[0,\overline{T}]\cap\mathbb{R}_+} e^{-\delta t}\left[X_t \circ_{\mathrm{s}} d\xi_t^{\mathrm{s}} - X_t \circ_{\mathrm{b}} d\xi_t^{\mathrm{b}} - C_{\mathrm{s}}\, d\xi_t^{\mathrm{s}} - C_{\mathrm{b}}\, d\xi_t^{\mathrm{b}}\right]\right]$$

over all admissible strategies $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}})$. Here, the constant $\delta \geq 0$ reflects the investor's impatience, while the constants $C_{\mathrm{s}}, C_{\mathrm{b}} \geq 0$ provide for a bid-ask spread or for proportional transaction costs. The choice $\delta = 0$ is the most natural one if the time horizon $\overline{T}$ is very short. We allow for choices $\delta > 0$ because these might be appropriate for execution problems lasting several days (see Lebedeva, Maug, and Schneider [22] for real-world examples of such executions) and are essential for a nontrivial solution if $\overline{T} = \infty$. Also, strictly positive values of C_{s}, C_{b} can arise from the existence of a bid-ask spread. Indeed, if we interpret X_t as the midprice of the stock price at time t, then we can view $X_t - C_{\mathrm{s}}$ (resp., $X_t + C_{\mathrm{b}}$) as the bid (resp., ask) price of the stock at time t. Such a modeling context has been considered in the literature, e.g., by Cont and de Larrard [10], who, based on empirical evidence, assume that the bid-ask spread is equal to one tick.

The performance criterion we have adopted is the expected revenue one featured in the models studied by, e.g., Bertsimas and Lo [8] and Gatheral [16]. Other choices of performance criteria that have been considered in the literature include the mean-variance criterion in Almgren and Chriss [5, 6], the expected utility criterion in Schied and Schöneborn [28], and the mean-quadratic variation criterion in Forsyth et al. [14]. Such alternative performance indices give rise to several variants of the model we study that could be the subject of future research. It is worth noting that Gatheral and Schied [18] have argued that a risk-neutral expected revenue or cost optimization objective is a reasonable choice, especially in contexts where market regularity conditions should be independent of investor preferences.

Mathematically, the optimization problem above takes the form of a singular stochastic control problem. Its HJB equation is a degenerate parabolic (if $\overline{T} < \infty$) or elliptic (if $\overline{T} = \infty$) PDE with state-dependent gradient constraints. Although the literature of singular stochastic control is rich and long, we are unaware of any results that characterize the value function or the optimal strategies in a context similar to the one we consider here; models that are closest to the one we analyze have been studied by Shreve and Soner [31] Soner and Shreve [32], Davis and Norman [12], Zhu [33], Ocone and Weerasinghe [26], and Dai and Yi [11].

Our analysis involves probabilistic as well as analytic techniques. A brief summary of our main results is as follows. First, we show that if we allowed for asymmetric price impact of buying and selling, then the market would present arbitrage opportunities (see Definition 3.2 and Proposition 3.4(I)–(II)). On the other hand, we prove that there are no arbitrage opportunities in the model with symmetric price impact that we consider (see Proposition 3.6(I)). In the spirit of Huberman and Stanzl [21], we define a price manipulation to be a round-trip trade, namely, a 0 net buying and selling trading strategy, that results in a strictly positive expected revenue (see Definition 3.3). It is worth noting here that the definitions of arbitrage and price manipulation that we have adopted involve no discounting, namely, $\delta = 0$, because the choice of a discounting rate characterizes specific investors rather than the market itself. We show that there is no price manipulation if and only if $\mu = 0$ (see Proposition 3.4(III) and Proposition 3.6(II)). This result is not surprising: given its definition and the symmetric nature of the market in terms of buying and selling, one can argue that a price manipulation cannot exist if and only if the stock price process is a martingale in the absence of transactions by a big investor. Indeed, in any model incorporating no price impact, the strategy that buys (resp., short-sells) one share of stock at time 0 and then sells it (resp., buys it back) at time 1 is a price manipulation if the stock price is a submartingale (resp., supermartingale) such as, e.g., a geometric Brownian motion with strictly positive (resp., negative) drift. Although the absence of a price manipulation is a desirable property of a model involving very short time scales (such as seconds or minutes), it could be viewed as rather restrictive for models involving long time scales (such as days or weeks) where the time-value of money and issues involving investor preferences come into play. From a mathematical perspective, our results are consistent with those of Huberman and Stanzl [21] and Gatheral [16], who showed that permanent price impact must be linear and symmetric to exclude price manipulation in the zero-drift models with additive price impact that they studied.

In our analysis of possible arbitrage opportunities and price manipulation, we naturally consider trading strategies that involve short-selling subject to the constraint that short positions are bounded by a constant. On the other hand, we assume that short-selling is not permitted in our analysis of the optimal execution problem itself (see, however, the paragraph above the statement of Proposition 3.5). In this context, we first prove that the investor would be able to realize arbitrarily high expected payoffs by means of simple round-trip trades if her discounting rate were strictly less than the drift of the stock price (see Proposition 3.4(IV)). To avoid unrealistic trivialities, we therefore assume that $\delta \geq \max\{\mu, 0\}$. In this case, we show that the optimal liquidation strategy involves no buying of shares (see Proposition 3.5(I)), namely, there is no transaction-triggered price manipulation in the sense of Alfonsi, Schied, and Slynko [3].

In the case when $\overline{T} < \infty$, we prove a verification theorem (Proposition 4.1) that relates an appropriate solution to the problem's HJB equation to the problem's value function. Such a solution to the HJB equation, which can be computed numerically offline, fully determines the optimal liquidation strategy. Indeed, its nature is such that the state space splits into two regions, the "waiting" one and the "selling" one. Beyond a possible sale of an appropriate amount of stock that positions the state process at the boundary of the two regions at time 0, the optimal strategy involves minimal action to keep the state process inside the closure of the waiting region and takes no action while the state process is in the interior of the waiting region.

If $\overline{T} = \infty$, then we derive the solution to the problem in an explicit form (see Proposition 5.1). An interesting feature of this solution is that an optimal strategy may not exist even though the value function is finite (see Proposition 5.1(II)). If it exists, the optimal strategy can be described informally as follows (see also Figure 1). If the stock price is below a critical level $F_\circ$, then it is optimal to take no action. If the stock price at time 0 is above $F_\circ$, then it is optimal to either sell all available shares immediately or liquidate an amount that would cause the stock price to drop to $F_\circ$ and then keep on selling until all shares are exhausted by

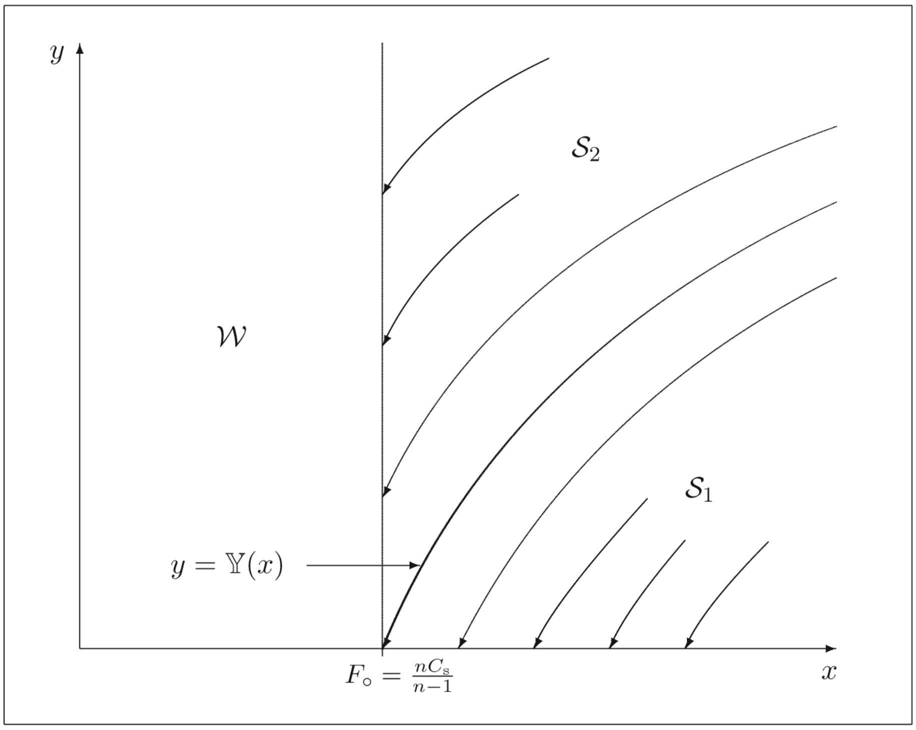

Figure 1. *The regions providing the optimal strategy when $\overline{T} = \infty$. If the stock price takes values in the "waiting" region $\mathcal{W}$, then it is optimal to take no action. If the stock price at time 0 is inside the "selling" region $\mathcal{S}_1$, then it is optimal to sell all available shares immediately. If the stock price at time 0 is inside the "selling" region $\mathcal{S}_2$, then it is optimal to liquidate an amount that would cause the stock price to drop to $F_\circ$ and then keep on selling until all shares are exhausted by just preventing the stock price from rising above $F_\circ$.*

just preventing the stock price from rising above $F_\circ$.

2. The market model and the control problem. We fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions and carrying a standard $(\mathcal{F}_t)$-Brownian motion W.

We denote by Y_t the total number of shares held by the investor at time t. Also, we denote by $\xi_t^{\rm s}$ (resp., $\xi_t^{\rm b}$) the total number of shares that the investor has sold (resp., bought) up to time t, so that $Y_t = y - \xi_t^{\rm s} + \xi_t^{\rm b}$, where $y \geq 0$ is the number of shares held by the investor at time 0. We assume that $\xi^{\rm s}$ and $\xi^{\rm b}$ are $(\mathcal{F}_t)$-adapted increasing càglàd processes such that $\xi_0^{\rm s} = \xi_0^{\rm b} = 0$. Also, we assume that the investor does not simultaneously buy and sell. In particular, we restrict our attention to strategies $(\xi^{\rm s}, \xi^{\rm b})$ such that

$$\text{if we define } \xi_t = \xi_t^{\rm s} - \xi_t^{\rm b}, \text{ then } \check{\xi}_t = \xi_t^{\rm s} + \xi_t^{\rm b} \text{ for all } t \geq 0, \tag{2.1}$$

where $\check{\xi}$ is the total variation process of ξ.

The investor's aim is to liquidate all share holdings by a time horizon $\overline{T} \in (0, \infty]$. We therefore consider trading strategies $(\xi^{\rm s}, \xi^{\rm b})$ such that

$$Y_{\overline{T}+} = 0 \text{ if } \overline{T} < \infty \quad \text{and} \quad \lim_{T \to \infty} Y_T = 0 \text{ if } \overline{T} = \infty. \tag{2.2}$$

In the absence of any transactions from the investor, we model the stock price by the geometric Brownian motion X^0 given by

$$dX_t^0 = \mu X_t^0\, dt + \sigma X_t^0\, dW_t, \quad X_0^0 = x > 0 \tag{2.3}$$

for some constants μ and $\sigma \neq 0$. We assume that small transactions made by the investor affect the share price proportionally to its value. In particular, if the investor sells (resp., buys) a small amount $\varepsilon > 0$ of shares at time t, then the share price exhibits a jump of size

$$\Delta X_t = X_{t+} - X_t = -\lambda\varepsilon X_t \quad (\text{resp., } \Delta X_t = X_{t+} - X_t = \lambda\varepsilon X_t)$$

for some constant $\lambda > 0$, where we have assumed that X is càglàd. In this context, a small sale (resp., buy) of size $\varepsilon > 0$ is associated with the expressions

$$X_{t+} = (1-\lambda\varepsilon)X_t \simeq e^{-\lambda\varepsilon} X_t \quad \left(\text{resp., } X_{t+} = (1+\lambda\varepsilon)X_t \simeq e^{\lambda\varepsilon} X_t\right).$$

If we view the sale of $\Delta\xi_t^{\mathrm{s}}$ shares as N individual sales of $\varepsilon = \Delta\xi_t^{\mathrm{s}}/N$ shares each, then, for N large enough, we obtain

$$X_{t+} = e^{-\lambda N\varepsilon} X_t = e^{-\lambda\Delta\xi_t^{\mathrm{s}}} X_t.$$

Similarly, we can see that buying $\Delta\xi_t^{\mathrm{b}}$ shares is associated with the jump $X_{t+} = e^{\lambda\Delta\xi_t^{\mathrm{s}}} X_t$.

