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Perpetual American options in diffusion-type models with running maxima and drawdowns

Pavel V. Gapeev* Neofytos Rodosthenous†

We study perpetual American option pricing problems in an extension of the Black-Merton-Scholes model in which the dividend and volatility rates of the underlying risky asset depend on the running values of its maximum and maximum drawdown. The optimal exercise times are shown to be the first times at which the underlying asset hits certain boundaries depending on the running values of the associated maximum and maximum drawdown processes. We obtain closed-form solutions to the equivalent free-boundary problems for the value functions with smooth fit at the optimal stopping boundaries and normal reflection at the edges of the state space of the resulting three-dimensional Markov process. The optimal exercise boundaries for the perpetual American options on the maximum of the market depth with fixed and floating strikes are determined as the minimal solutions of certain first-order nonlinear ordinary differential equations.

1. Introduction

The main aim of this paper is to present closed-form solutions to the discounted optimal stopping problem of (2.4) for the running maximum S and the running maximum drawdown Y associated with the continuous process X defined in (2.1)-(2.2). This problem is related to the

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option pricing theory in mathematical finance, where the process X can describe the price of a risky asset (e.g. a stock) on a financial market. The value of (2.4) can therefore be interpreted as the rational (or no-arbitrage) price of a perpetual American option in a diffusion-type extension of the Black-Merton-Scholes model (see, e.g. Shiryaev [37; Chapter VIII; Section 2a], Peskir and Shiryaev [33; Chapter VII; Section 25], and Detemple [5], for an extensive overview of other related results in the area).

Optimal stopping problems for running maxima of some diffusion processes given linear costs were studied by Jacka [19], Dubins, Shepp, and Shiryaev [7], and Graversen and Peskir [14]-[15] among others, with the aim of determining the best constants in the corresponding maximal inequalities. A complete solution of a general version of the same problem was obtained in Peskir [29], by means of the established maximality principle, which is equivalent to the superharmonic characterisation of the value function. Discounted optimal stopping problems for certain payoff functions depending on the running maxima of geometric Brownian motions were initiated by Shepp and Shiryaev [35]-[36] and then considered by Pedersen [28] and Guo and Shepp [16] among others, with the aim of computing rational values of perpetual American lookback (Russian) options. More recently, Guo and Zervos [17] derived solutions for discounted optimal stopping problems related to the pricing of perpetual American options with certain payoff functions depending on the running values of both the initial diffusion process and its associated maximum. Glover, Hulley, and Peskir [13] provided solutions to optimal stopping problems for integrals of functions depending on the running values of both the initial diffusion process and its associated minimum. The main feature of the resulting optimal stopping problems is that the normal-reflection condition holds for the value function at the diagonal of the state space of the two-dimensional continuous Markov process having the initial process and the running extremum as its components, which implies the characterisation of the optimal boundaries as the extremal solutions of one-dimensional first-order nonlinear ordinary differential equations.

Asmussen, Avram, and Pistorius [1] considered perpetual American options with payoffs depending on the running maximum of some Lévy processes with two-sided jumps having phase-type distributions in both directions. Avram, Kyprianou, and Pistorius [2] studied exit problems for spectrally negative Lévy processes and applied the results to solving optimal stopping problems for payoff functions depending on the running values of the initial processes or their associated maxima. Optimal stopping games with payoff functions of such type were considered by Baurdoux and Kyprianou [3] and Baurdoux, Kyprianou, and Pardo [4] within the framework of models based on spectrally negative Lévy processes. Other complicated optimal stopping problems for the running maxima were considered in [11] for a jump-diffusion model with compound Poisson processes with exponentially distributed jumps and by Ott [26] and Kyprianou and Ott [25] (see also Ott [27]) for a model based on spectrally negative Lévy processes. More recently, Peskir [31]-[32] studied optimal stopping problems for three-dimensional Markov processes having the initial diffusion process as well as its maximum and minimum as the state space components. It was shown that the optimal boundary surfaces depending on the maximum and minimum of the initial process provide the maximal and minimal solutions of the associated systems of first-order non-linear partial differential equations.

In this paper, we obtain closed-form solutions to the problems of rational valuation of the perpetual American options on the maximum of the market depth with fixed and floating strikes in an extension of the Black-Merton-Scholes model with path-dependent coefficients. Such options represent protections for the holders of particularly risky assets, the prices of which can fall deeply, after achieving their historic maxima. These contracts should be exercised when the maximum drawdown of the underlying asset price rises above the difference of the running maximum and either a certain fixed value or the current value of a certain number of assets. The closed-form expressions for the rational prices of perpetual American standard put and call options on the underlying assets in this model were recently computed in [12]. The maximum drawdown process represents the maximum of the difference between the running values of the underlying asset price and its maximum and can therefore be interpreted as the maximum of the market depth. We assume that the price dynamics of the underlying asset are described by a geometric diffusion-type process X with local drift and diffusion coefficients, which essentially depend on the running values of the maximum process S and the maximum drawdown process Y. Such dependence of the dividend and volatility rates on the past dynamics of the asset in the financial market is often used in practice, although it has not been well captured by local or stochastic dividend and volatility models studied in the literature.

It is shown that the optimal exercise times for these options are the first times at which the process X hits certain boundaries depending on the running values of S and Y. We derive closed-form expressions for the value functions as solutions of the equivalent free-boundary problems and apply the maximality principle from [29] to describe the optimal boundary surfaces as the *minimal* solutions of first-order nonlinear ordinary differential equations. The starting conditions for these surfaces at the edges of the three-dimensional state space of (X, S, Y) are specified from the solutions of the corresponding optimal stopping problems in the model with the coefficients of the process X depending only on the running maximum process S. We also present an explicit solution of the ordinary differential equation corresponding to the case of options with floating strikes in the latter particular model.

The Laplace transforms of the drawdown process and other related characteristics associated with certain classes of the initial processes such as some diffusion models and spectrally positive and negative Lévy processes were studied by Pospisil, Vecer, and Hadjiliadis [34] and by Mijatović and Pistorius [23], respectively. Diffusion-type processes with given joint laws for the terminal level and supremum at an independent exponential time were constructed in Forde [9], by allowing the diffusion coefficient to depend on the running values of the initial process and its running minimum. Other important characteristics for such diffusion-type processes were recently derived by Forde, Pogudin, and Zhang [10].

The paper is organized as follows. In Section 2, we formulate the associated optimal stopping problems for a necessarily three-dimensional continuous Markov process, which has the underlying asset price and the running values of its maximum and maximum drawdown as the state space components. The resulting optimal stopping problems are reduced to their equivalent free-boundary problems for the value functions which satisfy the smooth-fit conditions at the stopping boundaries and the normal-reflection conditions at the edges of the state space of the three-dimensional process. In Section 3, we obtain closed-form solutions of the associated free-boundary problems and derive first-order nonlinear ordinary differential equations for the candidate stopping boundaries. We specify the starting conditions for the latter and provide a recursive algorithm to determine the value functions and the optimal boundaries along with their intersection lines with the edges of the three-dimensional state space. In Section 4, by applying the change-of-variable formula with local time on surfaces from Peskir [30], we verify that the resulting solutions of the free-boundary problems provide the expressions for the value functions and the optimal stopping boundaries for the underlying asset price process in the initial problems. Applying an extension of the maximality principle from [29] to the three-dimensional optimal stopping problems, we show that the optimal stopping boundaries provide the minimal solutions of the associated ordinary differential equations (see also [32] for another three-dimensional case). The main results of the paper are stated in Theorem 4.1.

2. Formulation of the problem

In this section, we introduce the setting and notation of the three-dimensional optimal stopping problems which are related to the pricing of certain perpetual American options and formulate the equivalent free-boundary problems.

2.1. Formulation of the problem. For a precise formulation of the problem, let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $B = (B_t)_{t \ge 0}$ and its natural filtration $(\mathcal{F}_t)_{t \ge 0}$. Assume that there exists a continuous process $X = (X_t)_{t \ge 0}$ solving the stochastic differential equation

$$dX_t = (r - \delta(S_t, Y_t))\, X_t\, dt + \sigma(S_t, Y_t)\, X_t\, dB_t \quad (X_0 = x) \tag{2.1}$$

where $r > 0$ is a given constant, $\delta(s, y), \sigma(s, y) > 0$ are some continuously differentiable and bounded functions on $[0, \infty]^2$, and $x > 0$ is fixed. Here, the associated with X *running maximum* process $S = (S_t)_{t \ge 0}$ and the corresponding *running maximum drawdown* process $Y = (Y_t)_{t \ge 0}$ are defined by

$$S_t = s \vee \max_{0 \le u \le t} X_u \quad \text{and} \quad Y_t = y \vee \max_{0 \le u \le t} (S_u - X_u) \tag{2.2}$$

for arbitrary $0 < s - y \le x \le s$. Observe that, since the functions $\delta(s, y)$ and $\sigma(s, y)$ are assumed to be bounded on $[0, \infty]^2$, it follows from the result of [21; Chapter IV, Theorem 4.8] that there exists a (pathwise) unique solution of the stochastic differential equation in (2.1) which admits the representation

$$X_t = x \, \exp\bigg(\int_0^t \Big(r - \delta(S_u, Y_u) - \frac{\sigma^2(S_u, Y_u)}{2} \Big) \, du + \int_0^t \sigma(S_u, Y_u) \, dB_u \bigg) \tag{2.3}$$

for all $t \geq 0$. We further assume that the resulting continuous diffusion-type process X describes the price of a risky asset on a financial market, where r is the riskless interest rate, $\delta(s, y)$ is the dividend rate paid to the asset holders, and $\sigma(s, y)$ is the volatility rate.

The main purpose of the present paper is to derive a closed-form solution to the optimal stopping problem for the continuous time-homogeneous (strong) Markov process $(X, S, Y) = (X_t, S_t, Y_t)_{t \geq 0}$ given by

$$V_*(x, s, y) = \sup_\tau E_{x,s,y} \big[e^{-r\tau} \, G(X_\tau, S_\tau, Y_\tau) \big] \tag{2.4}$$

for any $(x, s, y) \in E^3$, where the supremum is taken over all stopping times τ with respect to the natural filtration of X, and the payoff function is either $G(x, s, y) = (K - s + y)^+$ or $G(x, s, y) = (Kx - s + y)^+$. Here $E_{x,s,y}$ denotes the expectation with respect to the (unique) martingale measure (see, e.g. [37; Chapter VII, Section 3g]), under the assumption that the (three-dimensional) process (X, S, Y) defined in (2.1)-(2.2) starts at $(x, s, y) \in E^3$, and $E^3 = \{(x, s, y) \in \mathbb{R}^3 \, | \, 0 < s - y \leq x \leq s\}$ provides the state space for the process (X, S, Y). The value of (2.4) is then actually a *rational* (or *no-arbitrage*) price of a perpetual American option on the maximum of the market depth with payoff function either $G(x, s, y) = (K - s + y)^+$ or $G(x, s, y) = (Kx - s + y)^+$, which corresponds to either the case of fixed strike $K > 0$ or floating strike $Kx > 0$, respectively. The values of perpetual American standard options were computed in [12] in the same diffusion-type model. The case of perpetual American lookback options with fixed and floating strikes with payoff functions $G(x, s, y) = (s - K)^+$ and $G(x, s, y) = (s - Kx)^+$ in the diffusion model of (2.1) with constant coefficients $\delta(s, y) = \delta$ and $\sigma(s, y) = \sigma$ was considered in [28] and [16], and more complicated π-options were studied in [17].

