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Spanning embeddings of arrangeable graphs with sublinear bandwidth*

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The Bandwidth Theorem of Böttcher, Schacht, and Taraz [Mathematische Annalen 343 (1), 175–205] gives minimum degree conditions for the containment of spanning graphs H with small bandwidth and bounded maximum degree. We generalise this result to a-arrangeable graphs H with $\Delta(H) \leq \sqrt{n}/\log n$, where n is the number of vertices of H.

Our result implies that sufficiently large n-vertex graphs G with minimum degree at least $(\frac{3}{4}+\gamma)n$ contain almost all planar graphs on n vertices as subgraphs. Using techniques developed by Allen, Brightwell, and Skokan [Combinatorica 33 (2), 125–160] we can also apply our methods to show that almost all planar graphs H have Ramsey number at most $12|H|$. We obtain corresponding results for graphs embeddable on different orientable surfaces.

1 Introduction

The existence of spanning subgraphs in dense graphs has been investigated very successfully over the past decades. Its early stages can be traced back to results by Dirac [9] in 1952, who showed that a minimum degree of $n/2$ forces a Hamilton cycle in graphs of order n, and Corrádi and Hajnal [8] in 1963 as well as Hajnal and Szemerédi [10] in

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1970, who proved that every graph G with $\delta(G) \geq \frac{r-1}{r}n$ must contain a family of $\lfloor n/r \rfloor$ vertex disjoint cliques, each of size r. The story gained new momentum when, in a series of papers in the 1990s, Komlós, Sarközy, and Szemerédi established a new methodology which, based on the Regularity Lemma and the Blow-up Lemma, paved the road to a series of results for spanning subgraphs with bounded maximum degree, such as powers of Hamilton cycles, trees, F-factors, and planar graphs (see the survey [19] for an excellent overview of these and related achievements).

During that period, Bollobás and Komlós [14] formulated a general conjecture which (approximately) included many of the results mentioned above. Böttcher, Schacht and Taraz proved this conjecture.

Theorem 1 (Böttcher, Schacht, Taraz [5])
For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds. If H is an r-colourable graph on n vertices with $\Delta(H) \leq \Delta$ and bandwidth at most βn and if G is a graph on n vertices with minimum degree $\delta(G) \geq \left(\frac{r-1}{r} + \gamma\right) n$, then G contains a copy of H.

Here a graph H has *bandwidth* at most b if there exists a labelling of the vertices by numbers $1, \ldots, n$ such that for every edge $\{i, j\} \in E(H)$ we have $|i - j| \leq b$. It is well known that the restriction on the bandwidth in Theorem 1 cannot be omitted (see [14]). On the other hand, powers of Hamilton cycles and F-factors have constant bandwidth. Moreover, bounded degree planar graphs and more generally any hereditary class of bounded degree graphs with small separators have bandwidth at most $O(n/\log n)$ (see [4]). Hence a rich class of graphs H is covered by Theorem 1.

However, a major constraint of this theorem is that it allows only H with constant maximum degree. In fact this is also true for most other results on spanning subgraphs mentioned above. There are only few exceptions, such as a result by Komlós, Sarközy, and Szemerédi [17], which shows that each sufficiently large graph with minimum degree at least $(\frac{1}{2} + \gamma)n$ contains all spanning trees of maximum degree $o(n/\log n)$.

One aim of this paper is to obtain a corresponding embedding result for a more general class of graphs with unbounded maximum degree. More precisely, we will generalise Theorem 1 to graphs with unbounded maximum degrees. We focus on arrangeable graphs.

Definition 2 (a-arrangeable)
Let a be an integer. A graph is called a-arrangeable if its vertices can be ordered as $(x_1, \ldots, x_n)$ in such a way that $\left|N\big(N(x_i) \cap \mathrm{Right}_i)\big) \cap \mathrm{Left}_i\right| \leq a$ for each $1 \leq i \leq n$, where $\mathrm{Left}_i = \{x_1, x_2, \ldots, x_i\}$ and $\mathrm{Right}_i = \{x_{i+1}, x_{i+2}, \ldots, x_n\}$.

Arrangeability was introduced by Chen and Schelp [7]. It generalises the concept of bounded maximum degree because graphs with maximum degree Δ are clearly $(\Delta^2 - \Delta + 1)$-arrangeable, and stars are 1-arrangeable. Moreover several important graph classes were shown to be constantly arrangeable: Kierstead and Trotter [13] showed that planar graphs are 10-arrangeable (see also [7]) and Rödl and Thomas [25] established that graphs without a K_p-subdivision are p^8-arrangeable.

Our main result asserts that we can replace the constant maximum degree bound in Theorem 1 by a-arrangeability and $\Delta(H) \le \sqrt{n}/\log n$.

Theorem 3 (The bandwidth theorem for arrangeable graphs)
For all $r, a \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ the following holds. If H is an r-colourable, a-arrangeable graph on n vertices with $\Delta(H) \le \sqrt{n}/\log n$ and bandwidth at most βn and if G is a graph on n vertices with minimum degree $\delta(G) \ge \left(\frac{r-1}{r} + \gamma\right) n$, then G contains a copy of H.

The key ingredient for generalising Theorem 1 to Theorem 3 is a variant of the Blow-up Lemma for arrangeable graphs, obtained recently by Böttcher, Kohayakawa, Taraz, and Würfl in [3] (see Theorem 13).

Applications. We give one direct application of Theorem 3 (Corollary 4), and one application which uses the techniques needed in the proof of Theorem 3 (Theorem 6). Both applications concern graphs of fixed genus.

Let S be an orientable surface and denote by $g(S)$ the genus of S. Let $\mathcal{H}_S(n)$ be the family of n-vertex graphs embeddable on S and let $\mathcal{H}_S(n, \Delta)$ be the family of those graphs in $\mathcal{H}_S(n)$ with maximum degree at most Δ. The celebrated Four Colour Theorem [2, 23] and the affirmative solution of Heawood's Conjecture [11, 22] guarantee that each graph in $\mathcal{H}_S(n)$ can be coloured with

$$r(S) := \left\lfloor \frac{7 + \sqrt{1 + 48g(S)}}{2} \right\rfloor \tag{1}$$

colours. Moreover, in [4] it was shown that graphs in $H \in \mathcal{H}_S(n, \Delta)$ have bandwidth at most

$$\mathrm{bw}(S, n, \Delta) := \frac{15n \log \Delta}{\log n - \log \max\big(1, g(S)\big)} . \tag{2}$$

Hence, as observed in [4], it is a direct consequence of Theorem 1 that large n-vertex graphs G with minimum degree at least $(\frac{r(S)-1}{r(S)} + \gamma)n$ contain all graphs from $\mathcal{H}_S(n, \Delta)$ as subgraphs, which extends results of Kühn, Osthus and Taraz [20] (see also [18]). With the help of Theorem 3 we are now able to say considerably more – namely, that in fact *almost all* graphs from $\mathcal{H}_S(n)$ are contained in each such graph G.

Indeed, McDiarmid and Reed [21] proved that for each fixed S there is a constant $C(S)$ such that, if we draw a graph H uniformly at random from $\mathcal{H}_S(n)$ then asymptotically almost surely H has maximum degree of order

$$\Delta(S, n) \le C(S) \log n . \tag{3}$$

Moreover, clearly $K_{r(S)+1}$ cannot be embedded in S and hence graphs from $\mathcal{H}_S(n)$ are $K_{r(S)+1}$-minor free. It thus follows from the result of Rödl and Thomas [25] mentioned above that the graphs in $\mathcal{H}_S(n)$ are $a(S)$-arrangeable with

$$a(S) := \big(r(S) + 1\big)^8 . \tag{4}$$

In conclusion, we immediately obtain the following corollary of Theorem 3.

Corollary 4
Let $\gamma > 0$, let S be an orientable surface and let G be an n-vertex graph with $\delta(G) \geq \big(\frac{r(S)-1}{r(S)} + \gamma\big)n$. If H is drawn uniformly at random from $\mathcal{H}_S(n)$, then G contains H almost surely.

In particular, if $\delta(G) \geq (\frac{3}{4} + \gamma)n$ then G contains almost all planar graphs on n vertices.

Our second application concerns Ramsey numbers of graphs in $\mathcal{H}_S(n)$. For a graph H we denote by $R(H)$ the two-colour Ramsey number of H. Allen, Brightwell, and Skokan [1] proved that graphs with bounded maximum degree and small bandwidth have small Ramsey numbers.

Theorem 5 (Allen, Brightwell and Skokan [1])
For all $\Delta \in \mathbb{N}$, there exist constants $\beta > 0$ and n_0 such that for every $n \geq n_0$ the following holds. If H is an n-vertex graph with maximum degree at most Δ and $\mathrm{bw}(H) \leq \beta n$, then $R(H) \leq (2\chi(H) + 4)n$.

With the help of (1) and (2) this implies that for any fixed orientable surface S and any fixed Δ each graph $H \in \mathcal{H}_S(n, \Delta)$ satisfies $R(H) \leq \big(2r(S) + 4\big)n$ if n is sufficiently large. In particular, large planar graphs H with bounded maximum degree have Ramsey number $R(H) \leq 12|H|$.

This together with the fact that planar graphs are known to have at most linear Ramsey number (see [7]) led Allen, Brightwell, and Skokan to conjecture that in fact *all* sufficiently large planar graphs H have Ramsey number at most $12|H|$. Combining their methods with ours we can now show that this is true for *almost all* planar graphs.