In view of the above considerations, we model the stock price dynamics by the stochastic equation

$$dX_t = \mu X_t\, dt - \lambda X_t \circ_{\mathrm{s}} d\xi_t^{\mathrm{s}} + \lambda X_t \circ_{\mathrm{b}} d\xi_t^{\mathrm{b}} + \sigma X_t\, dW_t, \tag{2.4}$$

where

$$X_t \circ_{\mathrm{s}} d\xi_t^{\mathrm{s}} = X_t\, d(\xi^{\mathrm{s}})_t^{\mathrm{c}} + \frac{1}{\lambda} X_t \left[1 - e^{-\lambda\Delta\xi_t^{\mathrm{s}}}\right] = X_t\, d(\xi^{\mathrm{s}})_t^{\mathrm{c}} + X_t \int_0^{\Delta\xi_t^{\mathrm{s}}} e^{-\lambda u}\, du \tag{2.5}$$

and

$$X_t \circ_{\mathrm{b}} d\xi_t^{\mathrm{b}} = X_t\, d(\xi^{\mathrm{b}})_t^{\mathrm{c}} + \frac{1}{\lambda} X_t \left[e^{\lambda\Delta\xi_t^{\mathrm{b}}} - 1\right] = X_t\, d(\xi^{\mathrm{b}})_t^{\mathrm{c}} + X_t \int_0^{\Delta\xi_t^{\mathrm{b}}} e^{\lambda u}\, du, \tag{2.6}$$

where the process $(\xi^{\mathrm{s}})^{\mathrm{c}}$ (resp., $(\xi^{\mathrm{b}})^{\mathrm{c}}$) is the continuous part of the process ξ^{s} (resp., ξ^{b}). Using Itô's formula, we can verify that the solution to (2.4) is given by

$$X_t = x \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t - \lambda\xi_t^{\mathrm{s}} + \lambda\xi_t^{\mathrm{b}} + \sigma W_t\right) = X_t^0 \exp\left(-\lambda\xi_t^{\mathrm{s}} + \lambda\xi_t^{\mathrm{b}}\right), \tag{2.7}$$

where X^0 is the solution to (2.3).

If we consider the sale of $\Delta\xi_t^{\mathrm{s}}$ shares at time t as equivalent to the sale of N packets of shares of small size $\varepsilon = \Delta\xi_t^{\mathrm{s}}/N$, then we can see that such a sale should result in a revenue of

$$\sum_{j=0}^{N-1} e^{-\lambda j\varepsilon} X_t \varepsilon \simeq \int_0^{\Delta\xi_t^{\mathrm{s}}} X_t e^{-\lambda u}\, du = \frac{1}{\lambda} X_t \left[1 - e^{-\lambda\Delta\xi_t^{\mathrm{s}}}\right].$$

In view of this observation and a similar one concerning the buying of $\Delta\xi_t^{\mathrm{b}}$ shares at time t, we associate the performance criterion

$$I_{\overline{T},x,y}(\xi^{\mathrm{s}},\xi^{\mathrm{b}}) = \begin{cases} J_{\overline{T},x,y}(\xi^{\mathrm{s}},\xi^{\mathrm{b}}) & \text{if } \overline{T}<\infty, \\ \limsup_{T\to\infty} J_{T,x,y}(\xi^{\mathrm{s}},\xi^{\mathrm{b}}) & \text{if } \overline{T}=\infty \end{cases} \tag{2.8}$$

with each liquidation strategy $(\xi^{\mathrm{s}},\xi^{\mathrm{b}})$, where $J_{T,x,y}(\xi^{\mathrm{s}},\xi^{\mathrm{b}})$ is defined by

$$J_{T,x,y}(\xi^{\mathrm{s}},\xi^{\mathrm{b}}) = \mathbb{E}\left[\int_{[0,T]} e^{-\delta t}\left[X_t \circ_{\mathrm{s}} d\xi_t^{\mathrm{s}} - X_t \circ_{\mathrm{b}} d\xi_t^{\mathrm{b}} - C_{\mathrm{s}}\, d\xi_t^{\mathrm{s}} - C_{\mathrm{b}}\, d\xi_t^{\mathrm{b}}\right]\right]$$

for $(T,x,y)\in\mathbb{R}_+\times\mathbb{R}_+^*\times\mathbb{R}_+$.[1] Here, the discounting rate $\delta\geq 0$ reflects the investor's "impatience," while the constants $C_{\mathrm{s}}, C_{\mathrm{b}}\geq 0$ may account for a constant bid-ask spread or provide for proportional transaction costs.

The investor's objective is to maximize $I_{\overline{T},x,y}(\xi^{\mathrm{s}},\xi^{\mathrm{b}})$ over all liquidation strategies $(\xi^{\mathrm{s}},\xi^{\mathrm{b}})$. Accordingly, we define the problem's value function v by

$$v(\overline{T},x,y) = \sup_{(\xi^{\mathrm{s}},\xi^{\mathrm{b}})\in\mathcal{A}_{\overline{T},y}} I_{\overline{T},x,y}(\xi^{\mathrm{s}},\xi^{\mathrm{b}}),$$

where $\mathcal{A}_{\overline{T},y}$ is the family of all admissible strategies, which is introduced by the following definition.

Definition 2.1. *Given a time horizon $\overline{T}\in(0,\infty]$ and an initial holding of $y\geq 0$ shares, the family $\mathcal{A}_{\overline{T},y}$ of all admissible liquidation strategies is the set of all pairs $(\xi^{\mathrm{s}},\xi^{\mathrm{b}})$ composed by $(\mathcal{F}_t)$-adapted increasing càglàd processes ξ^{s} and ξ^{b} such that $\xi_0^{\mathrm{s}}=\xi_0^{\mathrm{b}}=0$, (2.1) and (2.2) hold true, and*

$$Y_t = y-\xi_t^{\mathrm{s}}+\xi_t^{\mathrm{b}}\geq 0 \quad \text{and} \quad \mathbb{E}\left[e^{4\lambda\xi_{t+}^{\mathrm{b}}}\right]<\infty \quad \text{for all } t\in[0,\overline{T}]\cap\mathbb{R}_+. \tag{2.9}$$

We denote by $\mathcal{A}_{\overline{T},y}^{\mathrm{s}}$ the family of all processes ξ^{s} such that $(\xi^{\mathrm{s}},0)\in\mathcal{A}_{\overline{T},y}$.

The integrability assumption that we make in (2.9) is quite general and ensures that the optimization problem is well-posed. In particular, it would plainly be satisfied if we imposed an upper bound on the process ξ^{b}, which would rule out unbounded total buying and selling over a finite time horizon. On the other hand, the inequality $Y_t\geq 0$ for all $t\geq 0$ reflects the idea that the possibility of short-selling is not permitted.

In the next assumption we summarize the possible values that the various constants we have considered may take.

Assumption 1. *μ, $\sigma\neq 0$, $\delta\geq\max\{\mu,0\}$, and $\lambda>0$ are constants.*

Remark 1. In the model that we have developed, transactions made by the investor have a permanent impact. There are several extensions of the model that can accommodate transient impact. For instance, we can replace the dynamics given by (2.7) by $X_t = X_t^0 e^{Z_t}$, where

$$Z_t = -\lambda\int_{[0,t[} G(t-s)\, d\left[d\xi_s^{\mathrm{s}} - d\xi_s^{\mathrm{b}}\right]$$

[1] Throughout the paper, we use the notation $\mathbb{R}_+=[0,\infty)$ and $\mathbb{R}_+^*=(0,\infty)$.

for some kernel G. In this context, if we choose $G(t-s) = e^{-\gamma t}e^{\gamma s}$ for some constant $\gamma > 0$, then

$$dZ_t = -\gamma Z_t\, dt - \lambda\, d\xi_t^{\mathrm{s}} + \lambda\, d\xi_t^{\mathrm{b}}.$$

In such extensions, the resulting optimization problem's state space would involve four variables (namely, t, x, y, and z) instead of three (namely, t, x, and y). We leave this as well as other extensions accommodating resilience of the stock price for future research.

3. Study of the market and preliminary results. In this section, we establish a range of results that characterize the market we study as well as some estimates we will need. To this end, we first consider the so-called round-trip trades, which are trading strategies that involve 0 net buying or selling of shares over a given finite time horizon. It is worth noting that the inequality in (3.1) is an admissibility condition that requires the maximum number of stock shares that a round trip can be short to be bounded by a constant.

Definition 3.1. *An admissible round-trip trade with time horizon $T \in \mathbb{R}_+^*$ is any pair $(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})$ of $(\mathcal{F}_t)$-adapted increasing càglàd processes such that $\zeta_0^{\mathrm{s}} = \zeta_0^{\mathrm{b}} = 0$ and*

$$\zeta_{T+}^{\mathrm{s}} = \zeta_{T+}^{\mathrm{b}} \quad \text{and} \quad \sup_{t\in[0,T]} \left(\zeta_{t+}^{\mathrm{s}} - \zeta_{t+}^{\mathrm{b}}\right) \le \Gamma \tag{3.1}$$

for some constant $\Gamma > 0$, which may depend on the trading strategy itself.

Our first result shows that the model we consider would be unviable if we allowed for asymmetric impact of buying and selling. In particular, we prove that if we model the stock price dynamics by

$$dX_t = \mu X_t\, dt - \lambda X_t \circ_{\mathrm{s}}^{\lambda} d\xi_t^{\mathrm{s}} + \kappa X_t \circ_{\mathrm{b}}^{\kappa} d\xi_t^{\mathrm{b}} + \sigma X_t\, dW_t, \tag{3.2}$$

where $\circ_{\mathrm{s}}^{\lambda}$ (resp., $\circ_{\mathrm{b}}^{\kappa}$) is defined by (2.5) (resp., (2.6) with κ in place of λ), for some $\kappa \neq \lambda$, then the market may present arbitrage opportunities in the following sense.

Definition 3.2. *The market allows for arbitrage opportunities if there exists a round-trip trade with resulting revenue that is positive and strictly positive with strictly positive probability, namely, if there exists a round-trip trade $(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})$ such that*

$$R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}}) = \int_{[0,T]} \left[X_t \circ_{\mathrm{s}}^{\lambda} d\zeta_t^{\mathrm{s}} - X_t \circ_{\mathrm{b}}^{\kappa} d\zeta_t^{\mathrm{b}} - C_{\mathrm{s}}\, d\zeta_t^{\mathrm{s}} - X_t\, d\zeta_t^{\mathrm{b}}\right] \ge 0 \tag{3.3}$$

and $\mathbb{P}\left(R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}}) > 0\right) > 0$.

We also prove that price manipulation exists if and only if $\mu \neq 0$.

Definition 3.3. *A price manipulation is a round-trip trade $(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})$ resulting in a strictly positive expected revenue, namely, $\mathbb{E}\left[R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})\right] > 0$, where R is defined by (3.3). An unbounded price manipulation is a sequence of round-trip trades $(\zeta^{\mathrm{s},n}, \zeta^{\mathrm{b},n})$ such that $\lim_{n\to\infty} \mathbb{E}\big[R(\zeta^{\mathrm{s},n}, \zeta^{\mathrm{b},n})\big] = \infty$.*

In these definitions, we have taken $\delta = 0$ because the choice of a discounting rate is investor specific. In this way, the existence or not of arbitrage and/or price manipulation characterizes a market of risk-neutral investors as a whole.

The next result, which is complemented by Proposition 3.6 below, is concerned with these issues. We also show here that the investor's optimization problem would be trivial if the investor's discounting rate δ were strictly less than the stock price drift μ, which we have excluded as a possibility in Assumption 1.