2.2. The structure of the optimal stopping times. It follows from the general theory of optimal stopping problems for Markov processes (see, e.g. [33; Chapter I, Section 2.2]) that the optimal stopping time in the problem of (2.4) is given by

$$\tau_* = \inf\{ t \geq 0 \, | \, V_*(X_t, S_t, Y_t) = G(X_t, S_t, Y_t) \} \tag{2.5}$$

for the payoff function being either $G(x, s, y) = (K - s + y)^+$ or $G(x, s, y) = (Kx - s + y)^+$. The following assertion specifies the structure of the optimal stopping time τ_* in (2.5) in the both cases of payoff functions.

Lemma 2.1 *Suppose that $\delta(s,y), \sigma(s,y) > 0$ are continuously differentiable bounded functions on $[0,\infty]^2$, and $r > 0$ in (2.1). Then, in the optimal stopping problem of (2.4) with the payoff function being either $G(x,s,y) = (K - s + y)^+$ or $G(x,s,y) = (Kx - s + y)^+$, the optimal stopping time from (2.5) has the structure*

$$\tau_* = \inf\{t \geq 0 \,|\, X_t \geq b_*(S_t, Y_t)\} \tag{2.6}$$

for some function $b_*(s,y)$ to be determined, such that:

(i) for $G(x,s,y) = (K - s + y)^+$, we have

$$b_*(s,y) > s - y \;\; \text{such that} \;\; s - y < K \,, \quad \text{and} \quad b_*(s,y) > s \geq K + y \tag{2.7}$$

for all $0 < y < s$;

(ii) for $G(x,s,y) = (Kx - s + y)^+$, we have

$$b_*(s,y) \geq \underline{b}(s,y) \vee (s-y) \vee \frac{s-y}{K} \quad \text{with} \;\; \underline{b}(s,y) = \frac{r(s-y)}{\delta(s,y)K} \tag{2.8}$$

for all $0 < y < s$.

Proof. (i) The case of fixed strike. In the case of $G(x,s,y) = (K - s + y)^+$, it is obvious from the structure of the payoff that it is never optimal to stop when $S_t - Y_t \geq K$, for any $t \geq 0$. In other words, the set

$$C' = \{(x,s,y) \in E^3 \,|\, 0 < K \leq s - y \leq x \leq s\} \tag{2.9}$$

belongs to the continuation region

$$C_* = \{(x,s,y) \in E^3 \,|\, V_*(x,s,y) > (K - s + y)^+\}. \tag{2.10}$$

It is seen from the solution below that $V_*(x,s,y)$ is continuous, so that C_* is open. Then, applying the change-of-variable formula from [30] to the function $e^{-rt}(K - s + y)^+$, we get

$$\begin{aligned} e^{-rt}\,(K - S_t + Y_t)^+ = (K - s + y)^+ + \int_0^t e^{-ru}\, I(K > S_u - Y_u)\, dY_u \qquad (2.11) \\ - \int_0^t e^{-ru}\, I(K > S_u - Y_u)\, dS_u - \int_0^t e^{-ru}\, r(K - S_u + Y_u)\, I(K > S_u - Y_u)\, du \end{aligned}$$

where $I(\cdot)$ denotes the indicator function. It thus follows from the expression in (2.11) that

$$\begin{aligned} E_{x,s,y}\big[e^{-r\tau}\,(K - S_\tau + Y_\tau)^+\big] = (K - s + y)^+ + E_{x,s,y}\bigg[\int_0^\tau e^{-ru}\, I(K > S_u - Y_u)\, dY_u \qquad (2.12) \\ - \int_0^\tau e^{-ru}\, I(K > S_u - Y_u)\, dS_u - \int_0^\tau e^{-ru}\, r(K - S_u + Y_u)\, I(K > S_u - Y_u)\, du\bigg] \end{aligned}$$

holds for any stopping time τ and all $0 < s - y \le x \le s$. Observe that the process S can increase only at the plane $d_1 = \{(x,s,y) \in \mathbb{R}^3 \,|\, 0 < x = s\}$, while the process Y can increase only at the plane $d_2 = \{(x,s,y) \in \mathbb{R}^3 \,|\, 0 < x = s - y\}$. This fact yields through (2.12) that it is never optimal to stop when $X_t = S_t - Y_t < K$ for $t \ge 0$, so that the plane $\{(x,s,y) \in \mathbb{R}^3 \,|\, 0 < x = s - y < K\}$ belongs to the continuation region in (2.10).

Since the process (X,S,Y) stays at the same level under the second and third coordinates, as long as it fluctuates between the planes d_1 and d_2, it is clear that we should not let the process X grow up too much, since it might hit the plane d_1, that happens when $X_t = S_t$ for $t \ge 0$, and thereby increase its running maximum S and thus decrease the payoff $(K-s+y)^+$. Moreover, it is not optimal to let the process X run too much away from the plane d_2, since the cost of waiting until it comes back to the plane d_2, that happens when $X_t = S_t - Y_t$ for $t \ge 0$, and thereby increase its running maximum drawdown Y and thus the payoff $(K - s + y)^+$ could be too large due to the negative last term in (2.12) and the discounting factor in (2.4). It follows from the structure of the value function in (2.4) with the processes of (2.1)-(2.2) and the convex payoff function $G(x,s,y)$ that $V_*(x,s,y)$ is convex in x on the interval $(s-y,s)$. Then, standard geometric arguments imply that there exists a function $b_*(s,y)$ such that the continuation region in (2.10) consists of (2.9) and the set

$$C'' = \{(x,s,y) \in E^3 \,|\, s - y \le x < b_*(s,y) \ \text{ and } \ s - y < K\} \tag{2.13}$$

while the corresponding stopping region is the closure of the set

$$D_* = \{(x,s,y) \in E^3 \,|\, b_*(s,y) < x \le s \ \text{ and } \ s - y < K\} \tag{2.14}$$

with $b_*(s,y) > s - y$ for all $0 < y < s$ such that $0 < s - y < K$.

(ii) The case of floating strike. In the case of $G(x,s,y) = (Kx - s + y)^+$, it is obvious from the structure of the payoff that it is never optimal to stop when $KX_t \le S_t - Y_t$, for any $t \ge 0$. In other words, this fact shows that the set

$$C' = \{(x,s,y) \in E^3 \,|\, 0 < Kx \le s - y\} \tag{2.15}$$

which exists for all $0 < K \le 1$, belongs to the continuation region

$$C_* = \{(x,s,y) \in E^3 \,|\, V_*(x,s,y) > (Kx - s + y)^+\}. \tag{2.16}$$

It is seen from the solution below that $V_*(x,s,y)$ is continuous, so that C_* is open. Then, applying the change-of-variable formula from [30] to the function $e^{-rt}(Kx - s + y)^+$, we get

$$\begin{aligned} &e^{-rt}\,(KX_t - S_t + Y_t)^+ = (Kx - s + y)^+ + N_t \qquad (2.17)\\ &+ \int_0^t e^{-ru}\, H(X_u,S_u,Y_u)\, I(KX_u > S_u - Y_u)\, du - \int_0^t e^{-ru}\, I(KX_u > S_u - Y_u)\, dS_u \\ &+ \int_0^t e^{-ru}\, I(KX_u > S_u - Y_u)\, dY_u + \frac{1}{2}\int_0^t e^{-ru}\, K\, I(KX_u = S_u - Y_u)\, d\ell_u^K(X) \end{aligned}$$

where we set $H(x,s,y) = r(s-y) - K\delta(s,y)x$, and the process $\ell^K(X) = (\ell^K_t(X))_{t\ge 0}$ is the local time of X at the plane $\{(x,s,y) \in \mathbb{R}^3 \,|\, 0 < Kx = s-y\}$ given by

$$\ell^K_t(X) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t I(-\varepsilon < KX_u - S_u + Y_u < \varepsilon)\, \sigma^2(S_u, Y_u)\, X_u^2\, du \tag{2.18}$$

as a limit in probability. Here, the process $N = (N_t)_{t\ge 0}$ defined by

$$N_t = \int_0^t e^{-ru}\, I(KX_u > S_u - Y_u)\, \sigma(S_u, Y_u)\, X_u\, dB_u \tag{2.19}$$

is a continuous square integrable martingale under $P_{x,s,y}$. Hence, applying Doob's optional sampling theorem (see, e.g. [20; Chapter I, Theorem 3.22]) and using the expression in (2.17), we get that

$$\begin{aligned} &E_{x,s,y}\big[e^{-r\tau}\,(KX_\tau - S_\tau + Y_\tau)^+\big] = (Kx-s+y)^+ && (2.20)\\ &+ E_{x,s,y}\bigg[\int_0^\tau e^{-ru}\, H(X_u,S_u,Y_u)\, I(KX_u > S_u - Y_u)\, du - \int_0^\tau e^{-ru}\, I(KX_u > S_u - Y_u)\, dS_u \\ &\qquad + \int_0^\tau e^{-ru}\, I(KX_u > S_u - Y_u)\, dY_u + \frac{1}{2}\int_0^\tau e^{-ru}\, K\, I(KX_u = S_u - Y_u)\, d\ell^K_u(X)\bigg] \end{aligned}$$

holds for any stopping time τ and all $0 < s-y \le x \le s$. It is seen from (2.20) that it is never optimal to stop when $KX_t > S_t - Y_t$ and either $H(X_t,S_t,Y_t) > 0$ or $X_t = S_t - Y_t$ holds for $t \ge 0$, where the latter condition is true since the process Y can only increase at the plane d_2. In other words, the set

$$C'' = \{(x,s,y) \in E^3 \,|\, (s-y)/K < x < \underline{b}(s,y) \;\text{ or }\; x = s-y > (s-y)/K\} \tag{2.21}$$

with $\underline{b}(s,y) = r(s-y)/(\delta(s,y)K)$ for $0 < y < s$, belongs to the continuation region in (2.16). Note that, the set in (2.21) exists only if $K > 1$ holds or if $0 < K \le 1$ and $\delta(s,y) < r$ holds.