Theorem 6
Let S be an orientable surface. If H is drawn uniformly at random from $\mathcal{H}_S(n)$, then almost surely $R(H) \leq (2r(S) + 4)n$.

In particular, for almost every planar graph H we have $R(H) \leq 12|H|$.

Organisation. In Section 2 we give an outline of our proof of Theorem 3. This proof builds on partitioning results for G and for H, which we present in Section 3, and on a variant of the Blow-up Lemma for arrangeable graphs, which we discuss in Section 4. We then present the actual proof of Theorem 3 in Section 5. We close with the proof of Theorem 6 in Section 6 and with some concluding remarks in Section 7.

2 Outline of the proof of Theorem 3

Many of the results concerning the embedding of spanning, bounded degree graphs follow a general agenda which is nicely described in the survey paper [14] by Komlós. This agenda consists of five main steps: firstly preparing H, secondly preparing G, thirdly assigning parts of H to parts of G, fourthly connecting those parts, and fifthly embedding the parts of H separately, via the Blow-up Lemma.

In the proof of Theorem 3 we follow a similar agenda. The preparation for G uses, as is usual, Szemerédi's Regularity Lemma and some additional work to produce a suitable partition of G. For this step we can make use of a lemma from [5] (see Lemma 7).

The preparation of H (see Lemma 8) makes use of the bandwidth of H and produces a partition of H which is compatible to the partition of G (in this way we implicitly obtain an assignment of the parts of H to the parts of G). This step is also similar to the methods used in [5] to partition bounded degree graphs H. However, we need to strengthen this approach because we now deal with graphs H whose degrees are no longer bounded by a constant. In other words, we need a slightly different partitioning lemma for H in order to make this partition suitable for the Blow-up lemma that we will use in the next step.

In a final step we use the two partitions obtained to embed H into G. Our approach here is slightly different from the steps described by Komlós which are usually used (connecting the parts and embedding the parts of H separately). We use the Blow-up Lemma for arrangeable graphs, which was recently established in [3], to formulate an embedding result (see Theorem 14) which enables us to embed H into G at once (instead of needing several Blow-up Lemma applications, as is usually the case).

3 Lemmas for G and H

In this section we formulate a partitioning lemma for G, which asserts that G has a regular partition suitable for our purposes, and a corresponding partitioning lemma for H. Both these lemmas are tailored to the application of the version of the Blow-up Lemma that we will give in the next section.

We first introduce some notation. Let G, H and R be graphs with vertex sets $V(G)$, $V(H)$, and $V(R) = \{1, \dots, s\} =: [s]$. For $v \in V(G)$ and $S \subseteq V(G)$ we define $N(v, S) := N(v) \cap S$. Let $A, B \subseteq V(G)$ be non-empty and disjoint, and let $0 \leq \varepsilon, \delta \leq 1$. The *density* of the pair (A, B) is $d(A, B) := e(A, B)/(|A||B|)$. The pair (A, B) is ε-*regular*, if $|d(A, B) - d(A', B')| \leq \varepsilon$ for all $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$. An ε-regular pair (A, B) is called (ε, δ)-*regular* if $d(A, B) \geq \delta$, and (ε, δ)-*super-regular* if $|N(v, B)| \geq \delta|B|$ for all $v \in A$ and $|N(v, A)| \geq \delta|A|$ for all $v \in B$.

Let G have the partition $V(G) = V_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} V_s$ and H have the partition $V(H) = W_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} W_s$. We say that $(V_i)_{i\in[s]}$ is (ε, δ)-*(super-)regular on* R if (V_i, V_j) is an (ε, δ)-(super-)regular pair for every $ij \in E(R)$. In this case R is also called a *reduced graph* of the (super-)regular partition. The partition classes V_i are also called *clusters*.

For all $n, k, r \in \mathbb{N}$, we call an integer partition $(n_{i,j})_{i\in[k],j\in[r]}$ of n r-*equitable*, if $|n_{i,j} - n_{i,j'}| \leq 1$ for all $i \in [k]$ and $j, j' \in [r]$. Let B_k^r be the kr-vertex graph obtained from a path on k vertices by replacing every vertex by a clique of size r and replacing every edge by a complete bipartite graph minus a perfect matching. More precisely, $V(B_k^r) = [k] \times [r]$ and

$$\{(i, j), (i', j')\} \in E(B_k^r) \qquad \text{iff} \qquad i = i' \quad \text{or} \quad \big(|i - i'| = 1 \text{ and } j \neq j'\big)\,. \tag{5}$$

Let K_k^r be the graph on vertex set $[k] \times [r]$ that is formed by the disjoint union of k complete graphs on r vertices. Obviously, $K_k^r \subseteq B_k^r$.

Now we can formulate the partition lemma for G, which we take from [5, Lemma 6].

Lemma 7 (Lemma for G [5])
For all $r \in \mathbb{N}$ and $\gamma > 0$ there exist $d > 0$ and $\varepsilon_0 > 0$ such that for every positive $\varepsilon \le \varepsilon_0$ there exist K_0 and $\xi_0 > 0$ such that for all $n \ge K_0$ and for every graph G on vertex set $[n]$ with $\delta(G) \ge (\frac{r-1}{r} + \gamma)n$ there exist $k \in [K_0]$ and a graph R_k^r on vertex set $[k] \times [r]$ with

- *(R1)* $K_k^r \subseteq B_k^r \subseteq R_k^r$,
- *(R2)* $\delta(R_k^r) \ge (\frac{r-1}{r} + \gamma/2)kr$, *and*
- *(R3) there is an r-equitable integer partition $(m_{i,j})_{i\in[k],j\in[r]}$ of n with $(1+\varepsilon)n/(kr) \ge m_{i,j} \ge (1-\varepsilon)n/(kr)$ such that the following holds.*[1]

For every integer partition $(n_{i,j})_{i\in[k],j\in[r]}$ of n with $m_{i,j} - \xi_0 n \le n_{i,j} \le m_{i,j} + \xi_0 n$ there exists a partition $(V_{i,j})_{i\in[k],j\in[r]}$ of V with

- *(G1)* $|V_{i,j}| = n_{i,j}$,
- *(G2)* $(V_{i,j})_{i\in[k],j\in[r]}$ *is (ε, d)-regular on R_k^r, and*
- *(G3)* $(V_{i,j})_{i\in[k],j\in[r]}$ *is (ε, d)-super-regular on K_k^r.*

The remainder of this section is dedicated to a corresponding partitioning lemma for H, which again will be similar to the Lemma for H in [5] (Lemma 8 in that paper). However, we need to strengthen the conclusion of this lemma. We shall point out the main differences below.

Again, we start with some definitions. Let H be a graph on n vertices and $\sigma : V(H) \to \{0, \dots, r\}$ be a proper $(r+1)$-colouring of H. A set $W \subseteq V(H)$ is called *zero free* if $\sigma^{-1}(0) \cap W = \emptyset$. Now assume that the vertices of H are labelled $1, \dots, n$ and that this labelling is a labelling of bandwidth at most βn for some $\beta > 0$. Given an integer ℓ, an $(r+1)$-colouring $\sigma : V(H) \to \{0, \dots, r\}$ of H is said to be (ℓ, β)-*zero free* with respect to such a labelling if any ℓ consecutive blocks contain at most one block with zeros. Here a block is a set of the form $B_t := \{(t-1)5r\beta n + 1, \dots, t5r\beta n\}$, $t = 1, \dots, 1/(5r\beta)$. More precisely, we round to integer values such that the sizes of the B_t differ by no more than 1. We remark that here and throughout the rest of the paper we omit floors and ceilings to simplify the presentation.

Lemma 8 (Lemma for H)
Let $r, k \ge 1$ be integers and let $\beta, \xi > 0$ satisfy $\beta \le \xi^2/(10000r)$. Let H be a graph on n vertices and assume that H has a labelling of bandwidth at most βn and an $(r+1)$-colouring that is $(100/\xi, \beta)$-zero free with respect to this labelling. Let R_k^r be a graph with $V(R_k^r) = [k] \times [r]$ such that

- *(R1*)* $K_k^r \subseteq B_k^r \subseteq R_k^r$, *and*
- *(R2*) for every $i \in [k]$ there is a vertex $s_i \in ([k] \setminus \{i\}) \times [r]$ with $\{s_i, (i,j)\} \in E(R_k^r)$ for every $j \in [r]$.*

Furthermore, suppose $(m_{i,j})_{i\in[k],j\in[r]}$ is an r-equitable integer partition of n with $m_{i,j} \ge 12\beta n$ for every $i \in [k]$ and $j \in [r]$. Then there exists a mapping $f : V(H) \to [k] \times [r]$

[1] The upper bound on $m_{i,j}$ is implicit in the proof of Lemma 7 in [5].

and a set of special vertices $X \subseteq V(H)$ with the following properties, where we set $W_{i,j} := f^{-1}(i,j)$.

(H1) $|X \cap W_{i,j}| \le \xi n$ *and* $|N_H(X \cap W_{i,j}) \cap W_{i',j'}| \le \xi n$ *for all* $i, i' \in [k]$, $j, j' \in [r]$,

(H2) $m_{i,j} - \xi n \le |W_{i,j}| \le m_{i,j} + \xi n$ *for every* $i \in [k]$ *and* $j \in [r]$,

(H3) for every edge $\{u,v\} \in E(H)$ *we have* $\{f(u), f(v)\} \in E(R_k^r)$, *and*

(H4) if $\{u,v\} \in E(H) \setminus E(H[X])$ *then* $\{f(u), f(v)\} \in E(K_k^r)$.