Proposition 3.4. *The following statements are true:*

(I) *If the price process dynamics are given by (3.2) for some $\kappa > \lambda > 0$, then the market presents arbitrage opportunities and arbitrarily high risk-free profits can be realized by simple round-trip strategies.*

(II) *If the price process dynamics are given by (3.2) for some $\lambda > \kappa > 0$, then the market may present arbitrage opportunities.*

(III) *Suppose that the price process dynamics are given by (3.2) for some $\kappa = \lambda > 0$, namely, by (2.4). If $\mu < 0$, then price manipulation may exist, while if $\mu > 0$, then unbounded price manipulation exists.*

(IV) *Consider the optimal execution problem formulated in section 2, in which the price process dynamics are given by (3.2) for some $\kappa = \lambda > 0$, namely, by (2.4). If $0 \le \delta < \mu$ in violation of Assumption 1, then round-trip trades involving no short-selling can realize arbitrarily high expected payoffs and*

$$v(\overline{T}, x, y) = \infty \quad \text{for all } (\overline{T}, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+. \tag{3.4}$$

Proof. Suppose that the price process dynamics are given by (3.2) for some $\kappa > \lambda$, and let (ϱ_n) be any sequence of strictly positive numbers such that $\lim_{n\to\infty} \varrho_n = \infty$. Given $\varepsilon \in (0, (\kappa - \lambda) \wedge \lambda)$, we define the $(\mathcal{F}_t)$-stopping time

$$\tau = \inf\left\{ t \ge 0 \,\middle|\, \frac{X_t^0}{x} \notin \left[\frac{\lambda + \varepsilon}{\kappa}, \frac{\lambda}{\lambda - \varepsilon}\right]\right\} \wedge 1 > 0,$$

and we note that

$$\frac{\kappa X_\tau^0 - \lambda x}{\lambda x} \ge \frac{\varepsilon}{\lambda} \quad \text{and} \quad -\frac{\kappa X_\tau^0}{\lambda x} \ge -\frac{\kappa}{\lambda - \varepsilon}. \tag{3.5}$$

The round-trip trade $(\zeta^{\mathrm{s},n}, \zeta^{\mathrm{b},n})$ that buys ϱ_n shares at time 0 and then sells them at time $\tau \in (0, 1]$ results in the revenue

$$\begin{aligned} R(\zeta^{\mathrm{s},n}, \zeta^{\mathrm{b},n}) &= -\frac{1}{\kappa} x \left[e^{\kappa\varrho_n} - 1\right] + \frac{1}{\lambda} X_\tau^0 e^{\kappa\varrho_n} \left[1 - e^{-\lambda\varrho_n}\right] - (C_\mathrm{b} + C_\mathrm{s})\, \varrho_n \\ &= \frac{1}{\kappa}\left[1 + \frac{\kappa X_\tau^0 - \lambda x}{\lambda x} e^{\kappa\varrho_n} - \frac{\kappa X_\tau^0}{\lambda x} e^{(\kappa-\lambda)\varrho_n}\right] x - (C_\mathrm{b} + C_\mathrm{s})\, \varrho_n \\ &\overset{(3.5)}{\ge} \frac{1}{\kappa}\left[1 + \frac{\varepsilon}{\lambda} e^{\kappa\varrho_n} - \frac{\kappa}{\lambda - \varepsilon} e^{(\kappa-\lambda)\varrho_n}\right] x - (C_\mathrm{s} + C_\mathrm{b})\, \varrho_n. \end{aligned}$$

The last expression tends to ∞ as $n \to \infty$, and (I) follows.

To show (II), we assume that the price process dynamics are given by (3.2) for some $\kappa < \lambda$, we define the $(\mathcal{F}_t)$-stopping time

$$\tau = \left\{ t \ge 0 \,\middle|\, \frac{X_t^0}{x} \notin \left[\frac{\kappa + \varepsilon}{\lambda}, \frac{\kappa}{\kappa - \varepsilon}\right]\right\} \wedge 1 > 0$$

for some $\varepsilon \in (0, (\lambda - \kappa) \wedge \kappa)$, and we note that

$$\frac{\lambda X^0_\tau - \kappa x}{\kappa x} \geq \frac{\varepsilon}{\kappa} \quad \text{and} \quad -\frac{\lambda X^0_\tau}{\kappa x} \geq -\frac{\lambda}{\kappa - \varepsilon}. \tag{3.6}$$

The round-trip trade $(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})$ that short-sells $\varrho > 0$ shares at time 0 and then buys them back at time $\tau \in (0,1]$ results in the revenue

$$\begin{aligned} R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}}) &= \frac{1}{\lambda} x \left[1 - e^{-\lambda \varrho}\right] - \frac{1}{\kappa} X^0_\tau e^{-\lambda\varrho} \left[e^{\kappa\varrho} - 1\right] - (C_{\mathrm{s}} + C_{\mathrm{b}})\, \varrho \\ &= \frac{1}{\lambda} \left[1 + \frac{\lambda X^0_\tau - \kappa x}{\kappa x} e^{-\lambda\varrho} - \frac{\lambda X^0_\tau}{\kappa x} e^{-(\lambda - \kappa)\varrho}\right] x - (C_{\mathrm{s}} + C_{\mathrm{b}})\, \varrho \\ &\overset{(3.6)}{\geq} \frac{1}{\lambda} \left[1 + \frac{\varepsilon}{\kappa} e^{-\lambda\varrho} - \frac{\lambda}{\kappa - \varepsilon} e^{-(\lambda-\kappa)\varrho}\right] x - (C_{\mathrm{s}} + C_{\mathrm{b}})\, \varrho. \end{aligned}$$

The coefficient of x in the last expression is strictly positive for all $\varrho > 0$ sufficiently large. Given any such ϱ, the revenue $R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})$ is strictly positive for all x sufficiently large, and the claim that the market may present arbitrage opportunities follows.

To see (IV), suppose that $\delta < \mu$ and consider the round-trip trade that buys $\varrho > 0$ shares at time 0 and sells them at time $T > 0$. This strategy has expected payoff

$$\begin{aligned} &-\frac{1}{\lambda} x \left[e^{\lambda\varrho} - 1\right] + \frac{1}{\lambda} \mathbb{E}\left[e^{-\delta T} X^0_T e^{\lambda\varrho}\right] \left[1 - e^{-\lambda\varrho}\right] - \left(C_{\mathrm{b}} + e^{-\delta T} C_{\mathrm{s}}\right) \varrho \\ &\qquad = \frac{1}{\lambda} x e^{\lambda\varrho} \left[1 - e^{-\lambda\varrho}\right] \left[e^{(\mu-\delta)T} - 1\right] - \left(C_{\mathrm{b}} + e^{-\delta T} C_{\mathrm{s}}\right) \varrho, \end{aligned}$$

which tends to ∞ as $\varrho \to \infty$. In particular, (3.4) holds true.

To prove (III), suppose first that $\mu < 0$. The round-trip trade $(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})$ that short-sells $\varrho > 0$ shares at time 0 and buys them back at time $T > 0$ results in the expected revenue

$$\begin{aligned} \mathbb{E}\left[R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})\right] &= \frac{1}{\lambda} x \left[1 - e^{-\lambda\varrho}\right] - \frac{1}{\lambda} \mathbb{E}\left[X^0_T e^{-\lambda\varrho}\right] \left[e^{\lambda\varrho} - 1\right] - (C_{\mathrm{s}} + C_{\mathrm{b}})\, \varrho \\ &= \frac{1}{\lambda} x e^{-\lambda\varrho} \left[e^{\lambda\varrho} - 1\right] \left[1 - e^{-|\mu| T}\right] - (C_{\mathrm{s}} + C_{\mathrm{b}})\, \varrho, \end{aligned}$$

which is strictly positive for all $\varrho > 0$ provided x is sufficiently large. On the other hand, if $\mu > 0$, then the round-trip trade $(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})$ we considered in the proof of (IV) above results in the expected revenue

$$\mathbb{E}\left[R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})\right] = \frac{1}{\lambda} x e^{\lambda\varrho} \left[1 - e^{-\lambda\varrho}\right] \left[e^{\mu T} - 1\right] - (C_{\mathrm{b}} + C_{\mathrm{s}})\, \varrho,$$

which tends to ∞ as $\varrho \to \infty$ for all x and T. ■

We now switch our attention to the actual optimal execution problem. It is worth recalling that, contrary to our analysis thus far, short-selling is not permitted in this problem. However, a simple inspection of the proof of part (IV) reveals that its conclusions remain true if short-selling is indeed allowed. Furthermore, it is worth noting that if the conditions of parts (III) and (IV) are both satisfied, then every admissible liquidation strategy is optimal.

Proposition 3.5. *Consider the optimal execution problem formulated in section 2. Given a time horizon $\overline{T} \in (0, \infty]$ and any $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$, the following statements are true:*

(I) *The optimal liquidation strategy involves no buying of shares, namely,*

$$v(\overline{T}, x, y) = \sup_{\xi^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}_{\overline{T},y}} I_{\overline{T},x,y}(\xi^{\mathrm{s}}, 0). \tag{3.7}$$

In particular, the market does not allow for transaction-triggered price manipulation.

(II) *The value function satisfies*

$$\frac{1}{\lambda} x \left[1 - e^{-\lambda y}\right] - C_{\mathrm{s}} y \le v(\overline{T}, x, y) \le \frac{1}{\lambda} x \left[1 - e^{-\lambda y}\right] \tag{3.8}$$

for all $\overline{T} \in (0, \infty]$ and $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$.

(III) *If $C_{\mathrm{s}} = 0$, then it is optimal to sell all shares at time 0, and the value function is given by $v(\overline{T}, x, y) = \frac{1}{\lambda} x \left[1 - e^{-\lambda y}\right]$ for all $\overline{T} \in (0, \infty]$ and $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$.*

(IV) *Suppose that $\delta = \mu \ge 0$. If $\overline{T} \in \mathbb{R}_+^*$, then it is optimal to sell all available shares at $\overline{T}$. On the other hand, if $\overline{T} = \infty$, then selling all available shares at time $n = 1, 2, \ldots$ provides a sequence of ε-optimal strategies. In this case, the value function is given by*

$$v(\overline{T}, x, y) = \begin{cases} \frac{1}{\lambda} x \left[1 - e^{-\lambda y}\right] - e^{-\delta \overline{T}} C_{\mathrm{s}} y & \text{if } \overline{T} \in \mathbb{R}_+^*, \\ \frac{1}{\lambda} x \left[1 - e^{-\lambda y}\right] & \text{if } \overline{T} = \infty \end{cases} \tag{3.9}$$

for all $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$.

