What Is the Consumption-CAPM Missing? An Information-Theoretic Framework for the Analysis of Asset Pricing Models

Anisha Ghosh
Tepper School of Business, Carnegie Mellon University

Christian Julliard
London School of Economics and CEPR

Alex P. Taylor
Alliance Manchester Business School

We consider asset pricing models in which the SDF can be factorized into an observable component and a potentially unobservable one. Using a relative entropy minimization approach, we nonparametrically estimate the SDF and its components. Empirically, we find the SDF has a business-cycle pattern and significant correlations with market crashes and the Fama-French factors. Moreover, we derive novel bounds for the SDF that are tighter and have higher information content than existing ones. We show that commonly used consumption-based SDFs correlate poorly with the estimated one, require high risk aversion to satisfy the bounds and understate market crash risk. (*JEL* G11, G12, G13, C52)

The absence of arbitrage opportunities implies the existence of a pricing kernel, also known as the stochastic discount factor (SDF), such that the equilibrium price of a traded security can be represented as the conditional expectation of the future payoff discounted by the pricing kernel. The standard consumption-based asset pricing model, within the representative agent and time-separable power utility framework, identifies the pricing kernel as a simple parametric function of consumption growth. However, pricing kernels based

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on consumption growth alone cannot explain either the historically observed levels of returns, giving rise to the equity premium and risk-free rate puzzles (e.g., Mehra and Prescott 1985; Weil 1989), or the cross-sectional dispersion of returns between different classes of financial assets (e.g., Hansen and Singleton 1983; Mankiw and Shapiro 1986; Breeden et al. 1989; Campbell 1996).[1]

Nevertheless, considerable empirical evidence suggests that consumption risk does matter for explaining asset returns (e.g., Lettau and Ludvigson 2001a, 2001b; Parker and Julliard 2005; Hansen et al. 2008; Savov 2011). Therefore, a burgeoning literature has developed based on modifying the preferences of investors and/or the structure of the economy. In such models the resultant pricing kernel can be factorized into an observable component consisting of a parametric function of consumption growth, and a potentially unobservable, model-specific component. Prominent examples in this class include the external habit model in which the additional component consists of a function of the habit level (Campbell and Cochrane 1999; Menzly et al. 2004), the long run risks model based on recursive preferences in which the additional component consists of the return on total wealth (Bansal and Yaron 2004), and models with housing risk in which the additional component consists of the growth in the expenditure share on nonhousing consumption (Piazzesi et al. 2007). The additional and potentially unobserved component may also capture deviations from rational expectations (e.g., Brunnermeier and Julliard 2007), models with robust control (e.g., Hansen and Sargent 2010), heterogeneous agents (e.g., Constantinides and Duffie 1996), ambiguity aversion (e.g., Ulrich 2010), and a liquidity factor arising from solvency constraints (e.g., Lustig and Nieuwerburgh 2005).

In this paper, we propose a new methodology to analyze dynamic asset pricing models, such as those described above, for which the SDF can be factorized into an observable component and a potentially unobservable one. Our no-arbitrage approach allows us to (1) nonparametrically estimate, from the data, the time series of the unobserved pricing kernel under a set of asset pricing restrictions, (2) construct entropy bounds to assess the empirical plausibility of candidate SDFs, and (3) estimate, given a fully observable pricing kernel, the minimum (in the information sense) adjustment of the SDF needed to correctly price asset returns. This methodology provides useful diagnostics tools for studying the ways in which various models might fail empirically, and allows us to characterize some properties that a successful model must satisfy.

First, we show that, given a set of asset returns and consumption data, a relative entropy minimization approach can be used to nonparametrically extract the time series of both the SDF and the unobservable component of

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[1] Recently, Julliard and Ghosh (2012) show that pricing kernels based on consumption growth alone cannot explain either the equity premium puzzle, or the cross-section of asset returns, even after taking into account the possibility of rare disasters.

the SDF (if any). This method is equivalent to maximising the expected risk-neutral likelihood under a set of no arbitrage restrictions. Moreover, given a fully observable pricing kernel, this procedure identifies the *minimum* amount of extra information that needs to be added to the SDF to enable it to price asset returns correctly. Along this dimension, our paper is close in spirit to the long tradition of using asset (mostly options) prices to estimate the risk-neutral probability measure (see, e.g., Jackwerth and Rubinstein 1996; Ait-Sahalia and Lo 1998) and use this information to extract an implied pricing kernel (see, e.g., Ait-Sahalia and Lo 2000; Rosenberg and Engle 2002; Ross 2011).

Empirically, our estimated time series for the unobservable pricing kernel is substantially (but far from perfectly) correlated with the Fama and French (1993) factors, for a variety of sample frequencies and assets used in the estimation (even using only assets, like the industry and momentum portfolios, which are not well priced by the Fama-French factors).[2] This suggests that our approach successfully identifies the pricing kernel and provides a rationalization of the empirical success of the Fama and French factors. The estimated SDF has a clear business-cycle pattern but also shows significant and sharp reactions to stock market crashes (even if these crashes do not result in economy-wide contractions). Moreover, we show that, while the SDFs of most of the equilibrium models tend to adequately account for business-cycle risk, they nevertheless fail to show significant reactions to market crashes, and this hampers their ability to price asset returns – that is, all models seem to be missing a market crash risk component.

Second, we construct entropy bounds that restrict the admissible regions for the SDF and the unobservable component of the SDF. Our results complement and improve on the seminal work by Hansen and Jagannathan (1991) that provide minimum variance bounds for the SDF, and Hansen and Jagannathan (1997) (the so-called *second* Hansen-Jagannathan distance) that identifies the minimum variance (linear) modification of a candidate pricing kernel needed for it to be consistent with asset returns. The use of an entropy metric is also closely related to the works of (Stutzer 1995, 1996), who first suggested to construct entropy bounds based on asset pricing restrictions, and Alvarez and Jermann (2005), who derive a lower bound for the volatility of the permanent component of investors' marginal utility of wealth (see also Backus, Chernov, and Zin 2011; Bakshi and Chabi-Yo 2012; Kitamura and Stutzer 2002). We show that a second-order approximation of the risk-neutral entropy bounds (Q-bounds) has the canonical Hansen-Jagannathan bounds as a special case, but are generally tighter since they naturally impose the nonnegativity restriction on the pricing kernel. Using the multiplicative structure of the pricing kernel, we are able to provide novel bounds (M-bounds) that have higher information content, and are tighter, than both the Hansen and Jagannathan (1991) and the

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[2] This correlation ranges from 0.45 to 0.81 when Fama-French portfolios are used in the estimation of the minimum entropy SDF, but is reduced to a range of 0.43–0.70 when considering only Industry or Momentum portfolios.

risk-neutral entropy bounds. Moreover, our approach improves on Alvarez and Jermann (2005) in that a decomposition of the pricing kernel into permanent and transitory components is not required (but is still possible), and we can accommodate an asset space of arbitrary dimension.

Our methodology also can be used to construct bounds (Ψ-bounds) for the potentially unobserved component of the pricing kernel. We show that for models in which the pricing kernel is only a function of observable variables, the Ψ-bounds are the tightest ones, and can be satisfied if and only if the model is actually able to correctly price assets. Moreover, when the pricing kernel is fully observable, our Ψ-bounds are closely related to the second Hansen-Jagannathan distance: HJ identify the minimum variance *linear* adjustment, and our approach identifies the minimum entropy *multiplicative* (or log-linear) adjustment, that would make a candidate pricing kernel consistent with observed asset returns. The key difference between the two approaches is that the entropy one focuses not only on the second moment deviations, but also on all other higher moments. In an empirical example using stock return data, we find that these higher moments play an important role in driving about 22%–26% of the entropy of the estimated pricing kernel.

Third, we demonstrate how our methodology provides useful diagnostic tools to assess the plausibility of some of the most well-known consumption-based asset pricing models, and lends new insights into their empirical performance. For the standard time-separable power utility model, we show that the pricing kernel satisfies the Hansen and Jagannathan (1991) bound for large values of the risk-aversion coefficient and the Q and M bounds for even higher levels of risk aversion. However, the Ψ-bound is tighter and is not satisfied for any level of risk aversion. We show that these findings are robust to the use of the long-run consumption risk measure of Parker and Julliard (2005), despite that this measure of consumption risk is able to explain a substantial share of the cross-sectional variation in asset returns with a small risk-aversion coefficient. Considering more general models of dynamic economies, such as models with habit formation, long-run risks in consumption growth, and complementarities in consumption, we find that the SDFs implied by all of them (1) correlate poorly with the filtered SDF, (2) require implausibly high levels of risk-aversion to satisfy the entropy bounds, and (3) tend to understate market crash risk, in particular, the risk associated with market crashes that do not result in recessions. Moreover, the empirical application illustrates that inference based on the entropy bounds delivers results that are much more stable in evaluating the plausibility of a given model across different sets of assets and data frequencies than the cross-sectional R^2 (that, instead, tends to vary wildly for the same model).

Compared with the previous literature, our nonparametric approach offers five main advantages: (1) it can be used to extract information not only from options, as is common in the literature, but also from any type of financial asset, (2) instead of exclusively relying on the information contained in financial data,

it allow us to also exploit the information about the pricing kernel contained in the time series of aggregate consumption, thereby connecting our results to macrofinance modeling, (3) the relative entropy extraction of the SDF is akin to a nonparametric maximum likelihood procedure and provides an estimate of its time series, (4) the methodology has considerable generality, and may be applied to any model that delivers well-defined Euler equations and for which the SDF can be factorized into an observable component and an unobservable one (these include investment-based asset pricing models, and models with heterogenous agents, limited stock market participation, and fragile beliefs), (5) it relies not only on the second moment of the pricing kernel but also on all higher moments.

1. Entropy and the Pricing Kernel

In the absence of arbitrage opportunities, a strictly positive pricing kernel, M_{t+1}, or stochastic discount factor exists, such that the equilibrium price, P_{it}, of any asset i delivering a future payoff, X_{it+1}, is given by

$$P_{it} = \mathbb{E}_t[M_{t+1}X_{it+1}], \tag{1}$$

where $\mathbb{E}_t$ is the rational expectation operator conditional on the information available at time t. For a broad class of models, the SDF can be factorized as follows

$$M_t = m(\theta,t) \times \psi_t, \tag{2}$$

where $m(\theta,t)$ denotes the time t value of a known, strictly positive, function of observable data and the parameter vector $\theta \in \Theta \subseteq \mathbb{R}^k$ with true value θ_0, and ψ_t is a potentially unobservable component. In the most common case, $m(\theta,t)$ is simply a function of consumption growth, that is, $m(\theta,t) = m(\theta,\Delta c_t)$, where $\Delta c_t \equiv \log \frac{C_t}{C_{t-1}}$ and C_t denotes the time t consumption flow.

Equations (1) and (2) imply that, for any set of tradable assets, the following vector of Euler equations must hold in equilibrium

$$\mathbf{0} = \mathbb{E}\big[m(\theta,t)\psi_t \mathbf{R}_t^e\big] \equiv \int m(\theta,t)\psi_t \mathbf{R}_t^e \, dP, \tag{3}$$

where $\mathbb{E}$ is the unconditional rational expectation operator,[3] $\mathbf{R}_t^e \in \mathbb{R}^N$ is a vector of excess returns on different tradable assets, and P is the unconditional physical probability measure. Under weak regularity conditions the above pricing restrictions for the SDF can be rewritten as

$$\mathbf{0} = \int m(\theta,t)\frac{\psi_t}{\bar{\psi}} \mathbf{R}_t^e \, dP = \int m(\theta,t)\mathbf{R}_t^e \, d\Psi \equiv \mathbb{E}^\Psi\big[m(\theta,t)\mathbf{R}_t^e\big],$$

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[3] Our setting can accomodate departures from rational expectations as long as the objective and subjective probability measures are absolutely continuos (that is, as long as the two measures have the same zero probability sets). If agents had subjective beliefs of this type, Equation (3) would still hold, with $\mathbb{E}$ denoting rational expectations, but ψ_t would contain a change of measure element capturing the discrepancy between subjective beliefs and the rational expectations (see e.g., Hansen 2014).

where $\bar{x} \equiv \mathbb{E}[x_t]$, and $\frac{\psi_t}{\bar{\psi}} = \frac{d\Psi}{dP}$ is the Radon-Nikodym derivative of Ψ with respect to P. For the above change of measure to be legitimate, we need absolute continuity of the measures Ψ and P.

Therefore, given a set of consumption and asset returns data, for any θ, one can obtain a nonparametric maximum likelihood estimate of the Ψ probability measure as follows:

$$\Psi^*(\theta) \equiv \underset{\Psi}{\arg\min} \, D(\Psi \,\|\, P) \equiv \underset{\Psi}{\arg\min} \int \frac{d\Psi}{dP} \ln \frac{d\Psi}{dP} dP \ \text{ s.t. } \ \mathbf{0} = \mathbb{E}^{\Psi}\big[m(\theta,t)\mathbf{R}_t^e\big],$$
(4)

where, for any two absolutely continuous probability measures A and B, $D(A \,\|\, B) := \int \ln \frac{dA}{dB} dA \equiv \int \frac{dA}{dB} \ln \frac{dA}{dB} dB$ denotes the relative entropy of A with respect to B, that is, the Kullback-Leibler Information Criterion (KLIC) divergence between the measures A and B (White 1982). Note that $D(A \,\|\, B)$ is always nonnegative, and has a minimum at zero reached when A is identical to B. This divergence measures the additional information content of A relative to B, and, as pointed out by Robinson (1991), it is very sensitive to any deviation of one probability measure from another. Therefore, the above equation is a relative entropy minimization under the asset pricing restrictions coming from the Euler equations. That is, the minimization in Equation (4) estimates the unknown measure Ψ as the one that adds the minimum amount of additional information needed for the pricing kernel to price assets.

To understand the information-theoretic interpretation of the estimator of Ψ, let F be the set of all probability measures on $\mathbb{R}^{N+N'}$, where N' denotes the dimensionality of the observables in $m(\theta,t)$, and for each parameter vector $\theta \in \Theta$, define the following set of probability measures

$$\Psi(\theta) \equiv \big\{ \psi \in F : \mathbb{E}^{\psi}\big[m(\theta,t)\mathbf{R}_t^e\big] = \mathbf{0} \big\},$$

which are also absolutely continuous with respect to the physical measure P in Equation (3). If the observable component of the SDF, $m(\theta,t)$, correctly prices assets at the given value of θ, we have that $P \in \Psi(\theta)$, and P solves Equation (4), delivering a KLIC value of 0. On the other hand, if $m(\theta,t)$ is not sufficient to price assets, P is not an element of $\Psi(\theta)$, and a positive KLIC distance $D(\Psi \,\|\, P) > 0$ is attained by the solution $\Psi^*(\theta)$. Thus, the estimation approach searches for a $\Psi^*(\theta)$ that adds the minimum amount of additional information needed for the pricing kernel to price asset returns.

The above approach also can be used, as first suggested by Stutzer (1995), to recover the risk-neutral probability measure (Q) from the data as

$$Q^* \equiv \underset{Q}{\arg\min} \, D(Q \,\|\, P) \equiv \underset{Q}{\arg\min} \int \frac{dQ}{dP} \ln \frac{dQ}{dP} dP \ \text{ s.t. } \ \mathbf{0} = \int \mathbf{R}_t^e dQ \equiv \mathbb{E}^{Q}\big[\mathbf{R}_t^e\big]$$
(5)

under the restriction that Q and P are absolutely continuous.

The definition of relative entropy, or KLIC, implies that this discrepancy metric is not symmetric; that is, generally $D(A \,\|\, B) \neq D(B \,\|\, A)$ unless A and

B are identical (hence their divergence is always zero).[4] This implies that for measuring the information divergence between Ψ and P, as well as between Q and P, we can also invert the roles of Ψ and P in Equation (4) and the roles of Q and P in Equation (5) to recover Ψ and Q as

$$\Psi^*(\theta) \equiv \underset{\Psi}{\arg\min}\, D(P \| \Psi) \equiv \underset{\Psi}{\arg\min} \int \ln \frac{dP}{d\Psi} dP \quad \text{s.t.} \quad \mathbf{0} = \mathbb{E}^{\Psi}\left[m(\theta, t)\mathbf{R}_t^e \right], \quad (6)$$

$$Q^* \equiv \underset{Q}{\arg\min}\, D(P \| Q) \equiv \underset{Q}{\arg\min} \int \ln \frac{dP}{dQ} dP \quad \text{s.t.} \quad \mathbf{0} = \mathbb{E}^{Q}\left[\mathbf{R}_t^e \right]. \quad (7)$$

The divergence $D(P \| \Psi)$ can be thought of as the information loss from measure Ψ to measure P (and similarly for $D(P \| Q)$). This alternative approach, once again, chooses Ψ and Q such that assets are priced correctly and such that the estimated probability measures are as close as possible (that is minimizing the information loss of moving from one measure to the other) to the physical probability measure P.

Note that the approaches in Equations (4) and (6) identify $\{\psi_t\}_{t=1}^T$ only up to a positive scale constant. Nevertheless, this scaling constant can be recovered from the Euler equation for the risk-free asset (if one is willing to assume that such an asset is observable).

But why should relative entropy minimization be an appropriate criterion for recovering the unknown measures Ψ and Q? We make this choice for several reasons.

First, as formally shown in Appendix A.1, the KLIC minimizations in Equations (4)–(7) are equivalent to maximizing the (expected) [5] Q and Ψ nonparametric likelihood functions in an unbiased procedure for finding the pricing kernel or its ψ_t component. Note that this is also the rationale behind the *principle of maximum entropy* (see, e.g., Jaynes 1957b, 1957a) in physical sciences and Bayesian probability that states that, subject to known testable constraints – the asset pricing Euler restrictions in our case – the probability distribution that best represents our knowledge is the one with maximum entropy or minimum relative entropy in our notation.

Second, the use of relative entropy, due to the presence of the logarithm in the objective functions in Equations (4)–(7), naturally imposes the nonnegativity of the pricing kernel. This, for example, is not imposed in the identification of the minimum variance pricing kernel of Hansen and Jagannathan (1991).[6]

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[4] Information theory provides an intuitive way of understanding the asymmetry of the KLIC: $D(A \| B)$ can be thought of as the expected minimum amount of extra information bits necessary to encode samples generated from A when using a code based on B (rather than using a code based on A). Hence, generally $D(A \| B) \neq D(B \| A)$ since the latter, by the same logic, is the expected information gain necessary to encode a sample generated from B using a code based on A.

[5] With expectations under the physical measures proxied by their sample analogs

[6] Hansen and Jagannathan (1991) offer an alternative bound that imposes this restriction, but it is computationally cumbersome (the minimum variance portfolio is basically an option in this case). See also Hansen, Heaton, and Luttmer (1995).

Third, our approach to uncover the ψ_t component of the pricing kernel satisfies Occam's razor, or the law of parsimony, since it adds the *minimum amount of information* needed for the pricing kernel to price assets. This is due to the fact that the relative entropy is measured in units of information.

Fourth, it is straightforward to add conditioning information to construct a conditional version of the entropy bounds presented in the next section: given a vector of conditioning variables $\mathbf{Z}_{t-1}$, one simply has to multiply (element by element) the argument of the integral constraints in Equations (4), (5), (6), and (7) by the conditioning variables in $\mathbf{Z}_{t-1}$.

Fifth, there is no ex ante restriction on the number of assets that can be used in constructing ψ_t, and the approach can naturally handle assets with negative expected rates of return (cf. Alvarez and Jermann 2005).

Sixth, as implied by the work of Brown and Smith (1990), the use of entropy is desirable if we think that tail events are an important component of the risk measure.[7]

Finally, this approach is numerically simple when implemented via duality (see, e.g., Csiszar 1975). That is, when implementing the entropy minimization in Equation (4) each element of the series $\{\psi_t\}_{t=1}^{T}$ can be estimated, up to a positive constant scale factor, as

$$\psi_t^*(\theta) = \frac{e^{\lambda(\theta)'m(\theta,t)\mathbf{R}_t^e}}{\displaystyle\sum_{t=1}^{T} e^{\lambda(\theta)'m(\theta,t)\mathbf{R}_t^e}}, \quad \forall t, \tag{8}$$

where $\lambda(\theta) \in \mathbb{R}^N$ is the solution to the following unconstrained convex problem

$$\lambda(\theta) \equiv \arg\min_{\lambda} \frac{1}{T} \sum_{t=1}^{T} e^{\lambda'm(\theta,t)\mathbf{R}_t^e}, \tag{9}$$

and this last expression is the dual formulation of the entropy minimization problem in Equation (4).