Let us now fix some $(x,s,y) \in C_*$ from the continuation region in (2.16) and let $\tau_* = \tau_*(x,s,y)$ denote the optimal stopping time in the problem of (2.4) with $G(x,s,y) = (Kx - s+y)^+$. Then, by means of the results of general optimal stopping theory for Markov processes (see, e.g. [33; Chapter I, Section 2.2]), we conclude from the expression in (2.20) that

$$\begin{aligned} &V_*(x,s,y) - (Kx-s+y)^+ && (2.22)\\ &= E_{x,s,y}\bigg[\int_0^{\tau_*} e^{-ru}\, H(X_u,S_u,Y_u)\, I(KX_u > S_u - Y_u)\, du - \int_0^{\tau_*} e^{-ru}\, I(KX_u > S_u - Y_u)\, dS_u \\ &\qquad + \int_0^{\tau_*} e^{-ru}\, I(KX_u > S_u - Y_u)\, dY_u + \frac{1}{2}\int_0^{\tau_*} e^{-ru}\, K\, I(KX_u = S_u - Y_u)\, d\ell^K_u(X)\bigg] > 0 \end{aligned}$$

holds. Hence, taking any x' such that $\underline{b}(s,y) \vee (s-y) < x' < x$ and using the explicit expression for the process X through its starting point in (2.3) and also the structure of the maximum drawdown process Y in (2.2), we obtain from (2.20) that the inequalities

$$
\begin{aligned}
& V_*(x', s, y) - (Kx' - s + y)^+ \qquad (2.23) \\
& \geq E_{x',s,y}\bigg[\int_0^{\tau_*} e^{-ru}\, H(X_u, S_u, Y_u)\, I(KX_u > S_u - Y_u)\, du - \int_0^{\tau_*} e^{-ru}\, I(KX_u > S_u - Y_u)\, dS_u \\
& \qquad + \int_0^{\tau_*} e^{-ru}\, I(KX_u > S_u - Y_u)\, dY_u + \frac{1}{2} \int_0^{\tau_*} e^{-ru}\, K\, I(KX_u > S_u - Y_u)\, d\ell_u^K(X) \bigg] \\
& \geq E_{x,s,y}\bigg[\int_0^{\tau_*} e^{-ru}\, H(X_u, S_u, Y_u)\, I(KX_u > S_u - Y_u)\, du - \int_0^{\tau_*} e^{-ru}\, I(KX_u > S_u - Y_u)\, dS_u \\
& \qquad + \int_0^{\tau_*} e^{-ru}\, I(KX_u > S_u - Y_u)\, dY_u + \frac{1}{2} \int_0^{\tau_*} e^{-ru}\, K\, I(KX_u = S_u - Y_u)\, d\ell_u^K(X) \bigg]
\end{aligned}
$$

are satisfied. Thus, by virtue of the inequality in (2.22), we see that $(x', s, y) \in C_*$. Taking into account the convexity of the function $G(x, s, y)$ and thus of $V_*(x, s, y)$ in x on the interval $(s - y, s)$, we may therefore conclude from the standard geometric arguments that there exists a function $b_*(s, y)$ such that the continuation region in (2.16), which consists of the regions in (2.15) and (2.21), takes the form

$$
C_* = \{(x, s, y) \in E^3 \mid s - y \leq x < b_*(s, y)\} \qquad (2.24)
$$

while the corresponding stopping region is the closure of the set

$$
D_* = \{(x, s, y) \in E^3 \mid b_*(s, y) < x \leq s\} \qquad (2.25)
$$

with $b_*(s, y) \geq \underline{b}(s, y) \vee (s - y) \vee ((s - y)/K)$ and $\underline{b}(s, y) = r(s - y)/(\delta(s, y)K)$ for $0 < y < s$. $\square$

2.3. The free-boundary problem. By means of standard arguments based on the application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}$ of the process (X, S, Y) acts on a function $F(x, s, y)$ from the class $C^{2,1,1}$ on the interior of E^3 according to the rule

$$
(\mathbb{L}F)(x, s, y) = (r - \delta(s, y))\, x\, \partial_x F(x, s, y) + \frac{\sigma^2(s, y)}{2}\, x^2\, \partial^2_{xx} F(x, s, y) \qquad (2.26)
$$

for all $0 < s - y < x < s$. In order to find analytic expressions for the unknown value function $V_*(x, s, y)$ from (2.4) and the unknown boundary $b_*(s, y)$ from (2.6), let us build on the results of general theory of optimal stopping problems for Markov processes (see, e.g. [33; Chapter IV, Section 8]). We can reduce the optimal stopping problem of (2.4) to the equivalent free-boundary problem for $V_*(x, s, y)$ and $b_*(s, y)$ given by

$$
(\mathbb{L}V)(x, s, y) = rV(x, s, y) \quad \text{for} \quad (x, s, y) \in C \qquad (2.27)
$$

$$
V(x, s, y)\big|_{x = b(s,y)-} = G(b(s, y), s, y) \qquad (2.28)
$$

$$
V(x, s, y) = G(x, s, y) \quad \text{for} \quad (x, s, y) \in D \qquad (2.29)
$$

$$
V(x, s, y) > G(x, s, y) \quad \text{for} \quad (x, s, y) \in C \qquad (2.30)
$$

$$
(\mathbb{L}V)(x, s, y) < rV(x, s, y) \quad \text{for} \quad (x, s, y) \in D \qquad (2.31)
$$

where C is defined as $C' \cup C''$ in (2.9) and (2.13) or C_* in (2.24), and D is defined as D_* in (2.14) or (2.25), for the payoff function $G(x,s,y) = (K-s+y)^+$ or $G(x,s,y) = (Kx - s+y)^+$, respectively, with $b(s,y)$ instead of $b_*(s,y)$. The instantaneous-stopping condition in (2.28) is satisfied, when $s-y \le b(s,y) \le s$ holds, for each $0<y<s$. Observe that the superharmonic characterisation of the value function (see [8] and [33; Chapter IV, Section 9]) implies that $V_*(x,s,y)$ is the smallest function satisfying (2.27)-(2.30), with the boundary $b_*(s,y)$. Moreover, we further assume that the normal-reflection and the smooth-fit conditions

$$\partial_y V(x,s,y)\big|_{x=(s-y)+} = 0 \quad \text{and} \quad \partial_x V(x,s,y)\big|_{x=b(s,y)-} = \partial_x G(b(s,y),s,y) \tag{2.32}$$

are satisfied, when $s-y < b(s,y) < s$ holds, for each $0<y<s$. Otherwise, we assume that the normal-reflection conditions

$$\partial_y V(x,s,y)\big|_{x=(s-y)+} = 0 \quad \text{and} \quad \partial_s V(x,s,y)\big|_{x=s-} = 0 \tag{2.33}$$

are satisfied, when $b(s,y) > s$ holds, for each $0<y<s$.

Observe that when the inequalities $S_t - Y_t < X_t < b(S_t, Y_t) < S_t$ are satisfied for $t \ge 0$, the process X can increase towards the boundary $b(S_t,Y_t)$ in a continuous way, so that we can assume that the smooth-fit condition of (2.32) holds for the candidate value function $V(x,s,y)$ at the boundary $b(s,y)$. Such assumptions are naturally applied to determine the solutions of the free-boundary problems, which provide the solutions of associated optimal stopping problems (see [33; Chapter IV, Section 9] for an explanation and proofs). On the other hand, when either the inequalities $S_t - Y_t < X_t < b(S_t,Y_t) < S_t$ or $S_t - Y_t < X_t < S_t < b(S_t,Y_t)$ are satisfied for $t \ge 0$, the process X can increase or decrease towards the planes d_1 or d_2, respectively, in a continuous way. In this case, it follows from the property of the infinitesimal operator of the process (X,S,Y) that the normal-reflection conditions of (2.32) and (2.33) hold for the candidate value function $V(x,s,y)$ at the planes d_1 and d_2. These conditions are used to derive first-order ordinary differential equations for the candidate boundaries of the corresponding optimal stopping problems (see [7], [14]-[15], [35]-[36], [28], [16], [17], and [13] among others, and [29] and [33; Chapter IV, Section 13] for an explanation and further references). We follow the classical approach and apply the smooth-fit and normal-reflection conditions from (2.32) and (2.33) to find closed-form expressions for the candidate value functions as well as ordinary differential equations for the boundaries, and then verify in Theorem 4.1 below that the obtained solutions to the free-boundary problem provide the value functions and the optimal stopping boundaries in the original problems.

2.4. Some remarks. Let us finally note some facts about the value function $V_*(x,s,y)$ in (2.4), which will then be used in order to specify the asymptotic behavior of the boundary $b_*(s,y)$ from (2.6).

(i) The case of fixed strike. In the case of $G(x,s,y) = (K-s+y)^+$, we observe from

(2.4) that the inequalities

$$0 \le (K - s + y)^+ \le \sup_{\tau} E_{x,s,y}\big[e^{-r\tau}\,(K - S_\tau + Y_\tau)^+\big] \le K \tag{2.34}$$

hold for all $(x, s, y) \in E^3$. Thus, setting $x = s - y$ into (2.34) and letting y increase to s, we get that the property

$$\liminf_{y \uparrow s} V_*(s - y, s, y) = \limsup_{y \uparrow s} V_*(s - y, s, y) = K \tag{2.35}$$

holds for all $s > 0$.

(ii) The case of floating strike. In the case of $G(x, s, y) = (Kx - s + y)^+$, we observe from (2.4) that the inequalities

$$0 \le (K\,x - s + y)^+ \le \sup_{\tau} E_{x,s,y}\big[e^{-r\tau}\,(K\,X_\tau - S_\tau + Y_\tau)^+\big] \le K\,\sup_{\tau} E_{x,s,y}\big[e^{-r\tau}\,X_\tau\big] \tag{2.36}$$

imply that the expressions

$$0 \le (K\,x - s + y)^+ \le V_*(x, s, y) \le K\,x \tag{2.37}$$

hold for all $(x, s, y) \in E^3$, where the third inequality in (2.37) follows from the optimal immediate stopping of the problem in the right-hand side of (2.36). Thus, setting $x = s - y$ in (2.37) and letting y increase to s, we obtain that the property

$$0 \le (K - 1)^+ \le \lim_{y \uparrow s} \frac{V_*(s - y, s, y)}{s - y} \le K \tag{2.38}$$

holds for all $s > 0$.

3. Solution of the free-boundary problem

In this section, we obtain closed-form expressions for the value functions $V_*(x, s, y)$ in (2.4) associated with the options on the maximum of the market depth with fixed and floating strikes, and derive first-order nonlinear ordinary differential equations for the optimal exercise boundaries $b_*(s, y)$ from (2.6), as solutions to the free-boundary problem in (2.27)-(2.33). The analysis performed in this section also provides a recursive algorithm to determine the candidate value functions and optimal stopping boundaries as well as their intersection lines with the edges of the three-dimensional state space.

3.1. The general solution of the ordinary differential equation. We first observe that the general solution of the equation in (2.27) has the form

$$V(x, s, y) = C_1(s, y)\,x^{\gamma_1(s,y)} + C_2(s, y)\,x^{\gamma_2(s,y)} \tag{3.1}$$

where $C_i(s,y)$, $i=1,2$, are some arbitrary continuously differentiable functions and $\gamma_i(s,y)$, $i=1,2$, are given by

$$\gamma_i(s,y) = \frac{1}{2} - \frac{r-\delta(s,y)}{\sigma^2(s,y)} - (-1)^i \sqrt{\left(\frac{1}{2} - \frac{r-\delta(s,y)}{\sigma^2(s,y)}\right)^2 + \frac{2r}{\sigma^2(s,y)}} \tag{3.2}$$

so that $\gamma_2(s,y) < 0 < 1 < \gamma_1(s,y)$ holds for all $0<y<s$. Hence, applying the instantaneous-stopping condition from (2.28) to the function in (3.1), we get that the equality

$$C_1(s,y)\, b^{\gamma_1(s,y)}(s,y) + C_2(s,y)\, b^{\gamma_2(s,y)}(s,y) = G(b(s,y),s,y) \tag{3.3}$$

is satisfied, when $s-y \le b(s,y) \le s$ holds, for each $0<y<s$. Moreover, using the smooth-fit condition from the right-hand part of (2.32), we obtain that the equality

$$C_1(s,y)\, \gamma_1(s,y)\, b^{\gamma_1(s,y)}(s,y) + C_2(s,y)\, \gamma_2(s,y)\, b^{\gamma_2(s,y)}(s,y) = \partial_x G(b(s,y),s,y)\, b(s,y) \tag{3.4}$$

is satisfied, when $s-y < b(s,y) < s$ holds, for each $0<y<s$. Finally, applying the normal-reflection conditions from (2.33) to the function in (3.1), we obtain that the equalities