Proof. Given a liquidation strategy $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \in \mathcal{A}_{\overline{T},y}$, we define

$$\tilde{\xi}^{\mathrm{s}}_t = \sup_{0 \le u \le t} \left(\xi^{\mathrm{s}}_u - \xi^{\mathrm{b}}_u\right)^+ \quad \text{for } t \ge 0, \tag{3.10}$$

and we note that

$$\tilde{\xi}^{\mathrm{s}}_t \le \sup_{0 \le u \le t} \xi^{\mathrm{s}}_u = \xi^{\mathrm{s}}_t \quad \text{for all } t \ge 0. \tag{3.11}$$

In view of (2.9), we can see that

$$y \ge \xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t \ \text{ for all } t \in [0, \overline{T}] \cap \mathbb{R}_+ \quad \Rightarrow \quad y \ge \sup_{0 \le u \le t} \left(\xi^{\mathrm{s}}_u - \xi^{\mathrm{b}}_u\right)^+ = \tilde{\xi}^{\mathrm{s}}_t \ \text{ for all } t \in [0, \overline{T}] \cap \mathbb{R}_+.$$

Also, (2.2) and this observation imply that if $\overline{T} < \infty$, then

$$y = \xi^{\mathrm{s}}_{\overline{T}+} - \xi^{\mathrm{b}}_{\overline{T}+} \le \tilde{\xi}^{\mathrm{s}}_{\overline{T}+} \le y \quad \Rightarrow \quad \tilde{Y}_{\overline{T}+} = y - \tilde{\xi}^{\mathrm{s}}_{\overline{T}+} = 0,$$

while if $\overline{T} = \infty$, then

$$y = \lim_{T \to \infty} \left(\xi^{\mathrm{s}}_T - \xi^{\mathrm{b}}_T\right) \le \lim_{T \to \infty} \tilde{\xi}^{\mathrm{s}}_T \le y \quad \Rightarrow \quad \lim_{T \to \infty} \tilde{Y}_T = \lim_{T \to \infty} \left(y - \tilde{\xi}^{\mathrm{s}}_T\right) = 0.$$

It follows that $(\tilde{\xi}^{\mathrm{s}}, 0) \in \mathcal{A}_{\overline{T},y}$, namely, $\tilde{\xi}^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}_{\overline{T},y}$. For future reference, we also note that (3.11) implies that

$$\int_{[0,T]} e^{-\delta t}\, d\left(\xi^{\mathrm{s}}_t - \tilde{\xi}^{\mathrm{s}}_t\right) = e^{-\delta T}\left(\xi^{\mathrm{s}}_{T+} - \tilde{\xi}^{\mathrm{s}}_{T+}\right) + \delta \int_0^T e^{-\delta t}\left(\xi^{\mathrm{s}}_t - \tilde{\xi}^{\mathrm{s}}_t\right) dt \geq 0. \tag{3.12}$$

In view of the observations that

$$de^{-\lambda\xi^{\mathrm{s}}_t} = -\lambda e^{-\lambda\xi^{\mathrm{s}}_t}\, d(\xi^{\mathrm{s}})^{\mathrm{c}}_t - e^{-\lambda\xi^{\mathrm{s}}_t}\left[1 - e^{-\lambda\Delta\xi^{\mathrm{s}}_t}\right] = -\lambda e^{-\lambda\xi^{\mathrm{s}}_t} \circ_{\mathrm{s}} d\xi^{\mathrm{s}}_t,$$
$$de^{\lambda\xi^{\mathrm{b}}_t} = \lambda e^{\lambda\xi^{\mathrm{b}}_t}\, d(\xi^{\mathrm{b}})^{\mathrm{c}}_t + e^{\lambda\xi^{\mathrm{b}}_t}\left[e^{\lambda\Delta\xi^{\mathrm{b}}_t} - 1\right] = \lambda e^{\lambda\xi^{\mathrm{b}}_t} \circ_{\mathrm{b}} \xi^{\mathrm{b}}_t,$$

which follow from Itô's formula and (2.5)–(2.6), and the calculation

$$\begin{aligned} d\left(e^{-\delta t} X_t\right) &= d\left(e^{-\delta t} X^0_t e^{-\lambda(\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t)}\right) \\ &= -(\delta-\mu) e^{-\delta t} X^0_t e^{-\lambda(\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t)}\, dt + e^{-\delta t} X^0_t e^{\lambda\xi^{\mathrm{b}}_t}\, de^{-\lambda\xi^{\mathrm{s}}_t} \\ &\quad + e^{-\delta t} X^0_t e^{-\lambda\xi^{\mathrm{s}}_t}\, de^{\lambda\xi^{\mathrm{b}}_t} + \sigma e^{-\delta t} X^0_t e^{-\lambda(\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t)}\, dW_t, \end{aligned}$$

which follows from an application of the integration by parts formula, we can see that, given any $T \in [0, \overline{T}] \cap \mathbb{R}_+$,

$$\begin{aligned} &\int_{[0,T]} e^{-\delta t}\left[X_t \circ_{\mathrm{s}} d\xi^{\mathrm{s}}_t - X_t \circ_{\mathrm{b}} d\xi^{\mathrm{b}}_t - C_{\mathrm{s}}\, d\xi^{\mathrm{s}}_t - C_{\mathrm{b}}\, d\xi^{\mathrm{b}}_t\right] \\ &\quad = -\int_{[0,T]} e^{-\delta t}\left[\frac{1}{\lambda} X^0_t e^{\lambda\xi^{\mathrm{b}}_t}\, de^{-\lambda\xi^{\mathrm{s}}_t} + \frac{1}{\lambda} X^0_t e^{-\lambda\xi^{\mathrm{s}}_t}\, de^{\lambda\xi^{\mathrm{b}}_t} + C_{\mathrm{s}}\, d\xi^{\mathrm{s}}_t + C_{\mathrm{b}}\, d\xi^{\mathrm{b}}_t\right] \\ &\quad = -\frac{\delta-\mu}{\lambda}\int_0^T e^{-\delta t} X^0_t e^{-\lambda(\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t)}\, dt + \frac{x}{\lambda} - \frac{1}{\lambda} e^{-\delta T} X^0_T e^{-\lambda(\xi^{\mathrm{s}}_{T+} - \xi^{\mathrm{b}}_{T+})} \\ &\qquad - \int_{[0,T]} e^{-\delta t}\left[C_{\mathrm{s}}\, d\xi^{\mathrm{s}}_t + C_{\mathrm{b}}\, d\xi^{\mathrm{b}}_t\right] + \frac{\sigma}{\lambda}\int_0^T e^{-\delta t} X^0_t e^{-\lambda(\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t)}\, dW_t. \end{aligned} \tag{3.13}$$

In view of these identities, the assumption that $\delta \geq \mu$, the definition (3.10) of $\tilde{\xi}^{\mathrm{s}}$, and (3.12), we can see that

$$\begin{aligned} &\int_{[0,T]} e^{-\delta t}\left[X_t \circ_{\mathrm{s}} d\xi^{\mathrm{s}}_t - X_t \circ_{\mathrm{b}} d\xi^{\mathrm{b}}_t - C_{\mathrm{s}}\, d\xi^{\mathrm{s}}_t - C_{\mathrm{b}}\, d\xi^{\mathrm{b}}_t\right] \\ &\quad \leq -\frac{\delta-\mu}{\lambda}\int_0^T e^{-\delta t} X^0_t e^{-\lambda\tilde{\xi}^{\mathrm{s}}_t}\, dt + \frac{x}{\lambda} - \frac{1}{\lambda} e^{-\delta T} X^0_T e^{-\lambda\tilde{\xi}^{\mathrm{s}}_{T+}} \\ &\qquad - \int_{[0,T]} e^{-\delta t} C_{\mathrm{s}}\, d\tilde{\xi}^{\mathrm{s}}_t + \frac{\sigma}{\lambda}\int_0^T e^{-\delta t} X^0_t e^{-\lambda(\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t)}\, dW_t \\ &\quad = \int_{[0,T]} e^{-\delta t}\left[\tilde{X}_t \circ_{\mathrm{s}} d\tilde{\xi}^{\mathrm{s}}_t - C_{\mathrm{s}}\, d\tilde{\xi}^{\mathrm{s}}_t\right] + \frac{\sigma}{\lambda}\int_0^T e^{-\delta t} X^0_t\left(e^{-\lambda(\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t)} - e^{-\lambda\tilde{\xi}^{\mathrm{s}}_t}\right) dW_t, \end{aligned} \tag{3.14}$$

where $\tilde{X}$ is the solution to (2.4) with $\tilde{\xi}^{\mathrm{s}}$ and 0 in place of ξ^{s} and ξ^{b}, respectively.

Using Itô's isometry, Hölder's inequality, and (2.9) in Definition 2.1, we calculate

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T e^{-\delta t} X_t^0 \left(e^{-\lambda(\xi_t^{\mathrm{s}}-\xi_t^{\mathrm{b}})} - e^{-\lambda\tilde{\xi}_t^{\mathrm{s}}}\right) dW_t\right)^2\right] \\
&\qquad = \int_0^T \mathbb{E}\left[\left(e^{-\delta t} X_t^0 \left(e^{-\lambda(\xi_t^{\mathrm{s}}-\xi_t^{\mathrm{b}})} - e^{-\lambda\tilde{\xi}_t^{\mathrm{s}}}\right)\right)^2\right] dt \\
&\qquad \le 2\int_0^T \mathbb{E}\left[\left(X_t^0 e^{\lambda\xi_{T+}^{\mathrm{b}}}\right)^2\right] dt + 2\int_0^T \mathbb{E}\left[\left(X_t^0\right)^2\right] dt \\
&\qquad \le 2\int_0^T \sqrt{\mathbb{E}\left[e^{4\lambda\xi_{T+}^{\mathrm{b}}}\right]\mathbb{E}\left[\left(X_t^0\right)^4\right]}\, dt + 2\int_0^T \mathbb{E}\left[\left(X_t^0\right)^2\right] dt \\
&\qquad < \infty. \qquad (3.15)
\end{aligned}
$$

Therefore, the stochastic integral in (3.14) defines a square-integrable martingale and has 0 expectation. Taking expectations in (3.14), we therefore obtain

$$
\begin{aligned}
J_{T,x,y}(\xi^{\mathrm{s}},\xi^{\mathrm{b}}) &\le \mathbb{E}\left[-\frac{\delta-\mu}{\lambda}\int_0^T e^{-\delta t} X_t^0 e^{-\lambda\tilde{\xi}_t^{\mathrm{s}}}\, dt + \frac{x}{\lambda} - \frac{1}{\lambda}e^{-\delta T} X_T^0 e^{-\lambda\tilde{\xi}_{T+}^{\mathrm{s}}} - \int_{[0,T]} e^{-\delta t} C_{\mathrm{s}}\, d\tilde{\xi}_t^{\mathrm{s}}\right] \\
&= \mathbb{E}\left[\int_{[0,T]} e^{-\delta t}\left[\tilde{X}_t \circ_{\mathrm{s}} d\tilde{\xi}_t^{\mathrm{s}} - C_{\mathrm{s}}\, d\tilde{\xi}_t^{\mathrm{s}}\right]\right] \\
&= J_{T,x,y}(\tilde{\xi}^{\mathrm{s}}, 0),
\end{aligned}
$$

and (3.7) follows (see also (2.8)). Furthermore, the expression for $J_{T,x,y}(\tilde{\xi}^{\mathrm{s}}, 0)$ provided here implies the upper bound in (3.8) as well as establishes (III) because if $C_{\mathrm{s}} = 0$, then it is plainly maximized by the choice $\tilde{\xi}_t^{\mathrm{s}} = y$ for all $t > 0$ that corresponds to selling all shares at time 0. On the other hand, the lower bound in (3.8) is just the payoff of the strategy that sells all shares at time 0.

Finally, suppose that $\delta = \mu \ge 0$. Taking expectations in (3.13), we obtain

$$
J_{T,x,y}(\xi^{\mathrm{s}},\xi^{\mathrm{b}}) = \mathbb{E}\left[\frac{x}{\lambda} - \frac{1}{\lambda}e^{-\delta T} X_T^0 e^{-\lambda(\xi_{T+}^{\mathrm{s}}-\xi_{T+}^{\mathrm{b}})} - \int_{[0,T]} e^{-\delta t}\left[C_{\mathrm{s}}\, d\xi_t^{\mathrm{s}} + C_{\mathrm{b}}\, d\xi_t^{\mathrm{b}}\right]\right].
$$

If $\overline{T} < \infty$, then this expression and the fact that $\xi_{\overline{T}+}^{\mathrm{s}} - \xi_{\overline{T}+}^{\mathrm{b}} = y$ imply that it is optimal to sell all available shares at time $\overline{T}$ and

$$
v(\overline{T},x,y) = \frac{x}{\lambda} - \frac{1}{\lambda}\mathbb{E}\left[e^{-\delta\overline{T}} X_{\overline{T}}^0\right] e^{-\lambda y} - e^{-\delta\overline{T}} C_{\mathrm{s}} y = \frac{1}{\lambda}x\left[1 - e^{-\lambda y}\right] - e^{-\delta\overline{T}} C_{\mathrm{s}} y.
$$

On the other hand, if $\overline{T} = \infty$, then we can see that selling all available shares at time $n = 1, 2, \dots$ provides a sequence of ε-optimal strategies once we combine the observation that these strategies have expected payoffs such that

$$
\lim_{n\to\infty}\left(\frac{1}{\lambda}\mathbb{E}\left[e^{-\delta n} X_n^0\right]\left[1 - e^{-\lambda\varrho}\right] - e^{-\delta n} C_{\mathrm{s}} y\right) = \frac{1}{\lambda}x\left[1 - e^{-\lambda\varrho}\right]
$$

with the upper bound in (3.8). ∎

Remark 2. For future reference, we note the following estimate that we can derive using the integration by parts formula and Itô's isometry in the same way as in (3.13) and (3.15): given a time horizon $\overline{T} \in (0, \infty]$, a strategy $\xi^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}_{\overline{T},y}$, and any time $T \in [0, \overline{T}] \cap \mathbb{R}_+$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_{[0,t]} e^{-\delta t}\left[X_t \circ_{\mathrm{s}} d\xi^{\mathrm{s}}_t - C_{\mathrm{s}}\, d\xi^{\mathrm{s}}_t\right]\right|\right] \\
&\qquad \leq \mathbb{E}\left[\int_{[0,T]} e^{-\delta t} X_t \circ_{\mathrm{s}} d\xi^{\mathrm{s}}_t\right] + C_{\mathrm{s}} y \\
&\qquad = \mathbb{E}\left[-\frac{\delta-\mu}{\lambda}\int_0^T e^{-\delta t} X^0_t e^{-\lambda \xi^{\mathrm{s}}_t}\, dt + \frac{x}{\lambda} - \frac{1}{\lambda} e^{-\delta T} X^0_T e^{-\lambda \xi^{\mathrm{s}}_{T+}}\right] + C_{\mathrm{s}} y \\
&\qquad \leq \frac{x}{\lambda} + C_{\mathrm{s}} y.
\end{aligned} \tag{3.16}$$

Using Proposition 3.5, we now establish the following result that complements Proposition 3.4.