This lemma differs from Lemma 8 in [5] in that the conclusion *(H4)* is stronger. In order to obtain this stronger conclusion we had to strengthen the notion of zero-freeness as well. Nevertheless the proof of this modified Lemma for H closely follows the proof in [5]. We use the following propositions.

Proposition 9 (Proposition 20 in [5])
Let $c_1, \dots, c_r$ *be such that* $c_1 \le c_2 \le \cdots \le c_{r-1} \le c_r \le c_1 + x$ *and* $c'_1, \dots, c'_r$ *be such that* $c'_r \le c'_{r-1} \le \cdots \le c'_2 \le c'_1 \le c'_r + x$. *If we set* $c''_i := c_i + c'_i$ *for all* $i \in [r]$ *then*

$$\max_i\{c''_i\} \le \min_i\{c''_i\} + x\,.$$

Proposition 10 (Proposition 22 in [5])
Assume that the vertices of H *are labelled* $1, \dots, n$ *with bandwidth at most* βn *with respect to this labelling. Let* $s \in [n]$ *and suppose further that* $\sigma\colon [n] \to \{0, \dots, r\}$ *is a proper* $(r+1)$*-colouring of* $V(H)$ *such that* $[s - 2\beta n, s + 2\beta n]$ *is zero free.*

Then for any two colours $l, l' \in [r]$ *the mapping* $\sigma'\colon [n] \to \{0, \dots, r\}$ *defined by*

$$\sigma'(v) := \begin{cases} l & \textit{if } \sigma(v) = l', s < v \\ l' & \textit{if } \sigma(v) = l, s + \beta n < v \\ 0 & \textit{if } \sigma(v) = l, s - \beta n \le v \le s + \beta n \\ \sigma(v) & \textit{otherwise} \end{cases}$$

is a proper $(r+1)$*-colouring of* H.

By repeatedly applying Proposition 10 we can transform a colouring of H into a balanced colouring by allowing some more vertices to be coloured with colour 0. This is a first step towards the proof of Lemma 8.

In order to make this precise we need the following definition. For $x \in \mathbb{N}$, a colouring $\sigma : [n] \to \{0, \dots, r\}$ is called *x-balanced*, if for each pair $a, b \in [n] \cup \{0\}$ and each $i \in [r]$, we have

$$\frac{b-a}{r} - x \le \left|\sigma^{-1}(i) \cap \{a+1, \dots, b\}\right| \le \frac{b-a}{r} + x$$

and $|\sigma^{-1}(0)| \le x$.

Proposition 11
Assume that the vertices of H *are labelled* $1, \dots, n$ *with bandwidth at most* βn *and that* H *has an* $(r+1)$*-colouring that is* $(2/\xi, \beta)$*-zero free with respect to this labelling. Let* $\beta \le \xi^2/(100r)$. *Then there exists a proper* $(r+1)$*-colouring* $\sigma : V(H) \to \{0, \dots, r\}$ *that is* $(1/\xi, \beta)$*-zero free and* $4\xi n$*-balanced.*

Proof. The idea of the proof is to split H into small parts and use Proposition 10 to switch colours in the parts. This allows us to even out differences in the sizes of the colour classes and obtain a balanced colouring.

Define $\ell = 1/\xi$. Recall that the blocks $B_1, \ldots, B_{1/5r\beta}$ of H are the vertex sets of the form $B_t = \{(t-1)5r\beta n + 1, \ldots, t5r\beta n\}$.

We start by identifying so called *switching blocks*, which do not contain the colour 0 in the original colouring. They will be used to exchange the colours between parts of H with the help of Proposition 10, which will colour some vertices in the switching blocks with 0. We choose the switching blocks in such a way that every ℓ consecutive blocks contain at most one block which either has zeros (in the original colouring) or one switching block (but not both). As the ordering of H is $(2\ell, \beta)$-zero free this can be done so that every consecutive 3ℓ blocks contain at least one switching block. We next explain how to use the switching blocks.

Claim 12
Let $\sigma : [n] \to \{0, \ldots, r\}$ be a proper $(r+1)$-colouring of H, B_s a zero free block and π any permutation of $[r]$. Then there exists a proper $(r+1)$-colouring σ' of H with $\sigma'(v) = \sigma(v)$ for all $v \in \bigcup_{i<s} B_i$ and $\sigma'(v) = \pi(\sigma(v))$ for all $v \in \bigcup_{i>s} B_i$.

Indeed, every permutation of $[r]$ is the concatenation of at most r transpositions, i.e., permutations that exchange only two elements. We split the block B_s into r disjoint intervals of length $5\beta n$ and decompose π into at most r transpositions. The claim then follows from Proposition 10.

Let $\{s_1, s_2, \ldots, s_p\}$ be the set of indices belonging to switching blocks. For ease of notation let $s_0 = 0$ and let $s_{p+1} = 1/(5r\beta)+1$. Further let $B^*(t) := \bigcup_{i\le t}\left(\bigcup_{s_{i-1}<j<s_i} B_j\right)$, i.e., we take the union of all blocks up to the t-th switching block but exclude all switching blocks. Moreover we define $c_i(t) := |\{v \in B^*(t) : \sigma(v) = i\}|$ and $\widetilde{c}_i(t) := |\{v \in B^*(t+1) \setminus B^*(t) : \sigma(v) = i\}|$ for $t \in [p]$. We inductively construct a proper $(r+1)$-colouring of H with

$$\max_i\{c_i(t)\} \le \min_i\{c_i(t)\} + \xi n \tag{6}$$

for every $t \in [p+1]$.

Note that any proper colouring of H satisfies (6) for $t = 1$ as $|B^*(1)| \le 3\ell 5r\beta n \le \xi n$ because $s_1 \le 3\ell$. So let σ be a proper $(r+1)$-colouring which satisfies (6) for all $t' \le t$. Without loss of generality we assume that $c_1(t) \le c_2(t) \le \cdots \le c_r(t) \le c_1(t) + \xi n$. We define the switching for block s_t to be any permutation π which satisfies $\widetilde{c}_{\pi(r)}(t) + \xi n \ge \widetilde{c}_{\pi(1)}(t) \ge \widetilde{c}_{\pi(2)}(t) \ge \cdots \ge \widetilde{c}_{\pi(r-1)}(t) \ge \widetilde{c}_{\pi(r)}(t)$. Such a permutation exists as $|B^*(t+1) \setminus B^*(t)| \le \xi n$. We apply Claim 12 to σ, the block B_{s_t} and the permutation π and obtain a new proper $(r+1)$-colouring σ'. Let $c'_i(t) := |\{v \in B^*(t) : \sigma'(v) = i\}|$ and hence $c'_i(t+1) = c_i(t) + \widetilde{c}_{\pi(i)}(t)$. It follows from Proposition 9 that

$$\max_i\{c'_i(t+1)\} \le \min_i\{c'_i(t+1)\} + \xi n\,. \tag{7}$$

Therefore, the colouring σ' satisfies (6) for every $t' \le t+1$. Let σ^* be a colouring of H which satisfies (6) for every $t \le p+1$. Then σ^* is a proper $(r+1)$-colouring and (ℓ, β)-zero free by construction. It remains to show that σ^* is also $4\xi n$-balanced.

For this purpose consider any set $[a+1,b] := \{a+1,\dots,b\} \subseteq [n]$. Let t_a be the minimum $t \in [p+1]$ such that $a \in \bigcup_{i<s_t} B_i$ and t_b be the minimum $t \in [p+1]$ such that $b \in \bigcup_{i<s_t} B_i$. In other words either a is in the switching block $B_{s_{(t_a-1)}}$ or in $B^*(t_a) \setminus B^*(t_a-1)$. The definition of t_a, t_b guarantees that

$$\max\{|(B^*(t_b) \setminus B^*(t_a)) \setminus [a+1,b]|\,, |[a+1,b] \setminus (B^*(t_b) \setminus B^*(t_a))|\} \le \xi n\,. \tag{8}$$

Fix a colour $i \in [r]$ and let $C_i := (\sigma^*)^{-1}(i)$. It follows from (6) that

$$c_i(t) = |C_i \cap B^*(t)| = \frac{|B^*(t)|}{r} \pm \xi n$$

for every $t \in [p+1]$. This and the fact that $B^*(t_b) \supseteq B^*(t_a)$ imply that

$$|C_i \cap (B^*(t_b) \setminus B^*(t_a))| = \frac{|B^*(t_b) \setminus B^*(t_a)|}{r} \pm 2\xi n\,. \tag{9}$$

We conclude that

$$\begin{aligned} |C_i \cap [a+1,b]| &\overset{(8)}{=} |C_i \cap (B^*(t_b) \setminus B^*(t_a))| \pm \xi n \\ &\overset{(9)}{=} \frac{|B^*(t_b) \setminus B^*(t_a)|}{r} \pm 3\xi n \\ &\overset{(8)}{=} \frac{|[a+1,b]| \pm \xi n}{r} \pm 3\xi n = \frac{b-a}{r} \pm 4\xi n\,. \end{aligned}$$ □

With the help of Proposition 11 and an appropriate method for "cutting up" a graph H with a balanced colouring we can now construct the homomorphism asserted by Lemma 8.