$$\sum_{i=1}^{2} \Big(\partial_s C_i(s,y)\, s^{\gamma_i(s,y)} + C_i(s,y)\, \partial_s \gamma_i(s,y)\, s^{\gamma_i(s,y)} \ln s \Big) = 0 \tag{3.5}$$

$$\sum_{i=1}^{2} \Big(\partial_y C_i(s,y)\, (s-y)^{\gamma_i(s,y)} + C_i(s,y)\, \partial_y \gamma_i(s,y)\, (s-y)^{\gamma_i(s,y)} \ln(s-y) \Big) = 0 \tag{3.6}$$

are satisfied, when $s < b(s,y)$ and $s-y < b(s,y)$ holds, respectively, for each $0<y<s$. Here, the partial derivatives $\partial_s \gamma_i(s,y)$ and $\partial_y \gamma_i(s,y)$ take the form

$$\partial_s \gamma_i(s,y) = \varphi(s,y) - (-1)^i \frac{\varphi(s,y)(\gamma_1(s,y)-\gamma_2(s,y))\sigma^3(s,y) - 2r\partial_s\sigma(s,y)}{\sigma^2(s,y)\sqrt{(\gamma_1(s,y)-\gamma_2(s,y))^2\sigma^2(s,y)+2r}} \tag{3.7}$$

$$\partial_y \gamma_i(s,y) = \psi(s,y) - (-1)^i \frac{\psi(s,y)(\gamma_1(s,y)-\gamma_2(s,y))\sigma^3(s,y) - 2r\partial_y\sigma(s,y)}{\sigma^2(s,y)\sqrt{(\gamma_1(s,y)-\gamma_2(s,y))^2\sigma^2(s,y)+2r}} \tag{3.8}$$

for $i=1,2$, and the functions $\varphi(s,y)$ and $\psi(s,y)$ are defined by

$$\varphi(s,y) = \frac{\sigma(s,y)\partial_s\delta(s,y) + 2(r-\delta(s,y))\partial_s\sigma(s,y)}{\sigma^3(s,y)} \tag{3.9}$$

$$\psi(s,y) = \frac{\sigma(s,y)\partial_y\delta(s,y) + 2(r-\delta(s,y))\partial_y\sigma(s,y)}{\sigma^3(s,y)} \tag{3.10}$$

for $0<y<s$.

3.2. The solution to the problem in the $\delta(s)$ and $\sigma(s)$-setting. We begin with the case in which $\delta(s,y)=\delta(s)$ and $\sigma(s,y)=\sigma(s)$ holds in (2.1), and thus, we can define the functions $\beta_i(s)=\gamma_i(s,y)$, $i=1,2$, as in (3.2). Then, the general solution $V(x,s,y)$

of the equation in (2.27) has the form of (3.1) with $\gamma_i(s,y) = \beta_i(s)$, for $i = 1,2$. Recall that the border planes of the state space $E^3 = \{(x,s,y) \in \mathbb{R}^3 \,|\, 0 < s-y \le x \le s\}$ are $d_1 = \{(x,s,y) \in \mathbb{R}^3 \,|\, 0 < x = s\}$ and $d_2 = \{(x,s,y) \in \mathbb{R}^3 \,|\, 0 < x = s-y\}$, as well as that the second and third components of the process (X,S,Y) can increase only at the planes d_1 and d_2, that is, when $X_t = S_t$ and $X_t = S_t - Y_t$ for $t \ge 0$, respectively.

(i) The case of fixed strike. Let us first consider the payoff function $G(x,s,y) = (K-s+y)^+$ in (2.4). In this case, solving the system of equations in (3.3)-(3.4), we obtain that the function in (3.1) admits the representation

$$V(x,s,y;b(s,y)) = C_1(s,y;b(s,y))\, x^{\beta_1(s)} + C_2(s,y;b(s,y))\, x^{\beta_2(s)} \tag{3.11}$$

for $0 < s-y \le x < b(s,y) \le s$, with

$$C_i(s,y;b(s,y)) = \frac{\beta_{3-i}(s)(K-s+y)}{(\beta_{3-i}(s)-\beta_i(s))b(s,y)^{\beta_i(s)}} \tag{3.12}$$

for all $0 < y < s$ and $i = 1,2$. Hence, assuming that the boundary function $b(s,y)$ is continuously differentiable, we apply the condition of (3.6) to the functions $C_i(s,y) = C_i(s,y;b(s,y))$, $i = 1,2$, in (3.12) to obtain that the boundary solves the first-order nonlinear ordinary differential equation

$$\partial_y b(s,y) = \sum_{i=1}^{2} \frac{b(s,y)}{\beta_i(s)(K-s+y)} \left(\frac{((s-y)/b(s,y))^{\beta_i(s)}}{((s-y)/b(s,y))^{\beta_i(s)} - ((s-y)/b(s,y))^{\beta_{3-i}(s)}} \right) \tag{3.13}$$

for $0 < y < s$. Taking into account the condition in (2.35) for the value function in (3.11)-(3.12), we conclude after some straightforward calculations that $b(s,y) \sim g_*(s)(s-y)$ should hold as $y \uparrow s$, where $g_*(s)$ is the unique solution of the equation

$$\sum_{i=1}^{2} \beta_i(s)\,(g^{-\beta_{3-i}(s)}(s) - 1) = 0 \tag{3.14}$$

which is given by $g_*(s) = 1$ for all $s > 0$. Thus, any candidate solution of the differential equation in (3.13) should satisfy the condition

$$\lim_{y \uparrow s} \frac{b(s,y)}{s-y} = 1 \tag{3.15}$$

for all $s > 0$.

For any $s > 0$ fixed, let us now consider a candidate solution $b(s,y)$ of the first-order ordinary differential equation of (3.13), satisfying the starting condition of (3.15), given that this solution stays strictly above the plane d_2 for all $0 < y < s$ such that $s-y < K$, and strictly above

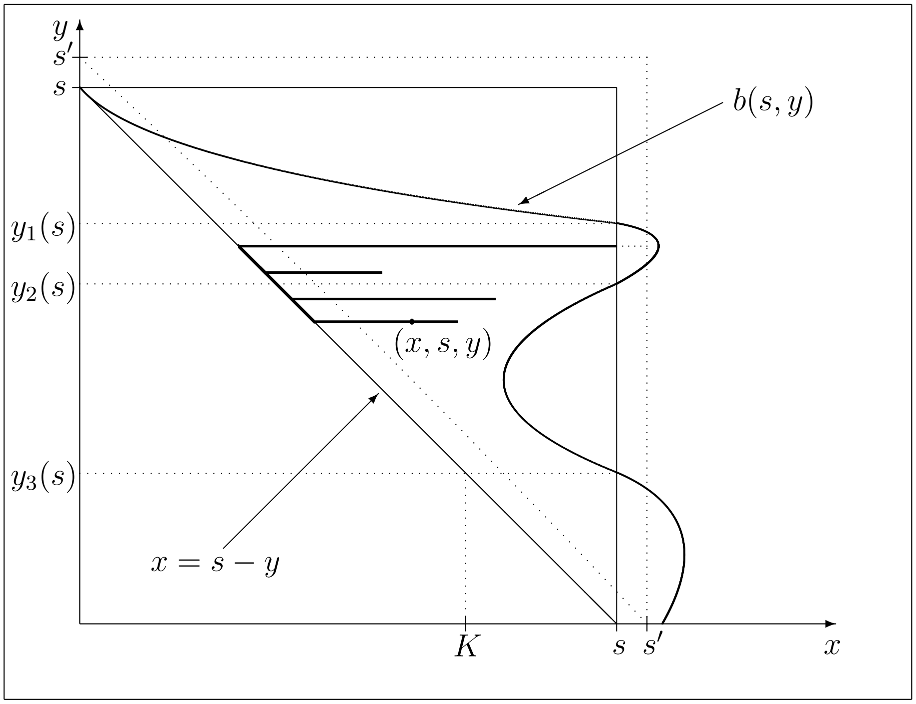

Figure 1. A computer drawing of the state space of the process (X, S, Y), for some s fixed, which increases to s', and the boundary function $b(s,y)$.

the plane d_1 for all $0 < y < s$ such that $s - y \geq K$. These assumptions for the boundary function $b(s,y)$ follow from the structure of the continuation region $C' \cup C''$ in (2.9) and (2.13), which results to the expressions of (2.7) in Lemma 2.1. Then, we put $y_0(s) = s$ and define a decreasing sequence $(y_n(s))_{n \in \mathbb{N}}$ such that the boundary $b(s,y)$ exits the region E^3 from the side of d_1 at the points $(s, s, y_{2l-1}(s))$ and enters E^3 downwards at the points $(s, s, y_{2l}(s))$. Namely, we define $y_{2l-1}(s) = \sup\{y < y_{2l-2}(s) \,|\, b(s,y) > s\}$ and $y_{2l}(s) = \sup\{y < y_{2l-1}(s) \,|\, b(s,y) \leq s\}$, whenever they exist, and put $y_{2l-1}(s) = y_{2l}(s) = 0$, $l \in \mathbb{N}$, otherwise. It follows from (2.7) that $s - K \leq y_{2l-1}(s) < y_{2l-2}(s) \leq s$, for $l = 1, \dots, l'$, where $l' = \max\{l \in \mathbb{N} \,|\, s - y_{2l-1}(s) \leq K\}$. Therefore, the candidate value function admits the expression of (3.11)-(3.12) in the regions

$$R_{2l-1} = \{(x,s,y) \in E^3 \,|\, y_{2l-1}(s) < y \leq y_{2l-2}(s)\} \tag{3.16}$$

for $l = 1, \dots, l'$ (see Figure 1 above).

On the other hand, the candidate value function takes the form of (3.1) with $C_i(s,y)$, $i = 1, 2$, solving the linear system of first-order partial differential equations in (3.5)-(3.6), in the regions

$$R_{2l} = \{(x,s,y) \in E^3 \,|\, y_{2l}(s) < y \leq y_{2l-1}(s)\} \tag{3.17}$$

for $l = 1, \dots, l'$, which belong to the continuation region $C' \cup C''$ given in (2.9) and (2.13). Note that, the process (X, S, Y) can enter the region R_{2l} in (3.17) from R_{2l+1} in (3.16), for some $l = 1, \dots, l'-1$, only through the point $(s - y_{2l}(s), s, y_{2l}(s))$ and can exit the region R_{2l} passing to the region R_{2l-1}, for some $l = 1, \dots, l'$, only through the point $(s - y_{2l-1}(s), s, y_{2l-1}(s))$, by hitting the plane d_2, so that increasing its third component Y. Thus, the candidate value function should be continuous at the points $(s - y_{2l}(s), s, y_{2l}(s))$ and $(s - y_{2l-1}(s), s, y_{2l-1}(s))$, that is expressed by the equalities

$$
\begin{aligned}
& C_1(s, y_{2l}(s)+) \left((s - y_{2l}(s))-\right)^{\beta_1(s)} + C_2(s, y_{2l}(s)+) \left((s - y_{2l}(s))-\right)^{\beta_2(s)} \\
& = V(s - y_{2l}(s), s, y_{2l}(s); b(s, y_{2l}(s)))
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
& C_1(s, y_{2l-1}(s)) \, (s - y_{2l-1}(s))^{\beta_1(s)} + C_2(s, y_{2l-1}(s)) \, (s - y_{2l-1}(s))^{\beta_2(s)} \\
& = V((s - y_{2l-1}(s))-, s, y_{2l-1}(s)+; b(s, y_{2l-1}(s)+))
\end{aligned}
\tag{3.19}
$$

for $s > 0$ and $l = 1, \ldots, l' - 1$, where the right-hand sides are given by (3.11)-(3.12) with $b(s, y_{2l-1}(s)+) = b(s, y_{2l}(s)) = s$. Moreover, in the region $R_{2l'}$, the condition of (3.19), for $l = l'$, changes its form to $C_2(\varepsilon, 0) \to 0$ as $\varepsilon \downarrow 0$, since otherwise $V(\varepsilon, \varepsilon, 0) \to \pm\infty$ as $\varepsilon \downarrow 0$, that must be excluded by virtue of the obvious fact that the value function in (2.4) is bounded at zero, while the condition of (3.18) holds for $l = l'$ as well.