Similarly, the entropy minimization in Equation (6) is solved by

$$\psi_t^*(\theta) = \frac{1}{T(1+\lambda(\theta)'m(\theta,t)\mathbf{R}_t^e)}, \quad \forall t, \tag{10}$$

where $\lambda(\theta) \in \mathbb{R}^N$ is the solution to

$$\lambda(\theta) \equiv \arg\min_{\lambda} -\sum_{t=1}^{T} \log(1+\lambda'm(\theta,t)\mathbf{R}_t^e), \tag{11}$$

and this last expression is the dual formulation of the entropy minimization problem in Equation (6).

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[7] Brown and Smith (1990) develop what they call "a Weak Law of Large Numbers for rare events;" that is, they show that the empirical distribution observed in a very large sample converges to the distribution that minimizes the relative entropy.

Note also that the above duality results imply that the number of free parameters available in estimating $\{\psi\}_{t=1}^T$ is equal to the dimension of (the Lagrange multiplier) λ; that is, it is simply equal to the number of assets considered in the Euler equation.

Moreover, since the $\lambda(\theta)$'s in Equations (9) and (11) are akin to extremum estimators (see, e.g., Hayashi 2000, Ch. 7), under standard regularity conditions (see, e.g., Amemiya 1985, Theorem 4.1.3), one can construct asymptotic confidence intervals for both $\{\psi_t\}_{t=1}^T$ and the entropy bounds presented in the next section.

To summarize, we estimate the ψ_t component of the SDF nonparametrically, using the relative entropy-minimizing procedures in Equations (4) and (6). The estimate $\{\psi_t^*(\theta)\}_{t=1}^T$ is then multiplied with the observable component $m(\theta, t)$ to obtain the overall SDF, $M_t^* = m(\theta, t)\psi_t^*(\theta)$. Since we have proposed two different relative entropy minimization approaches, we obtain two different estimates of the SDF given the data. Asymptotically, the two should be identical given the MLE property of these procedures. Nevertheless, in any finite sample they potentially could be very different. As shown in our empirical analysis, the two estimates are very close to each other, suggesting that their asymptotic behavior is well approximated in our sample.

1.1 Entropy bounds

Based on the relative entropy estimation of the pricing kernel and the component ψ outlined in the previous section, we now turn our attention to the derivation of a set of entropy bounds for the SDF, M, and the components of the SDF.

Dynamic equilibrium asset pricing models identify the SDF as a parametric function of variables determined by the consumers' preferences and the state variables driving the economy. Substantial research efforts have been devoted to developing diagnostic methods to assess the empirical plausibility of candidate SDFs, as well as to provide guidance for the construction and testing of other – more realistic – asset pricing theories.

The seminal work by Hansen and Jagannathan (1991) identifies, in a model-free no-arbitrage setting, a variance-minimizing benchmark SDF, $M_t^*(\bar{M})$, whose variance places a lower bound on the variances of other admissible SDFs:

Definition 1 (Canonical *HJ*-bound). For each $E[M_t] = \bar{M}$, the Hansen and Jagannathan (1991) minimum variance SDF is

$$M_t^*(\bar{M}) \equiv \underset{\{M_t(\bar{M})\}_{t=1}^T}{\arg\min} \sqrt{Var(M_t(\bar{M}))} \text{ s.t. } \mathbf{0} = \mathbb{E}\left[\mathbf{R}_t^e M_t(\bar{M})\right]. \qquad (12)$$

The solution to the above minimization is $M_t^*(\bar{M}) = \bar{M} + (\mathbf{R}_t^e - \mathbb{E}[\mathbf{R}_t^e])'\beta_{\bar{M}}$, where $\beta_{\bar{M}} = Cov(\mathbf{R}_t^e)^{-1}(-\bar{M}\mathbb{E}[\mathbf{R}_t^e])$, and any candidate stochastic discount factor M_t must satisfy $Var(M_t(\bar{M})) \geq Var(M_t^*(\bar{M}))$.

The *HJ*-bound offers a natural benchmark for evaluating the potential of an equilibrium asset pricing model since, by construction, any SDF consistent with observed data should have a variance not smaller than that of $M_t^*(\bar{M})$. However, the identified minimum variance SDF does not impose the nonnegativity constraint on the pricing kernel. In fact, since $M_t^*(\bar{M})$ is a linear function of returns, the restrction is not generally natisfied.[8]

As noticed in Stutzer (1995), using the Kullback-Leibler Information Criterion minimization in Equation (5), one can construct an entropy bound for the risk-neutral probability measure that naturally imposes the nonnegativity constraint on the pricing kernel. We generalize the idea of using an entropy minimization approach to construct risk-neutral bounds – Q-bounds – for the pricing kernel. For a given risk-neutral probability measure Q with Radon-Nikodym derivative $\frac{dQ}{dP} = \frac{M_t}{\bar{M}}$, we use $D(P \parallel Q)$ and $D(P \parallel \frac{M_t}{\bar{M}})$ interchangeably, that is, $D(P \parallel \frac{M_t}{\bar{M}}) \equiv D(P \parallel Q) \equiv \int \ln(\frac{dP}{dQ})dP \equiv -\int \ln(\frac{M_t}{\bar{M}})dP$. Similarly, $D(\frac{M_t}{\bar{M}} \parallel P) \equiv D(Q \parallel P) \equiv \int \ln(\frac{dQ}{dP})dQ \equiv \int \frac{dQ}{dP} \ln(\frac{dQ}{dP})dP \equiv \int \frac{M_t}{\bar{M}} \ln(\frac{M_t}{\bar{M}})dP$.

Definition 2 (Q-bounds). We define the following risk-neutral probability bounds for any candidate stochastic discount factor M_t:

1. Q1-bound:

$$D\left(P \parallel \frac{M_t}{\bar{M}}\right) \equiv \int -\ln \frac{M_t}{\bar{M}} dP \geqslant D(P \parallel Q^*),$$

   where Q^* solves Equation (7).

2. Q2-bound (Stutzer 1995):

$$D\left(\frac{M_t}{\bar{M}} \parallel P\right) \equiv \int \frac{M_t}{\bar{M}} \ln \frac{M_t}{\bar{M}} dP \geqslant D(Q^* \parallel P),$$

   where Q^* solves Equation (5).

These bounds, like the *HJ*-bound, use only the information contained in asset returns, but, differently from the latter, they impose the restriction that the pricing kernel must be positive. Moreover, under mild regularity conditions, we show that (see Remark 2 in Appendix A.2), to a second-order approximation, the problem of constructing canonical *HJ*-bounds and Q-bounds are equivalent, in the sense that approximated Q-bounds identify the minimum variance bound for the SDF.[9] The intuition behind this result is simple: (1) a second-order

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[8] We call the bound in Definition 1 the "canonical" *HJ*-bound since (Hansen and Jagannathan 1991, 1997) also provide an alternative bound, that imposes the non-negativity of the pricing kernel, but that is computationally more complex.

[9] The (sufficient, but not necessary) regularity conditions required for the approximation result are typically satisfied in consumption-based asset pricing models.

approximation of (the log of) a smooth pdf delivers an approximately Gaussian distribution (see, e.g., Schervish 1995), (2) the relative entropy of a Gaussian distribution is proportional to its variance, and (3) the *diffusion invariance principle* (see, e.g., Duffie 2005, Appendix D) implies that in the continuous time limit the (equivalent) change of measure does not change the volatility.

Both the *HJ* and *Q* bounds described above use only information about asset returns, but not information about consumption growth or the structure of the pricing kernel. Instead, we propose a novel approach that, while also imposing the nonnegativity of the pricing kernel, (1) takes into account more information about the form of the pricing kernel, therefore delivering sharper bounds, and (2) allows us to construct information bounds for both the pricing kernel as a whole and for its individual components.

Consider an SDF that, as in Equation (2), can be factorized into two components, that is, $M_t = m(\theta, t) \times \psi_t$, where $m(\theta, t)$ is a known nonnegative function of observable variables (generally consumption growth) and the parameter vector θ, and ψ_t is a potentially unobservable component. A large class of equilibrium asset pricing models, including ones with time-separable power utility with a constant coefficient of relative-risk aversion, external habit formation, recursive preferences, durable consumption goods, housing, and disappointment aversion fall into this framework. Based on the above factorization of the SDF, we can define the following bounds.

Definition 3 (M-bounds). For any candidate stochastic discount factor of the form in Equation (2), and given any choice of the parameter vector θ, we define the following bounds:

1. $M1$-bound:

$$D\left(P \parallel \frac{M_t}{\bar{M}}\right) \equiv \int -\ln \frac{M_t}{\bar{M}} dP \geqslant D\left(P \parallel \frac{m(\theta,t)\psi_t^*}{\overline{m(\theta,t)\psi_t^*}}\right)$$

$$\equiv \int -\ln \frac{m(\theta,t)\psi_t^*}{\overline{m(\theta,t)\psi_t^*}} dP,$$

   where ψ_t^* solves Equation (6) and $\overline{m(\theta,t)\psi_t^*} \equiv \mathbb{E}[m(\theta,t)\psi_t^*]$.

2. $M2$-bound:

$$D\left(\frac{M_t}{\bar{M}} \parallel P\right) \equiv \int \frac{M_t}{\bar{M}} \ln \frac{M_t}{\bar{M}} dP \geqslant D\left(\frac{m(\theta,t)\psi_t^*}{\overline{m(\theta,t)\psi_t^*}} \parallel P\right)$$

$$\equiv \int \frac{m(\theta,t)\psi_t^*}{\overline{m(\theta,t)\psi_t^*}} \ln \frac{m(\theta,t)\psi_t^*}{\overline{m(\theta,t)\psi_t^*}} dP,$$

   where ψ_t^* solves Equation (4).

The above bounds for the SDF are tighter than the Q-bounds since we have that (the minimum entropy risk-neutral probability measure is denoted Q^*)

$$D\left(P \parallel \frac{m(\theta,t)\psi_t^*}{m(\theta,t)\psi_t^*}\right) \geq D(P \parallel Q^*) \text{ and } D\left(\frac{m(\theta,t)\psi_t^*}{m(\theta,t)\psi_t^*} \parallel P\right) \geq D(Q^* \parallel P),$$
(13)

by construction, and are also more informative since not only is the information contained in asset returns used in their construction but also (1) the structure of the pricing kernel in Equation (2) and (2) the information contained in $m(\theta,t)$.

Information about the SDF also can be elicited by constructing bounds for the ψ_t component itself. Given the $m(\theta,t)$ component, these bounds identify the minimum amount of information that ψ_t should add for the pricing kernel M_t to be able to price asset returns.[10]

Definition 4 (Ψ-bounds). For any candidate stochastic discount factor of the form in Equation (2), and given any choice of the parameter vector θ, two lower bounds for the relative entropy of ψ_t are defined as

1. $\Psi 1$-bound:

$$D\left(P \parallel \frac{\psi_t}{\bar\psi}\right) \equiv -\int \ln \frac{\psi_t}{\bar\psi} dP \geqslant D\left(P \parallel \frac{\psi_t^*}{\bar\psi^*}\right),$$

  where ψ_t^* solves Equation (6);

2. $\Psi 2$-bound

$$D\left(\frac{\psi_t}{\bar\psi} \parallel P\right) \equiv \int \frac{\psi_t}{\bar\psi} \ln \frac{\psi_t}{\bar\psi} dP \geqslant D\left(\frac{\psi_t^*}{\bar\psi^*} \parallel P\right),$$

  where ψ_t^* solves Equation (4).

Besides providing an additional check for any candidate SDF, the Ψ-bounds are useful in that a simple comparison of $D\left(\frac{\psi_t^*}{\bar\psi^*} \parallel P\right)$, $D\left(\frac{m(\theta,t)}{m(\theta,t)} \parallel P\right)$, and $D(Q^* \parallel P)$ can provide a very informative decomposition in terms of the entropy contribution to the pricing kernel logically similar to the widely used variance decomposition analysis. For example, if $D\left(\frac{\psi_t^*}{\bar\psi^*} \parallel P\right)$ happens to be close to $D(Q^* \parallel P)$, while $D\left(\frac{m(\theta,t)}{m(\theta,t)} \parallel P\right)$ is substantially smaller, the decomposition implies that most of the ability of the candidate SDF to price assets comes from the ψ_t component.

Note also that, in principle, a volatility bound, similar to the Hansen and Jagannathan (1991) bound for the pricing kernel, can be constructed for the ψ_t

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[10] As for the Q and M bounds, we use $D(P \parallel \Psi)$ and $D(P \parallel \frac{\psi_t}{\bar\psi})$, as well as $D(\Psi \parallel P)$ and $D(\frac{\psi_t}{\bar\psi} \parallel P)$, interchangeably.

component. Such a bound, presented in Definition 5 of Appendix A.2, identifies a minimum variance $\psi_t^*(\bar{\psi}^*)$ component with standard deviation given by

$$\sigma_{\psi^*} = \bar{\psi}^* \sqrt{\mathbb{E}[\mathbf{R}_t^e m(\theta,t)]' \, Var(\mathbf{R}_t^e m(\theta,t))^{-1} \mathbb{E}[\mathbf{R}_t^e m(\theta,t)]}. \qquad (14)$$

This bound, as the entropy-based Ψ-bounds in Definition 4, uses information about the structure of the SDF but, differently from the latter, does not constrain ψ_t and M_t to be nonnegative as implied by economic theory. Moreover, using the same approach employed in Remark 2, this last bound can be obtained as a second-order approximation of the entropy-based Ψ-bounds in Definition 4.

Equation (14), viewed as a second-order approximation to the entropy Ψ-bounds, also makes clear why bounds based on the decomposition of the pricing kernel as $M_t = m(\theta,t)\psi_t$ offer sharper inference than do bounds based on only M_t. Consider, for example, the case in which the candidate SDF takes the form $M_t = m(\theta,t)$, that is, $\psi_t = 1$ for any t. In this case, a $\tilde{\theta}$ can easily exist such that

$$Var\left(M_t\left(\tilde{\theta}\right)\right) \equiv Var\left(m\left(\tilde{\theta},t\right)\right) \geq Var\left(M_t^*\left(\bar{M}\right)\right),$$

where $Var\left(M_t^*\left(\bar{M}\right)\right)$ is the Hansen and Jagannathan (1991) bound in Definition 1; that is, a $\tilde{\theta}$ exists such that the HJ-bound is satisfied. Nevertheless, the existence of such $\tilde{\theta}$ does not imply that the candidate SDF is able to price asset returns. This would be the case if and only if the volatility bound for ψ_t is also satisfied since, from Equation (14), we have that under the assumption of constant ψ_t the bound can be satisfied only if $\mathbb{E}\left[\mathbf{R}_t^e m(\theta_0,t)\right] \equiv \mathbb{E}\left[\mathbf{R}_t^e M_t(\theta_0)\right] = \mathbf{0}$, that is, only if the candidate SDF is able to price asset returns.

1.1.1 Residual ψ and the second Hansen-Jagannathan distance.
If we want to evaluate a model of the form $M_t = m(\theta,t)$ – that is, a model without an unobservable component – the Ψ-bounds will offer a tight selection criterion since, under the null of the model being true, we should have $D\left(\frac{\psi_t^*}{\bar{\psi}^*} \parallel P\right) = D\left(P \parallel \frac{\psi_t^*}{\bar{\psi}^*}\right) = 0$, and this is a tighter bound than the HJ, Q, and M bounds defined above. The intuition for this is simple: Q-bounds (and HJ-bounds) require the model under test to deliver at least as much relative entropy (variance) as the minimum relative entropy (variance) SDF, but they do not require that the $m(\theta,t)$ under scrutiny also should be able to price the assets. That is, it might be the case – as in practice we will show is the case – that for some values of θ both the Q-bounds and the HJ-bounds will be satisfied, but nevertheless the SDF grossly violates the pricing restrictions in the Euler Equation (3).

Note that when considering a model of the form $M_t = m(\theta,t)$, the estimated ψ^* component is a residual one – that is, it captures what is missed, for pricing assets correctly, by the pricing kernel under scrutiny. The residual ψ^* and the entropy bounds are also closely related to the *second* Hansen and Jagannathan bound. Given a model that identifies a SDF M, Hansen and Jagannathan (1997) assume that portfolio payoffs are elements of an Hilbert space and consider the

minimum squared deviation between M and a pricing kernel $q \in \mathcal{M}$ (or $\mathcal{M}^+$ if nonnegativity is imposed), where $\mathcal{M}$ denotes the set of all admissible SDFs. That is, the second HJ distance is defined as

$$d_{HJ}^2 := \min_{q \in \mathcal{M}} \mathbb{E}\big[(M_t - q_t)^2\big].$$

Note that $q \in \mathcal{M}$ can be rewritten as $q \in L^2$, which satisfies the pricing restriction (1), that is,

$$d_{HJ}^2 \equiv \min_{q \in L^2} \mathbb{E}\big[(M_t - q_t)^2\big] \ \text{ s.t. } \ \mathbf{0} = \mathbb{E}\big[q_t \mathbf{R}_t^e\big] \equiv \mathbb{E}^Q\big[\mathbf{R}_t^e\big].$$

Note that the constraint in the above formulation is the same one that we impose for constructing our entropy bounds.

In practice, the second HJ bound looks for the minimum – in a least squares sense – linear adjustment that makes $M_t - \lambda' \mathbf{R}_t^e$ an admissible SDF (where λ arises from the linear projection of M on the space of returns). This idea of minimum adjustment of the second HJ distance is strongly connected to our M and Ψ bounds and residual ψ.