Proposition 3.6. *Consider the market model developed in section 2, and recall Definitions 3.2 and 3.3. The following statements are true:*

(I) *The market does not allow for arbitrage opportunities.*

(II) *If $\mu = 0$, then there exists no price manipulation.*

Proof. To establish (II), we assume that $\mu = 0$ and we show that every round-trip trade has negative expected execution payoff. To this end, we consider any round-trip trade $(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})$ with time horizon $T \in \mathbb{R}^*_+$ and we define the liquidation strategy $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \in \mathcal{A}_{\overline{T},\Gamma}$ by

$$\xi^{\mathrm{s}}_t = \begin{cases} \zeta^{\mathrm{s}}_t & \text{if } t \leq T, \\ \zeta^{\mathrm{s}}_{T+} & \text{if } t \in (T, \overline{T}], \\ \Gamma & \text{if } t > \overline{T} \end{cases} \quad \text{and} \quad \xi^{\mathrm{b}}_t = \begin{cases} \zeta^{\mathrm{b}}_t & \text{if } t \leq T, \\ \zeta^{\mathrm{b}}_{T+} & \text{if } t > T, \end{cases} \tag{3.17}$$

where $\overline{T} = T + \varepsilon$ for some $\varepsilon > 0$, and $\Gamma > 0$ is any bound as in (3.1). This strategy puts us in the context of an investor who starts with Γ shares, follows the round-trip trade up to time T, and then sells all available shares Γ at a later time $\overline{T}$. The expected revenue resulting from the execution of the round-trip trade is

$$\begin{aligned}
\mathbb{E}\left[R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})\right] &\leq \mathbb{E}\left[\int_{[0,T]}\left[X_t \circ^{\lambda}_{\mathrm{s}} d\zeta^{\mathrm{s}}_t - X_t \circ^{\lambda}_{\mathrm{b}} d\zeta^{\mathrm{b}}_t\right]\right] \\
&= \mathbb{E}\left[\int_{[0,\overline{T}]}\left[X_t \circ^{\lambda}_{\mathrm{s}} d\xi^{\mathrm{s}}_t - X_t \circ^{\lambda}_{\mathrm{b}} d\xi^{\mathrm{b}}_t\right]\right] - \frac{1}{\lambda}\mathbb{E}\left[X_{\overline{T}}\right]\left[1 - e^{-\lambda\Gamma}\right] \\
&\overset{(3.8)}{\leq} \frac{1}{\lambda} x \left[1 - e^{-\lambda\Gamma}\right] - \frac{1}{\lambda}\mathbb{E}\left[X^0_{\overline{T}} e^{-\lambda(\zeta^{\mathrm{s}}_{T+} - \zeta^{\mathrm{b}}_{T+})}\right]\left[1 - e^{-\lambda\Gamma}\right] \\
&= \frac{1}{\lambda} x \left[1 - e^{-\lambda\Gamma}\right] - \frac{1}{\lambda}\mathbb{E}\left[X^0_{\overline{T}}\right]\left[1 - e^{-\lambda\Gamma}\right] \\
&= 0,
\end{aligned}$$

and (II) follows.

To show (I), we argue by contradiction and we assume that there exists a round-trip trade $(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})$ with time horizon $T \in \mathbb{R}_+^*$ satisfying the requirements of Definition 3.2. We then define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left(-\frac{\mu^2}{2\sigma^2}T - \frac{\mu}{\sigma}W_T\right),$$

and we note that

$$dX_t = -\lambda X_t \circ_{\mathrm{s}} d\xi_t^{\mathrm{s}} + \lambda X_t \circ_{\mathrm{b}} d\xi_t^{\mathrm{b}} + \sigma X_t \, dW_t^{\mathbb{Q}},$$

where $(W_t^{\mathbb{Q}},\ t \in [0,T])$ is the $\mathbb{Q}$-Brownian motion defined by $W_t^{\mathbb{Q}} = \frac{\mu}{\sigma}t + W_t$. The equivalence of $\mathbb{P}$ and $\mathbb{Q}$ implies that

$$R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}}) \geq 0,\ \mathbb{Q}\text{-a.s.} \quad \text{and} \quad \mathbb{Q}\left(R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}}) > 0\right) > 0.$$

Therefore, $\mathbb{E}_{\mathbb{Q}}\left[R(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})\right] > 0$. It follows that $(\zeta^{\mathrm{s}}, \zeta^{\mathrm{b}})$ is a price manipulation in a setting with $\mu = 0$, which contradicts (II), and the proof is complete. ■

4. The finite time horizon case ($\overline{T} < \infty$). In view of Proposition 3.5(I), we expect that the value function v of the stochastic control problem formulated in section 2 identifies with an appropriate solution $w : \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$ to the HJB equation

$$\max\left\{-w_t(t,x,y) + \mathcal{L}w(t,x,y),\ -\lambda x w_x(t,x,y) - w_y(t,x,y) + x - C_{\mathrm{s}}\right\} = 0, \tag{4.1}$$

with boundary condition

$$w(0,x,y) = \frac{1}{\lambda}x\left[1 - e^{-\lambda y}\right] - C_{\mathrm{s}}y, \tag{4.2}$$

where

$$\mathcal{L}w(t,x,y) = \frac{1}{2}\sigma^2 x^2 w_{xx}(t,x,y) + \mu x w_x(t,x,y) - \delta w(t,x,y). \tag{4.3}$$

To obtain qualitative understanding of this equation, we consider the following heuristic arguments. Suppose that, at a given time, the investor's horizon is $t > 0$, the share price is $x > 0$, and the investor holds an amount $y > 0$ of shares. At that time, the investor is faced with two possible actions. The first one is to wait for a short time Δt and then continue optimally. Bellman's principle of optimality implies that this possibility, which is not necessarily optimal, is associated with the inequality

$$v(t,x,y) \geq \mathbb{E}\left[e^{-\delta\Delta t}v(t - \Delta t, X_{\Delta t}, y)\right].$$

Applying Itô's formula and dividing by Δt before letting $\Delta t \downarrow 0$, we obtain

$$-v_t(t,x,y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) + \mu x v_x(t,x,y) - \delta v(x,y) \leq 0. \tag{4.4}$$

The second possibility is to sell a small amount $\varepsilon > 0$ of shares and then continue optimally. This action is associated with the inequality

$$v(t,x,y) \geq v(t, x - \lambda x\varepsilon, y - \varepsilon) + (x - C_{\mathrm{s}})\varepsilon.$$

Rearranging terms and letting $\varepsilon \downarrow 0$, we obtain

$$-\lambda x v_x(t,x,y) - v_y(t,x,y) + x - C_{\mathrm{s}} \leq 0. \tag{4.5}$$

The Markovian character of the problem implies that one of these two possibilities should be optimal and one of (4.4)–(4.5) should hold with equality at any point in the state space. It follows that the problem's value function v should identify with an appropriate solution w of the HJB equation (4.1). Also, the boundary condition in (4.2) follows from the requirement that the investor must liquidate all share holdings at the end of the planning horizon.

We now prove a verification theorem that associates a smooth solution to the HJB equation (4.1)–(4.2) with the control problem's value function and can be used to identify an optimal liquidation strategy. To this end, we consider the sets

$$\begin{aligned} \mathcal{W} &= \left\{(t,x,y) \in [0,T] \times \mathbb{R}_+^* \times \mathbb{R}_+ \mid\ -w_t(t,x,y) + \mathcal{L}w(t,x,y) = 0\right\}, \\ \mathcal{S} &= \left\{(t,x,y) \in [0,T] \times \mathbb{R}_+^* \times \mathbb{R}_+ \mid\ \lambda x w_x(t,x,y) + w_y(t,x,y) - x + C_{\mathrm{s}} = 0\right\}, \end{aligned}$$

and we call them the "waiting" region and the "selling" region, respectively, consistent with the heuristics that we have discussed above. Also, we note that the inequalities in (4.6) are consistent with the bounds (3.8) that the value function satisfies.

Proposition 4.1. *Consider the optimal execution problem formulated in section 2. Given a time horizon $\overline{T} \in (0,\infty)$, suppose that a function $w : [0,\overline{T}] \times \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$ is a $C^{1,2,1}$ solution to the HJB equation (4.1)–(4.2) such that*

$$-C_{\mathrm{s}} y \leq w(t,x,y) \leq \frac{1}{\lambda} x \quad \textit{for all } (t,x,y) \in [0,\overline{T}] \times \mathbb{R}_+^* \times \mathbb{R}_+. \tag{4.6}$$

If, for all initial conditions $(x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+$, there exists $\xi^{\mathrm{s}\star} \in \mathcal{A}^{\mathrm{s}}_{\overline{T},y}$ such that

$$(X_t^\star, Y_t^\star) \in \mathcal{W} \quad \textit{for all } t \geq 0,\ \mathbb{P}\textit{-a.s.}, \tag{4.7}$$

$$\xi^{\mathrm{s}\star}_{t+} = \int_{[0,t]} \mathbf{1}_{\{(X_t^\star, Y_t^\star) \in \mathcal{S}\}}\, d\xi^{\mathrm{s}\star}_t \quad \textit{for all } t \geq 0,\ \mathbb{P}\textit{-a.s.}, \tag{4.8}$$

where $X^\star$ and $Y^\star$ are the share price and shares held processes associated with the liquidation strategy $(\xi^{\mathrm{s}\star}, 0)$, then w identifies with the value function v of the stochastic control problem formulated in section 2. In particular,

$$v(\overline{T},x,y) = \sup_{\xi^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}_{\overline{T},y}} J_{\overline{T},x,y}(\xi^{\mathrm{s}}, 0) = w(\overline{T},x,y) \quad \textit{for all } (x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+, \tag{4.9}$$

and $(\xi^{\mathrm{s}\star}, 0)$ is an optimal liquidation strategy.