Proof of Lemma 8. Given r, k and β, let ξ, H and $R^r_k \supseteq B^r_k \supseteq K^r_k$ be as required. Assume without loss of generality that the vertices of R^r_k are labelled $[k] \times [r]$ corresponding to this copy of B^r_k, that is, so that the edges of this copy of B^r_k are the edges specified in (5). Assume moreover that the vertices of H are labelled $1,\dots,n$ with bandwidth at most βn and that H has a $(100/\xi, \beta)$-zero free $(r+1)$-colouring with respect to this labelling. Let $B_1, \dots, B_{1/(5r\beta)}$ be the corresponding blocks of H. Set $\xi' = \xi/10$ and note that $\beta \le \xi^2/(10000r) = (\xi')^2/(100r)$. Therefore, by Proposition 11 with input β, ξ', and H, there is an $(1/\xi', \beta)$-zero free and $4\xi' n$-balanced colouring $\sigma : V(H) \to \{0,\dots,r\}$ of H.

Given an r-equitable integer partition $(m_{i,j})_{i\in[k],j\in[r]}$ of n, set $M_i := \sum_{j\in[r]} m_{i,j}$ for $i \in [k]$. Now choose indices $0 = t_0 \le t_1 \le \dots \le t_{k-1} \le t_k = 1/(5r\beta)$ such that B_{t_i} and B_{t_i+1} are zero free blocks and

$$\sum_{i'\le t_i} |B_{i'}| \le \sum_{i'\le i} M_{i'} < 12r\beta n + \sum_{i'\le t_i} |B_{i'}|\,. \tag{10}$$

Indeed, such t_i exist as σ is $(1/\xi', \beta)$-zero free and, in particular, two out of every three consecutive blocks are zero free. Furthermore, the t_i are distinct because $m_{i,j} \geq 12\beta n$. The last βn vertices of the blocks B_{t_i} and the first βn vertices of the blocks B_{t_i+1} will be called *boundary vertices* of H. Observe that the choice of the t_i implies that boundary vertices are never assigned colour 0 by σ.

Using σ, we will now construct $f : V(H) \to [k] \times [r]$ and $X \subseteq V(H)$. For each $i \in [k]$, and each $v \in \bigcup_{t_{i-1} < i' \leq t_i} B_{i'}$ we set

$$f(v) := \begin{cases} s_i & \text{if } \sigma(v) = 0, \\ (i, \sigma(v)) & \text{otherwise,} \end{cases}$$

where s_i is the vertex which exists by property *(R2*)*. Further let

$$\begin{aligned} X_1 &:= \bigcup_{v \in \sigma^{-1}(0)} \big(\{v\} \cup N_H(v)\big), \\ X_2 &:= \big\{v \in V(H) \,:\, v \text{ is a boundary vertex}\big\}. \end{aligned}$$

It remains to show that f and $X := X_1 \cup X_2$ satisfy properties *(H1)*–*(H4)* of Lemma 8.

Recall that there are $1/(5r\beta)$ many blocks in the $(1/\xi', \beta)$-zero free colouring σ. The bandwidth-ordering implies that all vertices from $X_1 \cup N(X_1)$ lie in blocks that either contain zeros or that are adjacent to blocks that contain zeros (because $|B_{i'}| \geq 5r\beta n$). Hence, at most $3\xi'/(5r\beta) + 3$ out of $1/(5r\beta)$ blocks contain vertices from $X_1 \cup N(X_1)$. Furthermore, every $W_{i,j} = f^{-1}(i,j)$ contains at most $2\beta n$ boundary vertices and at most $4\beta n$ vertices adjacent to boundary vertices. Thus

$$\begin{aligned} |X \cap W_{i,j}| \leq |X_1| + |X_2 \cap W_{i,j}| &\leq \left(\frac{3\xi'}{5r\beta} + 3\right) 5r\beta\, n + 2\beta n \\ &\leq \frac{4\xi'}{5r\beta} 5r\beta n + 2\beta n = \frac{4}{10}\xi n + 2\beta n \leq \xi n\,, \end{aligned}$$

and

$$|N(X) \cap W_{i,j}| \leq |N(X_1)| + |N(X_2) \cap W_{i,j}| \leq \left(\frac{3\xi'}{5r\beta} + 3\right) 5r\beta\, n + 4\beta n \leq \xi n$$

and property *(H1)* holds.

It follows from (10) that $M_i - 12r\beta n \leq |\bigcup_{t_{i-1} < i' \leq t_i} B_{i'}| \leq M_i + 12r\beta n$. As $(m_{i,j})_{i \in [k], j \in [r]}$ is an r-equitable integer partition of n and since σ is $4\xi' n$-balanced this implies

$$m_{i,j} - \xi n \leq \frac{M_i}{r} - 12\beta n - 4\xi' n \leq |f^{-1}(i,j)| \leq \frac{M_i}{r} + 12\beta n + 4\xi' n \leq m_{i,j} + \xi n$$

for every $j \in [r]$. Hence property *(H2)* is satisfied.

Let $\{u, v\} \in E(H) \setminus E(H[X])$ with $u \notin X$. Since vertices with colour 0 and their neighbours lie in X, we know that therefore $\sigma(u) \neq 0 \neq \sigma(v)$. Hence $f(u) = (i, \sigma(u))$

and $f(v) = (i', \sigma(v))$ for some $i, i' \in [r]$. If $i \neq i'$, u and v must both be boundary vertices, which contradicts $u \notin X$. Hence $i = i'$ and property *(H4)* follows.

Let $\{u, v\} \in E(H[X])$. As σ is a proper $(r+1)$-colouring, $\sigma(u) \neq \sigma(v)$. First assume that $\sigma(u) = 0$. Then there is an index $i \in [k]$ such that $f(u) = s_i$ and $f(v) = (i, \sigma(v))$. But $\{s_i, (i, \sigma(v))\} \in E(R_k^r)$ by condition *(R2*)* and so *(H3)* holds in this case. It remains to consider the case $\sigma(u) \neq 0 \neq \sigma(v)$. This implies that both u, v are boundary vertices or neighbours of vertices of colour 0. Moreover, u and v are of different colour. Since we started with an ordering of bandwidth at most βn we have $f(u) = (i, \sigma(u))$ and $f(v) = (i', \sigma(v))$ with $|i - i'| \leq 1$. Hence $\{f(u), f(v)\} \in E(B_k^r) \subseteq E(R_k^r)$ by condition *(R1*)* and so property *(H3)* also holds in this case. $\square$

4 A Blow-up Lemma for arrangeable graphs

In this section we provide a Blow-up Lemma type result which we shall apply to prove Theorem 3 and Theorem 6. This result builds on the following Blow-up Lemma for arrangeable graphs from [3].

Theorem 13 (Arrangeable Blow-up Lemma, full version [3])
For all $C, a, \Delta_R, \kappa \in \mathbb{N}$ and for all $\delta', c > 0$ there exist $\varepsilon', \alpha' > 0$ such that for every integer s there is n_0 such that the following is true for every $n \geq n_0$. Assume that we are given

(*i*) *a graph R on vertex set $[s]$ with $\Delta(R) < \Delta_R$,*

(*ii*) *an a-arrangeable n-vertex graph H with maximum degree $\Delta(H) \leq \sqrt{n}/\log n$, together with a partition $V(H) = W_1 \dot\cup \ldots \dot\cup W_s$ such that $uv \in E(H)$ implies $u \in W_i$ and $v \in W_j$ with $ij \in E(R)$,*

(*iii*) *a graph G with a partition $V(G) = V_1 \dot\cup \ldots \dot\cup V_s$ that is (ε', δ')-super-regular on R and has $|W_i| \leq |V_i| =: n_i$ and $n_i \leq \kappa \cdot n_j$ for all $i, j \in [s]$,*

(*iv*) *for every $i \in [s]$ a set $S_i \subseteq W_i$ of at most $|S_i| \leq \alpha' n_i$* image restricted vertices, *such that $|N_H(S_i) \cap W_j| \leq \alpha' n_j$ for all $ij \in E(R)$,*

(*v*) *and for every $i \in [s]$ a family $\mathcal{I}_i = \{I_{i,1}, \ldots, I_{i,C}\} \subseteq 2^{V_i}$ of permissible* image restrictions, *of size at least $|I_{i,j}| \geq cn_i$ each, together with a mapping $I \colon S_i \to \mathcal{I}_i$, which assigns a permissible image restriction to each image restricted vertex.*

Then there exists an embedding $\varphi \colon V(H) \to V(G)$ such that $\varphi(W_i) \subseteq V_i$ and $\varphi(x) \in I(x)$ for every $i \in [s]$ and every $x \in S_i$.

This theorem requires super-regularity for all pairs used in the embedding. However, in applications this can usually not be guaranteed: Lemma 7 for example provides a partition of G where we know only for very few regular pairs that they are also super-regular.

The standard approach to deal with a situation like this is to apply the Blow-up Lemma only *locally* to small groups of clusters where super-regularity is guaranteed (such as the K_r-copies within K_k^r in Lemma 7) and to use image restrictions to connect these local embeddings into an embedding of the whole graph H.

Instead, here we combine Theorem 13 with a randomisation step in order to obtain the following version of the Blow-up Lemma for arrangeable graphs that can handle super-regular pairs and merely regular pairs at once.