Consider the decomposition $M_t = m(\theta, t)\psi_t$ in its extreme form: $M_t \equiv m(\theta, t)$; that is, the case in which the candidate SDF is fully observable and, under the null of the model under scrutiny, ψ^m (the model-implied ψ) should simply be a constant. In this case, we can estimate a *residual* $\{\psi_t^*\}_{t=1}^T$ that should be constant if the model is correct. In this case, the $M1$-bound defines the distance

$$d_{M1} = \min_{\{\psi_t\}_{t=1}^T} D(P \parallel M_t \psi_t) - D(P \parallel M_t) \equiv \min_{\{\psi_t\}_{t=1}^T} D(P \parallel \psi_t) \ \text{ s.t. } \ \mathbf{0} = \mathbb{E}\big[q_t \mathbf{R}_t^e\big],$$

where $q_t := M_t \psi_t$, and we have normalized ψ_t to have unit mean to simplify exposition, and note that the second equality is nothing but the $\Psi 1$ bound. Note that in this case we have $\log \psi_t \equiv \log q_t - \log M_t$. That is, while the second HJ distance focuses on the deviation between q and M, our entropy approach focuses on the log deviations. By construction, $M_t \psi_t^* \in \mathcal{M}$ (or $\mathcal{M}^+$ if M is nonnegative); that is, once again the relative entropy minimization identifies an admissible SDF in the Hansen and Jagannathan (1997) sense. To illustrate the link between the second HJ distance and the d_{M1} distance above, we follow the cumulant expansion approach of Backus, Chernov, and Zin (2014). Recall that the cumulant-generating function (that is, the log of the moment-generating function) of a random variable $\ln x_t$ is

$$k^x(s) = \ln \mathbb{E}\big[e^{s \ln x_t}\big],$$

and, with appropriate regularity conditions, it admits the power series expansion

$$k^x(s) = \sum_{j=1}^{\infty} \kappa_j^x \frac{s^j}{j!},$$

where the j-th cumulant, κ_j, is the j-th derivative of $k^x(s)$ evaluated at $s = 0$. That is, κ_j^x captures the j-th moment of the variable $\ln x_t$; that is, κ_1^x reflects

the mean of the variable, κ_2^x the variance, κ_3^x the skewness, κ_4^x the kurtosis, and so on.[11]

Using the cumulant expansion, the d_{M1} distance above can be rewritten as

$$d_{M1} = \frac{\kappa_2^{\psi^*}}{2!} + \frac{\kappa_3^{\psi^*}}{3!} + \frac{\kappa_4^{\psi^*}}{4!} + ..., \tag{15}$$

where $\kappa_j^{\psi^*}$ denotes the j-th cumulant of (log) ψ^*, and ψ^* solves

$$\underset{\{\psi_t\}_{t=1}^T}{\arg\min} \left(\frac{\kappa_2^\psi}{2!} + \frac{\kappa_3^\psi}{3!} + \frac{\kappa_4^\psi}{4!} + ... \right) \quad \text{s.t.} \quad \mathbf{0} = \mathbb{E}^\Psi\left[m(\theta,t)\mathbf{R}_t^e \right]. \tag{16}$$

The above implies that the ψ^* component identified by our $M1$ (and $\Psi1$) bound has a very similar interpretation to the second HJ distance: it provides the minimum – in the entropy sense – multiplicative (or log linear) adjustment that would make $m(\theta,t)\psi_t^*$ an admissible SDF. The key difference between the second HJ bound and our $M1$ bound is that the former only focuses on the minimum second moment deviation, that is, on the variance of $q_t - M_t$, and our bound takes into consideration not only the second moment (captured by the κ_2^ψ cumulant in Equation (15)) but also all other moments (captured by the $\kappa_{j>2}^\psi$ cumulants) of the log deviation $\log q_t - \log M_t \equiv \log \psi_t$. This implies that if skewness, kurtosis, tail probabilities, etc., are relevant for asset pricing, our approach would be more likely to capture these higher moments more effectively than the least squares one. Moreover, note that the cumulant-generating function cannot be a finite-order polynomial of degree greater than two (see Theorem 7.3.5 of Lukacs 1970). That is, if the mean and variance are not sufficient statistics for the distribution of the true SDF, then *all* the other higher moments become relevant for characterizing the SDF, and their relevance for asset pricing is captured by our entropy approach given the one-to-one mapping between relative entropy and cumulants. In Table A1 of Appendix A.3, we compute the minimum adjustment to the CCAPM SDF required to make it an admissible pricing kernel using both of the above approaches. The results show that, for a wide variety of test assets, the HJD adjustment leads to an SDF that has a close to Gaussian distribution. The relative entropy adjustment, on the other hand, results in an SDF having substantial skewness and kurtosis.

The cumulant decomposition also allows us to assess the relevance of higher moments for pricing asset returns. In particular, with the estimated $\{\ln \psi_t^*\}_{t=1}^T$ at hand, we can estimate its moments using sample analogs, use these moments to compute the cumulants, and, finally, compute the contribution of the j-th cumulant to the total entropy of ψ^* as

$$\frac{\kappa_j^{\psi^*}/j!}{\sum_{s=2}^\infty \kappa_s^{\psi^*}/s!} \equiv \frac{\kappa_j^{\psi^*}/j!}{D(P \| \Psi^*)}, \tag{17}$$

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[11] For instance, if $\ln x_t \sim N(\mu_x; \sigma_x^2)$, we have $\kappa_1^x = \mu_x$, $\kappa_2^x = \sigma_x^2$, $\kappa_{j>2}^x = 0$.

as well as the total contribution of cumulants of order larger than j as

$$\frac{\sum_{s=j+1}^{\infty} \kappa_s^{\psi^*}/s!}{\sum_{s=2}^{\infty} \kappa_s^{\psi^*}/s!} \equiv \frac{D(P \| \Psi^*) - \sum_{s=2}^{j} \kappa_s^{\psi^*}/s!}{D(P \| \Psi^*)}. \tag{18}$$

These statistics are important for comparing the informativeness of our bounds relative to the second HJ distance since, if the minimum variance deviation had all the relevant information for pricing asset returns, we would expect

$$\frac{D(P \| \Psi^*) - \kappa_2^{\psi^*}/2!}{D(P \| \Psi^*)} \cong 0 \text{ and } \frac{\kappa_j^{\psi^*}/j!}{D(P \| \Psi^*)} \cong 0 \quad \forall j > 2.$$

As we will show in the empirical section below, this is not the case.

2. An Illustrative Example: The C-CAPM with Power Utility

We first illustrate our methodology for the Consumption-CAPM (C-CAPM) of Breeden (1979), Lucas (1978), and Rubinstein (1976), when the utility function is time and state separable with a constant coefficient of relative-risk aversion. For this specification of preferences, the SDF takes the form

$$M_{t+1} = \delta(C_{t+1}/C_t)^{-\gamma}, \tag{19}$$

where δ denotes the subjective time discount factor, γ is the coefficient of relative-risk aversion, and C_{t+1}/C_t denotes the real per capita aggregate consumption growth. Empirically, the above pricing kernel fails to explain (1) the historically observed levels of returns, giving rise to the equity premium and risk-free rate Puzzles (e.g., Mehra and Prescott 1985; Weil 1989), and (2) the cross-sectional dispersion of returns between different classes of financial assets (e.g., Mankiw and Shapiro 1986; Breeden et al. 1989; Campbell 1996; Cochrane 1996).

Parker and Julliard (2005) argue that the covariance between contemporaneous consumption growth and asset returns understates the true consumption risk of the stock market if consumption is slow to respond to return innovations. They propose measuring the risk of an asset by its ultimate risk to consumption, defined as the covariance of its return and consumption growth over the period of the return and many following periods. They show that, while the ultimate consumption risk would correctly measure the risk of an asset if the C-CAPM were true, it may be a better measure of the true risk if consumption responds with a lag to changes in wealth. The ultimate consumption risk model implies the following SDF:

$$M_{t+1}^S = \delta^{1+S}(C_{t+1+S}/C_t)^{-\gamma} R_{t+1,t+1+S}^f, \tag{20}$$

where S denotes the number of periods over which the consumption risk is measured and $R_{t+1,t+1+S}^f$ is the risk-free rate between periods $t+1$ and $t+1+S$. Note that the standard C-CAPM obtains when $S=0$. Parker and Julliard (2005)
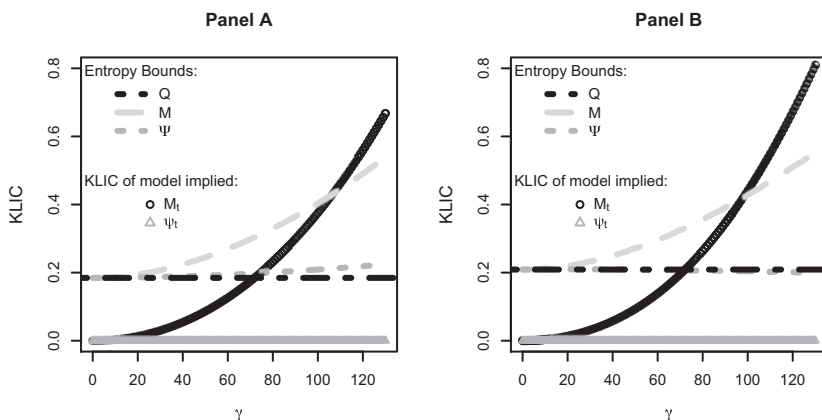
Panel A   **Panel B**

Figure 1
The figure plots the KLIC of the model SDF, $M_t = \delta \left(\frac{C_t}{C_{t-1}} \right)^{-\gamma}$, and the model ψ (equal to zero in this case), as well as the Q, M, and Ψ bounds as a function of the risk-aversion coefficient. The Q (M) bound is satisfied when the KLIC of M_t is above it, and the Ψ bound is satisfied when the KLIC of ψ_t is above it. Results when ψ_t^* is estimated using the relative entropy minimization procedures in Equations (6) and (4), respectively, using quarterly data for 1947:Q1-2009:Q4 and the 25 Fama-French portfolios as test assets are shown (*A* and *B*).

show that the specification of the SDF in Equation (20), unlike the one in Equation (19), explains a large fraction of the variation in expected returns across assets for low levels of the risk-aversion coefficient.

The functional forms of the above two SDFs fit into our framework in Equation (2). For the contemporaneous consumption risk model, $\theta = \gamma$, $m(\theta, t) = (C_t/C_{t-1})^{-\gamma}$, and $\psi_t^m = \delta$, a constant, for all t. For the ultimate consumption risk model, $\theta = \gamma$, $m(\theta, t) = (C_{t+S}/C_{t-1})^{-\gamma}$, and $\psi_t^m = \delta^{1+S} R_{t,t+S}^f$. Therefore, for each model, we construct entropy bounds for the SDF and the components of the SDF using quarterly data on per capita real personal consumption expenditures on nondurable goods and returns on the 25 Fama-French portfolios over the postwar period 1947:1-2009:4 and compare them with the *HJ* bound.[12, 13] We also obtain the nonparametrically extracted (called "filtered" hereafter) SDF and the components of the SDF for $\gamma = 10$. For the ultimate consumption risk model, we set $S = 11$ quarters because the fit of the model is the greatest at this value as shown in Parker and Julliard (2005).

Figure 1, panel A, plots the relative entropy (or KLIC) of the filtered and model-implied SDFs and their ψ components as a function of the risk-aversion coefficient γ and the *HJ*, $Q1$, $M1$, and $\Psi1$ bounds for the contemporaneous consumption risk model in Equation (19). The black curve with circles shows the relative entropy of the model-implied SDF as a function of the risk-aversion

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[12] See Appendix A.4 for a thorough description of the data.

[13] We use the 25 Fama-French portfolios as test assets because they have been used extensively in the literature to test the C-CAPM and also constituted the set of base assets in Parker and Julliard (2005).

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coefficient. For this model, the missing component of the SDF, ψ_t, is a constant, and hence, it has zero relative entropy for all values of γ, as shown by the gray straight line with triangles. The gray dashed curve and the gray dotted curve show, respectively, the relative entropy as a function of the risk-aversion coefficient of the filtered SDF and its missing component. The model satisfies the *HJ* bound for very high values of $\gamma \geqslant 64$. It satisfies the $Q1$ bound for even higher values of $\gamma \geqslant 72$, as shown by the intersection of the horizontal dotted-dashed line and the black curve with circles. The minimum value of γ, at which the $M1$ bound is satisfied, is given by the value corresponding to the intersection of the gray dashed curve and the black curve with circle; that is, it is the minimum value of γ for which the relative entropy of the model-implied SDF exceeds that of the filtered SDF. The figure shows that this corresponds to $\gamma = 107$. Finally, the $\Psi1$ bound identifies the minimum value of γ for which the missing component of the model-implied SDF has a higher relative entropy than the missing component of the filtered SDF. Since the former has zero relative entropy, while the latter has a strictly positive value for all values of γ, the model fails to satisfy the $\Psi1$ bound for any value of γ.[14]

Panel B shows that very similar results are obtained for the $Q2$, $M2$, and $\Psi2$ bounds. The $Q2$ and $M2$ bounds are satisfied for values of γ at least as large as 73 and 99, respectively, while the $\Psi2$ bound is not satisfied for any value of γ. Overall, as suggested by the theoretical predictions, the Q-bounds are tighter than the *HJ*-bound, the M-bounds are tighter than the Q-bounds, and the Ψ-bounds are tighter than the M-bounds.

We also construct confidence bands for the above relative entropy bounds using $1,000$ bootstrapped samples. The 95% confidence bands for the $Q1$ and $Q2$ bounds extend over the intervals $[70.0, 109.0]$ and $[69.5, 109.0]$, respectively, and those for the $M1$ and $M2$ bounds cover the intervals $[94.5, 157.5]$ and $[86.0, 150.0]$, respectively. Finally, the $\Psi1$ and $\Psi2$ bounds are not satisfied for any finite value of the risk-aversion coefficient in any of the bootstrapped samples. The bootstrap results reveal two points. First, it demonstrates the robustness of our approach: the two different definitions of relative entropy produce very similar results. Second, the confidence bands are quite tight in contrast with the large values of the standard error typically obtained when using GMM-type approaches to estimate the risk-aversion parameter.

Figure 2 presents analogous results to Figure 1 for the ultimate consumption risk model in Equation (20). Panel A shows that the $Q1$, and $M1$ bounds

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[14] Note that Figure 1 plots the relative entropy of the different components of the SDF as functions of the CRRA. The Q, M, and Ψ bounds are directly expressed in terms of the risk-aversion coefficient (vertical lines). The Q-bound could have been alternatively expressed in terms of entropy, that is, as a horizontal line at $D(Q^* \| P)$ and $D(P \| Q^*)$ in panels A and B, respectively. One could then have determined what the required minimum CRRA was to satisfy these bounds by computing the minimum CRRA such that the relative entropy of the resulting SDF was at least as large as $D(Q^* \| P)$ or $D(P \| Q^*)$. However, note that the M and Ψ bounds depend on the CRRA and, therefore, cannot be expressed as horizontal lines. We, therefore, choose to represent all the bounds directly in terms of the CRRA (as vertical lines).
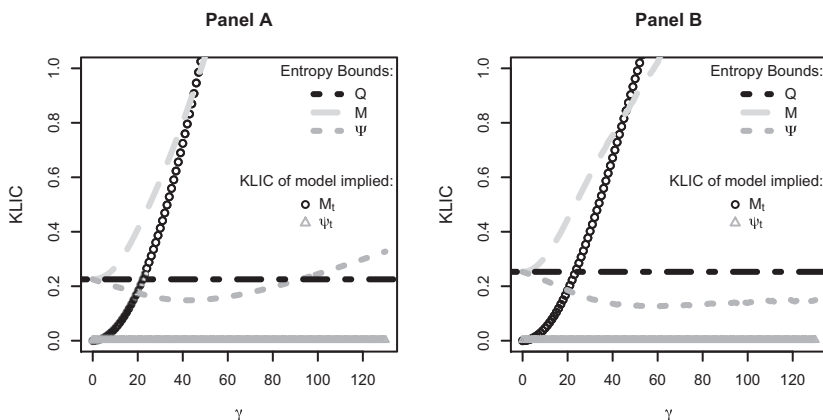
Figure 2

The figure plots the KLIC of the model SDF, $M_t = \delta^{1+S}\left(\frac{C_{t+S}}{C_{t-1}}\right)^{-\gamma} R_{t,t+S}^f$, and their unobservable components (ψ_t^* and the model ψ equal to zero in this case), as well as the Q, M and Ψ bounds as function of the risk-aversion coefficient. The Q (M) bound is satisfied when the KLIC of M_t is above it, and the Ψ bound is satisfied when the KLIC of ψ_t is above it. Results when ψ_t^* is estimated using the relative entropy minimization procedures in Equations (6) and (4), respectively, using quarterly data for 1947:Q1-2009:Q4 and the 25 Fama-French portfolios as test assets are shown (*A* and *B*).

are satisfied for $\gamma \geqslant 22$, 23, and 46, respectively. These are almost three times, more than three times, and more than two times smaller, respectively, than the corresponding values in Figure 1, panel A, for the contemporaneous consumption risk model. As for the latter model, the $\Psi 1$ bound is not satisfied for any value of γ. Panel B shows that the $Q2$ and $M2$ bounds are satisfied for $\gamma \geqslant 24$ and 47, respectively, while the $\Psi 2$ bound is not satisfied for any value of γ. The bootstrapped 95% confidence bands for the $Q1$ and $Q2$ bounds extend over the intervals $[23.0, 35.0]$ and $[24.0, 37.0]$, respectively, and those for the $M1$ and $M2$ bounds cover the intervals $[36.0, 60.0]$ and $[40.0, 74.0]$, respectively. Also, similar to the contemporaneous consumption risk model, the $\Psi 1$ and $\Psi 2$ bounds are not satisfied for any finite value of the risk-aversion coefficient in any of the bootstrapped samples.

It is important to notice that, even though the best fitting level for the RRA coefficient for the ultimate consumption risk model is smaller than 10 ($\hat{\gamma} = 1.5$), and at this value of the coefficient the model is able to explain about 60% of the cross-sectional variation in returns across the 25 Fama-French portfolios, all the bounds reject the model for low RRA, and the Ψ bounds are not satisfied for any level of RRA. This stresses the power of the proposed approach.

The above results indicate that our entropy bounds are not only theoretically tighter but also are empirically tighter than the HJ variance bounds. Using the cumulants decomposition introduced in the previous section, we can identify the information content added by taking into account higher moments of the
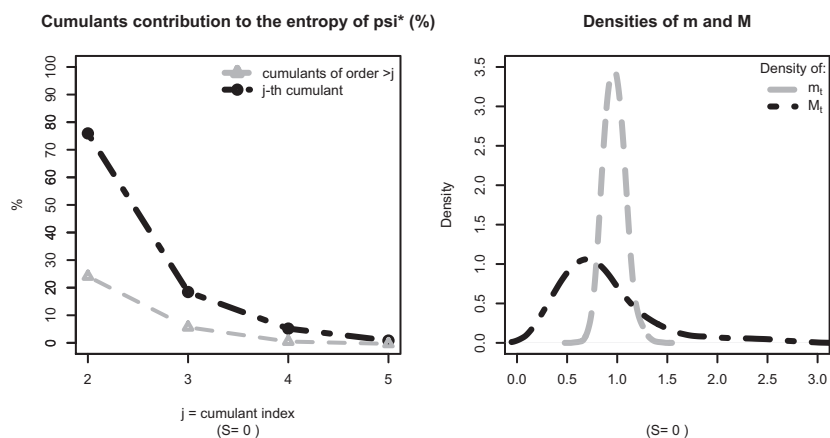
Figure 3
The relative contribution of the cumulants of ψ_t^* to $D(P \parallel \Psi^*)$ is plotted (A). The densities of $m_t := \left(\frac{C_t}{C_{t-1}}\right)^{-\gamma}$ and $M_t^* := \left(\frac{C_t}{C_{t-1}}\right)^{-\gamma} \psi_t^*$ are plotted (B). ψ_t^* is estimated using the relative entropy minimization procedure in Equation (6), using quarterly data for 1947:Q1-2009:Q4 and the 25 Fama-French portfolios as test assets, for the standard CCAPM with $\gamma = 10$.

SDF and the components of SDF. In particular, the statistics in Equations (17) (dashed-dotted line) and (18) (dashed line) are plotted in the left panels of Figures 3 (for $S = 0$) and 4 (for $S = 11$).

The figures show that the contribution of the second moment to $D(P \parallel \Psi^*)$ is large – in the 74%-78% range – but that higher moments also play a very important role, with their cumulated contribution in the 22% − 26% range. Among these higher moments, the lion's share goes to skewness, with individual contribution of skewness about 18% for both $S = 0$ and $S = 11$.

The relevance of skewness is also outlined in the right panels of Figures 3 (for $S = 0$) and 4 (for $S = 11$), where the (Epanechnikov kernel estimates of the) densities of $m_t := \left(\frac{C_{t+S}}{C_{t-1}}\right)^{-10} R_{t,t+S}^f$ and $M_t^* := \left(\frac{C_{t+S}}{C_{t-1}}\right)^{-10} R_{t,t+S}^f \psi_t^*$ are reported. The figures illustrate, besides the increase in variance generated by ψ^*, a substantial increase in the skewness of our estimated pricing kernel. This point is also outlined in Figures 5 (for $S = 0$) and 6 (for $S = 11$), where the left panels report the cumulant decomposition of the entropy of $m_t := \left(\frac{C_{t+s}}{C_{t-1}}\right)^{-10} R_{t,t+S}^f$, while the right panel reports the cumulant decomposition for $M_t^* := m_t \psi_t^*$. The figures show that the sources of entropy of our filtered pricing kernel ($m_t \psi_t^*$) are very different than the ones for the consumption growth component alone (m_t): almost all (99%) the entropy of m_t is generated by its second moment, while higher cumulants have basically no role; instead, about a quarter (24% − 25%) of the entropy of $m_t \psi_t^*$ is generated by the third and higher cumulants.