In addition, the process (X, S, Y) can exit the region R_{2l} in (3.17) passing to the stopping region D_* from (2.14) only through the point $(s(y), s(y), y)$, by hitting the plane d_1, so that increasing its second component S until it reaches the value $s(y) = \inf\{q > s \,|\, b(q, y) \le q\}$. Then, the candidate value function should be continuous at the point $(s(y), s(y), y)$, that is expressed by the equality

$$
\begin{aligned}
& C_1(s(y)-, y) \, (s(y)-)^{\beta_1(s(y)-)} + C_2(s(y)-, y) \, (s(y)-)^{\beta_2(s(y)-)} \\
& = V(s(y), s(y), y; \, b(s(y), y)) \equiv s(y) - K
\end{aligned}
\tag{3.20}
$$

for each $y_{2l}(s) < y \le y_{2l-1}(s)$, $l = 1, \ldots, l'-1$. However, in the region $R_{2l'}$, we have $s(y) = \infty$, since for the points $(x, s, y) \in R_{2l'}$ satisfying $s - y \ge K$, we have $b(s, y) > s$ which will hold irrespective of how large s becomes.

Thus, the condition of (3.20) changes its form to $C_1(\infty, y) = 0$, since otherwise $V(x, \infty, y) \to \pm\infty$ as $x \uparrow \infty$, that must be excluded by virtue of the obvious fact that the value function in (2.4) is bounded at infinity. We can therefore conclude that the candidate value function admits the representation

$$
\begin{aligned}
& V(x, s, y; s(y), y_{2l-1}(s), y_{2l}(s)) \\
& = C_1(s, y; s(y), y_{2l-1}(s), y_{2l}(s)) \, x^{\beta_1(s)} + C_2(s, y; s(y), y_{2l-1}(s), y_{2l}(s)) \, x^{\beta_2(s)}
\end{aligned}
\tag{3.21}
$$

in the regions R_{2l} given by (3.17), where $C_i(s, y; s(y), y_{2l-1}(s), y_{2l}(s))$, $i = 1, 2$, provide a solution of the two-dimensional coupled system of first-order linear partial differential equations in (3.5)-(3.6) with the boundary conditions of (3.18)-(3.20), for $l = 1, \ldots, l'$.

In order to argue the existence and uniqueness of solutions of the boundary value problem formulated above, let us use the classical results of the general theory of linear systems of first-order partial differential equations (see, e.g. [6; Chapter I] or [18; Chapter VII]). For this, we first observe that the system of first-order linear partial differential equations in (3.5)-(3.6) does not admit characteristic curves, so that the considered system is of elliptic type. In this respect, we can consider an invertible analytic function $y = \Upsilon(s)$ such that the corresponding non-characteristic curve coincides with the boundary curve $y_{2l-1}(s)$ or $y_{2l}(s)$ defined above for some $l = 1, \ldots, l'$. Then, we may apply an appropriate affine transformation, which takes the point $(s, y_{2l-1}(s))$ or $(s, y_{2l}(s))$ into the origin, and introduce the change of coordinates from (s, y) to (s, z) with $z = y - \Upsilon(s)$, in order to reduce the system of (3.5)-(3.6) to the normal form. On the other hand, we can consider an invertible analytic function $s = \Gamma(y)$ such that the corresponding non-characteristic curve coincides with the boundary curve $s(y)$ defined above. In that case, we may apply an appropriate affine transformation, which takes the point $(s(y), y)$ into the origin, and introduce the change of coordinates from (s, y) to (q, y) with $q = s - \Gamma(y)$, in order to reduce the system of (3.5)-(3.6) to the normal form. In both cases, taking into account the assumption of continuity of the partial derivatives of $\delta(s, y)$ and $\sigma(s, y)$ on $[0, \infty]^2$, we can conclude by means of a version of the Cauchy-Kowalewski theorem from [6; Chapter I, Theorem 5.1] or [18; Theorem 7.2.9] (also in connection with Holmgren's uniqueness theorem) that there exists a (locally) unique solution of the system (3.5)-(3.6), satisfying the boundary conditions of (3.18)-(3.20). The obtained solution can admit an analytic continuation into the appropriate parts of the state space E^3 (see, e.g. [39], [38], [22], [24] and the references therein).

Note that such coupled systems of first-order linear partial differential equations have recently arisen in [32] under the study of certain other optimal stopping problems for the running extremal processes. However, we may observe that the system in (3.5)-(3.6) above turns out to be essentially more complicated than the corresponding system in [32; Equations (3.42)-(3.43)], because the latter system can be decoupled. The difficulty for the former system also arises from the specific form of the boundary conditions of (3.18)-(3.20) formulated above or (3.30)-(3.32) below. The complicated structure of these conditions can be explained by the fact that the running maximum process S plays a crucial role in the definition of the running maximum drawdown process Y in (2.2), but not in the definition of the running minimum process I, which is the counterpart coordinate process contained in [32].

(ii) The case of floating strike. Let us now consider the payoff function $G(x, s, y) = (Kx - s + y)^+$ in (2.4). Then, solving the system of equations in (3.3)-(3.4), we obtain that the function in (3.1) admits the representation

$$V(x, s, y; b(s, y)) = C_1(s, y; b(s, y))\, x^{\beta_1(s)} + C_2(s, y; b(s, y))\, x^{\beta_2(s)} \tag{3.22}$$

for $0 < s - y \le x < b(s, y) \le s$, with

$$C_i(s, y; b(s, y)) = \frac{(\beta_{3-i}(s) - 1) K b(s, y) - \beta_{3-i}(s)(s - y)}{(\beta_{3-i}(s) - \beta_i(s)) b(s, y)^{\beta_i(s)}} \tag{3.23}$$

for all $0 < y < s$ and $i = 1, 2$. Hence, assuming that the boundary function $b(s, y)$ is continuously differentiable, we apply the condition of (3.6) to the functions $C_i(s, y) = C_i(s, y; b(s, y))$, $i = 1, 2$, in (3.23) to obtain that the boundary solves the following first-order nonlinear ordinary differential equation

$$\partial_y b(s, y) = \sum_{i=1}^{2} \frac{\beta_{3-i}(s) b(s, y)}{(\beta_i(s) - 1)(\beta_{3-i}(s) - 1) K b(s, y) - \beta_i(s)\beta_{3-i}(s)(s - y)} \times \frac{((s - y)/b(s, y))^{\beta_i(s)}}{((s - y)/b(s, y))^{\beta_i(s)} - ((s - y)/b(s, y))^{\beta_{3-i}(s)}} \tag{3.24}$$

for $0 < y < s$. Taking into account the condition in (2.38) for the value function in (3.22)-(3.23), we conclude that $b(s, y) \sim g_*(s)(s - y)$ should hold as $y \uparrow s$. Here, the function $g_*(s)$ is the unique solution of the arithmetic equation

$$\sum_{i=1}^{2} (-1)^i \Big(\beta_i(s)(\beta_{3-i}(s) - 1) - (\beta_i(s) - 1)(\beta_{3-i}(s) - 1) K g(s) \Big) \, g^{\beta_i(s)}(s) = 0 \tag{3.25}$$

so that any candidate solution of the differential equation in (3.24) should satisfy the condition

$$\lim_{y \uparrow s} \frac{b(s, y)}{s - y} = g_*(s) \tag{3.26}$$

for each $s > 0$. (The proof of uniqueness of the solution of the equation in (3.25) is given in the Appendix.)

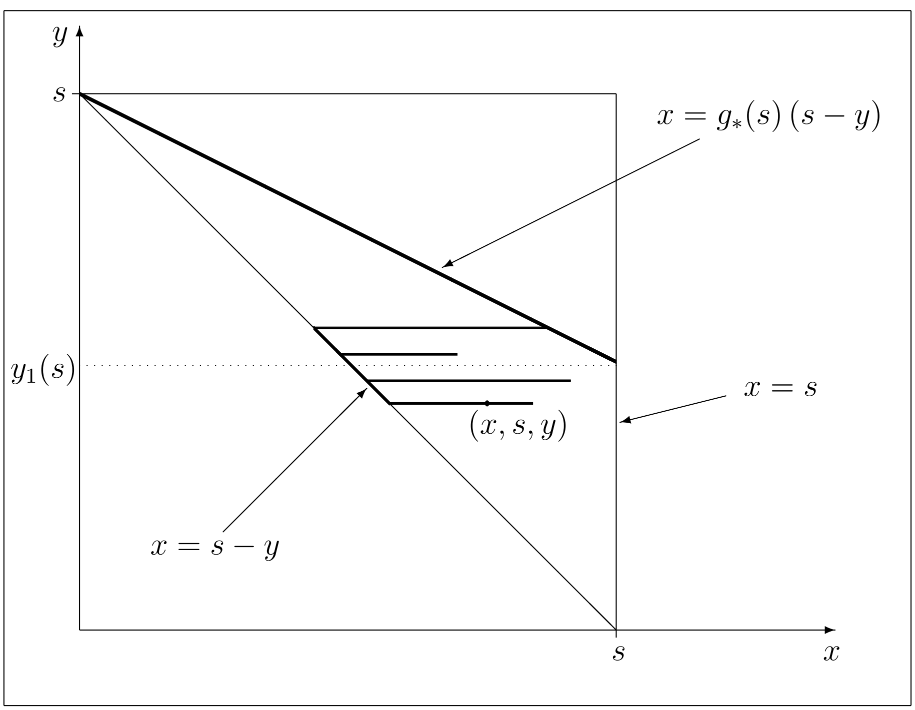

Figure 2. A computer drawing of $b_*(s, y) = g_*(s)(s - y)$ for some $s > 0$ fixed, in the case when $\gamma_i(s, y) = \beta_i(s)$, for $i = 1, 2$.

For any $s > 0$ fixed, we observe that the ordinary differential equation in (3.24) is equivalent to a one with separable variables and admits the explicit solution $b_*(s, y) = g_*(s)(s - y)$ which satisfies the starting condition of (3.26) and stays strictly above the plane d_2 and the surface $\{(x, s, y) \in E^3 \,|\, x = (s-y)/K \vee \underline{b}(s, y)\}$. These assumptions for the boundary function $b_*(s, y)$ follow from the structure of the continuation region C_* in (2.24), which results to the expressions of (2.8) in Lemma 2.1. Then, we put $y_0(s) = s$, $y_1(s) = (s(g_*(s) - 1)/g_*(s))-$ and $y_2(s) = 0$, and observe that the boundary $b_*(s, y)$ exits the region E^3 from the side of d_1 at the point $(s, s, y_1(s))$ and never returns back. Hence, the candidate value function admits the expression in (3.22)-(3.23) in the region R_1 in (3.16) and the boundary $b_*(s, y) = g_*(s)(s - y)$ provides the explicit solution of the equation in (3.24), satisfying the condition of (3.26) and such that (2.8) holds (see Figure 2 above).