Proof. We have established the first identity in (4.9) in Proposition 3.5(I). To prove the second one, we fix any initial condition $(x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+$ and any process $\xi^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}_{\overline{T},y}$. In view of the Itô–Tanaka–Meyer formula and the left-continuity of the processes X, Y, we can see that

$$
\begin{aligned}
e^{-\delta t} w(\overline{T}-t, X_{t+}, Y_{t+}) = w(\overline{T}, x, y) &+ \int_0^t e^{-\delta s} \left[-w_t(\overline{T}-s, X_s, Y_s) + \mathcal{L}w(\overline{T}-s, X_s, Y_s) \right] ds \\
&+ M_t - \int_0^t e^{-\delta s} \left[\lambda X_s w_x(\overline{T}-s, X_s, Y_s) + w_y(\overline{T}-s, X_s, Y_s) \right] d(\xi^{\mathrm{s}})^{\mathrm{c}}_s \\
&+ \sum_{0 \le s \le t} e^{-\delta s} \left[w(\overline{T}-s, X_{s+}, Y_{s+}) - w(\overline{T}-s, X_s, Y_s) \right],
\end{aligned}
$$

where

$$
M_t = \sigma \int_0^t e^{-\delta s} X_s w_x(\overline{T}-s, X_s, Y_s)\, dW_s.
$$

Combining this calculation with the observation that

$$
\begin{aligned}
&w(\overline{T}-s, X_{s+}, Y_{s+}) - w(\overline{T}-s, X_s, Y_s) \\
&\qquad = w(\overline{T}-s, e^{-\lambda \Delta \xi^{\mathrm{s}}_s} X_s, Y_s - \Delta \xi^{\mathrm{s}}_s) - w(\overline{T}-s, X_s, Y_s) \\
&\qquad = \int_0^{\Delta \xi^{\mathrm{s}}_s} \frac{\partial w(\overline{T}-s, e^{-\lambda u} X_s, Y_s - u)}{\partial u}\, du \\
&\qquad = -\int_0^{\Delta \xi^{\mathrm{s}}_s} \left[\lambda e^{-\lambda u} X_s w_x(\overline{T}-s, e^{-\lambda u} X_s, Y_s - u) + w_y(\overline{T}-s, e^{-\lambda u} X_s, Y_s - u) \right] du
\end{aligned}
$$

and (2.5), we obtain

$$
\begin{aligned}
&\int_{[0,t]} e^{-\delta s} \left[X_s \circ_{\mathrm{s}} d\xi^{\mathrm{s}}_s - C_{\mathrm{s}}\, d\xi^{\mathrm{s}}_s \right] + e^{-\delta t} w(\overline{T}-t, X_{t+}, Y_{t+}) \\
&\qquad = w(\overline{T}, x, y) + \int_0^t e^{-\delta s} \left[-w_t(\overline{T}-s, X_s, Y_s) + \mathcal{L}w(\overline{T}-s, X_s, Y_s) \right] ds + M_t \\
&\qquad\quad + \int_0^t e^{-\delta s} \left[-\lambda X_s w_x(\overline{T}-s, X_s, Y_s) - w_y(\overline{T}-s, X_s, Y_s) + X_s - C_{\mathrm{s}} \right] d(\xi^{\mathrm{s}})^{\mathrm{c}}_s \\
&\qquad\quad + \sum_{0 \le s \le t} e^{-\delta s} \int_0^{\Delta \xi^{\mathrm{s}}_s} \Big[-\lambda e^{-\lambda u} X_s w_x(\overline{T}-s, e^{-\lambda u} X_s, Y_s - u) \\
&\qquad\qquad\qquad - w_y(\overline{T}-s, e^{-\lambda u} X_s, Y_s - u) + e^{-\lambda u} X_s - C_{\mathrm{s}} \Big] du.
\end{aligned} \tag{4.10}
$$

Since w satisfies (4.1) and admits the lower bound in (4.6), this calculation implies that

$$
\int_{[0,t]} e^{-\delta s} \left[X_s \circ_{\mathrm{s}} d\xi^{\mathrm{s}}_s - C_{\mathrm{s}}\, d\xi^{\mathrm{s}}_s \right] - C_{\mathrm{s}} e^{-\delta t} Y_{t+} \le w(\overline{T}, x, y) + M_t \quad \text{for all } t \in [0, T]. \tag{4.11}
$$

In particular,

$$
\int_{[0,\overline{T}]} e^{-\delta s} \left[X_s \circ_{\mathrm{s}} d\xi^{\mathrm{s}}_s - C_{\mathrm{s}}\, d\xi^{\mathrm{s}}_s \right] \le w(\overline{T}, x, y) + M_{\overline{T}} \tag{4.12}
$$

because $Y_{\overline{T}+} = 0$. In view of (3.16) in Remark 2, (4.11), and the fact that $Y_t \in [0, y]$ for all $t \in [0, \overline{T}]$, we can see that $\inf_{0 \le t \le \overline{T}} M_t$ is an integrable random variable. Therefore, the stochastic integral M is a supermartingale. In light of this observation, we can take expectations in (4.12) to obtain

$$J_{\overline{T},x,y}(\xi^{\mathrm{s}}, 0) = \mathbb{E}\left[\int_{[0,\overline{T}]} e^{-\delta s} \left[X_s \circ_{\mathrm{s}} d\xi^{\mathrm{s}}_s - C_{\mathrm{s}}\, d\xi^{\mathrm{s}}_s\right]\right] \le w(\overline{T}, x, y).$$

This inequality and the first identity in (4.9) imply that

$$v(\overline{T}, x, y) \le w(\overline{T}, x, y) \tag{4.13}$$

because $\xi^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}_{\overline{T},y}$ has been arbitrary.

If a strategy $\xi^{\mathrm{s}\star} \in \mathcal{A}^{\mathrm{s}}_{\overline{T},y}$ is such that (4.7)–(4.8) hold true, then we can check that (4.1)–(4.2), the upper bound in (4.6), (4.10), and the fact that $Y^\star_{\overline{T}+} = 0$ imply that

$$\begin{aligned}\int_{[0,t]} & e^{-\delta s} \left[X^\star_s \circ_{\mathrm{s}} d\xi^{\mathrm{s}\star}_s - C_{\mathrm{s}}\, d\xi^{\mathrm{s}\star}_s\right] + \frac{1}{\lambda} e^{-\delta t} X^0_t e^{-\lambda \xi^{\mathrm{s}\star}_{t+}} \\ & \ge \int_{[0,t]} e^{-\delta s} \left[X^\star_s \circ_{\mathrm{s}} d\xi^{\mathrm{s}\star}_s - C_{\mathrm{s}}\, d\xi^{\mathrm{s}\star}_s\right] + e^{-\delta t} w(\overline{T} - t, X^\star_{t+}, Y^\star_{t+}) = w(\overline{T}, x, y) + M^\star_t\end{aligned}$$

and

$$\begin{aligned}\int_{[0,\overline{T}]} & e^{-\delta s} \left[X^\star_s \circ_{\mathrm{s}} d\xi^{\mathrm{s}\star}_s - C_{\mathrm{s}}\, d\xi^{\mathrm{s}\star}_s\right] \\ & = \int_{[0,\overline{T}]} e^{-\delta s} \left[X^\star_s \circ_{\mathrm{s}} d\xi^{\mathrm{s}\star}_s - C_{\mathrm{s}}\, d\xi^{\mathrm{s}\star}_s\right] + e^{-\delta \overline{T}} w(0, X^\star_{\overline{T}+}, 0) = w(\overline{T}, x, y) + M^\star_{\overline{T}},\end{aligned}$$

instead of (4.11) and (4.12), respectively. The inequality here and (3.16) in Remark 2 imply that $\sup_{0 \le t \le \overline{T}} M_t$ is an integrable random variable. Therefore, M is a submartingale and we can take expectations in the identity to obtain

$$J_{\overline{T},x,y}(\xi^{\mathrm{s}\star}, 0) = \mathbb{E}\left[\int_{[0,\overline{T}]} e^{-\delta s} \left[X^\star_s \circ_{\mathrm{s}} d\xi^{\mathrm{s}\star}_s - C_{\mathrm{s}}\, d\xi^{\mathrm{s}\star}_s\right]\right] \ge w(\overline{T}, x, y),$$

which, combined with the first identity in (4.9) and (4.13), implies the second identity in (4.9) as well as the optimality of $(\xi^{\mathrm{s}\star}, 0)$. ■

5. The infinite time horizon case ($\overline{T} = \infty$).

Throughout this section, we write $v(x, y)$ instead of $v(\infty, x, y)$ and we assume that

$$\mu < \delta \quad \text{and} \quad C_{\mathrm{s}} > 0 \tag{5.1}$$

(we have solved the cases arising in the context of Assumption 1 when the problem data does not satisfy these inequalities in Proposition 3.5(III)–(IV)).

In light of the heuristics we considered in the previous section that explain the structure of the HJB equation (4.1)–(4.2), we solve the stochastic control problem that arises when $\overline{T} = \infty$ and (5.1) holds true by constructing an appropriate solution $w : \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$ to the HJB equation

$$\max \{\mathcal{L}w(x,y),\ -\lambda x w_x(x,y) - w_y(x,y) + x - C_{\mathrm{s}}\} = 0, \tag{5.2}$$

where $\mathcal{L}$ is defined by (4.3), with boundary condition

$$w(x,0) = 0 \quad \text{for all } x > 0. \tag{5.3}$$

To this end, we look for a solution w to (5.2)–(5.3) that is characterized by a function $F : \mathbb{R}_+ \to \mathbb{R}_+$ that partitions the state space $\mathbb{R}_+^* \times \mathbb{R}_+$ into two regions, the "waiting" region $\mathcal{W}$ and the "selling" region $\mathcal{S}$, defined by

$$\mathcal{W} = \left\{(x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+ \mid\ y > 0 \text{ and } x < F(y)\right\} \cup \left(\mathbb{R}_+^* \times \{0\}\right), \tag{5.4}$$

$$\mathcal{S} = \left\{(x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+ \mid\ y > 0 \text{ and } x \geq F(y)\right\}. \tag{5.5}$$

Inside $\mathcal{W}$, w should satisfy the differential equation

$$\frac{1}{2}\sigma^2 x^2 w_{xx}(x,y) + \mu x w_x(x,y) - \delta w(x,y) = 0.$$

The only solution to this ODE that remains bounded as $x \downarrow 0$ is given by

$$w(x,y) = A(y) x^n \tag{5.6}$$

for some function $A : \mathbb{R}_+ \to \mathbb{R}$, where n is the positive solution to the quadratic equation

$$\frac{1}{2}\sigma^2 \ell(\ell-1) + \mu\ell - \delta \equiv \frac{1}{2}\sigma^2\ell^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\ell - \delta = 0. \tag{5.7}$$

For future reference, we note that $n > 1$ if and only if $\delta > \mu$. On the other hand, w should satisfy

$$-\lambda x w_x(x,y) - w_y(x,y) + x - C_{\mathrm{s}} = 0 \quad \text{for } (x,y) \in \mathcal{S}, \tag{5.8}$$

which implies that

$$-\lambda x w_{xx}(x,y) - \lambda w_x(x,y) - w_{yx}(x,y) + 1 = 0 \quad \text{for } (x,y) \in \mathcal{S}. \tag{5.9}$$

To proceed further, we look for A and F such that w is $C^{2,1}$. Such a requirement, (5.6), and (5.8)–(5.9) yield the system of equations

$$-\lambda n A(y) x^n - \dot{A}(y) x^n + x - C_{\mathrm{s}}\Big|_{x=F(y)} = 0,$$

$$-\lambda n^2 A(y) x^{n-1} - n\dot{A}(y) x^{n-1} + 1\Big|_{x=F(y)} = 0,$$

which is equivalent to

$$F(y) = \frac{nC_{\mathrm{s}}}{n-1} =: F_\circ, \tag{5.10}$$

$$\dot{A}(y)F_\circ^n = -\lambda n A(y) F_\circ^n + F_\circ - C_{\mathrm{s}}. \tag{5.11}$$

In view of the boundary condition (5.3) and (5.6), we require that $A(0) = 0$ and we solve (5.11) to obtain

$$A(y) = e^{-\lambda n y} \int_0^y e^{\lambda n u} \frac{1}{n} \left(\frac{n-1}{nC_{\mathrm{s}}} \right)^{n-1} du = \frac{1}{\lambda n^2} \left(\frac{n-1}{nC_{\mathrm{s}}} \right)^{n-1} (1 - e^{-\lambda n y}). \tag{5.12}$$