We now turn to the analysis of the time-series properties of the candidate SDFs considered. Figure 7, panel A, plots the time series of the filtered
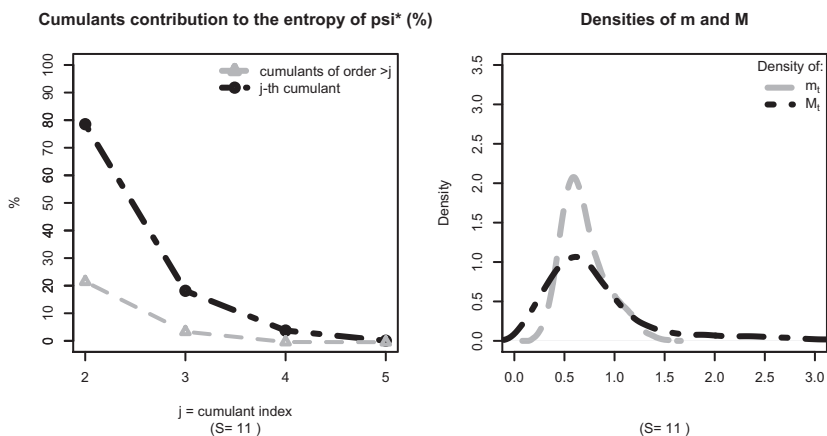
Figure 4
The relative contribution of the cumulants of ψ_t^* to $D(P \| \Psi^*)$ is plotted (A). The densities of $m_t :=$ $\left(\frac{C_{t+S}}{C_{t-1}}\right)^{-\gamma} R_{t,t+S}^f$ and $M_t^* := \left(\frac{C_{t+S}}{C_{t-1}}\right)^{-\gamma} R_{t,t+S}^f \psi_t^*$ are plotted (B). ψ_t^* is estimated using the relative entropy minimization procedure in Equation (6), using qarterly data for 1947:Q1-2009:Q4 and the 25 Fama-French portfolios as test assets, for the ultimate consumption risk CCAPM of Parker and Julliard (2005) with $S = 11$ and $\gamma = 10$.
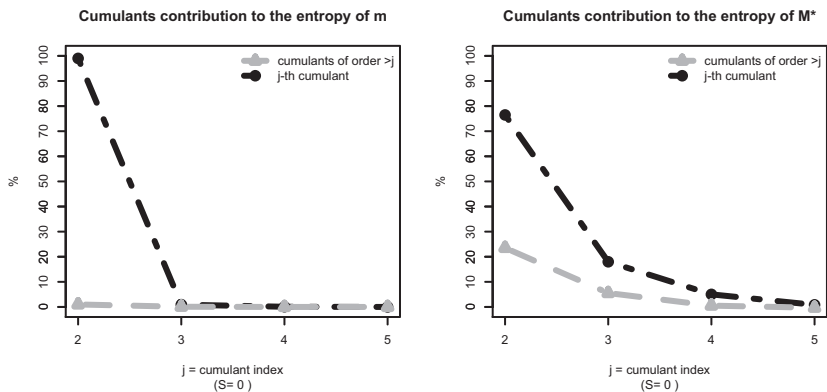
Figure 5
The contribution of the cumulants of $\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma}$ to $D\left(P \| \left(\frac{C_t}{C_{t-1}}\right)^{-\gamma}\right)$ is plotted (A). The contribution of

the cumulants of $\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma} \psi_t^*$ to $D\left(P \| \left(\frac{C_t}{C_{t-1}}\right)^{-\gamma} \psi_t^*\right)$ is plotted (B). ψ_t^* is estimated using the relative

entropy minimization procedure in Equation (6), using qarterly data for 1947:Q1-2009:Q4 and the 25 Fama-French portfolios as test assets, for the standard CCAPM with $\gamma = 10$.

SDF and the components of SDF estimated using Equation (6) for $\gamma = 10$ for the contemporaneous consumption risk model ($S = 0$). The dashed line plots the component of the SDF that is a parametric function of consumption

Cumulants contribution to the entropy of m **Cumulants contribution to the entropy of M***
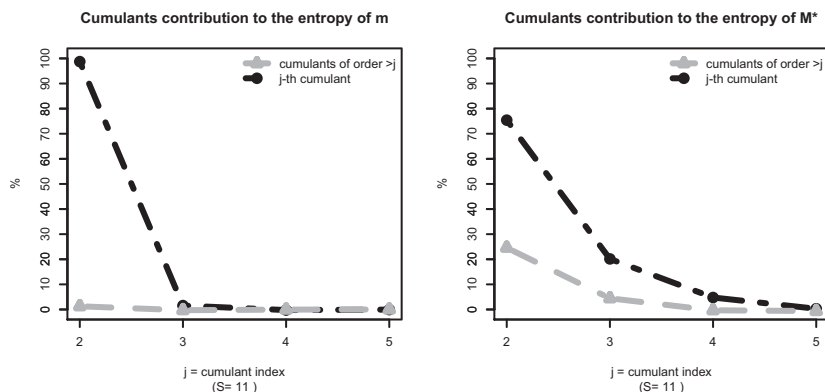
Figure 6

The contribution of the cumulants of $\left(\frac{C_{t+S}}{C_{t-1}}\right)^{-\gamma} R^f_{t,t+S}$ to $D\left(P \parallel \left(\frac{C_{t+S}}{C_{t-1}}\right)^{-\gamma} R^f_{t,t+S}\right)$ is plotted (*A*). The

cumulants of $\left(\frac{C_{t+S}}{C_{t-1}}\right)^{-\gamma} R^f_{t,t+S}\psi^*_t$ to $D\left(P \parallel \left(\frac{C_{t+S}}{C_{t-1}}\right)^{-\gamma} R^f_{t,t+S}\psi^*_t\right)$ is plotted (*B*). ψ^*_t is estimated using the

relative entropy minimization procedure in Equation (6), using qarterly data for 1947:Q1–2009:Q4 and the 25 Fama-French portfolios as test assets, for the ultimate consumption risk CCAPM of Parker and Julliard (2005) with $S = 11$ and $\gamma = 10$.

growth, $m(\theta,t) = (C_t/C_{t-1})^{-\gamma}$. The dotted line with circles plots the filtered unobservable component of the SDF, ψ^*_t, estimated using Equation (6). The black solid line plots the filtered SDF, $M^*_t = (C_t/C_{t-1})^{-\gamma}\psi^*_t$. The gray shaded areas represent NBER-dated recessions, and the dashed-dotted vertical lines correspond to the major stock market crashes identified in Mishkin and White (2002).[15] The figure reveals two main points. First, the estimated SDF has a clear business-cycle pattern, but also shows significant and sharp reactions to financial market crashes that do not result in economy-wide contractions. Second, the time series of the SDF almost coincides with that of the unobservable component. In fact, the correlation between the two time series is 0.996. The observable consumption growth component of the SDF, on the other hand, has a correlation of only 0.06 with the SDF. Therefore, most of the variation in the SDF comes from variation in the unobservable component, ψ, and not from the consumption growth component. In fact, the volatility of the SDF and its unobservable component are very similar, with the latter explaining about 99% of the volatility of the former, while the volatility of the consumption growth component accounts for only about 1% of the volatility of the filtered SDF. Similar results are obtained in panel B that plots the time series

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[15] Mishkin and White (2002) identify a stock market crash as a period in which the Dow Jones Industrial, the S&P 500, or the NASDAQ index drops by at least 20% in a time window of either one day, five days, one month, three months, or one year. Consequently, in yearly figures, we classify a given year as having a stock market crash if any such event was recorded in that year. Similarly, in quarterly figures, we identify a given quarter as being a crash period if either a crash was registered in that quarter or the entire year (containing the quarter) was identified by Mishkin and White as a stock market crash year.

Figure 7
The figure plots the (demeaned) time series of the filtered SDF, $M_t^* = m(\theta;t)\psi_t^*$, and its components for the standard CCAPM for $\gamma = 10$. Results when ψ_t^* is estimated using the relative entropy minimization procedures in Equations (6) and (4), respectively, using qarterly data for 1947:Q1-2009:Q4 and the 25 Fama-French portfolios as test assets are shown (*A* and *B*). Shaded areas are NBER recession periods. Vertical dot-dashed lines are the stock market crashes identified by Mishkin and White (2002).

of the filtered SDF and the components of SDF estimated using Equation (4) for $\gamma = 10$.

Finally, Figure 8, panel A, plots the time series of the filtered SDF and the components of SDF estimated using Equation (6) for $\gamma = 10$ for the ultimate consumption risk model ($S = 11$). The figure shows that, as in the contemporaneous consumption risk model, the estimated SDF has a clear business-cycle pattern, but also shows significant and sharp reactions to financial market crashes that do not result in economy-wide contractions. However, different from the latter model, the time series of the consumption growth component is much more volatile and more highly correlated with the SDF. The volatility of the consumption growth component is 21.7%, more than 2.5 times higher than that for the standard model. The correlation between the filtered SDF and the consumption growth component of the SDF is 0.37, an order of magnitude larger than the correlation of 0.06 in the contemporaneous consumption risk model. This explains the ability of the model to account for
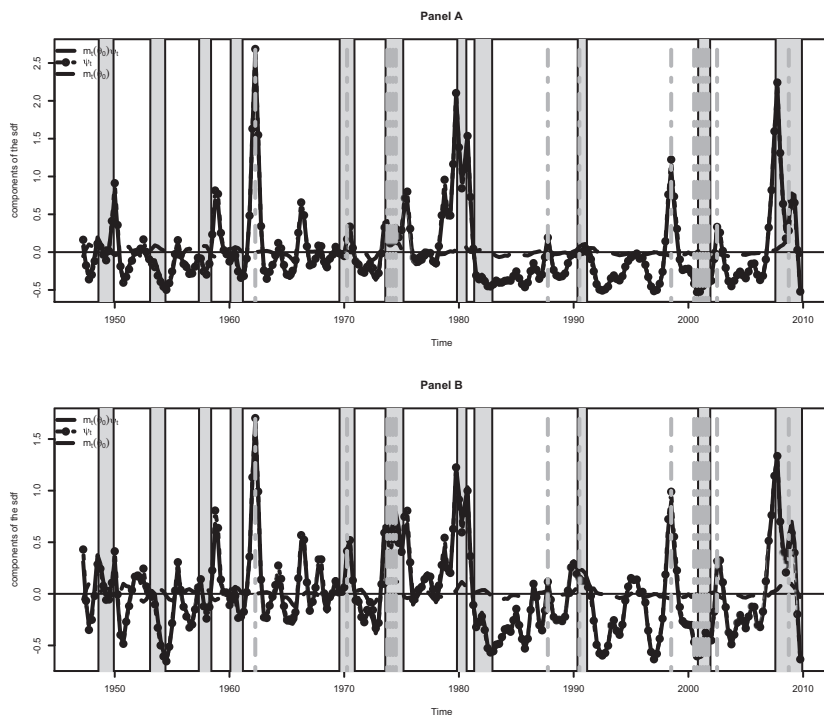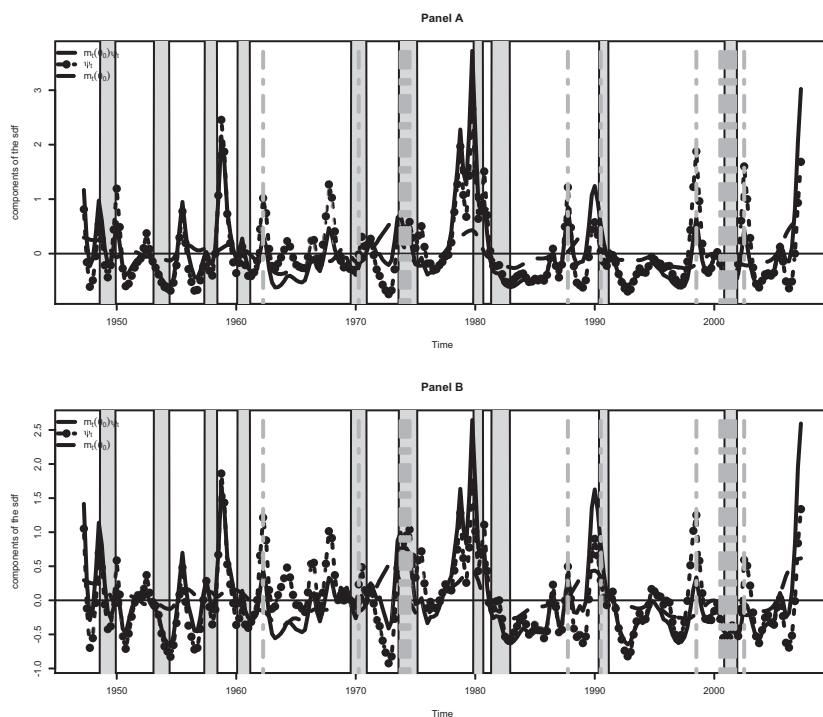
Figure 8
The figure plots the (demeaned) time series of the filtered SDF, $M_t^* = m(\theta; t)\psi_t^*$, and its components for the ultimate consumption risk CCAPM of Parker and Julliard (2005) for $\gamma = 10$. Results when ψ_t^* is estimated using the relative entropy minimization procedures in Equations (6) and (4), respectively, using qarterly data for 1947:Q1-2009:Q4 and the 25 Fama-French portfolios as test assets are shown (*A* and *B*). Shaded areas are NBER recession periods. Vertical dot-dashed lines are the stock market crashes identified by Mishkin and White (2002).

a much larger fraction of the variation in expected returns across the 25 Fama-French portfolios for low levels of the risk-aversion coefficient. In fact, the cross-sectional R^2 of the model is 54.1% (for $\gamma = 10$), an order of magnitude higher than the value of 5.2% for the standard model. However, the correlation between the ultimate consumption risk SDF and the unobservable component of the SDF is still very high at 0.92, showing that the model is missing important elements that would further improve its ability to explain the cross-section of returns. Similar results are obtained in panel B that plots the time series of the filtered SDF and the components of the SDF estimated using Equation (4) for $\gamma = 10$.

Overall, the results show that our methodology provides useful diagnostics for dynamic asset pricing models. Moreover, the very similar results obtained using the two different types of relative entropy minimization in Equations (4) and (6) suggest robustness of our approach.

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3. Application to More General Models of Dynamic Economies

Our methodology provides useful diagnostics to assess the empirical plausibility of a large class of consumption-based asset pricing models in which the SDF, M_t, can be factorized into an observable component consisting of a parametric function of consumption, C_t, as in the standard time-separable power utility model and a potentially unobservable one, ψ_t, that is model-specific. In this section, we apply it to a set of "winner" asset pricing models, that is, frameworks that can successfully explain the equity premium and the risk-free rate puzzles with "reasonable" calibrations. In particular, we consider the external habit formation models of Campbell and Cochrane (1999) and Menzly et al. (2004), the long-run risks model of Bansal and Yaron (2004), and the housing model of Piazzesi et al. (2007). We apply our methodology to assess the empirical plausibility of these models in two ways. First, since our approach delivers an estimate of the time series of the SDF, for each model considered we compare the estimated time series with the model-implied one. Second, for each model we compute the values of the power coefficient, γ, at which the model-implied SDF satisfies the HJ, Q, M, and Ψ bounds.

In the next subsection we present the models considered. The reader familiar with these models can go directly to Section 3.2, which reports the empirical results, without loss of continuity. A detailed data description is presented in Appendix A.4.

3.1 The models considered

3.1.1 External habit formation model: Campbell and Cochrane (1999).　In this model, identical agents maximize power utility, defined over the difference between consumption and a slow-moving habit or time-varying subsistence level. The SDF is given by

$$M_t^m = \underbrace{(C_t/C_{t-1})^{-\gamma}}_{m(\theta,t)} \underbrace{\delta (S_t/S_{t-1})^{-\gamma}}_{\psi_t^m}, \tag{21}$$

where δ is the subjective time discount factor, γ is the curvature parameter that provides a lower bound on the time-varying coefficient of relative-risk aversion, $S_t = \frac{C_t - X_t}{C_t}$ denotes the surplus consumption ratio, and X_t is the habit component. Note that the ψ^m component depends on the surplus consumption ratio, S, not directly observed. To obtain the time series of ψ^m, we extract the surplus consumption ratio from observed data using two different procedures.

First, we extract the time series of the surplus consumption ratio from consumption data. In this model, the aggregate consumption growth is assumed to follow an $i.i.d.$ process:

$$\Delta c_t = g + \upsilon_t, \quad \upsilon_t \sim i.i.d.N\left(0, \sigma^2\right).$$

The log surplus consumption ratio evolves as a heteroscedastic $AR(1)$ process:

$$s_t = (1 - \phi)\bar{s} + \phi s_{t-1} + \lambda(s_{t-1})\upsilon_t, \tag{22}$$

where $s_t := \ln S_t$ and $\bar{s}$ is the steady-state log surplus consumption ratio and

$$\lambda(s_t) = \begin{cases} \frac{1}{\bar{S}}\sqrt{1 - 2(s_t - \bar{s})} - 1, & \text{if } s_t \leq s_{max} \\ 0, & \text{if } s_t > s_{max} \end{cases},$$

$$s_{max} = \bar{s} + \frac{1}{2}\left(1 - \bar{S}^2\right), \quad \bar{S} = \sigma\sqrt{\frac{\gamma}{1 - \phi}}.$$

For each value of γ, we use the calibrated values of the model preference parameters (δ, ϕ) in Campbell and Cochrane (1999), the sample mean (g) and volatility (σ) of the consumption growth process, and the innovations in real consumption growth, $\widehat{\upsilon}_t = \Delta c_t - g$, to extract the time series of the surplus consumption ratio using Equation (22) and, thereby, obtain the time series of the model-implied SDF and its ψ^m component.

Second, in this model, the equilibrium market-wide price-dividend ratio is a function of the surplus consumption ratio alone, although the form of the function is not available in closed form. Using numerical methods, we invert this function to extract the time series of the surplus consumption ratio from the historical time series of the price-dividend ratio and thereby obtain the time series of the model-implied SDF and its ψ^m component from Equation (21).

3.1.2 External habit formation model: Menzly et al. (2004).

In this model, the SDF is analogous to the Campbell and Cochrane (1999) one disussed above. The aggregate consumption growth is also assumed to follow an *i.i.d.* process:

$$dc_t = \mu_c dt + \sigma_c dB_t,$$

where μ_c is the mean consumption growth, $\sigma_c > 0$ is a scalar, and B_t is a Brownian motion. The point of departure from the Campbell and Cochrane (1999) framework is that Menzly et al. (2004) assume that the *inverse* surplus consumption ratio, $Y_t := \frac{1}{S_t}$, follows a mean-reverting process perfectly negatively correlated with innovations in consumption growth:

$$dY_t = k\left(\bar{Y} - Y_t\right)dt - \alpha(Y_t - \lambda)[dc_t - E(dc_t)], \tag{23}$$

where $\bar{Y}$ is the long-run mean of the inverse surplus consumption ratio and k controls the speed of mean reversion. To obtain the time series of ψ^m (the model-implied ψ component), we extract the surplus consumption ratio from observed data using two different procedures.

First, for each value of γ,[16] we use the calibrated values of the model parameters $(\delta, k, \bar{Y}, \alpha, \lambda)$ in Menzly et al. (2004), the sample values of μ_c

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[16] Note that the Menzly et al. (2004) model assumes that the representative agent has log utility, that is, γ is set equal to 1, to derive the closed-form solution for the price-consumption ratio. For other values of γ, the model does not admit a closed-form solution. Nevertheless, the pricing kernel is well defined even if γ is different than one; hence we will be considering this more general case.

and σ_c, and the innovations in real consumption growth, $\widehat{dB_t} = \frac{[dc_t - E(dc_t)]}{\sigma_c}$, to extract the time series of the surplus consumption ratio, and this allows us to compute the time series of the model-implied SDF.

Second, in this model, the equilibrium price-consumption ratio of the total wealth portfolio is a function of the surplus consumption ratio alone. However, this function is not available in closed form except for $\gamma = 1$. Therefore, we rely on log-linear approximations to the return on the total wealth portfolio to express the equilibrium log price-consumption ratio as an affine function of the log surplus consumption ratio for all values of γ. Details of this procedure are described in Appendix A.5. We then invert this affine function to extract the time series of the surplus consumption ratio from the historical time series of the market-wide price-dividend ratio and thereby obtain the time series of the model-implied SDF and its ψ^m component from Equation (21). Note that approximating the total wealth price-consumption ratio by the market-wide price-dividend ratio is the approach used by Menzly et al. (2004).