This result will allow us to embed a spanning graph H at once by imposing the additional restriction that edges which are embedded into pairs that are regular but not necessarily super-regular are confined to a small subpair in this pair (see *(H1)*).

Theorem 14 (Arrangeable Blow-up Lemma, mixed version)
For all a, Δ_R, κ and for all $\delta > 0$ there exist $\varepsilon, \alpha > 0$ such that for every s there is n_0 such that the following is true for every $n_1, \dots, n_s$ with $n_0 \le n = \sum n_i$ and $n_i \le \kappa \cdot n_j$ for all $i, j \in [s]$. Assume that we are given graphs R, R^* with $V(R) = [s]$, $\Delta(R) < \Delta_R$ and $R^* \subseteq R$, and graphs G, H on $V(G) = V_1 \dot\cup \dots \dot\cup V_s$, $V(H) = W_1 \dot\cup \dots \dot\cup W_s$ with

- *(G1) $|V_i| = n_i$ for every $i \in [s]$,*
- *(G2) $(V_i)_{i\in[s]}$ is (ε, δ)-regular on R, and*
- *(G3) $(V_i)_{i\in[s]}$ is (ε, δ)-super-regular on R^*.*

Further let H be a-arrangeable, $\Delta(H) \le \sqrt{n}/\log n$, and let there be a function $f : V(H) \to [s]$ and a set $X \subseteq V(H)$ with

- *(H1) $|X \cap W_i| \le \alpha n_i$ and $|N_H(X \cap W_i) \cap W_j| \le \alpha n_j$ for every $i \in [s]$ and every $ij \in E(R)$,*
- *(H2) $|W_i| \le n_i$ for every $i \in [s]$,*
- *(H3) for every edge $\{u, v\} \in E(H)$ we have $\{f(u), f(v)\} \in E(R)$,*
- *(H4) for every edge $\{u, v\} \in E(H) \setminus E(H[X])$ we have $\{f(u), f(v)\} \in E(R^*)$.*

Then $H \subseteq G$.

The idea of the proof is as follows. If $R = R^*$, that is, if all edges in R correspond to super-regular pairs in G, we are done by Theorem 13. In general of course this will not be the case. However, we will artificially create a situation like that: we carefully construct an auxiliary graph $G' \supseteq G$ which also has R as a reduced graph, but which has super-regular pairs for *all* edges in R. We then use Theorem 13 to embed H into G'. It will then remain to show that we constructed G' (and the image restrictions used in the application of Theorem 13) sufficiently carefully that this embedding in fact uses only edges from G.

As a preparatory step, the next proposition shows that by adding edges to vertices with insufficient degree we can make a regular pair super-regular.

Proposition 15
Let $\varepsilon, \delta > 0$ and let $G = (V_1 \dot\cup V_2, E)$ with $|V_1|, |V_2| \ge m$ be an ε-regular pair with density δ and set $T_i := \{v \in V_i : |N_G(v) \cap V_{3-i}| < \delta^* |V_{3-i}|\}$ with $\delta^* := \min\{\delta - \varepsilon, 1/2\}$.

Now assume that in a first round, for each $v \in T_1$ and $w \in V_2 \setminus N_G(v)$ we add the edge vw to G uniformly at random with probability $(\delta|V_2| - \deg_G(v))/(|V_2| - \deg_G(v))$ to obtain G'. Then, in a second round, for each $v \in T_2$ and $w \in V_1 \setminus N_{G'}(v)$ we add the edge vw to G' uniformly at random with probability $(\delta|V_1| - \deg_{G'}(v))/(|V_1| - \deg_{G'}(v))$ to obtain G''.

Then asymptotically almost surely (as m tends to infinity) the resulting graph G'' is $(4\varepsilon, \delta^*)$-super-regular.

Proof. We first prove that asymptotically almost surely $\deg_{G'}(v) \geq (\delta - \varepsilon)|V_2|$ for all $v \in T_1$. For $v \in T_1, w \in V_2 \setminus N_G(v)$ we define the Bernoulli variable

$$X_{v,w} := \begin{cases} 1 & \text{edge } (v,w) \text{ is added} \\ 0\,. \end{cases}$$

Then $\mathbb{P}[X_{v,w} = 1] = (\delta|V_2| - \deg_G(v))/(|V_2| - \deg_G(v))$ by construction. Setting $D_v := \sum_{w \in V_2 \setminus N_G(v)} X_{v,w}$ we have

$$\mathbb{E}[D_v] = |V_2 \setminus N_G(v)|(\delta|V_2| - \deg_G(v))/(|V_2| - \deg_G(v)) = \delta|V_2| - \deg_G(v)\,.$$

It follows from a Chernoff bound (Theorem 2.1 in [12]) that

$$\begin{aligned}\mathbb{P}\left[D_v < (\delta - \varepsilon)|V_2| - \deg_G(v)\right] &= \mathbb{P}\left[D_v < \mathbb{E}[D_v] - \varepsilon|V_2|\right] \\ &\leq \exp\left(-\frac{\varepsilon^2|V_2|^2}{2\,\mathbb{E}[D_v]}\right) \leq \exp\left(-\frac{\varepsilon^2}{2\delta}|V_2|\right)\,.\end{aligned}$$

Thus, asymptotically almost surely, $D_v \geq (\delta - \varepsilon)|V_2| - \deg_G(v)$ and, by taking a union bound over all choices of $v \in T_1$, we have $\deg_{G''}(v) \geq \deg_{G'}(v) \geq (\delta - \varepsilon)|V_2| \geq \delta^*|V_2|$ for all $v \in T_1$. A similar argument gives $\deg_{G''}(v) \geq (\delta - \varepsilon)|V_1| \geq \delta^*|V_1|$ for all $v \in T_2$ asymptotically almost surely. All vertices in $V_i \setminus T_i$ have $\deg(v) \geq \delta^*|V_{3-i}|$ by definition of T_i.

Finally we show that asymptotically almost surely G'' is 4ε-regular. For this, observe first that for $i = 1, 2$

$$\sum_{v \in T_i} (\delta|V_{3-i}| - \deg(v)) \leq 2\varepsilon^2|V_i||V_{3-i}|\,. \tag{11}$$

Indeed, denote by $\overline{T}_i \subseteq V_i$ the $\varepsilon|V_i|$ vertices of smallest degree in V_i. Observe that $T_i \subseteq \overline{T}_i$ and that $\deg(v) \leq (\delta + \varepsilon)|V_{3-i}|$ for all $v \in \overline{T}_i$. Since G is ε-regular we have $\sum_{v \in \overline{T}_i} \deg(v) = e(\overline{T}_i, V_{3-i}) \geq (\delta - \varepsilon)|\overline{T}_i||V_{3-i}| \geq \delta|\overline{T}_i||V_{3-i}| - \varepsilon^2|V_i||V_{3-i}|$. Hence

$$\begin{aligned}\sum_{v \in T_i} (\delta|V_{3-i}| - \deg(v)) &\leq \sum_{v \in T_i} (\delta|V_{3-i}| - \deg(v)) + \sum_{v \in \overline{T}_i \setminus T_i} (\delta|V_{3-i}| - \deg(v) + \varepsilon|V_{3-i}|) \\ &\leq \sum_{v \in \overline{T}_i} (\delta|V_{3-i}| - \deg(v)) + |\overline{T}_i \setminus T_i|\varepsilon|V_{3-i}| \leq \varepsilon^2|V_i||V_{3-i}| + \varepsilon^2|V_i||V_{3-i}|\,,\end{aligned}$$

which gives (11). Now let $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ with $|W_i| \geq 4\varepsilon|V_i|$ and denote the number of edges between W_1 and W_2 in $G' \setminus G$ by

$$D_{W_1,W_2} := \sum_{v \in T_1 \cap W_1, w \in W_2 \setminus N_G(v)} X_{v,w}\,.$$

Then

$$\begin{aligned}\mathbb{E}[D_{W_1,W_2}] &= \sum_{v\in T_1\cap W_1} |W_2\setminus N_G(v)|\frac{\delta|V_2|-\deg_G(v)}{|V_2|-\deg_G(v)}\\ &\le \sum_{v\in T_1} (\delta|V_2|-\deg_G(v))\,|\frac{|W_2|}{|V_2|-\frac{1}{2}|V_2|}\\ &\overset{(11)}{\le} 4\varepsilon^2|V_1||V_2|\frac{|W_2|}{|V_2|} \le \varepsilon|W_1||W_2|\end{aligned}$$

Using another Chernoff bound (Remark 2.5 in [12]) we obtain

$$\mathbb{P}\left[D_{W_1,W_2} > \mathbb{E}[D_{W_1,W_2}] + \frac{\varepsilon}{2}|W_1||W_2|\right] \le \exp\left(-\frac{2\varepsilon^2|W_1|^2|W_2|^2}{4|W_1|\,|W_2|}\right) \le \exp\left(-8\varepsilon^4|V_1||V_2|\right)\ .$$