The analysis thus far has fully characterized w inside the waiting region $\mathcal{W}$. To determine w inside the selling region $\mathcal{S}$, we consider the function $\mathbb{Y}$ defined by

$$\mathbb{Y}(x) = \frac{1}{\lambda} \ln \frac{x}{F_\circ} \quad \text{for } x > 0, \tag{5.13}$$

and we note that

$$F_\circ - x = -x \left[1 - e^{-\lambda \mathbb{Y}(x)} \right] \quad \text{and} \quad y - \mathbb{Y}(x) > 0 \iff x < F_\circ e^{\lambda y}. \tag{5.14}$$

In particular, we note that the restriction of $\mathbb{Y}$ in $(F_\circ, \infty)$ partitions the selling region into

$$\mathcal{S}_1 = \left\{ (x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \mid \ x \geq F_\circ \text{ and } y \leq \mathbb{Y}(x) \right\},$$
$$\mathcal{S}_2 = \left\{ (x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \mid \ x \geq F_\circ \text{ and } y > \mathbb{Y}(x) \right\}$$

(see also Figure 1). The region $\mathcal{S}_1$ is the part of the state space where it is optimal to sell all available shares at time 0. On the other hand the region $\mathcal{S}_2$ is the part of the state space where it is optimal to sell an amount $\mathbb{Y}(x)$ of shares at time 0 and then sell continuously in a manner such that the optimal joint process $(X^\star, Y^\star)$ is reflected in the line $x = F_\circ$ in an appropriate oblong way until all shares are exhausted. These considerations and the structure of the performance criterion that we maximize suggest that

$$w(x,y) = \frac{1}{\lambda} x \left[1 - e^{-\lambda y} \right] - C_{\mathrm{s}} y \quad \text{if } (x,y) \in \mathcal{S}_1,$$
$$w(x,y) = w(F_\circ, y - \mathbb{Y}(x)) + \frac{1}{\lambda} x \left[1 - e^{-\lambda \mathbb{Y}(x)} \right] - C_{\mathrm{s}} \mathbb{Y}(x) \quad \text{if } (x,y) \in \mathcal{S}_2.$$

We conclude this discussion with the candidate for a solution to the HJB equations (5.2)–(5.3) given by

$$w(x,y) = \begin{cases} 0 & \text{if } y = 0 \text{ and } x > 0, \\ A(y) x^n & \text{if } y > 0 \text{ and } x \leq F_\circ, \\ A(y - \mathbb{Y}(x)) F_\circ^n + \frac{x - F_\circ}{\lambda} - C_{\mathrm{s}} \mathbb{Y}(x) & \text{if } y > 0 \text{ and } F_\circ < x < F_\circ e^{\lambda y}, \\ \frac{1}{\lambda} x \left[1 - e^{-\lambda y} \right] - C_{\mathrm{s}} y & \text{if } y > 0 \text{ and } F_\circ e^{\lambda y} \leq x. \end{cases} \tag{5.15}$$

We can now prove the main result of the section, which shows that this function is indeed the control problem's value function and identifies an optimal liquidation strategy.

Proposition 5.1. *Consider the optimal execution problem formulated in section 2, and suppose that $\overline{T} = \infty$ and that (5.1) holds true. The function w defined by (5.15), where $F_\circ$, A are given by (5.10), (5.12), is a $C^{2,1}$ solution to the HJB equation (5.2) that identifies with the value function v of the stochastic control problem. In particular,*

$$v(x,y) = \sup_{\xi^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}_{\infty,y}} I_{\infty,x,y}(\xi^{\mathrm{s}}, 0) = w(x,y) \quad \textit{for all } (x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+. \tag{5.16}$$

Furthermore, if we define

$$\xi^{\mathrm{s}\star}_t = y \wedge \sup_{0 \le s \le t} \frac{1}{\lambda} \left[\ln x + B_s - \ln F_\circ\right]^+ \quad \textit{for } t > 0, \tag{5.17}$$

where

$$B_t = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t, \tag{5.18}$$

then the following statements are true:

(I) *If $\mu - \frac{1}{2}\sigma^2 \ge 0$, then $(\xi^{\mathrm{s}\star}, 0)$ is an optimal liquidation strategy.*

(II) *If $\mu - \frac{1}{2}\sigma^2 < 0$, then $(\xi^{\mathrm{s}\star}, 0)$ is not an admissible liquidation strategy. In this case, if we define*

$$\xi^{\mathrm{s}\star j}_t = \xi^{\mathrm{s}\star}_t \mathbf{1}_{\{t \le j\}} + y \mathbf{1}_{\{j < t\}} \quad \textit{for } t > 0 \textit{ and } j \ge 1, \tag{5.19}$$

then $(\xi^{\mathrm{s}\star j}, 0)$ gives rise to a sequence of ε-optimal strategies.

Proof. In view of its construction, we will prove that w is $C^{2,1}$ if we show that w_y, w_x, and w_{xx} are continuous along the free-boundary F as well as along the restriction of $\mathbb{Y}$ in $(F_\circ, \infty)$. To this end, we consider any $(x,y) \in \mathcal{S}_2$ and we use the ODE (5.11) that A satisfies as well as the definition (5.13) of $\mathbb{Y}$ to calculate

$$w_y(x,y) = \dot{A}(y - \mathbb{Y}(x)) F_\circ^n, \tag{5.20}$$

$$\begin{aligned} w_x(x,y) &= \left[-\dot{A}(y - \mathbb{Y}(x)) F_\circ^n - C_{\mathrm{s}}\right] \frac{1}{\lambda x} + \frac{1}{\lambda} \\ &= nA(y - \mathbb{Y}(x)) \frac{F_\circ^n}{x} + \frac{1}{\lambda}\left[1 - \frac{F_\circ}{x}\right] \end{aligned} \tag{5.21}$$

and

$$\begin{aligned} w_{xx}(x,y) =& -n\dot{A}(y - \mathbb{Y}(x)) F_\circ^n \frac{1}{\lambda x^2} - nA(y - \mathbb{Y}(x)) \frac{F_\circ^n}{x^2} + \frac{F_\circ}{\lambda x^2} \\ &= n(n-1) A(y - \mathbb{Y}(x)) \frac{F_\circ^n}{x^2} - \frac{1}{\lambda x^2}\left[(n-1)F_\circ - nC_{\mathrm{s}}\right] \\ &= n(n-1) A(y - \mathbb{Y}(x)) \frac{F_\circ^n}{x^2}, \end{aligned} \tag{5.22}$$

where the last identity follows thanks to (5.10). These calculations imply the required continuity results along F because $\lim_{x\downarrow F_\circ} \mathbb{Y}(x) = 0$. Also, these calculations, the observation that

$$\dot{A}(0) = \lim_{y\downarrow 0} \dot{A}(y) = (F_\circ - C_{\rm s})F_\circ^{-n},$$

which follows from (5.11), and the fact that $A(0) = 0$ imply that given any point $x > F_\circ$ and any sequence $(x_n, y_n) \in \mathcal{S}_2$ converging to $(x, \mathbb{Y}(x))$,

$$\lim_{n\to\infty} w_y(x_n, y_n) = F_\circ - C_{\rm s},$$

$$\lim_{n\to\infty} w_x(x_n, y_n) = \frac{1}{\lambda}\left[1 - \frac{F_\circ}{x}\right] = \frac{1}{\lambda}\left[1 - e^{-\lambda\mathbb{Y}(x)}\right] \quad \text{and} \quad \lim_{n\to\infty} w_{xx}(x_n, y_n) = 0.$$

These expressions are the same as the corresponding ones that we derive using the definition (5.15) of w for a point $x > F_\circ$ and any sequence $(x_n, y_n) \in \mathcal{S}_1$ converging to $(x, \mathbb{Y}(x))$, and the required continuity results along the restriction of $\mathbb{Y}$ in $(F_\circ, \infty)$ follow.

By the construction and the $C^{2,1}$ continuity of w, we will show that w satisfies the HJB equation (5.2) if we prove that

$$-\lambda x w_x(x, y) - w_y(x, y) + x - C_{\rm s} \le 0 \quad \text{for all } (x, y) \in \mathcal{W}, \tag{5.23}$$

$$\frac{1}{2}\sigma^2 x^2 w_{xx}(x, y) + \mu x w_x(x, y) - \delta w(x, y) \le 0 \quad \text{for all } (x, y) \in \mathcal{S}. \tag{5.24}$$

In view of (5.11), we can see that (5.23) is equivalent to

$$\frac{x - C_{\rm s}}{x^n} \le \frac{F_\circ - C_{\rm s}}{F_\circ^n} \quad \text{for all } x \le F_\circ,$$

which is true thanks to the calculation

$$\frac{d}{dx}\left(\frac{x - C_{\rm s}}{x^n}\right) = \frac{n-1}{x^{n+1}}\left(\frac{nC_{\rm s}}{n-1} - x\right) > 0 \quad \text{for all } x < F_\circ = \frac{nC_{\rm s}}{n-1}.$$

To prove (5.24), we first note that the quadratic equation (5.7), which $n > 1$ satisfies, and the definition of $F_\circ$ in (5.10) imply that

$$\delta C_{\rm s} - (\delta - \mu)F_\circ = -\frac{C_{\rm s}(\delta - \mu n)}{n-1} = -\frac{1}{2}\sigma^2 n C_{\rm s} < 0. \tag{5.25}$$

Given any $(x, y) \in \mathcal{S}_1$, we use the fact that $x \ge F_\circ e^{\lambda y}$ to calculate

$$\begin{aligned}\frac{1}{2}\sigma^2 x^2 w_{xx}(x, y) + \mu x w_x(x, y) - \delta w(x, y) &= -\frac{\delta - \mu}{\lambda} x\left[1 - e^{-\lambda y}\right] + \delta C_{\rm s} y\\ &\le -\frac{\delta - \mu}{\lambda} F_\circ\left[e^{\lambda y} - 1\right] + \delta C_{\rm s} y =: Q_1(y).\end{aligned} \tag{5.26}$$

Also, we use the fact that n satisfies (5.7) to see that, given any $(x, y) \in \mathcal{S}_2$,

$$\frac{1}{2}\sigma^2 x^2 w_{xx}(x, y) + \mu x w_x(x, y) - \delta w(x, y) = -\frac{\delta - \mu}{\lambda}(x - F_\circ) + \delta C_{\rm s}\mathbb{Y}(x) =: Q_2(x). \tag{5.27}$$

In view of (5.25) and the definition (5.13) of $\mathbb{Y}$, we can see that

$$Q_1(0) = 0 \quad \text{and} \quad Q_1'(y) = \delta C_{\rm s} - (\delta - \mu) F_\circ e^{\lambda y} < 0 \quad \text{for all } y \geq 0$$

and

$$Q_2(F_\circ) = 0 \quad \text{and} \quad Q_2'(x) = -\frac{\delta - \mu}{\lambda} + \frac{\delta C_{\rm s}}{\lambda x} \leq \frac{\delta C_{\rm s} - (\delta - \mu) F_\circ}{\lambda F_\circ} < 0 \quad \text{for all } x \geq F_\circ.$$

It follows that the right-hand side of (5.26) (resp., (5.27)) is negative for all $y \geq 0$ (resp., $x \geq F_\circ$), and (5.24) has been established.

We have established the first identity in (5.16) in Proposition 3.5(I). To derive the second one, we consider any strategy $\xi^{\rm s} \in \mathcal{A}^{\rm s}_{\infty,y}$. Arguing in the same way as in the proof of Proposition 4.1 up to (4.11)–(4.12) and using the positivity of w instead of the lower bound in (4.6), we can show that

$$\int_{[0,T]} e^{-\delta t} \left[X_t \circ_{\rm s} d\xi^{\rm s}_t - C_{\rm s}\, d\xi^{\rm s}_t \right] \leq w(x,y) + M_T, \tag{5.28}$$

where

$$M_T = \sigma \int_0^T e^{-\delta t} X_t w_x(X_t, Y_t)\, dW_t. \tag{5.29}$$

This result and (3.16) in Remark 2 imply that the random variable $\inf_{t \in [0,T]} M_t$ is integrable for all $T > 0$. Therefore, the stochastic integral M is a supermartingale. In light of this observation, we can take expectations in (5.28) to obtain

$$I_{\infty,x,y}(\xi^{\rm s}, 0) = \limsup_{T \to \infty} J_{T,x,y}(\xi^{\rm s}, 0) = \limsup_{T \to \infty} \mathbb{E}\left[\int_{[0,T]} e^{-\delta t} \left[X_t \circ_{\rm s} d\xi^{\rm s}_t - C_{\rm s}\, d\xi^{\rm s}_t \right] \right] \leq w(x,y).$$

It follows that $v(x,y) \leq w(x,y)$ because $\xi^{\rm s} \in \mathcal{A}^{\rm s}_{\infty,y}$ has been arbitrary.