3.1.3 Long-run Risks model: Bansal and Yaron (2004). Bansal and Yaron's (2004) long-run risks model assumes that the representative consumer has the version of Kreps and Porteus (1978) preferences adopted by Epstein and Zin (1989) and Weil (1989), for which the SDF is given by

$$M_{t+1}^m = \delta^\theta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\rho}} R_{c,t+1}^{\theta-1},$$

where $R_{c,t+1}$ is the unobservable gross return on an asset that delivers aggregate consumption as its dividend each period, δ is the subjective time discount factor, ρ is the elasticity of intertemporal substitution, $\theta := \frac{1-\gamma}{1-1/\rho}$, and γ is the relative-risk aversion coefficient.

The aggregate consumption and dividend growth rates, Δc_{t+1} and Δd_{t+1}, respectively, are modeled as containing a small persistent expected growth rate component, x_t, which follows an AR(1) process with stochastic volatility, and fluctuating variance, σ_t^2, which evolves according to a homoscedastic linear mean-reverting process.

Appendix A.6 shows that, for the log-linearized model, the log of the SDF and its ψ^m component are given by

$$\ln M_{t+1}^m = \underbrace{c_2 \Delta c_{t+1}}_{\ln m(\theta, t+1)} + \underbrace{c_1 + c_3 x_{t+1} + c_4 \sigma_{t+1}^2 + c_5 x_t + c_6 \sigma_t^2}_{\ln \psi_{t+1}^m} \qquad (24)$$

where the parameters $(c_1, c_2, c_3, c_4, c_5, c_6)$ are known functions of the underlying time series and preference parameters of the model.

To obtain the time series of the SDF and ψ^m, we extract the state variables, x_t and σ_t^2, from observed data using two different procedures. First, we extract them from consumption data. Second, we extract them from asset market data, in particular, from the market-wide price-dividend ratio and the risk-free rate.

The extraction of the state variables using these two procedures is described in Appendix A.6. Finally, for each value of γ, we use the calibrated parameter values from Bansal and Yaron (2004) and the time series of the state variables to obtain the time series of the SDF and its ψ^m component from Equation (24).

3.1.4 Housing: Piazzesi et al. (2007).

In this model, the pricing kernel is given by

$$M_t^m = \delta (C_t/C_{t-1})^{-\gamma} (A_t/A_{t-1})^{\frac{\gamma\rho-1}{\rho-1}},$$

where A_t is the expenditure share on nonhousing consumption, γ^{-1} is the intertemporal elasticity of substitution, and ρ is the intratemporal elasticity of substitution between housing services and nonhousing consumption.

Taking the logs, we have

$$\ln M_t^m = \underbrace{-\gamma\,\Delta c_t}_{\ln m(\theta,t)} + \underbrace{\ln\delta + \frac{\gamma\rho-1}{\rho-1}\Delta a_t}_{\ln\psi_t^m}. \tag{25}$$

Note that, in this model, ψ^m depends on observable variables alone and, therefore, does not need to be extracted from consumption or asset market data. For each value of γ, we use the calibrated values of the model parameters (δ, ρ) in Piazzesi et al. (2007) to obtain the time series of the model-implied SDF and its ψ^m component from Equation (25).

3.2 Empirical results

For our empirical analysis, we focus on two data samples: an annual data sample starting at the onset of the Great Depression ($1929-2009$), and a quarterly data sample starting in the post-World War II period ($1947\!:\!Q1-2009\!:\!Q4$). A detailed data description is presented in Appendix A.4. Note that, in any finite sample, the extracted time series of the SDF, as well as the information bounds on the SDF and the unobservable component of the SDF, depend on the set of test assets used for their construction. Since the Euler equation holds for any traded asset, as well as for any adapted portfolio of assets, this gives an infinitely large number of moment restrictions. Nevertheless, econometric considerations necessitate the choice of only a subset of assets to be used. As a consequence, in our empirical analysis, we compute bounds and filter the time series of the SDF and the components of the SDF, using a broad cross-section of test assets. In particular, at the quarterly frequency, the test assets include the 6 size and book-to-market-equity-sorted portfolios of Fama-French, 10 industry-sorted portfolios, and 10 momentum-sorted portfolios. Due to the smaller available time series at the annual frequency, we restrict the cross-section of test assets to include the 6 size and book-to-market-equity-sorted portfolios, 5 industry-sorted portfolios, and the smallest and largest deciles of the 10 momentum-sorted portfolios.

Table 1
Correlation of filtered and model SDFs, 1947:Q1-2009:Q4

	Correlation of filtered and model SDF		Cross-sectional R^2	
	$\rho\left(\ln\psi_t^*, \ln\psi_t^m\right)$	$\rho\left(\ln M_t^*, \ln M_t^m\right)$	No intercept	Free intercept
A. State variables extracted from consumption				
CC	0.10 / 0.07	0.13 / 0.09	−1.19	0.002
	[−0.09,0.18] [−0.11,0.18]	[−0.07,0.20] [−0.09,0.19]	[−3.14,0.02]	[0.00,0.38]
MSV	−0.01 / 0.003	0.05 / 0.04	−0.79	0.002
	[−0.07,0.18] [−0.09,0.18]	[−0.07,0.20] [−0.09,0.19]	[−2.72,0.06]	[0.000,0.37]
BY	−0.02 / 0.03	0.16 / 0.09	−0.71	0.005
	[−0.14,0.12] [−0.11,0.18]	[−0.03,0.25] [−0.12,0.18]	[−2.83,0.02]	[0.00,0.35]
PST	−0.12 / −0.14	−0.03 / −0.04	−0.91	0.03
	[−0.24,0.02] [−0.24,0.03]	[−0.16,0.09] [−0.21,0.09]	[−3.21,0.14]	[0.00,0.36]
B. State variables extracted from asset prices				
CC	0.17 / 0.16	0.18 / 0.17	−0.77	0.31
	[−0.10,0.18] [−0.10,0.18]	[−0.10,0.18] [−0.10,0.19]	[−3.13,0.08]	[0.00,0.39]
MSV	0.18/ 0.23	0.19 / 0.24	−0.46	0.04
	[−0.10,0.19] [−0.10,0.22]	[−0.10,0.20] [−0.10,0.22]	[−3.78,0.00]	[0.00,0.48]
BY	0.03 / 0.06	0.04 / 0.07	−1.26	0.24
	[−0.11,0.17] [−0.11,0.21]	[−0.11,0.17] [−0.10,0.21]	[−3.23,−0.39]	[0.00,0.52]

The table reports the correlation between the filtered and the model-implied ψ-components of the SDFs (Column 1), the correlation between the filtered and the model-implied SDFs (Column 2), the cross-sectional R^2 implied by the model-specific SDFs when no intercept is allowed in the cross-sectional regression (Column 3), and the cross-sectional R^2 when an intercept is allowed in the regression (Column 4), using quarterly data for 1947:Q1-2009:Q4. The bootstrapped 95% confidence intervals are reported in brackets. Each cell in Columns 1 and 2 has two entries corresponding to whether the filtered ψ^*-component and, therefore, the filtered SDF is estimated using Equation (6), reported on the left, or Equation (4), reported on the right. Panel A reports results when the models' state variables and, therefore, the model-implied SDFs are extracted from consumption data, and panel B reports the same when the state variables are extracted from asset prices. *CC*, *MSV*, *BY*, and *PST* denote, respectively, the models of Campbell and Cochrane (1999), Menzly et al. (2004), Bansal and Yaron (2004), and Piazzesi et al. (2007).

3.2.1 The time series of the filtered SDF.
Our first approach for assessing the empirical plausibility of these models is based on the observation that our method identifies the minimum entropy time series of the SDF, which we call the *filtered SDF*. That is, given a candidate SDF with observable component $m(\theta, t)$, we use the relative entropy-minimizing procedures in Equations (4) and (6) to estimate a time series for the unobservable (or residual, if the SDF is fully observable) component $\{\psi_t^*(\theta)\}_{t=1}^T$ and obtain the filtered SDF as $m(\theta, t)\psi_t^*$.

Note that the filtered SDF and its missing component depend on the local curvature of the utility function γ, since changing γ modifies the constraints in Equations (4) and (6). Therefore, for each model, we fix γ at the authors' calibrated value and extract the time series of the filtered SDF and the components of the SDF. We compare the filtered SDF $(m(\theta, t)\psi_t^*)$ with the model-implied SDF $(m(\theta, t)\psi_t^m)$ for each model.

Table 1 reports the results at the quarterly frequency. Panel A reports results when the model-implied SDF and its components are obtained by extracting the state variable(s) from consumption data, and panel B presents results when asset market data are used to extract the state variable(s). The first column reports

the correlation between the filtered time series of the missing component, $\{\ln \psi_t^*\}_{t=1}^T$, of the SDF, and of the corresponding model-implied time series, $\{\ln \psi_t^m\}_{t=1}^T$. The second column shows the correlation between the filtered SDF, $\{\ln M_t^* = \ln(m(\theta,t)\psi_t^*)\}_{t=1}^T$, where $m(\theta,t) = (C_t/C_{t-1})^{-\gamma}$, and the model-implied SDF, $\{\ln M_t^m = \ln(m(\theta,t)\psi_t^m)\}_{t=1}^T$. The 95% confidence intervals for these correlations are obtained by bootstrapping with replacement from the data.

Consider first the results for the CC external habit model presented in the first row of each panel. For this model, the utility curvature parameter is set to the calibrated value of $\gamma = 2$. Column 1 in panel A shows that when the model-implied state variable is extracted from consumption, the correlation between the filtered and model-implied ψ is only 0.10 when ψ^* is estimated using Equation (6). Column 2 shows that the correlation between the filtered and model-implied SDFs is marginally higher at 0.13. When ψ^* is estimated using Equation (4), the correlations are very similar at 0.07 and 0.09, respectively. Panel B shows that the correlations between the filtered and model-implied SDFs and ψ's remain small when the model state variable is extracted from the market-wide price-dividend ratio.

The second row in each panel presents the results for the MSV external habit model. In this case, γ is set equal to 1, which is the calibrated value in the model. Row 2 in each panel shows that the results for the MSV model are similar to those for the CC model. When ψ^* is estimated using Equation (6), the correlations between the filtered and model-implied ψ components of the SDFs are small, varying from -0.01, when the surplus consumption ratio is extracted from consumption data, to 0.18, when the state variable is extracted using the price-dividend ratio. The correlations between the filtered and model-implied SDFs are marginally higher, varying from 0.05, when the surplus consumption ratio is extracted from consumption data, to 0.19 when it is extracted using the price-dividend ratio. Similar results are obtained when ψ^* is estimated using Equation (4).

The third row in each panel presents the results for the BY long-run risks model. The parameter γ is set equal to the BY calibrated value of 10. Row 3 and Column 1 of panel A, shows that when the state variables are extracted from consumption, the correlation between the filtered and model-implied ψ components is -0.02 (0.03) when ψ^* is estimated using Equation (6) (Equation (4)). Column 2 shows that the correlation between the filtered and model-implied SDFs is 0.16 (0.09). Similar results are obtained in panel B, where the state variables are extracted from the market-wide price-dividend ratio.

The fourth row in panel A presents the results for the PST housing model. Note that, in this model, the SDF and its ψ^m component are directly observable and thereby do not need to be extracted from either consumption or asset market data. Therefore, we do not have a fourth row in panel B. The risk-aversion parameter, γ, is set equal to 16, which is the calibrated value in the original

Table 2
Correlation of filtered and model SDFs, 1929–2009

	Correlation of filtered and model SDF		Cross-sectional R^2	
	$\rho\left(\ln\psi_t^*,\ln\psi_t^m\right)$	$\rho\left(\ln M_t^*,\ln M_t^m\right)$	No intercept	Free intercept
A. State variables extracted from consumption				
CC	0.35 / 0.31	0.39 / 0.34	0.082	0.504
	[−0.04,0.44] [−0.04,0.41]	[−0.00,0.48] [−0.02,0.92]	[−2.19,0.74]	[0.00,0.81]
MSV	0.33 / 0.22	0.41 / 0.34	0.76	0.82
	[−0.02,0.41] [−0.04,0.37]	[0.06,0.46] [−0.02,0.96]	[−2.05,0.76]	[0.00,0.80]
BY	−0.17 / −0.028	0.27 / 0.20	0.45	0.47
	[−0.31,0.22] [−0.44,0.21]	[−0.03,0.48] [−0.16,0.77]	[−2.25,0.75]	[0.00,0.80]
PST	−0.09 / −0.001	−0.004/ −0.013	−0.73	0.09
	[−0.23,0.24] [−0.25,0.25]	[−0.20,0.21] [−0.26,0.26]	[−2.39,0.09]	[0.00,0.40]
B. State variables extracted from asset prices				
CC	0.19/ 0.14	0.24/ 0.17	−0.20	0.60
	[−0.12,0.35] [−0.10,0.28]	[−0.11,0.37] [−0.08,0.29]	[−2.86,0.53]	[0.00,0.63]
MSV	−0.04 / 0.13	0.01 / 0.18	−0.16	0.001
	[−0.10,0.33] [−0.10,0.27]	[−0.08,0.35] [−0.09,0.28]	[−2.69,0.27]	[0.00,0.52]
BY	−0.01 / 0.10	−0.02 / 0.09	−0.15	0.005
	[−0.21,0.34] [−0.23,0.31]	[−0.21,0.29] [−0.29,0.32]	[−0.77,0.25]	[0.00,0.27]

The table reports the correlation between the filtered and the model-implied ψ-components of the SDFs (Column 1), the correlation between the filtered and the model-implied SDFs (Column 2), the cross-sectional R^2 implied by the model-specific SDFs when no intercept is allowed in the cross-sectional regression (Column 3), and the cross-sectional R^2 when an intercept is allowed in the regression (Column 4), using annual data for 1929-2009. The bootstrapped 95% confidence intervals are reported in brackets. Each cell in Columns 1 and 2 has two entries corresponding to whether the filtered ψ^*-component and, therefore, the filtered SDF is estimated using Equation (6), reported on the left, or Equation (4), reported on the right. Panel A reports results when the models' state variables and, therefore, the model-implied SDFs are extracted from consumption data, and panel B reports the same when the state variables are extracted from asset prices. CC, MSV, BY, and PST denote, respectively, the models of Campbell and Cochrane (1999), Menzly et al. (2004), Bansal and Yaron (2004), and Piazzesi et al. (2007).

paper. Column 1 shows that the correlations between the filtered and model-implied ψ components of the SDFs are very small and have the wrong sign, varying from −0.12 to −0.14, when ψ^* is estimated using Equations (6) and (4). The correlations between the filtered and model-implied SDFs are marginally higher varying from −0.03 to −0.04.

Table 2 reports results analogous to those in Table 1 at the annual frequency. The results are largely similar to those in Table 1. The two habit models, when the state variable is extracted from consumption data, are notable exceptions. In these two cases the correlations between filtered and model-implied SDFs and ψ components are much higher than at the quarterly frequency, being in the 0.31-0.39 range for CC and 0.22-0.41 for MSV.

The last two columns of Tables 1 and 2 report the cross-sectional R^2's, along with 95% confidence bands, in brackets, implied by the model-specific SDFs at the quarterly and annual frequencies, respectively. The cross-sectional R^2 are obtained by performing a cross-sectional regression of the historical average returns on the model-implied expected returns. Column 3 reports the cross-sectional R^2 when there is no intercept in the regression, and Column 4 presents

results when an intercept is included. The results reveal that the cross-sectional R^2's wildly vary for the same model, and often take on large negative values when an intercept is not allowed in the cross-sectional regression, or when the model-implied state variables are extracted using either consumption or asset market data. Moreover, they have very wide confidence intervals. As we show in the next subsection, this is in stark contrast with the results based on entropy bounds in Tables 6 and 7 that tend instead to give consistent results and tighter confidence bands for each model across different samples and procedures used to extract the model state variables.

Overall, Tables 1 and 2 make two main points. First, they demonstrate the robustness of our estimation method – very similar results are obtained using either Equation (4) or (6) to filter ψ^* and M^*. Second, they show that, regardless of the data frequency and the procedure used to extract the model-implied SDFs, all the asset pricing models considered imply SDFs that tend to have low correlation with the filtered ones. While the results in Tables 1 and 2 are obtained using the combined set of size and book-to-market-equity-sorted, momentum-sorted, and industry-sorted portfolios, very similar results are obtained using the 25 Fama-French portfolios as test assets.[17]

The correlations between model-specific SDFs and filtered SDFs discussed above would have little significance if the filtered discount factors had no clear economic interpretation. To address this concern, we show below that our filtered pricing kernel has clear economic content since (1) it is always highly correlated with the Fama-French factors, that can be interpreted as proxies for the true unknown sources of systematic risk, (2) it implies that the SDF should have a strong business-cycle pattern, and (3) it significantly reacts to financial market crashes.

Tables 3 and 4 report the correlations between the filtered and model-implied log SDFs and the three Fama-French (FF) factors at the quarterly and annual frequencies, respectively. Column 1 presents the correlation between the model-implied SDF, when the state variables are extracted from consumption data, and the three FF factors. This is computed by performing a linear regression of the model-implied time series of the SDF, $\left\{\ln\left(M_t^m\right)\right\}_{t=1}^T$, on the three FF factors and computing the correlation between $\ln(M^m)$ and the fitted value from the regression. Column 2 reports the correlation when the model-implied state variables are extracted from asset market data. Columns 3 and 4 present the correlations of the filtered SDF and its missing component with the three FF factors, respectively.

Consider Table 3 first. Column 3, panel A, shows that the log of the filtered SDF, $M_t^* \equiv m(\theta, t)\psi_t^*$, strongly correlates with the FF factors, having correlation coefficients ranging from 0.49 to 0.59 when the set of test assets

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[17] The results are available from the authors on request.

Table 3
Correlations with FF3, 1947:Q1-2009:Q4

	$(\ln M_t^m)_{cons}$	$(\ln M_t^m)_{prices}$	$\ln M_t^*$	$\ln \psi_t^*$
			Correlation with FF3	
A. 25 Fama-French				
CC	0.18	0.20	0.54/0.59	0.54/0.59
MSV	0.21	0.95	0.54/0.59	0.54/0.59
BY	0.25	0.45	0.54/0.58	0.52/0.57
PST	0.07	–	0.49/0.52	0.45/0.50
B. 10 Momentum				
CC	0.18	0.20	0.52/0.52	0.51/0.51
MSV	0.21	0.95	0.52/0.52	0.51/0.51
BY	0.25	0.45	0.55/0.53	0.50/0.50
PST	0.07	–	0.53/0.51	0.43/0.43
C. 10 Industry				
CC	0.18	0.20	0.65/0.69	0.64/0.68
MSV	0.21	0.95	0.65/0.69	0.65/0.68
BY	0.25	0.45	0.66/0.69	0.62/0.65
PST	0.07	–	0.53/0.55	0.47/0.51

The table reports the correlations between the 3 Fama-French factors and (*1*) the model-implied SDF with state variables extracted from consumption (Column 1) and stock market (Column 2) data, (*2*) the filtered SDF (Column 3), and (*3*) the filtered ψ^* component of the SDF (Column 4), using quarterly data for 1947:Q1-2009:Q4 and a different set of portfolios in each panel. Each cell in Columns 3 and 4 has two entries corresponding to whether the filtered ψ^*-component and, therefore, the filtered SDF is estimated using Equation (6), reported on the left, or Equation (4), reported on the right. *CC*, *MSV*, *BY*, and *PST* denote, respectively, the models of Campbell and Cochrane (1999), Menzly et al. (2004), Bansal and Yaron (2004), and Piazzesi et al. (2007).

consists of the 25 size and book-to-market-equity-sorted portfolios of Fama-French. Column 4 reveals that this high correlation is almost entirely due to the ψ^* component, and not to $m(\theta, t)$, since the correlation between the filtered SDF and the FF factors is the same as that between the filtered missing component of the SDF and the FF factors.