As there are no more than $2^{|V_1|+|V_2|}$ choices for $W_1\subseteq V_1$, $W_2\subseteq V_2$ we apply a union bound to infer that $D_{W_1,W_2}\le\frac{3\varepsilon}{2}|W_1||W_2|$ holds asymptotically almost surely for all $W_1\subseteq V_1$, $W_2\subseteq V_2$. This in turn implies that

$$d_{G'}(W_1,W_2)-d_G(W_1,W_2) = \frac{D_{W_1,W_2}}{|W_1||W_2|} \le \frac{3\varepsilon|W_1||W_2|}{2|W_1||W_2|} = \frac{3\varepsilon}{2}$$

holds asymptotically almost surely for all $W_1\subseteq V_1$, $W_2\subseteq V_2$. Similarly,

$$d_{G''}(W_1,W_2)-d_{G'}(W_1,W_2) \le \frac{3\varepsilon}{2}$$

holds asymptotically almost surely for all $W_1\subseteq V_1$, $W_2\subseteq V_2$. As G is ε-regular, we conclude that G'' is 4ε-regular asymptotically almost surely. □

Proof of Theorem 14. Let a,Δ_R,κ and $\delta>0$ be given. Let $\varepsilon',\alpha'>0$ as in Theorem 13 with $C:=1,a,\Delta_R,\kappa,\delta':=\delta/2$, and $c:=1/2$ and set $\varepsilon:=\min\{\varepsilon'/4,1/(2\Delta_R),\delta/2\}$, $\alpha:=\alpha'$. Let s be given and choose n_0 as given by Theorem 13. Now let R,R^*,G,H have the required properties. In particular, let $V(G)=V_1\mathbin{\dot\cup}\dots\mathbin{\dot\cup}V_s$, $V(H)=W_1\mathbin{\dot\cup}\dots\mathbin{\dot\cup}W_s$ be partitions such that $(V_i)_{i\in[s]}$ is (ε,δ)-regular on R and (ε,δ)-super-regular on R^*.

For $i\in[s]$ define U_i to be the set of all vertices $v\in V_i$ with $|N_G(v)\cap V_j|\ge(\delta-\varepsilon)n_j$ for all $j\in N_R(i)$. Since $\Delta(R)<\Delta_R$ and all pairs (V_i,V_j) with $j\in N_R(i)$ are (ε,δ)-regular we have

$$|U_i| \ge |V_i| - \Delta_R\varepsilon|V_i| \ge \tfrac{1}{2}|V_i|\ . \tag{12}$$

In the next step we construct a graph G' which is super-regular on all pairs (V_i,V_j) with $ij\in E(R)$. For every $ij\in E(R)\setminus E(R^*)$ we apply Proposition 15 to $G[V_i\cup V_j]$. Let G' be the resulting graph. With positive probability, all pairs (V_i,V_j) with $ij\in E(R)$ are now $(4\varepsilon,\min\{\delta-\varepsilon,1/2\})$-super-regular in G'. Note that every $(4\varepsilon,\min\{\delta-\varepsilon,1/2\})$-super-regular pair is also $(4\varepsilon,\delta')$-super-regular as $\delta'=\delta/2\le\min\{\delta-\varepsilon,1/2\}$. In particular, there exists at least one graph G' with (V_i, V_j) being an $(4\varepsilon, \delta')$-super-regular pair in G' for every $ij \in E(R)$ and

$$G[V_i \cup V_j] = G'[V_i \cup V_j] \qquad \text{if } ij \in E(R^*), \tag{13}$$
$$G[U_i \cup U_j] = G'[U_i \cup U_j] \qquad \text{if } ij \in E(R). \tag{14}$$

As G' is (ε', δ')-super-regular for every $ij \in E(R)$ we have $H \subseteq G'$ by Theorem 13 even if, for every $i \in [s]$, we restrict the embedding of vertices in $S_i := W_i \cap X$ to $U_i \in \mathcal{I}_i := \{U_i\}$. This is possible by (12) and the fact that $|W_i \cap X| \leq \alpha n_i$ and $|N_H(W_i \cap X) \cap W_j| \leq \alpha n_j$ for all $i \in [s]$ and all $ij \in E(R)$.

Moreover, every $uv \in E(H) \cap W_i \times W_j$ with $ij \in E(R) \setminus E(R^*)$ has $u, v \in X$. Therefore, the embedding of H into G' also is an embedding of H into G by (13) and (14). $\square$

5 Proof of Theorem 3

Our strategy for this proof is as follows. We use the Lemma for G (Lemma 7) and the Lemma for H (Lemma 8) to get a partition of H and a matching regular partition of G which is (ε, δ)-(super-)regular wherever edges of H are to be embedded. Given these partitions, the Blow-up Lemma (Theorem 14) guarantees an embedding of H into G.

Proof of Theorem 3. We first set up the constants. Given r, a, $\gamma > 0$, let d, ε_0 be given by Lemma 7. Set $\Delta_R := 3r + 2/\gamma + 1$, $\kappa := 2$ and $\delta := d$ and let ε_{T14} and $0 < \alpha \leq 1$ be given by Theorem 14. Plug this $\varepsilon := \min\{\varepsilon_0, 1/4, \varepsilon_{\text{T14}}\}$ into Lemma 7 and obtain K_0, ξ_0. If necessary decrease ξ_0 such that $\xi_0 \leq \alpha/(2rK_0)$. Choose β, ξ such that $\xi \leq \xi_0$ and $\beta \leq \xi^2/(10000r)$. Finally for every $s \leq r \cdot K_0$ let $n_0 \geq K_0$ be sufficiently large for the application of Theorem 14.

Now let G be any graph on $n \geq n_0$ vertices with $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$. Then Lemma 7 returns a $k \leq K_0$ and a graph $\widetilde{R}^r_k$ on vertex set $[k] \times [r]$ and an r-equitable integer partition $(m_{i,j})_{i \in [k], j \in [r]}$ with properties *(R1)*–*(R3)*. In particular,

$$m_{i,j} \geq \frac{n}{2kr} \geq \frac{n}{2k} \frac{2K_0\xi_0}{\alpha} \geq \xi n \geq \sqrt{10000r\beta}n \geq 12\beta n$$

for all $i \in [k]$, $j \in [r]$.

With this integer partition we return to Lemma 8. Let H satisfy the conditions of Theorem 3, in particular H is r-colourable and has bandwidth at most βn. Hence, clearly there is a labelling of bandwidth at most βn with a $(100/\xi, \beta)$-zero free $(r+1)$-colouring. Furthermore, we need to show that there is a graph R^r_k with $B^r_k \subseteq R^r_k \subseteq \widetilde{R}^r_k$ which satisfies conditions *(R1*)* and *(R2*)* of Lemma 8 and additionally has $\Delta(R^r_k) < \Delta_R$. Indeed, R^r_k can be obtained as follows. Recall that $\delta(\widetilde{R}^r_k) \geq \left(\frac{r-1}{r} + \gamma/2\right)kr$ by property *(R2)*. Thus for every $i \in [k]$ there are at least $\frac{\gamma}{2}kr$ vertices $v \in ([k] \setminus \{i\}) \times [r]$ with $\{v, (i,j)\} \in E(\widetilde{R}^r_k)$ for all $j \in [r]$. We say that such a vertex v covers i. Now, consecutively choose for each $i = 1, \ldots, k$ a vertex $v_i \in [k] \times [r]$ among those vertices covering i which has been used as $v_{i'}$ as few times as possible for $i' < i$. Then the edges of R^r_k only consist of edges of

B_k^r in $\widetilde{R}_k^r$ and all edges $\{v_i, (i,j)\} \in E(\widetilde{R}_k^r)$. Since $\Delta(B_k^r) \leq 3r$ we have by the choice of the v_i that $\Delta(R_k^r) \leq 3r + 2/\gamma < \Delta_R$. Hence R_k^r satisfies conditions *(R1*)* and *(R2*)* of Lemma 8.

As r, k, β, ξ and R_k^r and the r-equitable integer partition $(m_{i,j})_{i\in[k],j\in[r]}$ satisfy the requirements of Lemma 8, we obtain a mapping $f : V(H) \to [k] \times [r]$ and a set X which satisfy conditions *(H1)*–*(H4)*. In the next step we will partition $V(G)$ into $(V_{i,j})_{i\in[k],j\in[r]}$. A vertex $x \in V(H)$ is then embedded into $V_{i,j} \subseteq V(G)$ if and only if $x \in f^{-1}(i,j)$.

Define $n_{i,j} := |f^{-1}(i,j)|$ and note that $m_{i,j} - \xi_0 n \leq n_{i,j} \leq m_{i,j} + \xi_0 n$ by property *(H2)*. Thus there exists a partition of $V(G)$ into $(V_{i,j})_{i\in[k],j\in[r]}$ with properties *(G1)*–*(G3)* by Lemma 7. Moreover, $n_{i,j} \leq 2n_{i',j'}$ for all $i, i' \in [k]$ and $j, j' \in [r]$ by property *(R3)* and property *(H2)* as

$$n_{i,j} \leq m_{i,j} + \xi_0 n \leq (1+\varepsilon)\frac{n}{kr} + \xi_0 n \leq 2\left((1-\varepsilon)\frac{n}{kr} - \xi_0 n\right) \leq 2\left(m_{i',j'} - \xi_0 n\right) \leq 2n_{i',j'}\,.$$

Now all conditions of Theorem 14 are satisfied and thus $H \subseteq G$. □

6 Proof of Theorem 6

The proof of Theorem 6 closely follows the methods of Allen, Brightwell and Skokan [1]. The restriction on $\Delta(H)$ in their result (Theorem 5) originates from the embedding result they use (Theorem 24 in [1], which follows from the proof of Theorem 1). This embedding result in turn relies on the Blow-up Lemma and the Lemma for H in [5]. The following Lemma 16 is a consequence of our Lemma for H (Lemma 8). We shall use this lemma together with the Blow-up Lemma for arrangeable graphs (Theorem 13) to extend the result of Allen, Brightwell and Skokan to arrangeable graphs.