On the other hand, the candidate value function takes the form of (3.1) with $C_i(s, y)$, $i = 1, 2$, solving the linear system of first-order partial differential equations in (3.5)-(3.6), in the region R_2 in (3.17), which belongs to the continuation region C_* in (2.24). We can therefore conclude by means of the arguments presented in part (i) above that the candidate value function admits the representation (3.21) for $l = 1$ in the region R_2 given by (3.17), where $C_i(s, y; s(y), y_1(s), y_2(s))$, $i = 1, 2$, provide a unique solution of the two-dimensional system of first-order linear partial differential equations in (3.5)-(3.6) with the boundary condition of (3.18), where the right-hand side is given by (3.22)-(3.23) with $b(s, y_1(s)) = s$, as well as the boundary conditions $C_2(\varepsilon, 0) \to 0$ as $\varepsilon \downarrow 0$ and $C_1(\infty, y) = 0$.

3.3. The solution to the problem in the general setting. We now continue with the general form of the coefficients $\delta(s, y)$ and $\sigma(s, y)$ in (2.1), and thus, of the functions $\gamma_i(s, y)$, $i = 1, 2$, from (3.2).

(i) The case of fixed strike. Let us now consider the payoff $G(x, s, y) = (K - s + y)^+$ in (2.4). In this case, solving the system of equations in (3.3)-(3.4), we obtain that the function in (3.1) admits the representation

$$V(x, s, y; b(s, y)) = C_1(s, y; b(s, y))\, x^{\gamma_1(s,y)} + C_2(s, y; b(s, y))\, x^{\gamma_2(s,y)} \tag{3.27}$$

for $0 < s - y \le x < b(s, y) \le s$, with

$$C_i(s, y; b(s, y)) = \frac{\gamma_{3-i}(s, y)(K - s + y)}{(\gamma_{3-i}(s, y) - \gamma_i(s, y))b(s, y)^{\gamma_i(s,y)}} \tag{3.28}$$

for all $0 < y < s$ and $i = 1, 2$. Hence, assuming that the boundary function $b(s, y)$ is continuously differentiable, we apply the condition of (3.6) to the functions $C_i(s, y) = C_i(s, y; b(s, y))$, $i = 1, 2$, in (3.28) to obtain that the boundary solves the first-order nonlinear ordinary differential equation

$$\partial_y b(s,y) = \sum_{i=1}^{2} \frac{b(s,y)}{\gamma_i(s,y)} \Bigg(\frac{((s-y)/b(s,y))^{\gamma_i(s,y)}}{((s-y)/b(s,y))^{\gamma_i(s,y)} - ((s-y)/b(s,y))^{\gamma_{3-i}(s,y)}} \times \left(\frac{1}{K-s+y} + \partial_y \gamma_i(s,y) \, \ln \frac{s-y}{b(s,y)} \right) + \frac{\partial_y \gamma_i(s,y)}{\gamma_{3-i}(s,y) - \gamma_i(s,y)} \Bigg) \tag{3.29}$$

for $0 < y < s$, where the partial derivatives $\partial_y \gamma_i(s,y)$, $i = 1,2$, are given by (3.8) with (3.10). Since the functions $\delta(s,y)$ and $\sigma(s,y)$ are assumed to be continuously differentiable and bounded, it follows that the limits $\delta(s,s-)$ and $\sigma(s,s-)$ exist for each $s > 0$. Then, the limits $\gamma_i(s,s-)$ can be identified with the functions $\beta_i(s)$, $i = 1,2$, from the solution of the problem in the particular setting considered in the previous subsection. Taking into account the condition of (2.35) for the value function in (2.4), we conclude after some straightforward calculations that $b(s,y) \sim g_*(s)(s-y)$ should hold as $y \uparrow s$, where $g_*(s)$ solves the equation in (3.14) with $\beta_i(s) = \gamma_i(s,s-)$, that eventually yields the unique solution $g_*(s) = 1$ for all $s > 0$. Thus, any candidate solution of the differential equation in (3.29) should satisfy the starting condition of (3.15).

For any $s > 0$ fixed, let us now consider a candidate solution $b(s,y)$ of the equation in (3.29), satisfying the starting condition of (3.15), given that this solution satisfies the expressions in (2.7). Then, we define a decreasing sequence $(y_n(s))_{n \in \mathbb{N}}$ as in part (i) of the previous subsection. Therefore, the candidate value function admits the expression of (3.27)-(3.28) in the regions R_{2l-1} from (3.16), for $l = 1, \ldots, l'$.

On the other hand, the candidate value function takes the form of (3.1) with $C_i(s,y)$, $i = 1,2$, solving the linear system of first-order partial differential equations in (3.5)-(3.6) in the regions R_{2l} from (3.17), for $l = 1, \ldots, l'$, which belong to the continuation region $C' \cup C''$ given in (2.9) and (2.13). Following arguments similar to the ones from part (i) of the previous subsection, we obtain that the value function satisfies the conditions

$$C_1(s, y_{2l}(s)+) \, ((s - y_{2l}(s))-)^{\gamma_1(s, y_{2l}(s)+)} + C_2(s, y_{2l}(s)+) \, ((s - y_{2l}(s))-)^{\gamma_2(s, y_{2l}(s)+)} = V(s - y_{2l}(s), s, y_{2l}(s); b_*(s, y_{2l}(s))) \tag{3.30}$$

$$C_1(s, y_{2l-1}(s)) \, (s - y_{2l-1}(s))^{\gamma_1(s, y_{2l-1}(s))} + C_2(s, y_{2l-1}(s)) \, (s - y_{2l-1}(s))^{\gamma_2(s, y_{2l-1}(s))} = V((s - y_{2l-1}(s))-, s, y_{2l-1}(s)+; b(s, y_{2l-1}(s)+)) \tag{3.31}$$

for $s > 0$, where the right-hand sides are given by (3.27)-(3.28) with $b(s, y_{2l-1}(s)+) = b(s, y_{2l}(s)) = s$, and

$$C_1(s(y)-, y) \, (s(y)-)^{\gamma_1(s(y)-, y)} + C_2(s(y)-, y) \, (s(y)-)^{\gamma_2(s(y)-, y)} = V(s(y), s(y), y; b(s(y), y)) \equiv s(y) - K \tag{3.32}$$

for each $y_{2l}(s) < y \le y_{2l-1}(s)$ and $l = 1, \ldots, l'-1$. Moreover, we similarly obtain the condition in (3.30) together with $C_2(\varepsilon, 0) \to 0$ as $\varepsilon \downarrow 0$, and $C_1(\infty, y) = 0$, instead of (3.31) and (3.32), respectively, for $l = l'$. We can therefore conclude that the candidate value function admits the representation

$$
\begin{aligned}
&V(x, s, y; s(y), y_{2l-1}(s), y_{2l}(s)) \qquad (3.33)\\
&= C_1(s, y; s(y), y_{2l-1}(s), y_{2l}(s))\, x^{\gamma_1(s,y)} + C_2(s, y; s(y), y_{2l-1}(s), y_{2l}(s))\, x^{\gamma_2(s,y)}
\end{aligned}
$$

in the regions R_{2l} given by (3.17), where $C_i(s, y; s(y), y_{2l-1}(s), y_{2l}(s))$, $i = 1, 2$, provide a unique solution of the two-dimensional coupled system of first-order linear partial differential equations in (3.5)-(3.6) with the boundary conditions of (3.30)-(3.32), for $l = 1, \ldots, l'$. The existence and uniqueness of the solution of the latter system follows from the arguments presented in part (i) of the previous subsection.

(ii) The case of floating strike. Let us now consider the payoff $G(x, s, y) = (Kx - s + y)^+$ in (2.4). Then, solving the system of equations in (3.3)-(3.4), we obtain that the function in (3.1) admits the representation

$$
V(x, s, y; b(s, y)) = C_1(s, y; b(s, y))\, x^{\gamma_1(s,y)} + C_2(s, y; b(s, y))\, x^{\gamma_2(s,y)} \qquad (3.34)
$$

for $0 < s - y \le x < b(s, y) \le s$, with

$$
C_i(s, y; b(s, y)) = \frac{(\gamma_{3-i}(s, y) - 1)Kb(s, y) - \gamma_{3-i}(s, y)(s - y)}{(\gamma_{3-i}(s, y) - \gamma_i(s, y))b(s, y)^{\gamma_i(s,y)}} \qquad (3.35)
$$

for all $0 < y < s$ and $i = 1, 2$. Hence, assuming that the boundary function $b(s, y)$ is continuously differentiable, we apply the condition of (3.6) to the functions $C_i(s, y) = C_i(s, y; b(s, y))$, $i = 1, 2$, in (3.35) to obtain that the boundary solves the first-order nonlinear ordinary differential equation

$$
\begin{aligned}
\partial_y b(s, y) = \sum_{i=1}^{2} \Bigg(&\frac{\gamma_{3-i}(s, y)b(s, y)}{(\gamma_i(s, y) - 1)(\gamma_{3-i}(s, y) - 1)Kb(s, y) - \gamma_i(s, y)\gamma_{3-i}(s, y)(s - y)} \qquad (3.36)\\
&\times \frac{((s - y)/b(s, y))^{\gamma_i(s,y)}}{((s - y)/b(s, y))^{\gamma_i(s,y)} - ((s - y)/b(s, y))^{\gamma_{3-i}(s,y)}}\\
&+ \frac{b(s, y)((\gamma_{3-i}(s, y) - 1)Kb(s, y) - \gamma_{3-i}(s, y)(s - y))}{(\gamma_1(s, y) - 1)(\gamma_2(s, y) - 1)Kb(s, y) - \gamma_1(s, y)\gamma_2(s, y)(s - y)}\, \partial_y \gamma_i(s, y)\\
&\times \left(\frac{1}{\gamma_{3-i}(s, y) - \gamma_i(s, y)} + \frac{((s - y)/b(s, y))^{\gamma_i(s,y)} \ln((s - y)/b(s, y))}{((s - y)/b(s, y))^{\gamma_i(s,y)} - ((s - y)/b(s, y))^{\gamma_{3-i}(s,y)}} \right) \Bigg)
\end{aligned}
$$

for $0 < y < s$, where the partial derivatives $\partial_y \gamma_i(s, y)$, $i = 1, 2$, are given by (3.8) with (3.10). Recall the assumption that the functions $\delta(s, y)$ and $\sigma(s, y)$ are continuously differentiable and bounded, so that the limits $\gamma_i(s, s-)$ can be identified with the functions $\beta_i(s)$, $i = 1, 2$, from the solution of the problem in the particular setting considered in the previous subsection. Therefore, the function in (3.34)-(3.35) should satisfy the property $V(x, s, y; b(s, y)) \to V(x, s, s-\varepsilon; b(s, s-\varepsilon))$ as $y \uparrow s-\varepsilon$, for each $s-y \le x < b(s, y)$ and any sufficiently small $\varepsilon > 0$, where $V(x, s, s-\varepsilon; b(s, s-\varepsilon))$ is given by the equation in (3.22)-(3.23), for $\varepsilon < x < b(s, s-\varepsilon)$, with $b(s, s-\varepsilon)$ being a solution of the differential equation in (3.24). Thus, letting $\varepsilon \downarrow 0$, we see that any candidate solution of the differential equation in (3.36) should satisfy the starting condition of (3.26).