The above results are perhaps not surprising because the FF factors are known to be quite successful in explaining a large fraction of the cross-sectional variation in the returns of the 25 size and book-to-market-equity-sorted portfolios. However, panels *B* and *C* reveal that the filtered SDF strongly correlates with the FF factors independently of the set of test assets used to extract the filtered SDF. When the set of test assets consists of the 10 momentum-sorted portfolios, the correlations vary from 0.51 to 0.55. For the 10 industry-sorted portfolios, the correlations vary from 0.53 to 0.69. Column 4 of panels *B* and *C* reveals that this high correlation is almost entirely driven by the missing component of the SDF, and not the consumption growth component.

Row 1, Column 1, of each panel shows that, for the CC model, while the filtered SDF correlates strongly with the FF factors, the model-implied SDF has a small correlation coefficient of 0.18, when the surplus consumption ratio is extracted from consumption data. Row 1, Column 2, shows that the correlation only rises marginally to 0.20 when the state variable is extracted from the market-wide price-dividend ratio.

Table 4
Correlations with FF3, 1929-2009

	Correlation with FF3			
	$(\ln M_t^m)_{cons}$	$(\ln M_t^m)_{prices}$	$\ln M_t^*$	$\ln \psi_t^*$
A. 6 Fama-French				
CC	0.19	0.12	0.73/0.78	0.72/0.77
MSV	0.26	0.87	0.73/0.78	0.72/0.77
BY	0.38	0.73	0.77/0.77	0.68/0.72
PST	0.35	–	0.81/0.76	0.65/0.67
B. 10 Momentum				
CC	0.19	0.12	0.55/0.63	0.58/0.61
MSV	0.26	0.87	0.55/0.62	0.57/0.61
BY	0.38	0.73	0.69/0.69	0.51/0.57
PST	0.35	–	0.73/0.70	0.50/0.55
C. 10 Industry				
CC	0.19	0.12	0.49/0.53	0.49/0.53
MSV	0.26	0.87	0.50/0.54	0.50/0.55
BY	0.38	0.73	0.42/0.39	0.38/0.42
PST	0.35	–	0.41/0.27	0.34/0.37

The table reports the correlations between the 3 Fama-French factors and (*1*) the model-implied SDF with state variables extracted from consumption (Column 1) and stock market (Column 2) data, (*2*) the filtered SDF (Column 3), and (*3*) the filtered ψ^* component of the SDF (Column 4), using annual data for 1929-2009 and a different set of portfolios in each panel. Each cell in Columns 3 and 4 has two entries corresponding to whether the filtered ψ^*-component and, therefore, the filtered SDF is estimated using Equation (6), reported on the left, or Equation (4), reported on the right. *CC*, *MSV*, *BY*, and *PST* denote, respectively, the models of Campbell and Cochrane (1999), Menzly et al. (2004), Bansal and Yaron (2004), and Piazzesi et al. (2007).

For the MSV model, the correlation between the model-implied SDF and the FF factors is small at 0.21 when the surplus consumption ratio is extracted from consumption data. However, when the state variable is extracted from the price-dividend ratio, the correlation between the model-implied SDF and the FF factors is very high at 0.95 - much higher than the correlation between the filtered SDF and the FF factors for each set of test assets.

Row 3 in each panel shows that for the BY model, the correlation between the model-implied SDF and the FF factors is 0.25 when the state variables are extracted from consumption data. The correlation increases to 0.45 when asset price data are used in the extraction of the model-implied state variables.

Finally, Row 4 in each panel shows that for the PST model, the correlation between the model-implied SDF and the FF factors is very small at 0.07.

Table 4 reveals that very similar results are obtained at the annual frequency. Tables 3 and 4 demonstrate the soundness of our estimation method: the filtered time series of the SDF and its ψ^* component are quite robust, in terms of their correlations with the FF factors, to the choice of the utility curvature parameter γ, the set of assets, and the data frequency considered. Moreover, our filtered SDF and ψ^* are consistently highly correlated with the FF factors independently of the sample frequency and the cross-section of assets used for the estimation (even assets, like the industry and momentum portfolios, that are not well priced by the FF factors). This finding has several important implications. First, it suggests that our estimation approach successfully identifies the unobserved

pricing kernel, since there is substantial empirical evidence that the FF factors do proxy for asset risk sources. Second, our finding provides a rationalization of the empirical success of the FF factors in pricing asset returns. Finally, although the filtered SDF is highly correlated with the FF factors, the correlation coefficient is substantially smaller than unity, particularly for the industry and momentum portfolios (see, e.g., Table 4), suggesting that the FF factors cannot fully capture all the underlying sources of systematic risk that are important in pricing these assets.

The reason behind the stable correlation results between our filtered SDFs and the three Fama French factors seems to be the fact that, independently of the set of assets used for the filtering, the filtered SDF tends to have a very similar time series behavior. In particular, it shows a clear business-cycle pattern, and significant and sharp reactions to stock market crashes (even if these crashes do not necessarily result in economy wide contractions). This feature of the filtered SDFs is illustrated in Figures 9 (annual frequency) and 10 (quarterly frequency). In each figure we report the business-cycle component (panel A) and the residual component of the filtered M^* for the different models.[18] At both data frequencies, of from the model considered, both the business-cycle and residual components are extremely similar across the models.

In Table 5 we compare the business-cycle and market crash properties of the filtered SDFs with the model-implied ones. For each model considered, and for both the filtered (M^*) and model-implied (M^m) pricing kernels, the table reports the risk-neutral probabilities of recessions (Column 1), and stock market crashes nonconcomitant with recessions (Column 2), as well as in the first row of each panel, the sample frequency of these events.[19] For the model-implied pricing kernels, we present the probabilities when the state variables are extracted using consumption data as well as using asset price data (in brackets below).

Focusing on quarterly data (panel A), Column 1 shows that the filtered SDFs (M^*) imply a risk-neutral probability of a recession in the 25%–26% range. Comparing this with the model-implied probabilities reveals that, whether the state variables are extracted using consumption or asset market data, all the model-implied pricing kernels deliver a similar risk-neutral probability of recessions that is similar to the one of our filtered SDFs (with the notable exception of the BY pricing kernel that, extracting the state variables using asset

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[18] The decomposition into a business-cycle and a residual component is obtained by applying the Hodrick and Prescott (1997) filter to the estimated M^*.

[19] To compute the risk-neutral probabilities, note that for any quantity A_t and function $f(.)$, we have that $\mathbb{E}^Q[f(A_t)] = \int f(A_t)\frac{dQ}{dP}dP = \int f(A_t)\frac{M_t}{M}dP$. Hence, given an SDF M_t (either filtered or model-implied) the risk-neutral expectation can be estimated (assuming ergodicity) using the sample analog $\widehat{\mathbb{E}^Q[f(A_t)]} = \frac{1}{T}\sum_t^T f(A_t)\frac{M_t}{M}$. For instance, to estimate the probability of a recession, we replace $f(A_t)$ with an index function that takes value 1 if the economy was in an NBER-designated recession at time t and zero otherwise. See also Remark 1 in Appendix A.1.
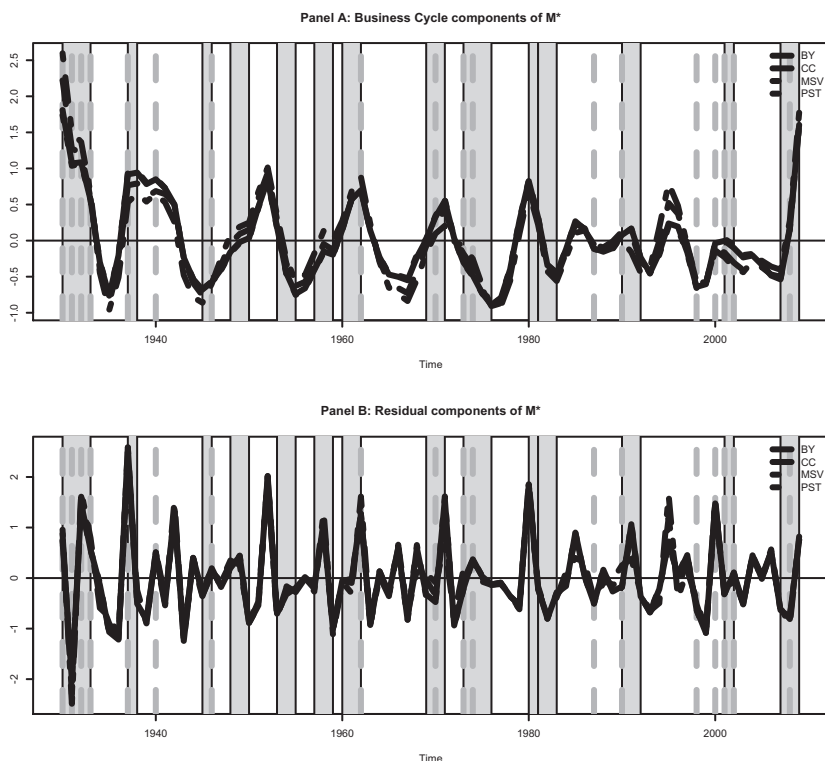
Panel A: Business Cycle components of M*

Panel B: Residual components of M*

Figure 9

Business-cycle (*A*) and residual (*B*) components of the filtered (log) SDF ($M_t^* = \left(\frac{C_t}{C_{t-1}} \right)^{-\gamma} \psi_t^*$) filtered using the relative entropy-minimizing procedure in Equation (6) using annual data over the period 1929–2009 for the different models considered: Bansal and Yaron (2004) (BY), Campbell and Cochrane (1999) (CC), Menzly et al. (2004) (MSV), and Piazzesi et al. (2007) (PST). The difference between the models is driven by the value of the utility curvature parameter γ that is set to the authors' original calibrations. The decomposition into a business-cycle and a residual component is obtained by applying the Hodrick and Prescott (1997) filter to the estimated M^*. The set of test assets used in the filtering consists of the six size and book-to-market-equity-sorted portfolios, ten industry-sorted portfolios, and the ten momentum-sorted portfolios. Shaded areas denote NBER recession years, and vertical dashed lines indicate the major stock market crashes identified by Mishkin and White (2002).

market data, implies a risk-neutral probability of recession of about 55%). More interestingly, Column 2 shows that the model-implied kernels fail to show the significant and sharp reaction to stock market crashes exhibited by the filtered SDFs: the probabilities of stock market crashes nonconcomitant with recessions implied by the filtered SDFs are between 104% and 207% higher than those implied by the model specific kernels when the model-implied state variables are extracted from consumption data and between 44% and 207% higher when the state variables are extracted from asset price data. Panel B reports similar findings at the annual frequency, but also shows that MSV and PST imply too
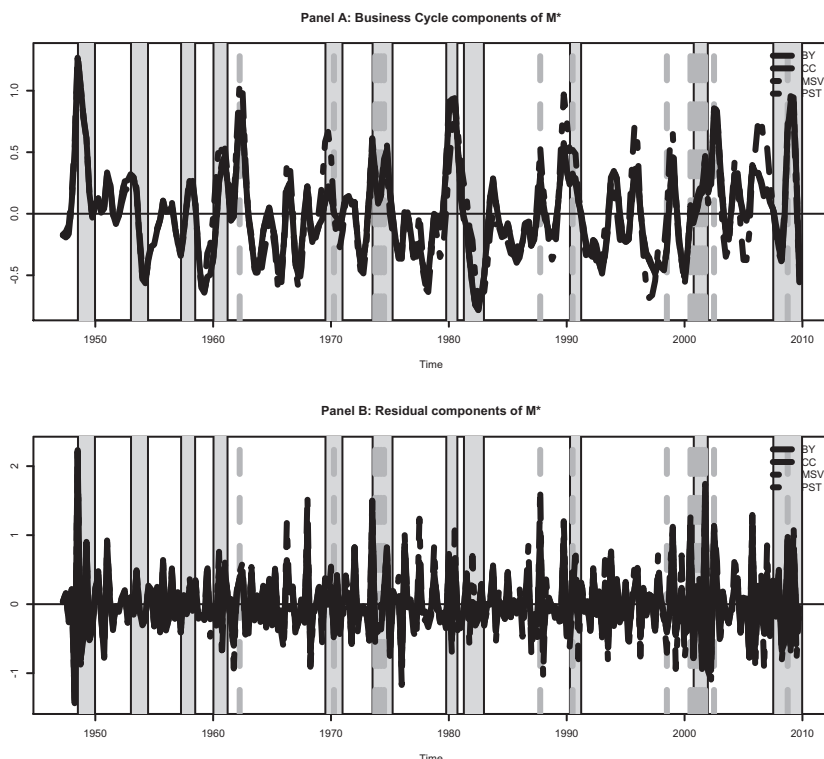
Panel A: Business Cycle components of M*

Panel B: Residual components of M*

Figure 10

Business-cycle (A) and residual (B) components of the filtered (log) SDF ($M_t^* = \left(\frac{C_t}{C_{t-1}} \right)^{-\gamma} \psi_t^*$) filtered using the relative entropy-minimizing procedure in Equation (6) using quarterly data over the period 1947:Q1-2009:Q4 for the different models considered: Bansal and Yaron (2004) (BY), Campbell and Cochrane (1999) (CC), Menzly et al. (2004) (MSV), and Piazzesi et al. (2007) (PST). The difference between the models is driven by the value of the utility curvature parameter γ that is set to the authors' original calibrations. The decomposition into a business-cycle and a residual component is obtained by applying the Hodrick and Prescott (1997) filter to the estimated M^*. The set of test assets used in the filtering consists of the six size and book-to-market-equity-sorted portfolios, ten industry-sorted portfolios, and the ten momentum-sorted portfolios. Shaded areas denote NBER recession years, and vertical dashed lines indicate the major stock market crashes identified by Mishkin and White (2002).

low of probablities for recessions and BY – only when extraxted from asset prices – implies a very high probability of market crash.[20]

Overall, the above results suggest that the explanatory power of these models for asset pricing would be improved by augmenting the pricing kernels with a component that exhibits sharp reactions to market crashes that are not perfectly correlated with the business cycle.

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[20] Note that, at the annual frequency, a year is designated as a recession year if at least one of its quarters is in an NBER recession period.

Table 5
Recession and market crash probabilities of M^m and M^*

		Recession Recession probability	Market crash without recession probability
A. Quarterly Data, 1947:Q1-2009:Q4			
Sample		.022	0.024
CC	M^m	0.25[0.24]	0.024[0.025]
	M^*	0.25/0.25	0.054/0.059
BY	M^m	0.22[0.55]	0.024[0.034]
	M^*	0.26/0.26	0.049/0.057
MSV	M^m	0.22[0.22]	0.024[0.028]
	M^*	0.25/0.25	0.055/0.059
PST	M^m	0.21	0.028
	M^*	0.25/0.25	0.086/0.065
B. Annual Data, 1929-2009			
Sample		0.375	0.088
CC	M^m	0.61[0.49]	0.055[0.068]
	M^*	0.58/0.56	0.092/0.119
BY	M^m	0.41[0.59]	0.083[0.227]
	M^*	0.59/0.59	0.085/0.097
MSV	M^m	0.38[0.39]	0.086[0.098]
	M^*	0.57/0.56	0.094/0.122
PST	M^m	0.37	0.067
	M^*	0.60/0.60	0.103/0.093

The table reports the risk-neutral probability of recessions (Column 1) and stock market crashes nonconcomitant with recessions (Column 2) implied by the model (M^m) and filtered (M^*) SDFs at quarterly (panel A) and annual (panel B) frequencies. Each cell in the rows corresponding to the model SDF has two entries corresponding to whether the models' state variables are extracted from consumption data, reported on the left, or from asset market data, reported on the right. Each cell in the rows corresponding to the filtered SDF has two entries corresponding to whether the filtered ψ^*-component and, therefore, the filtered SDF is estimated using Equation (6), reported on the left, or Equation (4), reported on the right.

3.2.2 Entropy Bounds Analysis.

Our second approach to assess the empirical plausibility of the asset pricing models considered relies on the entropy bounds derived in Section 1.1. For each model we compute the minimum values of the power coefficient, γ, at which the model-implied SDF satisfies the HJ, Q, M, and Ψ bounds. We also compute 95% confidence bands via bootstrap. Table 6 reports the results at the quarterly frequency. Panels A and B report results when the state variables needed to construct the time series of the model-implied SDF and its components are extracted from consumption (panel A) and asset market data (panel B). Consider first the results for the HJ, $Q1$, $M1$, and $\Psi1$ bounds. The first row in each panel presents the bounds for the CC model. Panel A shows that when the surplus consumption ratio is extracted from consumption data, the minimum values of γ at which the pricing kernel satisfies the HJ, $Q1$, $M1$, and $\Psi1$ bounds are 10.2, 16.1, 16.4, and 23.2, respectively. Therefore, as suggested by the theoretical predictions, the Q-bound is tighter than the HJ-bound, and the M-bound is tighter than the Q-bound. Note that in this model, the curvature of the utility function is $\frac{\gamma}{S_t}$, where S_t is the surplus consumption ratio, and this ratio is almost identical to the coefficient of relative-risk aversion (see, e.g., the discussion in Campbell and Cochrane (1999)). For $\gamma = 2$, the calibrated value in CC, the curvature varies over $[19.7, \infty)$. Panel A reveals that

Table 6
Bounds for RRA, quarterly data, 1947:Q1-2009:Q4

	HJ-bound	*Q*1/*Q*2-bounds	*M*1/*M*2-bounds	Ψ1/Ψ2-bounds
A. State variables extracted from consumption				
CC	10.2	16.1 / 15.7	16.4 / 16.0	23.2 / 23.9
		[16.0,38.0][14.4,34.8]	[16.0,38.0][14.6,36.8]	[23.0,>100][21.2,>100]
MSV	32.6	40.8 / 40.4	43.4 / 43.5	61.3 / 62.8
		[38.0,62.0][38.0,59.0]	[40.0,64.0][40.0,64.0]	[59,113][59.0,>100]
BY	>100	>100 / >100	>100 / >100	>100 / >100
		[>100,>100][>100,>100]	[>100,>100][>100,>100]	[>100,>100][>100,>100]
PST	73.8	99.0 / 92.6	111.1 / 102.2	96.2 / 90.5
		[96.0,172.0][88.0,161.0]	[102.0,183.0][93.0,172,1]	[94.0,187.0][86.0,176.0]
B. State variables extracted from asset prices				
CC	19	43 / 46	46 / 46	47 / 48
		[43.0,50.0][46.0,49.0]	[46.0,50.0][46.0,49.0]	[47.0,51.0][48.0,50.0]
MSV	73.3	90.3 / 90.0	>100 / >100	>100 / >100
		[92.0,>100][89.5,>100]	[>100,>100][>100,>100]	[>100,>100][>100,>100]
BY	4.0	5 / 5	5 / 5	5 / 5
		[5.0,6.0][5.0,6.0]	[5.0,6.0][5.0,6.0]	[5.0,6.0][5.0,6.0]

The table reports the minimum values of the utility curvature parameter γ at which the model-implied SDF satisfies the HJ (Column 1), Q (Column 2), M (Column 3), and Ψ (Column 4) bounds using quarterly data for 1947:Q1-2009:Q4. The bootstrapped 95% confidence intervals are reported in brackets below. Columns 2-4 has two entries in each cell that correspond to whether the filtered ψ^*-component of the SDF and, therefore, the filtered SDF are estimated using equation (6), reported on the left, or Equation (4), reported on the right. Panels A and B present results when the models' state variables are extracted from consumption data and asset market data, respectively. *CC*, *MSV*, *BY*, and *PST* denote, respectively, the models of Campbell and Cochrane (1999), Menzly et al. (2004), Bansal and Yaron (2004), and Piazzesi et al. (2007).

the Q-bound is satisfied for $\gamma \geqslant 16.1$, implying that the curvature varies over $[56.6, \infty)$, the M-bound is satisfied for $\gamma \geqslant 16.4$, implying that the curvature varies over $[57.2, \infty)$, and the Ψ-bound is satisfied for $\gamma \geqslant 23.2$, implying that the curvature varies over $[68.5, \infty)$. A similar ordering of the bounds is obtained when the surplus consumption ratio is extracted from the market-wide price-dividend ratio in panel B, except that, in this case, even higher values of risk aversion are needed to satisfy the bounds. Also, very similar results are obtained for the $Q2$, $M2$, and Ψ2 bounds, stressing the robustness of our approach.