We denote by P_m^r the r-th power of a path P_m, that is, P_m^r has vertex set $[m]$ and edge set $\{uv : |u - v| \leq r\}$. Analogously, C_m^r is the r-th power of the cycle C_m.

Lemma 16
For any $\xi > 0$ and for any natural numbers r, m_0 there exists $\beta > 0$ such that the following is true. Let H be a graph on n vertices that is r-colourable and has $\mathrm{bw}(H) \leq \beta n$. Then for any m with $2r \leq m \leq m_0$ there exists a homomorphism $f : H \to C_m^r$ with $|f^{-1}(i)| \leq \frac{n}{m}(1+\xi)$ for every $i \in [m]$.

Proof. Let $\xi > 0$ and r, m_0 be given. We choose k' sufficiently large so that $m_0/k' \leq \xi/3$ and so that $(k' + r - 1)/m$ is an integer for each $m \in [m_0]$. We set

$$\xi' := \frac{\xi}{3k'r} \qquad \text{and} \qquad \beta := \min\left(\frac{\xi'^2}{10000r}, \frac{\xi}{6k'r}\right).$$

Assume that H satisfies the requirements of the lemma. Observe that by the definition of β we can assume that the number of vertices n of H satisfies $n \geq 6k'r/\xi$ and hence

$$1 + \xi' n = \frac{n}{k'r}\left(\frac{k'r}{n} + k'r\xi'\right) = \frac{n}{k'r}\left(\frac{k'r}{n} + \frac{\xi}{3}\right) \leq \frac{n}{k'r} \cdot \frac{\xi}{2}\,. \tag{15}$$

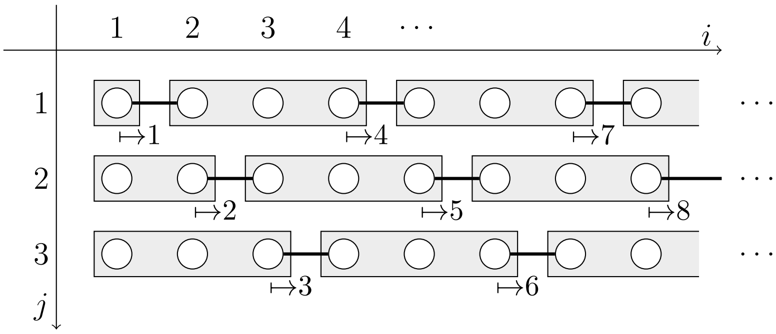

Figure 1: An illustration of f^* for $r = 3$. The white circle in column i and row j represents the set $f'^{-1}(i, j)$. The homomorphism f^* groups these sets as indicated. The thick horizontal edges indicate the additional edges $\{(i, j), s_i\}$ of the reduced graph $R^r_{k'}$. For example $f^*(4, 2) = 5$.

Let m with $2r \le m \le m_0$ be given.

We would now like to start by applying Lemma 8 with parameters r, k' and β, ξ'. For this purpose let $R^r_{k'}$ be the graph obtained from $B^r_{k'}$ (defined in the beginning of Section 3) by adding all edges of the form $\{(i, j), (i+1, j)\}$ where $i \in [k'-1]$ and $i-j \equiv 0 \mod r$ (see Figure 1). These additional edges ensure that for every $i \in [k']$ there is a vertex $s_i = (i + 1, i')$ or $s_i = (i - 1, i')$ (where $i' \in [r]$ satisfies $i - i' \equiv 0 \mod r$) such that $\{s_i, (i, j)\} \in E(R^r_{k'})$ for all $j \in [r]$. Hence the graph $R^r_{k'}$ satisfies conditions *(R1*)* and *(R2*)* of Lemma 8.

Furthermore let $\lfloor n/(k'r)\rfloor =: m_{1,1} \le m_{1,2} \le \cdots \le m_{k',r} := \lceil n/(k'r)\rceil$. Then Lemma 8 guarantees a mapping $f' : V(H) \to [k'] \times [r]$ and a set $X \subseteq V(H)$ with properties *(H1)*–*(H4)*. In the following we call each set $f'^{-1}(i, j)$ with $i \in [k']$, $j \in [r]$ an *f'-class* and use these classes to define a homomorphism $f : V(H) \to C^r_m$ with the properties promised by Lemma 16.

We will construct f in two further steps. Recall that $V(R^r_{k'}) = [k'] \times [r]$ and consider the r-th power of a path $P^r_{k'+r-1}$ on vertex set $V(P^r_{k'+r-1}) = [k' + r - 1]$. First we now define a mapping $f^* \colon [k'] \times [r] \to [k'+r-1]$ whose purpose is to group the f'-classes and which is a homomorphism from $R^r_{k'}$ to $P^r_{k'+r-1}$. Let $(i, j) \in [k'] \times [r]$. Observe that there are unique non-negative integers ℓ and x such that $x \in [r]$ and $i = -(r - j) + r \cdot \ell + x$. Then set $f^*(i, j) := r \cdot \ell + j$ (see also Figure 1). This guarantees for all $y \in [k'+r-1]$ that at most r pairs (i, j) are mapped to y, all of which have the same j-coordinate. In fact only the first and the last $r-1$ values y have less than r such pairs mapped to y, which we call the *exceptional preimages*. Moreover it is easy to verify that $|f^*(i, j) - f^*(i', j')| \le r$ whenever $|i - i'| \le 1$, that $f^*(i, j) = f^*(i', j')$ only if $j = j'$, and that $f^*(i, j) \ne f^*(s_i)$ for all $i, i' \in [k']$ and $j, j' \in [r]$. Hence f^* is a homomorphism from $R^r_{k'}$ to $P^r_{k'+r-1}$.

Our second step is to define the mapping $f^{**} \colon [k'+r-1] \to [m]$ by setting $f^{**}(y) := (y \mod m) + 1$ for all $y \in [k' + r - 1]$. Clearly f^{**} is a homomorphism from $P^r_{k'+r-1}$ to C^r_m. In conclusion, $f := f^{**} \circ f^* \circ f'$ is a homomorphism from H to C^r_m.

It remains to verify that also $|f^{-1}(i)| \le \frac{n}{m}(1 + \xi)$ for every $i \in [m]$. Indeed, by *(H2)* of Lemma 8 we have $|(f')^{-1}(i,j)| = m_{i,j} \pm \xi' n = \frac{n}{k'r} \pm 1 \pm \xi' n$ for all $i \in [k'], j \in [r]$. Moreover, by construction the preimages of f^* are all of size at most r and only $2(r-1)$ of these preimages, the exceptional preimages, are smaller than r. The preimages of f^{**} are all of the same size and f^{**} maps at most one vertex with exceptional preimage under f^* to each vertex of C^r_m. Thus, because $f^{**} \circ f^*$ is a mapping from $[k'] \times [r]$ to $[m]$, the preimages of $f^{**} \circ f^*$ are all of size $\frac{k'r}{m} \pm r$. Hence, in total for each $i \in [m]$ we have

$$|f^{-1}(i)| = \left(\frac{n}{k'r} \pm 1 \pm \xi' n\right) \cdot \left(\frac{k'r}{m} \pm r\right) = \left(\frac{n}{k'r} \pm 1 \pm \xi' n\right) \cdot \frac{k'r}{m}\left(1 \pm \frac{m}{k'}\right)$$
$$\overset{(15)}{=} \frac{n}{k'r}\left(1 \pm \frac{\xi}{2}\right) \cdot \frac{k'r}{m}\left(1 \pm \frac{\xi}{3}\right) = \frac{n}{m}(1 \pm \xi)\,,$$

where we used $m_{i,j} = \frac{n}{k'r} \pm 1$ in the second equality and $\frac{m}{k'} \le \frac{m_0}{k'} \le \frac{1}{3}\xi$ in the third. $\square$

For the proof of Theorem 6 we additionally need the following lemma, which is implicit in the proof of Theorem 5 that is given in [1]. Before we can state this lemma we need some further definitions.

Assume we are given a complete graph K_n whose edges are red/blue-coloured. Let A and B be disjoint vertex sets in K_n. Then (A,B) is a *coloured ε-regular* pair if (A,B) is an ε-regular pair in the subgraph of K_n formed by the red edges. It is easy to see that such a pair is also ε-regular in blue. A vertex partition $(V_i)_{i\in[s]}$ of $V(K_n)$ is called coloured ε-regular if all but at most $\varepsilon\binom{s}{2}$ of the pairs (V_i, V_j) with $\{i,j\} \in \binom{s}{2}$ are not coloured ε-regular. The *coloured reduced graph R* corresponding to this partition is the graph with vertex set $[s]$ and an edge for exactly each coloured ε-regular pair. Each edge ij of R is coloured in the majority-colour of the edges of (V_i, V_j). This clearly implies that if ij is a red edge of R, then the subgraph of (V_i, V_j) formed by the red edges is $(\varepsilon, \frac{1}{2})$-regular.