For any $s > 0$ fixed, let us now consider a candidate solution $b(s, y)$ of the equation in (3.36), satisfying the starting condition of (3.26), given that this solution satisfies the expressions in (2.8). Then, we define a decreasing sequence $(y_n(s))_{n\in\mathbb{N}}$ as in part (i) of the previous subsection. Therefore, the candidate value function admits the expression in (3.34)-(3.35) in the regions R_{2l-1} defined in (3.16) for $l = 1, \dots, l'$.

On the other hand, the candidate value function takes the form of (3.1) with $C_i(s, y)$, $i = 1, 2$, solving the linear system of first-order partial differential equations in (3.5)-(3.6) in the regions R_{2l} defined in (3.17), for $l = 1, \dots, l'$, which belong to the continuation region C_* in (2.24). Using arguments similar to the ones of part (i) of this subsection, we can obtain the same conditions as in (3.30)-(3.32) above, where the right-hand sides are given by (3.34)-(3.35) with $b(s, y_{2l-1}(s)+) = b(s, y_{2l}(s)) = s$. By means of the arguments from part (i) of the previous subsection, we can therefore conclude that the candidate value function admits the representation in (3.33) in the regions R_{2l} given by (3.17), where $C_i(s, y; s(y), y_{2l-1}(s), y_{2l}(s))$, $i = 1, 2$, provide a unique solution of the two-dimensional coupled system of first-order linear partial differential equations in (3.5)-(3.6) satisfying the boundary conditions of (3.30)-(3.32), for $l = 1, \dots, l'$.

4. Main result and proof

In this section, we formulate and prove the main result of the paper, using the facts proved above. The proof of this assertion is based on a development of the maximality principle established in [29] and its extension to an optimal stopping problem for a three-dimensional Markov process (X, S, Y) from (2.1)-(2.2) (see also [32] for another three-dimensional problem).

Theorem 4.1 *In the perpetual American fixed-strike or floating-strike option on the maximum of market depth with payoff $G(x, s, y) = (K - s + y)^+$ or $G(x, s, y) = (Kx - s + y)^+$, the value function of the optimal stopping problem of (2.4) for the process (X, S, Y) from (2.1)-(2.2) has the expression*

$$V_*(x, s, y) = \begin{cases} V(x, s, y; b_*(s, y)), & \text{if} \quad s - y \le x < b_*(s, y) \le s \\ V(x, s, y; s(y), y_{2l-1}(s), y_{2l}(s)), & \text{if} \quad s - y \le x \le s < b_*(s, y) \\ G(x, s, y), & \text{if} \quad (s-y) \vee b_*(s, y) \le x \le s \end{cases} \tag{4.1}$$

and the optimal stopping time is given by (2.6), where the functions $V(x,s,y;b_*(s,y))$ *and* $V(x,s,y;s(y),y_{2l-1}(s),y_{2l}(s))$ *as well as the boundary function* $b_*(s,y)$ *are specified as follows:*

(i) if $G(x,s,y)=(K-s+y)^+$ *then the function* $V(x,s,y;b_*(s,y))$ *is given by (3.27)-(3.28) and the boundary* $b_*(s,y)$ *provides the minimal solution of the equation in (3.29) satisfying the starting condition of (3.15) and such that (2.7) holds for* $(x,s,y)\in R_{2l-1}$ *defined in (3.16), and* $V(x,s,y;s(y),y_{2l-1}(s),y_{2l}(s))$ *is given by (3.33), whenever* $(x,s,y)\in R_{2l}$ *defined in (3.17), with* $C_i(s,y)$, $i=1,2$, *solving the coupled system of equations in (3.5)-(3.6) and satisfying the conditions of (3.30)-(3.32),* $l=1,\dots,l'$, *where (3.31)-(3.32) change their form to* $C_2(\varepsilon,0)\to 0$ *as* $\varepsilon\downarrow 0$, *and* $C_1(\infty,y)=0$, *for the case* $l=l'$;

(ii) if $G(x,s,y)=(Kx-s+y)^+$ *then the function* $V(x,s,y;b_*(s,y))$ *is given by (3.34)-(3.35) and the boundary* $b_*(s,y)$ *provides the minimal solution of the equation in (3.36) satisfying the starting condition of (3.26) and such that (2.8) holds for* $(x,s,y)\in R_{2l-1}$ *defined in (3.16), and* $V(x,s,y;s(y),y_{2l-1}(s),y_{2l}(s))$ *is given by (3.33), whenever* $(x,s,y)\in R_{2l}$ *defined in (3.17), with* $C_i(s,y)$, $i=1,2$, *solving the coupled system of equations in (3.5)-(3.6) and satisfying the conditions of (3.30)-(3.32),* $l=1,\dots,l'$, *where (3.31)-(3.32) change their form to* $C_2(\varepsilon,0)\to 0$ *as* $\varepsilon\downarrow 0$, *and* $C_1(\infty,y)=0$, *for the case* $l=l'$;

(iii) if $G(x,s,y)=(Kx-s+y)^+$ *as well as* $\delta(s,y)=\delta(s)$ *and* $\sigma(s,y)=\sigma(s)$ *then the function* $V(x,s,y;b_*(s,y))$ *is given by (3.22)-(3.23) and the boundary is given by* $b_*(s,y)=g_*(s)(s-y)$ *as the explicit solution of the equation in (3.24) satisfying the starting condition of (3.26) and such that (2.8) holds, where* $g_*(s)$ *provides the unique solution of (3.25) for* $(x,s,y)\in R_1$ *defined in (3.16), and* $V(x,s,y;s(y),y_1(s),y_2(s))$ *is given by (3.21), whenever* $(x,s,y)\in R_2$ *defined in (3.17), with* $C_i(s,y)$, $i=1,2$, *solving the coupled system of equations in (3.5)-(3.6) and satisfying the conditions of (3.18),* $C_2(\varepsilon,0)\to 0$ *as* $\varepsilon\downarrow 0$, *and* $C_1(\infty,y)=0$, *for* $l=l'=1$.

Since all the assertions formulated above are proved using similar arguments, we only give a proof for the general optimal stopping problem related to the perpetual American fixed-strike option on the maximum of market depth in part (i) of Theorem 4.1.

Proof of part (i) of Theorem 4.1. In order to verify the assertion stated above, it remains to show that the function defined in (4.1) coincides with the value function in (2.4) with payoff $(K-s+y)^+$ and that the stopping time τ_* in (2.6) is optimal with the boundary $b_*(s,y)$ specified above. For this, let $b(s,y)$ be any solution of (3.29) with the starting condition in (3.15) and satisfying (2.7). Let us also denote by $V_b(x,s,y)$ the right-hand side of the expression in (4.1) associated with this $b(s,y)$. It then follows using straightforward calculations and the assumptions presented above that the function $V_b(x,s,y)$ solves the system of (2.27)-(2.29), while the normal-reflection and smooth-fit conditions are satisfied in (2.32)-(2.33). Hence, taking into account the fact that the function $V_b(x,s,y)$ is $C^{2,1,1}$ and the boundary $b(s,y)$ is assumed to be continuously differentiable for all $0<y<s$, by applying the change-of-variable

formula from [30; Theorem 3.1] to $e^{-rt}V_b(X_t,S_t,Y_t)$, we obtain

$$
\begin{aligned}
&e^{-rt}\,V_b(X_t,S_t,Y_t) = V_b(x,s,y) + M_t && (4.2)\\
&+\int_0^t e^{-ru}\,(\mathbb{L}V_b - rV_b)(X_u,S_u,Y_u)\,I(X_u\neq S_u-Y_u, X_u\neq b(S_u,Y_u), X_u\neq S_u)\,du\\
&+\int_0^t e^{-ru}\,\partial_s V_b(X_u,S_u,Y_u)\,I(X_u=S_u)\,dS_u + \int_0^t e^{-ru}\,\partial_y V_b(X_u,S_u,Y_u)\,I(X_u=S_u-Y_u)\,dY_u
\end{aligned}
$$

where the process $M=(M_t)_{t\geq 0}$ given by

$$
M_t = \int_0^t e^{-ru}\,\partial_x V_b(X_u,S_u,Y_u)\,I(X_u\neq S_u-Y_u, X_u\neq S_u)\,\sigma(S_u,Y_u)\,X_u\,dB_u \tag{4.3}
$$

is a square integrable martingale under $P_{x,s,y}$. Note that, since the time spent by the process X at the boundary surface $\{(x,s,y)\in E^3 \,|\, x=b(s,y)\}$ as well as at the planes $d_1=\{(x,s,y)\in \mathbb{R}^3 \,|\, 0<x=s\}$ and $d_2=\{(x,s,y)\in\mathbb{R}^3 \,|\, 0<x=s-y\}$ is of Lebesgue measure zero, the indicators in the second line of the formula (4.2) as well as in the formula (4.3) can be ignored. Moreover, since the process S increases only on the plane d_1 and the process Y increases only on the plane d_2, the indicators in the third line of (4.2) can be set equal to one.

By using straightforward calculations and the arguments from the previous section, it is verified that $(\mathbb{L}V_b - rV_b)(x,s,y)\leq 0$ for all $(x,s,y)\in E^3$ such that $x\neq b(s,y)$, $x\neq s-y$, and $x\neq s$. Moreover, it is shown by means of standard arguments that the properties in (2.30)-(2.31) also hold, which together with (2.28)-(2.29) imply that the inequality $V_b(x,s,y)\geq (K-s+y)^+$ is satisfied for all $(x,s,y)\in E^3$. It therefore follows from the expression (4.2) that the inequalities

$$
e^{-r\tau}\,(K-S_\tau+Y_\tau)^+ \leq e^{-r\tau}\,V_b(X_\tau,S_\tau,Y_\tau) \leq V_b(x,s,y) + M_\tau \tag{4.4}
$$

hold for any finite stopping time τ with respect to the natural filtration of X.

Taking the expectation with respect to $P_{x,s,y}$ in (4.4), by means of the optional sampling theorem (see, e.g. [20; Chapter I, Theorem 3.22]), we get

$$
\begin{aligned}
E_{x,s,y}\big[e^{-r(\tau\wedge t)}\,(K-S_{\tau\wedge t}+Y_{\tau\wedge t})^+\big] &\leq E_{x,s,y}\big[e^{-r(\tau\wedge t)}\,V_b(X_{\tau\wedge t},S_{\tau\wedge t},Y_{\tau\wedge t})\big] \qquad (4.5)\\
&\leq V_b(x,s,y) + E_{x,s,y}\big[M_{\tau\wedge t}\big] = V_b(x,s,y)
\end{aligned}
$$

for all $(x,s,y)\in E^3$. Hence, letting t go to infinity and using Fatou's lemma, we obtain that the inequalities

$$
E_{x,s,y}\big[e^{-r\tau}\,(K-S_\tau+Y_\tau)^+\big] \leq E_{x,s,y}\big[e^{-r\tau}\,V_b(X_\tau,S_\tau,Y_\tau)\big] \leq V_b(x,s,y) \tag{4.6}
$$

are satisfied for any finite stopping time τ and all $(x,s,y)\in E^3$. Taking first the supremum over all stopping times τ and then the infimum over all b, we conclude that

$$
E_{x,s,y}\big[e^{-r\tau_*}\,(K-S_{\tau_*}+Y_{\tau_*})^+\big] \leq \inf_b V_b(x,s,y) = V_{b_*}(x,s,y) \tag{4.7}
$$

where $b_*(s, y)$ is the minimal solution of (3.29) with the starting condition in (3.15) and satisfying (2.7). Using the fact that the function $V_b(x, s, y)$ is increasing in b, satisfying $b(s, y) > s-y$ for $s - y < K$ and $b(s, y) > s$ for $s - y \geq K$ under any $0 < y < s$ fixed, we see that the infimum in (4.7) is attained over any sequence of solutions $(b_n(s, y))_{n\in\mathbb{N}}$ to (3.29) with the starting condition in (3.15) and satisfying (2.7), and such that $b_n(s, y) \downarrow b_*(s, y)$ as $n \to \infty$. Since the inequalities in (4.6) hold also for $b_*(s, y)$, we see that (4.7) holds for $b_*(s, y)$ and $(x, s, y) \in E^3$ as well. Note that $V_b(x, s, y)$ in (4.5) is superharmonic for the Markov process (X, S, Y) on E^3. Taking into account the fact that $V_b(x, s, y)$ is increasing in b and that the inequality $V_b(x, s, y) \geq (K - s + y)^+$ holds for all $(x, s, y) \in E^3$, we observe that the selection of the minimal solution $b_*(s, y)$, which satisfies (2.7) whenever such a choice exists, is equivalent to invoking the superharmonic characterisation of the value function as the smallest superharmonic function dominating the payoff function (see, e.g. [29] or [33; Chapter I, Section 2]).