To prove the reverse inequality and establish the optimality claims associated with $(\xi^{\rm s\star}, 0)$, where $\xi^{\rm s\star}$ is given by (5.17), we first note that apart from a jump of size $\min\{y, \frac{1}{\lambda}(\ln \frac{x}{F_\circ})^+\} = \min\left\{y, (\mathbb{Y}(x))^+\right\}$ at time 0, the process $(\ln x + B - \lambda \xi^{\rm s\star}, Y^\star - \xi^{\rm s\star})$ is reflecting in the line $x = \ln F_\circ$ in the direction defined by the vector $(-\lambda, -1)$. In particular,

$$\ln x + B_t - \lambda \xi^{\rm s\star}_t \leq \ln F_\circ \quad \text{and} \quad \xi^{\rm s\star}_t - \xi^{\rm s\star}_0 = \int_{]0,t[} \mathbf{1}_{\{\ln x + B_t - \lambda \xi^{\rm s\star}_t = \ln F_\circ\}}\, d\xi^{\rm s\star}_s \quad \text{for all } t \leq \tau^*,$$

where $\tau^* = \inf\{t \geq 0 \mid \ \xi^{\rm s\star}_t = y\}$. In view of this observation and (2.7), if we denote by $X^\star$ and $Y^\star$ the price process and the remaining amount of shares process associated with $(\xi^{\rm s\star}, 0)$, then

$$(X^\star_t, Y^\star) \in \mathcal{W} \quad \text{and} \quad \xi^{\rm s\star}_{t+} = \int_{[0,t]} \mathbf{1}_{\{(X^\star_t, Y^\star_t) \in \mathcal{S}\}}\, d\xi^{\rm s\star}_s \quad \text{for all } t \geq 0,$$

where the waiting region $\mathcal{W}$ and the selling region $\mathcal{S}$ are given by (5.4) and (5.5). Also, we can check that the strategy $(\xi^{\rm s\star}, 0)$ is admissible provided $\lim_{T \to \infty} Y^\star_T = 0$. In view of (5.17)–(5.18), we can see that this is indeed the case if and only if $\mu - \frac{1}{2}\sigma^2 \geq 0$ because a Brownian motion with strictly negative drift has a supremum over time that is an exponentially distributed random variable.

In the same way as in the proof of Proposition 4.1, we now see that

$$\int_{[0,T]} e^{-\delta t}\left[X_t^\star \circ_{\mathrm{s}} d\xi_t^{\mathrm{s}\star} - C_{\mathrm{s}}\, d\xi_t^{\mathrm{s}\star}\right] + e^{-\delta T} w(X_{T+}^\star, Y_{T+}^\star) = w(x,y) + M_T^\star, \tag{5.30}$$

where the local martingale $M^\star$ is defined as in (5.29). In view of this identity, (3.16), and the inequality

$$0 \leq w(X_t^\star, Y^\star) \leq \frac{1}{\lambda n^2}\left(\frac{n-1}{nC_{\mathrm{s}}}\right)^{n-1} F_\circ^n, \tag{5.31}$$

which follows from (5.12) and the definition (5.15) of w, we can see that the random variable $\sup_{t\in[0,T]} M_t$ is integrable for all $T > 0$. Therefore, $M^\star$ is a submartingale, and we can take expectations in (5.30) to obtain

$$\begin{aligned} J_{T,x,y}(\xi^{\mathrm{s}\star}, 0) &= \mathbb{E}\left[\int_{[0,T]} e^{-\delta t}\left[X_t^\star \circ_{\mathrm{s}} d\xi_t^{\mathrm{s}\star} - C_{\mathrm{s}}\, d\xi_t^{\mathrm{s}\star}\right]\right] \\ &\geq w(x,y) - \mathbb{E}\left[e^{-\delta T} w(X_{T+}^\star, Y_{T+}^\star)\right]. \end{aligned} \tag{5.32}$$

These identities and (5.31) imply that

$$I_{\infty,x,y}(\xi^{\mathrm{s}\star}, 0) = \limsup_{T\to\infty} J_{T,x,y}(\xi^{\mathrm{s}\star}, 0) \geq w(T,x,y).$$

Combining this result with the inequality $v(x,y) \leq w(x,y)$ that we have established above, we derive (5.16) as well as the optimality of $(\xi^{\mathrm{s}\star}, 0)$, which is admissible if and only if $\mu - \frac{1}{2}\sigma^2 \geq 0$.

If $\mu - \frac{1}{2}\sigma^2 < 0$, then we can use (5.32) to check that the strategy $(\xi^{\mathrm{s}\star j}, 0)$ given by (5.19) has payoff

$$\begin{aligned} I_{\infty,x,y}(\xi^{\mathrm{s}\star j}, 0) &= \limsup_{T\to\infty} J_{T,x,y}(\xi^{\mathrm{s}\star j}, 0) \\ &= \mathbb{E}\left[\int_{[0,j]} e^{-\delta t}\left[X_t^\star \circ_{\mathrm{s}} d\xi_t^{\mathrm{s}\star} - C_{\mathrm{s}}\, d\xi_t^{\mathrm{s}\star}\right]\right] + \frac{1}{\lambda}\mathbb{E}\left[X_j^\star\left[1 - e^{-\lambda(y - Y_j^\star)}\right]\right] \\ &\geq w(x,y) - \mathbb{E}\left[e^{-\delta j} w(X_{j+}^\star, Y_{j+}^\star)\right]. \end{aligned}$$

The inequality $v(x,y) \leq w(x,y)$ and the fact that the right-hand side of this expression converges to $w(x,y)$ as $j \to \infty$ imply (5.16) and establish that $(\xi^{\mathrm{s}\star j}, 0)$ is a sequence of ε-optimal strategies. ■

Acknowledgments. We thank the three anonymous referees for their constructive comments. We are also grateful to the associate editor, who handled the paper for numerous detailed comments and suggestions that enhanced the motivation and the presentation of the model we study.

REFERENCES

[1] A. Alfonsi, A. Fruth, and A. Schied, *Constrained portfolio liquidation in a limit order book model*, in Advances in Mathematics of Finance, Banach Center Publ. 83, Polish Academy of Sciences, Warsaw, 2008, pp. 9–25.
[2] A. Alfonsi, A. Fruth, and A. Schied, *Optimal execution strategies in limit order books with general shape functions*, Quant. Finance, 10 (2010), pp. 143–157.
[3] A. Alfonsi, A. Schied, and A. Slynko, *Order book resilience, price manipulation, and the positive portfolio problem*, SIAM J. Financial Math., 3 (2012), pp. 511–533.
[4] R. Almgren, *Optimal execution with nonlinear impact function and trading enhanced risk*, Appl. Math. Finance, 10 (2003), pp. 1–18.
[5] R. Almgren and N. Chriss, *Value under liquidation*, Risk, 12 (1999), pp. 61–63.
[6] R. Almgren and N. Chriss, *Optimal execution of portfolio transactions*, J. Risk, 3 (2000), pp. 5–39.
[7] R. Almgren and J. Lorenz, *Adaptive arrival price*, Institutional Investor Journals, 1 (2007), pp. 59–66.
[8] D. Bertsimas and A. W. Lo, *Optimal control of execution costs*, J. Financ. Market, 1 (1998), pp. 1–50.
[9] D. Bertsimas, A. W. Lo, and P. Hummel, *Optimal control of execution costs for portfolios*, Comput. Sci. Eng., 1 (1999), pp. 40–53.
[10] R. Cont and A. de Larrard, *Order Book Dynamics in Liquid Markets: Limit Theorem and Diffusion Approximations*, http://ssrn.com/abstract=1757861 (February 1, 2012).
[11] M. Dai and F. Yi, *Finite horizon optimal investment with transaction costs: A parabolic double obstacle problem*, J. Differential Equations, 246 (2009), pp. 1445–1469.
[12] M. H. A. Davis and A. R. Norman, *Portfolio selection with transaction costs*, Math. Oper. Res., 15 (1990), pp. 676–713.
[13] R. Engle and R. Ferstenberg, *Execution risk*, J. Portfolio Manage., 33 (2007), pp. 34–44.
[14] P. A. Forsyth, J. S. Kennedy, S. T. Tse, and H. Windcliff, *Optimal trade execution: A mean-quadratic-variation approach*, J. Econom. Dynam. Control, 36 (2011), pp. 1971–1991.
[15] P. A. Forsyth, J. S. Kennedy, S. T. Tse, and H. Windcliff, *Comparison between the mean-variance optimal and the mean-quadratic-variation optimal trading strategies*, Appl. Math. Finance, 20 (2013), pp. 415–449.
[16] J. Gatheral, *No-dynamic-arbitrage and market impact*, Quant. Finance, 10 (2010), pp. 749–759.
[17] J. Gatheral and A. Schied, *Optimal trade execution under geometric Brownian motion in the Almgren and Chriss framework*, Int. J. Theor. Appl. Finance, 14 (2011), pp. 353–368.
[18] J. Gatheral and A. Schied, *Dynamical models of market impact and algorithms for order execution*, in Handbook on Systemic Risk, Cambridge University Press, Cambridge, UK, 2013, pp. 579–599.
[19] J. Gatheral, A. Schied, and A. Slynko, *Transient linear price impact and Fredholm integral equations*, Math. Finance, 22 (2012), pp. 445–474.
[20] H. He and H. Mamaysky, *Dynamic trading with price impact*, J. Econom. Dynam. Control, 29 (2005), pp. 891–930.
[21] G. Huberman and W. Stanzl, *Price manipulation and quasi-arbitrage*, Econometrica, 72 (2004), pp. 1247–1275.
[22] O. Lebedeva, E. Maug, and C. Schneider, *Trading Strategies of Corporate Insiders*, http://ssrn.com/abstract=2118110 (January 16, 2015).
[23] A. Løkka, *Optimal liquidation in a limit order book for a risk-averse investor*, Math. Finance, 24 (2014), pp. 696–727.
[24] C. Moallemi, B. Park, and B. Van Roy, *The Execution Game*, Columbia Business School Research Paper Forthcoming; available from http://ssrn.com/abstract=1916298 (April 28, 2008).
[25] A. Obizhaeva and J. Wang, *Optimal trading strategy and supply/demand dynamics*, J. Financ. Market, 16 (2013), pp. 1–32.
[26] D. Ocone and A. Weerasinghe, *Degenerate variance control of a one-dimensional diffusion*, SIAM J. Control Optim., 39 (2000), pp. 1–24.
[27] S. Predoiu, G. Shaikhet, and S. Shreve, *Optimal execution in a general one-sided limit-order book*, SIAM J. Financial Math., 2 (2011), pp. 183–212.
[28] A. Schied and T. Schöneborn, *Liquidation in the Face of Adversity: Stealth vs. Sunshine Trading*, EFA 2008 Athens Meetings Paper; available from http://ssrn.com/abstract=1007014 (March 18, 2009).

[29] A. Schied and T. Schöneborn, *Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets*, Finance Stoch., 13 (2009), pp. 181–204.

[30] A. Schied, T. Schöneborn, and M. Tehranchi, *Optimal basket liquidation for CARA investors is deterministic*, Appl. Math. Finance, 17 (2010), pp. 471–489.

[31] S. Shreve and H. M. Soner, *Optimal investment and consumption with transaction costs*, Ann. Appl. Probab., 4 (1994), pp. 609–692.

[32] H. M. Soner and S. E. Shreve, *Regularity of the value function for a two-dimensional singular stochastic control problem*, SIAM J. Control Optim., 27 (1989), pp. 876–907.

[33] H. Zhu, *Generalized solution in singular stochastic control: The nondegenerate problem*, Appl. Math. Optim., 25 (1992), pp. 225–245.