Lemma 17 (Implicit in [1][2])
For every $\varepsilon > 0$, r, and $\widetilde{m}$ there exists k_0 and n_0 such that the following is true for every $n \ge n_0$. Let the edges of K_n be red/blue-coloured.

(a) *The graph K_n has a coloured ε-regular partition $(V_i)_{i\in[k]}$ with $(2r+3)\widetilde{m} \le k \le k_0$ and $|V_1| \le |V_2| \le \ldots |V_k| \le |V_1| + 1$.*

Let R be the coloured reduced graph corresponding to this partition and let m be any multiple of $r+1$ with $k \ge (2r+3)m$.

(b) *The graph R contains a monochromatic copy of C^r_m.*

We now apply Lemma 17, Lemma 16 and Theorem 13 to derive the following result.

Theorem 18
Given $a \ge 1$, there exists n_0 and $\beta > 0$ such that, whenever $n \ge n_0$ and H is an a-arrangeable n-vertex graph with maximum degree at most $\sqrt{n}/\log n$ and $\mathrm{bw}(H) \le \beta n$, we have $R(H) \le (2\chi(H) + 4)n$.

[2] Lemma 17(b) is not used as such in [1]. However, the straightforward modification of the proof of [1, Theorem 11] that proves [1, Lemma 31] also gives Lemma 17(b).

Proof. The statement of Theorem 18 requires the definition of n_0 and β to be independent of H and its chromatic number. However, consider the following, seemingly weaker statement: *Given $a \geq 1$ and $r \in [a+1]$, there exists n_0 and $\beta > 0$ such that, whenever $n \geq n_0$ and H is an a-arrangeable, r-chromatic n-vertex graph with maximum degree at most $\sqrt{n}/\log n$ and $\mathrm{bw}(H) \leq \beta n$, we have $R(H) \leq (2r+4)n$.*

Since every a-arrangeable graph is $(a+1)$-colourable, we can infer Theorem 18 by applying the above statement for each value $r \in [a+1]$ and taking the maximum over the values n_0 and the minimum over the values β obtained.

So let a and $r \in [a+1]$ be given and set $\xi := 1/(100r)$. Choose ε' as given by Theorem 13 with $C := 0$, a, $\Delta_R := 2r+1$, $\kappa := 2$ and $\delta' := 1/4$, $c := 1$. Set $\varepsilon := \varepsilon'/2$. If necessary decrease ε such that $\varepsilon \leq \xi/(4r)$. Further set $\widetilde{m} := 100r^2$. Let n_0' and k_0 be as returned by Lemma 17 for these $\varepsilon, r, \widetilde{m}$. Then, for each $s \in [k_0]$ continue the application of Theorem 13 with s and obtain $n_0''(s)$. Set $m_0 := k_0$ and

$$n_0 := \max\left(\{100 m_0 r, n_0'\} \cup \{n_0''(s) \colon s \in [k_0]\}\right).$$

Let $\beta > 0$ be as given by Lemma 16 with parameters ξ, r, and m_0. Finally, let n and H with $r = \chi(H)$ be given, and assume we have a red/blue-colouring of the edges of $K_{(2r+4)n}$.

Lemma 17(*a*) asserts that there is a coloured ε-regular partition $(V_i')_{i\in[k]}$ of $K_{(2r+4)n}$ with $(2r+3)\widetilde{m} \leq k \leq k_0$ whose clusters differ in size by at most 1. Let R' be the coloured reduced graph of the partition $(V_i')_{i\in[k]}$. Let m be the multiple of $r+1$ which satisfies $(2r+3)m \leq k < (2r+3)(m+r+1)$. Observe that this and $k \geq (2r+3)\widetilde{m}$ implies $m \geq \widetilde{m} - r$ and thus

$$\frac{1}{2}m \geq \frac{1}{2}(\widetilde{m} - r) \geq 2r^2 + 5r + 3 \tag{16}$$

because $\widetilde{m} = 100r^2$. Further, $m \leq k \leq k_0 = m_0$ and so

$$\frac{m}{n} \leq \frac{m_0}{n_0} \leq \frac{1}{100r}. \tag{17}$$

We conclude that we have

$$\begin{aligned} |V_i'| &\geq \frac{(2r+4)n}{k} - 1 \geq \frac{(2r+4)n}{(2r+3)(m+r+1)} - 1 \overset{(16)}{\geq} \frac{(2r+4)n}{(2r+3.5)m} - 1 \\ &= \left(1 + \frac{0.5}{2r+3.5} - \frac{m}{n}\right)\frac{n}{m} \overset{(17)}{\geq} \left(1 + \frac{1}{20r} - \frac{1}{100r}\right)\frac{n}{m} \geq (1+2\xi)\frac{n}{m} \end{aligned}$$

because $\xi = 1/(100r)$. In addition, by Lemma 17(*b*) there is a monochromatic C_m^r in R', without loss of generality a red C_m^r. Let $U \subseteq V(K_{(2r+4)n})$ be the set of all vertices contained in clusters of this C_m^r.

Our next step is to apply Lemma 16 to the graph H with parameters ξ, r, m_0, β and m. This lemma guarantees a homomorphism $f \colon H \to C_m^r$ with $|f^{-1}(i)| \leq (1+\xi)\frac{n}{m}$ for every $i \in [m]$. By setting $W_i := f^{-1}(i)$ we obtain a partition $(W_i)_{i\in V(C_m^r)}$ of H.

We finish the proof with an application of Theorem 13. In this application we will not have image restricted vertices and we will use $R := C_m^r$. Observe that $\Delta(R) = 2r < \Delta_R$ and thus *(i)* of Theorem 13 is satisfied. The partition $(W_i)_{i\in V(C_m^r)}$ and the conditions on H guarantee that also condition *(ii)* of Theorem 13 is satisfied.

Now let G' be the subgraph of K_n with vertices U and all red edges of $K_{(2r+4)n}$ in U. In the following we consider this graph as an uncoloured graph. Clearly the partition $(V_i')_{i\in[k]}$ induces a partition $(V_i')_{i\in V(C_m^r)}$ of G' which is $(\varepsilon, \frac{1}{2})$-regular on C_m^r. Moreover, since C_m^r has maximum degree $2r$, by deleting from each of these clusters V_i' at most $2r\varepsilon|V_i'| \le \frac{1}{2}\xi|V_i'|$ vertices we can obtain a partition $(V_i)_{i\in V(C_m^r)}$ of a subgraph G of G' which is $(2\varepsilon, \frac{1}{4})$-super-regular on C_m^r and satisfies $|V_i| \ge (1+\xi)\frac{n}{m} \ge |W_i|$ (see, e.g., [5, Proposition 13]). Hence for G and $(V_i)_{i\in V(C_m^r)}$ also condition *(iii)* of Theorem 13 is satisfied.

Thus Theorem 13 implies that there is a copy of H in G. This copy corresponds to a red copy of H in the red/blue-coloured $K_{(2r+4)n}$. □

Proof Theorem 6. The statement follows now easily from Theorem 18 as graphs in $\mathcal{H}_S(n)$

- have arrangeability bounded by $(r(S)+1)^8$ by (4),
- almost surely have maximum degree at most $C(S)\log n$ by (3),
- almost surely have bandwidth $O(n\log\log n/\log n)$ by (2)
- and can be $r(S)$ coloured by (1). □

7 Concluding remarks

Optimality of Theorem 3. The degree bound $\Delta(H) \le \sqrt{n}/\log n$ in Theorem 3 arises from our proof method: For the Blow-up Lemma, Theorem 13, such a degree bound is necessary (see [3, Proposition 35]). For trees H, however, the corresponding result of Komlós, Sarközy, and Szemerédi [17] requires only the weaker condition $\Delta(H) = o(n/\log n)$. It is thus well possible that our maximum degree condition is not best possible and could be improved to $o(n/\log n)$.

Blow-up Lemmas. In the original formulation of the Blow-up Lemma [15, 16, 24] the regularity ε required for the super-regular pairs depends on the number of clusters k' used in an application. Consequently, this lemma can never be used on the *whole cluster graph* obtained from an application of the Regularity Lemma: the number of clusters k the Regularity Lemma produces depends on the required regularity ε. Moreover, all pairs used in the embedding have to be super-regular.

The Blow-up Lemma for arrangeable graphs formulated in [3] overcomes the first difficulty: Here ε only depends on the maximum degree of the reduced graph of the super-regular partition that is used. (In fact, fairly straight-forward modifications of the original Blow-up Lemma proof from [15] would also allow for a corresponding result for bounded degree graphs.)

In Theorem 14 we also overcome the second difficulty: Pairs into which we only want to embed few edges (concentrated on few vertices) are now allowed to be merely ε-regular. This allows us to avoid the occasionally tedious procedure of setting up suitable image restrictions and then applying the Blow-up Lemma several times. This might turn out could be useful for other applications as well.

Degeneracy. Though by now many important graph classes were shown to be a-arrangeable for some constant a, the notion of arrangeability has the disadvantage of seeming somewhat artificial at first sight. The notion of degeneracy is more natural (and more general): A graph H is d-degenerate if there is an ordering of its vertices such that each vertex has at most d neighbours to its left.

It would be very interesting to obtain an analogue of Theorem 3 for d-degenerate graphs. However, most likely this problem is very hard. Indeed, a version of the Blow-up Lemma for d-degenerate graphs would imply the difficult and long-standing Burr-Erdös conjecture [6], which states that degenerate graphs have linear Ramsey number.

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