The second row in each panel presents the bounds for the MSV model. When the surplus consumption ratio is extracted from consumption data, the HJ, $Q1$, $M1$, and Ψ1 bounds are satisfied for a minimum value of $\gamma = 32.6$, 40.8, 43.4, and 61.3, respectively. Very similar results are obtained for the $Q2$, $M2$, and Ψ2 bounds. Therefore, this model requires much higher values of risk aversion than CC to be consistent with observed asset returns. Note, however, that for both models and both procedures used to extract the model-implied SDFs, the risk-aversion coefficients at which the models satisfy the bounds are very high.

The third row in each panel presents the bounds for the BY model. Panel A shows that when the model-implied state variables are extracted from consumption data, the model-implied pricing kernel fails to satisfy the HJ, Q, M, and Ψ bounds for any value of the risk-aversion parameter smaller than 100. On the contrary, when the model-implied state variables are extracted from

asset market data (panel B), the HJ bound is satisfied for a minimum value of $\gamma = 4.0$, while the $Q1$, $M1$, and $\Psi1$ bounds are all satisfied by a relative risk-aversion as small as 5. Similar results are obtained for the $Q2$, $M2$, and $\Psi2$ bounds. Therefore, the results reveal that the empirical performance of the BY framework crucially depends on how the latent state variables are extracted from the data.

Finally, the fourth row of panel A presents the bounds for the PST model. Note that, in this model, the SDF is a function of observable data alone, hence there is no need to extract any state variable from asset market data. Therefore, we do not have a fourth row in panel B. The model satisfies the HJ, $Q1(Q2)$, $M1(M2)$, and $\Psi1(\Psi2)$ bounds for minimum values of $\gamma = 73.8$, 99.0(92.6), 111.1(102.2), and 96.2(90.5), respectively. Therefore, this model requires very high levels of risk-aversion to be consistent with observed asset returns.

Overall, Table 6 demonstrates that, in line with the theoretical underpinnings of the various bounds, the Q-bound is generally tighter than the HJ-bound because it naturally exploits the restriction that the SDF is a strictly positive random variable. The M-bound is tighter than the Q-bound because it formally takes into account the ability of the SDF to price assets and the dependency of the pricing kernel on consumption. Furthermore, the results suggest that all the models considered require very high levels of risk-aversion to satisfy the bounds, with the only exception being the long-run risks model of BY (but only when the model state variables are extracted from asset price data).

Table 7 reports analogous bounds as in Table 6 at the annual frequency. The table shows that, at this frequency, all the bounds tend to be satisfied with smaller values of the utility curvature parameter, suggesting that the models considered can more easily rationalize asset pricing dynamics at the annual, rather than quarterly, frequency. However, once again in line with the theoretical predictions, the Q-bound is tighter than the HJ-bound, and the M-bound is tighter than the Q-bound.

Note that the above bound results have tight confidence bands and are much more consistent, in evaluating the plausibility of a given model across different procedures used to extract the model-implied SDF and its components, than the cross-sectional R^2 measures reported in Tables 1 and 2 that wildly vary for the same model and have very wide confidence intervals.

Note that the results in Tables 6 and 7 are obtained by allowing only the utility curvature parameter, γ, to vary while holding constant all the other model parameters at the authors' calibrated values. Note that most consumption based asset pricing models, including the ones considered in this paper, are highly parameterized. Since the state variables are not directly observed in many of the models, the parameters governing their dynamics are typically chosen to match some moments of the data. Consequently, the properties of the SDF are quite sensitive to not only γ but also the values of all the other parameters. Therefore, we also compute the minimum values of the power coefficient, γ,

Table 7
Bounds for RRA, annual data, 1929-2009

	HJ-bound	*Q*1/*Q*2-bounds	*M*1/*M*2-bounds	Ψ1/Ψ2-bounds
A. State variables extracted from consumption				
CC	0.7	5.1/ 2.7	5.2 / 2.7	7.6/ 3.6
		[4.0,41.0][3.0,8.0]	[4.0,41.0][3.0,8.0]	[5.0,>100][4.0,23.2]
MSV	17	28.7/ 24.4	30.3/ 26.6	>100/ 76.5
		[19.0,53.3][23.7,35.0]	[20.0,53.3][24.7,35.4]	[>100,>100][81.0,>100]
BY	50	53/ 71	60 / >80	55/ >80
		[22.0,71.0][69.7,>80]	[24.0,72.0][>80,>80]	[49.0,>80][2.0,>80]
PST	17.1	28.6/ 24.1	31.4/ 27.0	22.0/ 18.6
		[19.0,51.7][23.0,35.4]	[20.0,51.3][24.0,35.4]	[14.0,42.7][19.7,29.0]
B. State variables extracted from asset prices				
CC	4	7/ 6	7/ 6	8/ 7
		[4.0,12.0][6.0,9.0]	[4.0,12.0][6.0,9.0]	[4.0,14.0][7.0,11.0]
MSV	23.7	39.1/ 33.4	42.2/ 37.0	>100/ >100
		[22.0,69.5][29.5,45.0]	[26.0,69.5][30.5,45.0]	[>100,>100][>100,>100]
BY	5	6 / 6	6 / 6	6 / 6
		[5.0,6.0][2.0,7.0]	[5.0,6.0][2.0,6.0]	[5.0,6.0][2.0,6.0]

The table reports the minimum values of the utility curvature parameter γ at which the model-implied SDF satisfies the HJ (Column 1), Q (Column 2), M (Column 3), and Ψ (Column 4) bounds using annual data for 1929-2009. The bootstrapped 95% confidence intervals are reported in brackets. Columns 2-4 has two entries in each cell that correspond to whether the filtered ψ^*-component of the SDF and, therefore, the filtered SDF are estimated using equation (6), reported on the left, or Equation (4), reported on the right. Panels A and B present results when the models' state variables are extracted from consumption data and asset market data, respectively. CC, MSV, BY, and PST denote, respectively, the models of Campbell and Cochrane (1999), Menzly et al. (2004), Bansal and Yaron (2004), and Piazzesi et al. (2007).

at which the model-implied SDFs satisfy the *HJ*, *Q*, *M*, and Ψ bounds while allowing the remaining model parameters to simultaneously vary over two standard-error intervals around their calibrated values. The results, reported in Table A2 of Appendix A.7.1, remain qualitatively unchanged. In particular, for each model, the *HJ*, *Q*, *M*, and Ψ bounds are satisfied for smaller values of γ when the other parameters are allowed to vary simultaneously compared to Tables 6 and 7 where the other parameters are held fixed. However, as in the latter tables, the *CC*, *MSV*, and *PST* models still require much larger values of risk-aversion to satisfy the bounds compared to the authors' calibrated values at the quarterly frequency.

Also note that we have used *excess* returns (in excess of the risk-free rate) on a broad cross section of risky assets to extract SDF and obtain entropy bounds on the SDF and its components. However, it is well known that the level of the risk-free asset constrains models quite dramatically. Therefore, in order to check the robustness of our results, we repeat the empirical exercise using as test assets the gross returns (instead of excess returns) on the same assets considered so far plus the risk-free asset. The methodology needs to be slightly modified in this case and is described in Appendix A.7.2. The results, reported in Table A3 of Appendix A.7.2, show that the inclusion of the risk-free rate as an additional asset leaves the *HJ*, *Q*, *M*, and Ψ bounds on the SDF and its components very similar to those obtained in Tables 6 and 7 without the risk-free rate, for all the models considered.

Table 8
Relative entropy of SDF and SDF components

	$KLIC(M_t^m)$	$KLIC\left(\psi_t^{resid}\right)$	$\dfrac{KLIC\left(\psi_t^{resid}\right)}{KLIC\left(M_t^m\psi_t^{resid}\right)}$
A. Quarterly, 1947:Q1-2009:Q4			
CC	0.035/0.037	0.26/0.32	0.772/0.786
	[0.018][0.019]	[0.30][0.33]	[0.909][0.859]
MSV	0.0002/0.0002	0.31/0.36	0.992/0.996
	[0.004][0.004]	[0.30][0.35]	[0.950][0.953]
BY	0.003/0.003	0.30/0.35	0.957/0.971
	[1.69][1.70]	[0.59][0.39]	[0.647][0.448]
PST	0.008/0.008	0.39/0.39	1.01/0.989
B. Annual, 1929–2009			
CC	0.379/0.660	0.66/0.69	0.688/0.676
	[0.164][0.169]	[0.76][0.73]	[0.815][0.767]
MSV	0.001/0.001	0.85/0.85	0.972/0.974
	[0.023][0.023]	[0.85][0.81]	[0.973][0.906]
BY	0.023/0.022	0.82/0.84	0.932/0.959
	[2.66][1.75]	[2.33][1.02]	[1.44][0.712]
PST	0.19/0.27	0.96/0.91	1.06/0.996

The table reports the KLIC of the model-implied SDF (Column 1), the KLIC of the residual psi (Column 2), and the ratio of the KLIC of the residual psi and the KLIC of the product of the model-implied SDF and the residual psi (Column 3) at the quarterly (panel A) and annual (panel B) frequencies. Each cell has four entries that correspond to whether the models' state variables are extracted from consumption data, reported at the top, or from asset market data, reported at the bottom, and to whether the KLIC between measure A and the physical measure P is computed as D(P ∥ A), reported on the left, or as D(A ∥ P), reported on the right. *CC*, *MSV*, *BY*, and *PST* denote, respectively, the models of Campbell and Cochrane (1999), Menzly et al. (2004), Bansal and Yaron (2004), and Piazzesi et al. (2007).

3.2.3 What Are The Consumption-Based Models Missing?. As shown in Section 1.1.1, modeling the SDF as fully observable, that is, setting $m(\theta,t)=M_t^m$, where M_t^m is the *entire* pricing kernel of the model under consideration (given in Equations (21), (24), and (25)), we can extract a residual ψ^{resid} component such that $M_t^* := M_t^m \times \psi_t^{resid}$ correctly prices assets. The ψ^{resid} component can once again be estimated using the relative entropy minimization procedures in Equations (4) and (6) replacing m with M^m. The ψ^{resid} multiplicative adjustment of the pricing kernel: (1) still has an maximum likelihood interpretation, (2) adds the minimum amount of information needed for M^* to be able to price assets correctly, and (3) most importantly, as the second Hansen-Jagannathan distance, it provides a useful diagnostic for detecting what the pricing kernels are missing to be consistent with observed asset returns.

We first examine the relative importance of the two components of M^*, M^m and ψ^{resid}, in pricing a broad cross-section of assets. We do this by computing the contribution of each component to the overall entropy of the pricing kernel. The results are reported in Table 8. Columns 1 and 2 present the relative entropy, or KLIC, of the model-implied SDF, M_t^m, and the residual component, ψ_t^{resid}, respectively. Column 3 reports the KLIC of ψ_t^{resid} as a fraction of the KLIC of the overall filtered kernel $M_t^m \times \psi_t^{resid}$.

Each row of Column 1 reports the KLIC, or relative entropy, of M_t^m. There are four numbers for this quantity since there are two possible ways of computing the KLIC (as $D(P \parallel M^m)$, reported on the left, and $D(M^m \parallel P)$, reported on the right), and two possible ways of extracting the models' state variables (from consumption, top numbers, and from asset market data, bottom numbers in brackets). Similarly, four numbers with the same ordering are reported in the remaining two columns. First, consider panel A, that presents results obtained at the quarterly frequency. Columns 1 and 2 show that, for the CC model, the relative entropy of ψ^{resid} is an order of magnitude bigger than that of M^m, regardless of whether ψ_t^{resid} is estimated using Equation (4) or (6), or whether M_t^m is obtained by extracting the state variable from consumption or asset market data. This point is further highlighted in Column 3 that shows that the KLIC of ψ_t^{resid} accounts for a lions share of the KLIC of the overall kernel: 77.2%-78.6% when the model-implied state variable is extracted from consumption data and 85.9%-90.9% when it is extracted from asset price data. Very similar results are obtained for the MSV, BY, and PST models in Rows 2-4, and also at the annual frequency in panel B. Overall, the results suggest that, for each model considered, most of the ability of the kernel to price assets comes from the residual component and very little from the model-implied component, that is all the pricing kernels under consideration seem to miss a substantial share of the information needed to price correctly the observed asset returns.

In order to assess whether these models are missing similar features of the data, Table 9 reports the correlations between the ψ^{resid} of different models at the quarterly (panel A) and annual (panel B) frequencies. As in the previous table, for all the entries we have four number given by the two ways of computing relative entropy (left and right numbers corresponding to Equations (4) and (6)) and the two ways of extracting the models' state variables (from consumption in the top numbers and from asset prices for the numbers below in brackets). Panel A shows that, when the models' state variables are extracted from consumption data, the correlations between the residual ψ's are extremely high, varying from 0.85 (between CC and PST) to (almost) 1.0 (between MSV and BY) when the ψ^{resid} component is estimated using Equation (6). When the ψ^{resid} component is estimated using Equation (4), the correlations are very similar, varying from 0.93 to (almost) 1.0. When the models' state variables are extracted from asset prices the correlations among the various ψ^{resid} are almost unchanged with one important exception: in this case the correlation between the residual component of the BY model and all other models becomes much smaller ranging from 0.1 to 0.41. This implies that the BY pricing kernel changes a lot depending on whether its state variables are extracted from market or consumption data. Similar results are obtained at the annual frequency in panel B, although the correlations are generally smaller at this frequency.[21]

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[21] Note that the estimates at the annual frequency are inherently more imprecise, due to the small available sample size, than those at the quarterly frequency.

Table 9
Correlation between residual ψ's

	CC	MSV	BY	PST
A. Quarterly, 1947:Q1-2009:Q4				
CC	1.0	0.96/0.93	0.97/0.96	0.85/0.93
	[0.96][0.95]	[0.32][0.10]	[0.93][0.94]	
MSV	–	1.0	1.0/1.0	0.91/0.97
			[0.41][0.20]	[0.89][0.94]
BY	–	–	1.0	0.91/0.97
				[0.26][.10]
PST	–	–	–	1.0
B. Annual, 1929:2009				
CC	1.0	0.87/0.66	0.88/0.77	0.80/0.51
		[0.91][0.78]	[0.40][0.22]	[0.83][0.53]
MSV	–	1.0	0.99/0.95	0.92/0.71
			[0.52][0.27]	[0.89][0.62]
BY	–	–	1.0	0.88/0.62
				[0.38][−0.03]
PST	–	–	–	1.0

The table reports the correlations between the residual ψ's of the different asset pricing models using quarterly data for 1947:Q1-2009:Q4 (panel A) and annual data for 1929-2009 (panel B). Each cell has four entries that correspond to whether the models' state variables are extracted from consumption data, reported at the top, or from asset market data, reported at the bottom, and to whether the residual psi is estimated using Equation (6), reported on the left, or using equation (4), reported on the right. *CC*, *MSV*, *BY*, and *PST* denote, respectively, the models of Campbell and Cochrane (1999), Menzly et al. (2004), Bansal and Yaron (2004), and Piazzesi et al. (2007).

Figure 11 plots the time series of the residual ψ's for the four models at the quarterly (panel A) and annual (panel B) frequencies, with state variables extracted from consumption data and ψ^{resid} estimated using Equation (6). The results suggest that these models are all missing a very similar component that would improve their ability to explain asset return dynamics. In particular, all the ψ^{resid} have a clear business-cycle pattern, but also show significant and sharp reactions to financial market crashes that do not result in economy wide contractions.

To further illustrate this point, Table 10 reports the changes in the model-implied risk-neutral probabilities need to rationalize stock returns according to ψ^{resid}, that is, the percentage change caused by replacing M^m with $M^m \times \psi^{resid}$. As before, we have four entries per model since we compute probabilities when state variables are extracted using consumption data as well as using asset price data (in brackets below), and two minimum entropy methods (left and right numbers). Focusing on quarterly data in panel A, three patterns emerge. First (Column 1), ψ^{resid} implies a relatively small increase in the risk-neutral probability of recessions, suggesting that the models considered tend to adequately capture business-cycle risk at this frequency (with the exception of BY, when the sate variables are extracted from asset prices, that seems to imply too much recession risk). Second (Column 2), all the models seem to imply a too low risk-neutral probability of market crash that is ψ^{resid} increases this quantity by about 53%-98% (with again the exception of BY that seems to imply
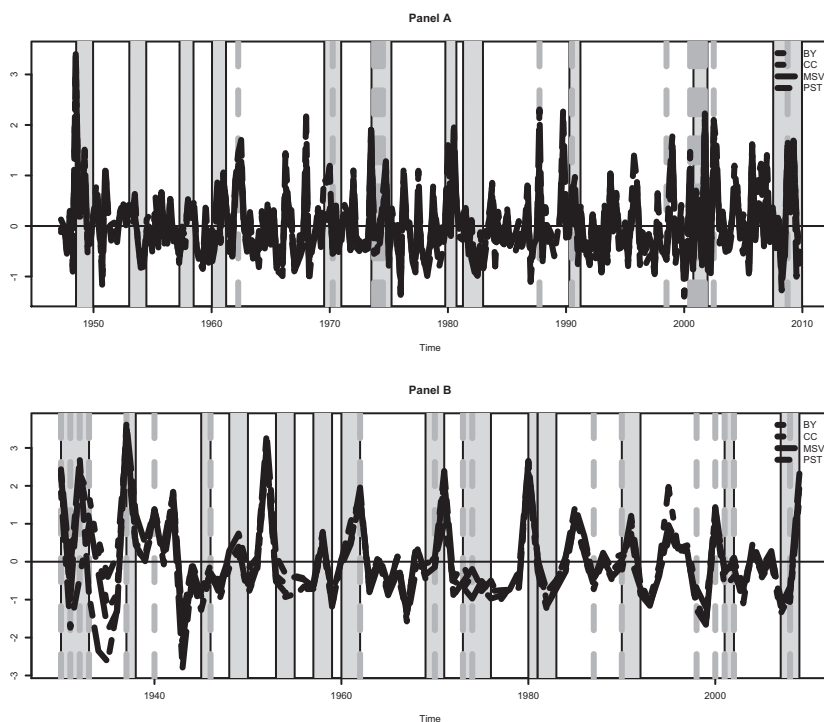
Panel A

Panel B

Figure 11

The (log) residual ψ components, $\ln\left(\psi_t^{resid}\right)$, of the SDFs ($M_t^* = M_t^m \psi_t^{resid}$) filtered using the relative entropy-minimizing procedure in Equation (6) using quarterly data for 1947:Q1-2009:Q4 (*A*) and annual data over the period 1929-2009 (*B*) for the different models considered: Bansal and Yaron (2004) (BY), Campbell and Cochrane (1999) (CC), Menzly et al. (2004) (MSV), and Piazzesi et al. (2007) (PST). Shaded areas denote NBER recession years, and vertical dashed lines indicate the major stock market crashes identified by Mishkin and White (2002).

too much crash risk). Third (Column 3), all the models imply a much too low probability of market crashes not concomitant with recessions: ψ^{resid} increases the risk-neutral likelihood of these events by about 72%-232%. Panel B shows a similar patter, albeit the probability of market crashes without recessions are harder to identify at this frequency. Overall, Table 10 suggests that the models do not seem to price correctly market crash risk, especially market crashes that do not lead to large real economic contractions.

To summarize, the results in this section suggest that the consumption based asset pricing models we have considered would benefit from being augmented with a component that exhibits significant reactions to financial market crashes, in particular crashes that do not result in macroeconomic contractions. Moreover, not only the standard C-CAPM with power utility, but also most of the more recent models that have been proposed in the literature, seem to be missing this component.