In order to clarify the (local) existence and uniqueness of the solution to the equation in (3.29), we recall the arguments of [32; Subsection 3.5] and denote the right-hand side of (3.29) by $\Psi(s, y, b(s, y))$. Since the function $\Psi(s, y, b)$ is (locally) continuous in y and (locally) Lipschitz in b, we may conclude from the general theory of first-order nonlinear ordinary differential equations that the one in (3.29) admits a (locally) unique solution.

In order to construct the minimal solution $b_*(s, y)$ of the equation in (3.29), satisfying the conditions mentioned above, and prove the fact that it is optimal in E^3, we provide an extension of the arguments from [29; Theorem 3.1] to the (X, S, Y)-setting (see also [32; Subsection 3.5] for another three-dimensional case). For this, let us consider the sequence of stopping times τ_n defined as in (2.6) with $b_n(s, y)$ instead of $b_*(s, y)$, where $b_n(s, y)$ is the solution of (3.29) with the starting condition of (3.15) and such that $b_n(s, y_n) = s - y_n < K$ holds for some $y_n \downarrow c$ as $n \to \infty$, where $c > 0$ is such that $b_\infty(s, y) > s - y$ for all $s - K < y \leq c$, and $b_\infty(s, y) > s$ holds for all $y \leq s-K$, whenever such a sequence exists. Otherwise, we consider as $b_n(s, y)$ the solution of (3.29) with the starting condition of (3.15) and such that $b_n(s, s - K) = c_n$ holds for some $c_n \uparrow c$ as $n \to \infty$, where $c > 0$ is such that $b_\infty(s, y) > s$ holds for all $y \leq s - K$. It follows from the uniqueness of the solution to (3.29) that no distinct solutions intersect, so that the sequence $(b_n(s, y))_{n\in\mathbb{N}}$ is increasing and the limit $b_*(s, y) = \lim_{n\to\infty} b_n(s, y)$ exists. By virtue of the fact that the function $V_{b_n}(x, s, y)$ from the expression in (4.1) associated with this $b_n(s, y)$, satisfies the system of (2.27)-(2.31) with (2.32) and taking into account the structure of τ_n given by (2.6) with $b_n(s, y)$ instead of $b_*(s, y)$, it follows from the equivalent expression of (4.2) that the equalities

$$e^{-r(\tau_n\wedge t)} (K - S_{\tau_n\wedge t} + Y_{\tau_n\wedge t})^+ = e^{-r(\tau_n\wedge t)} V_{b_n}(X_{\tau_n\wedge t}, S_{\tau_n\wedge t}, Y_{\tau_n\wedge t}) = V_{b_n}(x, s, y) + M_{\tau_n\wedge t} \quad (4.8)$$

hold for all $(x, s, y) \in E^3$. Observe that $\tau_n \uparrow \tau_*$ ($P_{x,s,y}$-a.s.), the property

$$E_{x,s,y}\Big[\sup_{t\geq 0} e^{-r(\tau_*\wedge t)}\, Y_{\tau_*\wedge t}\Big] \leq E_{x,s,y}\Big[\sup_{t\geq 0} e^{-r(\tau_*\wedge t)}\, S_{\tau_*\wedge t}\Big] = E_{x,s,y}\Big[\sup_{t\geq 0} e^{-r(\tau_*\wedge t)}\, X_{\tau_*\wedge t}\Big] < \infty \quad (4.9)$$

holds for all $(x, s, y) \in E$ and the variable $e^{-r\tau_*}(K - S_{\tau_*} + Y_{\tau_*})^+$ is bounded on the set $\{\tau_* = \infty\}$. Note also that, by using the asymptotic behavior of $b_*(s, y)$ when $y \uparrow s$ in (3.15), the property $P_{x,s,y}(\tau_* < \infty) = 1$ holds for all $(x, s, y) \in E^3$. Hence, letting t and n go to infinity and using the conditions in (2.28) and (2.32), as well as the fact that $\tau_n \uparrow \tau_*$ ($P_{x,s,y}$-a.s.), we can apply the Lebesgue dominated convergence theorem for (4.8) to obtain the equality

$$E_{x,s,y}\big[e^{-r\tau_*}\,(K - S_{\tau_*} + Y_{\tau_*})^+\big] = V_{b_*}(x, s, y) \tag{4.10}$$

for all $(x, s, y) \in E^3$, which together with (4.7) directly implies the desired assertion. $\square$

5 Appendix

In this section, we prove the existence of a unique solution $g_*(s)$ to the equation in (3.25). For this, we first rewrite the latter equation in the form

$$F_1(g(s)) = F_2(g(s)) \quad \text{with} \quad F_i(x) = \big((\beta_1(s)-1)\,(\beta_2(s)-1)\,K\,x - \beta_i(s)\,(\beta_{3-i}(s)-1)\big)\,x^{\beta_i(s)} \tag{5.1}$$

for $i = 1, 2$ and some arbitrary function $g(s)$, for some $s > 0$ fixed. Then, the derivatives of the functions $F_i(x)$ from (5.1) take the form

$$F_i'(x) = (\beta_{3-i}(s) - 1)\,\big((\beta_i^2(s) - 1)\,K\,x - \beta_i^2(s)\big)\,x^{\beta_i(s)-1} \tag{5.2}$$

for all $x > 0$ and $i = 1, 2$, and some $s > 0$ fixed. It therefore follows that the function $F_1(x)$ is increasing on the interval $(0, \beta_1^2(s)/((\beta_1^2(s) - 1)K))$ with $F_1(0+) = 0$ and

$$F_1\bigg(\frac{\beta_1^2(s)}{(\beta_1^2(s) - 1)K}\bigg) = -\frac{\beta_1(s)(\beta_2(s) - 1)}{\beta_1(s) + 1}\bigg(\frac{\beta_1^2(s)}{(\beta_1^2(s) - 1)K}\bigg)^{\beta_1(s)} > 0 \tag{5.3}$$

and then decreasing on the interval $(\beta_1^2(s)/((\beta_1^2(s) - 1)K), \infty)$ with $F_1(\infty) = -\infty$, for any $s > 0$ fixed.

Let us now specify the structure of the function $F_2(x)$ based on the value of $\beta_2(s)$. For this, we first assume that $\beta_2(s) < -1$ holds, that is equivalent to $2r - \delta(s) - \sigma^2(s) > 0$ in the model in (2.1)-(2.2), for some $s > 0$ fixed. In this case, it follows that $F_2(x)$ is decreasing on the interval $(0, \beta_2^2(s)/((\beta_2^2(s) - 1)K))$, with $F_2(0+) = +\infty$ and

$$F_2\bigg(\frac{\beta_2^2(s)}{(\beta_2^2(s) - 1)K}\bigg) = -\frac{\beta_2(s)(\beta_1(s) - 1)}{\beta_2(s) + 1}\bigg(\frac{\beta_2^2(s)}{(\beta_2^2(s) - 1)K}\bigg)^{\beta_2(s)} < 0 \tag{5.4}$$

and increasing on the interval $(\beta_2^2(s)/((\beta_2^2(s) - 1)K), \infty)$, with $F_2(\infty) = 0$. Thus, taking also into account the fact that $h_1(s) > h_2(s)$ holds as well, where $h_i(s)$ is such that $F_i(h_i(s)) = 0$ for $i = 1, 2$, we conclude that there exist exactly two solutions of the equation in (5.1). Let us now assume that $-1 \le \beta_2(s) < 0$ holds, that is equivalent to $2r - \delta(s) - \sigma^2(s) \le 0$ in the model of (2.1)-(2.2). In this case, the value $\beta_2^2(s)/(\beta_2^2(s)-1)$ is negative, that yields the fact that the function $F_2(x)$ is strictly decreasing in the interval $(0,\infty)$, with $F_2(0+)=\infty$ and $F_2(\infty)=-\infty$. Hence, the same arguments as above guarantee the existence of at least one solution of the equation in (5.1). Moreover, by computing the second-order derivatives of the functions $F_i(x)$, $i=1,2$, we get

$$F_i''(x)=\beta_i(s)\,(\beta_1(s)-1)\,(\beta_2(s)-1)\,\big((\beta_i(s)+1)Kx-\beta_i(s)\big)\,x^{\beta_i(s)-2} \tag{5.5}$$

for $s>0$ fixed and $i=1,2$, we observe that $F_1''(x)<0$ for all $x>h_1(s)>\beta_1^2(s)/((\beta_1^2(s)-1)K)>\beta_1(s)/((\beta_1(s)+1)K)>0$ and $F_2''(x)>0$ for all $x>h_2(s)$, under the assumption that $\beta_2(s)\geq-1$ holds. It follows that the function $F_1(x)$ is concave, when it becomes negative for $x\geq h_1(s)>h_2(s)$, while the function $F_2(x)$ is convex, when it becomes negative for $x\geq h_2(s)$. It therefore follows that there exist exactly two solutions of the equation in (5.1).

Let us finally consider the two solutions $g_1(s)$ and $g_2(s)$ of the equation in (5.1), which satisfy the inequalities $g_2(s)<h_2(s)<1/K<h_1(s)<g_1(s)$, for all possible choices of the parameters of the model. The fact that $b_*(s,y)$ satisfies (2.8) in Lemma 2.1, for all $0<y<s$, implies that

$$g_*(s)>\frac{r}{\delta(s)K}\equiv\frac{\beta_1(s)\beta_2(s)}{(\beta_1(s)-1)(\beta_2(s)-1)K}>\frac{\beta_2(s)}{(\beta_2(s)-1)K}\equiv h_2(s) \tag{5.6}$$

holds for all $s>0$. We therefore reject $g_2(s)<h_2(s)$ and accept $g_1(s)>h_1(s)>r/(\delta(s)K)$. Furthermore, taking into account the fact that $F_1(1)>F_2(1)$ and $F_1(x)\leq F_2(x)$ holds for all $x\geq g_*(s)$, it follows that the function $g_*(s)\equiv g_1(s)$ satisfying $F_1(g_*(s))=F_2(g_*(s))$, is greater than 1, for any choice of fixed $s>0$.

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