Table 10
Percentage change in risk-neutral probabilities due to residual ψ's

	Recession probability	Market crash probability	Market crash without recession probability
A. Quarterly Data, 1947:Q1-2009:Q4			
CC	10/11	60/59	72/105
	[9/14]	[78/78]	[133/144]
BY	14/15	69/68	107/136
	[−65/−67]	[−31/−32][84/144]	
MSV	15/15	78/74	124/144
	[12/11]	[53/53]	[93/126]
PST	17/20	98/75	232/148
B. Annual Data, 1929-2009			
CC	−1/−1	−2/1	10/36
	[21/17]	[73/85]	[11/81]
BY	42/37	84/86	−2/37
	[−22/−8]	[−45/43]	[5/−24]
MSV	50/46	92/92	7/39
	[43/39]	[61/63]	[3/33]
PST	58/57	64/71	−3/69

The table reports the percentage changes in risk-neutral probabilities generated by the the residual ψ component. recession probabilities. Columns 1 to 3 focus, respectively, on recession, market, and market crash without recession, probabilities. Each cell has four entries that correspond to whether the models' state variables are extracted from consumption data, reported at the top, or from asset market data, reported at the bottom, and to whether the residual psi is estimated using Equation (6), reported on the left, or using Equation (4), reported on the right. *CC*, *MSV*, *BY*, and *PST* denote, respectively, the models of Campbell and Cochrane (1999), Menzly et al. (2004), Bansal and Yaron (2004), and Piazzesi et al. (2007).

4. Conclusion

In this paper, we propose an information-theoretic approach as a diagnostic tool for dynamic asset pricing models. The models we consider are characterized by having a pricing kernel that can be factorized into an observable component, consisting of a parametric function of observable variables, and a potentially unobservable one that is model specific. Based on this decomposition of the pricing kernel, we provide three major contributions.

First, using a relative entropy minimization approach, we show how to extract the time series of both the SDF and its unobservable component nonparametrically. Moreover, given a fully observable pricing kernel, this procedure delivers the minimal (in the entropy sense) modification of the SDF that would enable it to price asset returns correctly. Applying this method to the data, we find that the estimated SDF has a clear business-cycle pattern, but also shows significant and sharp reactions to financial market crashes that do not result in economy wide contractions. Moreover, we find that the nonparametrically extracted SDF, independently of the set of assets used for its construction, is substantially (yet not perfectly) correlated with the risk factors proposed in Fama and French (1993). This provides a rationalization of the empirical success of the Fama-French factors in pricing asset returns and suggests that our filtering procedure does successfully identify the unobserved component of the SDF.

Second, we construct a new set of entropy bounds that build on and improve the ones suggested in the previous literature in that (1) they naturally impose the nonnegativity of the pricing kernel, (2) they are generally tighter and have higher information content, and (3) allow us to utilize the information contained in consumption data and a large cross-section of asset returns jointly.

Third, applying the methodology developed in this paper to a large class of dynamic asset pricing models, we find that the SDFs implied by all of these models poorly correlate with our filtered SDF, require implausibly high levels of risk-aversion to satisfy our entropy bounds, and are all missing a similar component that exhibits significant reactions to financial market crashes that do not result in economy-wide macroeconomic contractions. These results are robust to the choice of test assets used, as well as the frequency of the data.

The methodology developed in this paper is considerably general, and may be applied to any model that delivers well-defined Euler equations like models with heterogenous agents, limited stock market participation, and fragile beliefs.

A. Appendix

A.1 Maximum Likelihood Analogy

To formally show the analogy between our estimation approach for the measures Ψ and Q and an MLE procedure, we have to consider the two definition of relative entropy (and corresponding estimators) separately.

First, consider the entropy minimization problem of the type $D(P \parallel x)$, with x being either the Q or the Ψ measures, used to contruct the estimators in Equations (7) and (6). Let the vector $\mathbf{z}_t$ be a sufficient statistic for the state of the economy at time t. That is, $\mathbf{z}_t$ can be thought of as an augmented state vector (e.g., containing the beginning of period state variables, as well as the time t realizations of the shocks and expectations about the future). Given $\mathbf{z}_t$, the equilibrium quantities, such as returns $\mathbf{R}^e$ and the sdf M, are just a mapping from $\mathbf{z}$ on to the real line, that is,

$$M(\mathbf{z}):\mathbf{z} \to \mathbb{R}_+, \quad \mathbf{R}^e(\mathbf{z}):\mathbf{z} \to \mathbb{R}^N, \quad M_t \equiv M(\mathbf{z}_t), \quad \mathbf{R}_t^e \equiv \mathbf{R}^e(\mathbf{z}_t),$$

where $\mathbf{z}_t$ is the time t realization of $\mathbf{z}$.

Equipped with the above definition, we can rewrite the Euler equation (3) as

$$0 = \mathbb{E}\left[\mathbf{R}_t^e M_t\right] \equiv \int \mathbf{R}_t^e M_t dP = \int \mathbf{R}^e(\mathbf{z}) M(\mathbf{z}) p(\mathbf{z}) d\mathbf{z}, \tag{A1}$$

where $p(\mathbf{z})$ is the pdf associated with the physical measure P. Moving to the risk-neutral measure we have

$$0 = \mathbb{E}\left[\mathbf{R}_t^e M_t\right] = \mathbb{E}^Q\left[\mathbf{R}_t^e\right] = \int \mathbf{R}^e(\mathbf{z}) q(\mathbf{z}) d\mathbf{z}, \tag{A2}$$

where $q(\mathbf{z})$ is the pdf associated with the risk-neutral measure Q and $M = dQ/dP$. Note that

$$D(P \parallel Q) = \int \ln \frac{dP}{dQ} dP = \int p(\mathbf{z}) \ln p(\mathbf{z}) d\mathbf{z} - \int p(\mathbf{z}) \ln q(\mathbf{z}) d\mathbf{z}.$$

Since the first term on the right-hand side of the above expression does not involve q, $D(P \parallel Q)$ is minimized, with respect to q, by choosing the distribution that maximizes the second term, that is,

$$Q^* \equiv \underset{Q}{\arg\min} D(P \parallel Q) \equiv \underset{q}{\arg\max} \mathbb{E}[\ln q(\mathbf{z})] \text{ s.t. } \mathbb{E}^Q\left[\mathbf{R}_t^e\right] = 0.$$

That is, the minimum entropy estimator in Equation (7) maximizes the expected – risk-neutral – log likelihood. Following (Owen 1988, 1991, 2001), approximating the continuous distribution $q(\mathbf{z})$

with a multinomial distribution $\{q_t\}_{t=1}^T$ that assigns probability weight q_t to the time t realizations of $\mathbf{z}$, a NPMLE of Q can be obtained as

$$\{q_t^*\}_{t=1}^T = \arg\max \quad \frac{1}{T} \sum_{t=1}^T \ln q_t \tag{A3}$$

$$\text{s.t. } q_t \in \Delta^T \equiv \left\{ (q_1, q_2, ..., q_T) : q_t \geqslant 0, \sum_{t=1}^T q_t = 1 \right\} \text{ and (A2) holds,}$$

provided that

$$\frac{1}{T} \sum_{t=1}^T \ln q_t \xrightarrow[T \to \infty]{p.} \mathbb{E}[\ln q(\mathbf{z})].$$

Note also that the NPMLE of $p(\mathbf{z})$ is simply $p_t = 1/T \ \forall t$ (see, e.g., (Owen, 1988, 1991, 2001)) which is the maximum entropy distribution. Therefore q^* contains all the necessary information to recover the state-price density from the Radon-Nykodin deivative dQ/dP.

Similarly, we have that

$$\Psi^* \equiv \underset{\Psi}{\arg\min}\, D(P \parallel \Psi) \equiv \underset{\psi}{\arg\min} \int p(\mathbf{z}) \ln p(\mathbf{z}) d\mathbf{z} - \int p(\mathbf{z}) \ln \psi(\mathbf{z}) d\mathbf{z}$$

$$\equiv \underset{\psi}{\arg\max} \quad \mathbb{E}[\ln \psi(\mathbf{z})] \text{ s.t. } \mathbb{E}^{\Psi}\left[\mathbf{R}_t^e m_t\right] = 0,$$

where $\psi(\mathbf{z})$ is the pdf associated with the measure Ψ. That is, the Ψ^* estimator in Equation (6) is also an MLE. Moreover, in a very similar fashion, one can show that $\psi^* m$ provides a MLE of q under the restriction that the pricing kernel has the multiplicative reppresentation $M = m\psi$.

Hence, the estimates Q^* and Ψ^* maximize the log likelihoods of the data, but not the physical ones: the risk-neutral log likelihood in the first case and an intermediate one in the second case (and Ψ^* also can be interpreted as maximizing the risk-neutral log likelihood under the constraint that $M_t = m_t \psi_t$).

Remark 1. The above implies that, for any equilibrium quantity A_t, we have that $A_t \equiv A(\mathbf{z}_t)$. Hence, the risk-neutral expectation of any function $f(.)$ of A, defined as

$$\mathbb{E}^Q[f(A_t)] \equiv \int f(A(\mathbf{z})) q(\mathbf{z}) d\mathbf{z},$$

can be estimated as (see, e.g., Kitamura 2006)

$$\widehat{\mathbb{E}^Q[f(A_t)]} = \sum_{t=1}^T f(A_t) q_t^*,$$

where q_t^* is the relative entropy-minimizing risk-neutral measure. For instance, the risk-neutral probability of a recession in a given year that is $\mathbb{E}^Q\left[\mathbf{1}_{\{\text{recession in year } t\}}\right]$, where $\mathbf{1}_{\{\text{recession in year } t\}}$ is an indicator function that takes the value one if time t was an NBER-designated recession and zero otherwise, can be estimated as $\sum_{t=1}^T \mathbf{1}_{\{\text{recession in year } t\}} q_t^*$.

Second, consider the entropy minimization problem of the type $D(x \parallel P)$ with x being either the Q or the Ψ measures. This alternative definition of relative entropy in Equations (5) and (4) also deliver nonparametric maximum likelihood estimates of the Q and Ψ measures, respectively. We establish this result for Ψ^* since for Q^* the same result can be shown by a simplified version of the same argument.

To see why the estimation problem in Equation (4) delivers an MLE of ψ_t, consider the following procedure for constructing (up to a scale) the series $\{\psi_t\}_{t=1}^T$. First, given an integer $N \gg 0$,

distribute to the various points in time $t = 1, \ldots, T$, at random and with equal probabilities, the value $1/N$ in N independent draws. That is, draw a series of values (probability weights) $\{\tilde{\psi}\}_{t=1}^{T}$ given by

$$\tilde{\psi}_t \equiv \frac{n_t}{N},$$

where n_t measures the number of times that the value $1/N$ has been assigned to time t. Second, check whether the drawn series $\{\tilde{\psi}\}_{t=1}^{T}$ satisfies the pricing restriction $\sum_{t=1}^{T} m(\theta, t) R_t^e \tilde{\psi}_t = 0$. If it does, use this series as the estimator of $\{\psi_t\}_{t=1}^{T}$, and if it does not draw another series. Obviously, a more efficient way of finding an estimate for ψ_t would be to choose the most likely outcome of the above procedure. Noticing that the distribution of the $\tilde{\psi}_t$ is, by construction, a multinomial distribution with support given by the data sample, we have that the likelihood of any particular sequence $\{\tilde{\psi}_t\}_{t=1}^{T}$ is

$$L\left(\{\tilde{\psi}_t\}_{t=1}^{T}\right) = \frac{N!}{n_1! n_2! \ldots n_T!} \times T^{-N} = \frac{N!}{N\tilde{\psi}_1! N\tilde{\psi}_2! \ldots N\tilde{\psi}_T!} \times T^{-N}.$$

Therefore, the most likely value of $\{\tilde{\psi}_t\}_{t=1}^{T}$ maximizes the log likelihood

$$\ln L\left(\{\tilde{\psi}_t\}_{t=1}^{T}\right) \propto \frac{1}{N}\left(\ln N! - \sum_{t=1}^{T} \ln\left(N\tilde{\psi}_t!\right)\right).$$

Since the above procedure of assigning probability weights will become more and more accurate as N grows bigger, we would ideally like to have $N \to \infty$. But in this case one can show that[22]

$$\lim_{N \to \infty} \ln L\left(\{\tilde{\psi}_t\}_{t=1}^{T}\right) = -\sum_{t=1}^{T} \tilde{\psi}_t \ln \tilde{\psi}_t.$$

Therefore, taking into account the constraint for the pricing kernel, the MLE of ψ_t solves

$$\left\{\hat{\psi}_t\right\}_{t=1}^{T} \equiv \arg\max -\sum_{t=1}^{T} \tilde{\psi}_t \ln \tilde{\psi}_t, \quad \text{s.t.} \ \left\{\tilde{\psi}_t\right\}_{t=1}^{T} \in \Delta^T, \ \sum_{t=1}^{T} m(\theta, t) \mathbf{R}_t^e \tilde{\psi}_t = \mathbf{0}.$$

But the solution of the above MLE problem is also the solution of the relative entropy minimization problem in Equation (4) (see, e.g., Csiszar 1975). That is, the KLIC minimization is equivalent to maximizing the likelihood in an unbiased procedure for finding the ψ_t component of the pricing kernel.

A.2 Additional Bounds and Derivations

Remark 2 (*HJ*-bounds as approximated *Q*-bounds). Let p and q denote the densities of the state x associated, respectively, with the physical, P, and the risk-neutral, Q, probability measures.[23] Assuming that

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[22] Recall that from Stirling's formula, we have

$$\lim_{N\tilde{\psi}_t \to \infty} \frac{N\tilde{\psi}_t!}{\sqrt{2\pi N\tilde{\psi}_t}\left(\frac{N\tilde{\psi}_t}{e}\right)^{N\tilde{\psi}_t}} = 1.$$

[23] For expositional simplicity, we focus on a scalar state variable, but the result is straightforward to extend to a vector state.

A.1 q and p are twice continuously differentiable

and that there exists a $\mu_p < \infty$ and a $\mu_q < \infty$ such that:

A.2 (Existence of maxima)

$$\left.\frac{\partial \ln p}{\partial x}\right|_{x=\mu_p} = 0, \quad \left.\frac{\partial \ln q}{\partial x}\right|_{x=\mu_q} = 0; \tag{A4}$$

A.3 (Finite second moments)

$$-\left[\left.\frac{\partial^2 \ln p}{\partial x^2}\right|_{x=\mu_p}\right]^{-1} \equiv \sigma_p^2 < \infty, \quad -\left[\left.\frac{\partial^2 \ln q}{\partial x^2}\right|_{x=\mu_q}\right]^{-1} \equiv \sigma_q^2 < \infty. \tag{A5}$$

We have that, in the limit of the small time interval, a second-order approximation of the Q-bounds yields[24]

$$D\left(P \,\|\, \frac{M_t}{\bar{M}}\right) \propto Var(M_t), \tag{A6}$$

$$D\left(\frac{M_t}{\bar{M}} \,\|\, P\right) \propto Var(M_t). \tag{A7}$$

Proof. [Proof of Remark 2] The densities associated, respectively, with the physical probability measure P and the risk-neutral measure Q are denoted p and q. We can then rewrite the $Q1$ and $Q2$ bounds, respectively, as

$$D\left(P \,\|\, \frac{M_t}{\bar{M}}\right) \equiv \int \ln \frac{dP}{dQ} dP = \int p \ln \frac{p}{q} dx \tag{A8}$$

and

$$D\left(\frac{M_t}{\bar{M}} \,\|\, P\right) \equiv \int \frac{dQ}{dP} \ln \frac{dQ}{dP} dP = \int \ln \frac{dQ}{dP} dQ = \int q \ln \frac{q}{p} dx. \tag{A9}$$

Given conditions $A.1$-$A.3$, we have from a second-order Taylor approximation that

$$\ln q \propto \frac{1}{2} \left.\frac{\partial^2 \ln q}{\partial x^2}\right|_{x=\mu_q} (x - \mu_q)^2 \equiv -\frac{1}{2} \frac{(x - \mu_q)^2}{\sigma_q^2}.$$

$$\ln p \propto \frac{1}{2} \left.\frac{\partial^2 \ln p}{\partial x^2}\right|_{x=\mu_p} (x - \mu_p)^2 \equiv -\frac{1}{2} \frac{(x - \mu_p)^2}{\sigma_p^2}.$$

That is, q and p are approximately (to a second-order) Gaussian

$$q \approx N\left(\mu_q; \sigma_q^2\right), \quad p \approx N\left(\mu_p; \sigma_p^2\right).$$

Note also that in the limit of the small time interval, by the diffusion invariance principle, we have $\sigma_q^2 = \sigma_p^2 = \sigma^2$. Therefore, plugging the above approximation into Equation (A8), we have that in the

---

[24] For the $Q2$ bound only, using the dual objective function of the entropy minimization problem, Stutzer (1995) provides an approximation result similar to the one in Equation (A7) that is valid when the variance bound is sufficiently small. Moreover, for the case of Gaussian iid returns, Kitamura and Stutzer (2002) show that the approximation of the $Q2$ bound in Equation (A7) is exact.

limit of the small time interval

$$\int p \ln \frac{p}{q} dx \approx \int \left[-\frac{1}{2} \frac{(x-\mu_p)^2}{\sigma^2} + \frac{1}{2} \frac{(x-\mu_q)^2}{\sigma^2} \right] p dx$$

$$= \frac{1}{2\sigma^2} \left[-\sigma^2 + \int (x-\mu_q)^2 p dx \right]$$

$$= \frac{1}{2\sigma^2} \{ -\sigma^2 + \int \left[(x-\mu_p)^2 + (\mu_p-\mu_q)^2 \right. \right.$$

$$\left. + 2(\mu_p-\mu_q)(x-\mu_p) \right] p dx \}$$

$$= \frac{1}{2\sigma^2} (\mu_p-\mu_q)^2 = \frac{1}{2\sigma^2} \sigma^2 \sigma_\xi^2 = \frac{1}{2} \sigma_\xi^2$$

where the density ξ is a (strictly positive) martingale defined by $\xi \equiv \frac{dQ}{dP}$, and the one to the last equality comes from the change of drift implied by the Girsanov's Theorem (see, e.g., (Duffie, 2005, Appendix D)).

Similarly, from Equation (A9) we have

$$\int q \ln \frac{q}{p} dx = \frac{1}{2} \sigma_\xi^2.$$

Since Q and P are equivalent measures, $M_t \propto \xi_t$. Therefore, in the limit of the small time interval $Var(M_t) \propto \sigma_\xi^2$, implying

$$D\left(P \parallel \frac{M_t}{\bar{M}} \right) \propto Var(M_t), \quad D\left(\frac{M_t}{\bar{M}} \parallel P \right) \propto Var(M_t).$$

∎

Definition 5 (Volatility bound for ψ_t). For each $E[\psi_t] = \bar{\psi}$, the minimum variance ψ_t is

$$\psi_t^*(\bar{\psi}) \equiv \underset{\{\psi_t(\bar{\psi})\}_{t=1}^T}{\arg\min} \sqrt{Var(\psi_t(\bar{\psi}))} \text{ s.t. } \mathbf{0} = \mathbb{E}\left[\mathbf{R}_t^e m(\theta,t) \psi_t(\bar{\psi}) \right],$$

and any candidate SDF must satisfy the condition $Var(\psi_t) \geq Var(\psi_t^*(\bar{\psi}))$.

The solution of the above minimization for a given θ is

$$\psi_t^*(\bar{\psi}) = \bar{\psi} + \left(\mathbf{R}_t^e m(\theta,t) - \mathbb{E}\left[\mathbf{R}_t^e m(\theta,t) \right] \right)' \beta_{\bar{\psi}},$$

where $\beta_{\bar{\psi}} = Var\left(\mathbf{R}_t^e m(\theta,t) \right)^{-1} \left(-\bar{\psi} \mathbb{E}\left[\mathbf{R}_t^e m(\theta,t) \right] \right)$ and the lower volatility bound is given by

$$\sigma_{\psi^*} \equiv \sqrt{Var(\psi_t^*(\bar{\psi}))} = \bar{\psi} \sqrt{\mathbb{E}\left[\mathbf{R}_t^e m(\theta,t) \right]' Var(\mathbf{R}_t^e m(\theta,t))^{-1} \mathbb{E}\left[\mathbf{R}_t^e m(\theta,t) \right]}.$$

A.3 HJ Kernel Versus Minimum Entropy Kernel

We compute the minimum adjustment to the CCAPM SDF required to make it an admissible pricing kernel using (1) the HJD approach and (2) the relative entropy minimization approach. Table A1 presents the moments of the resultant SDFs extracted using both these approaches for a variety of different sets of test assets.

Table A1
Moments of SDF, 1947:Q1-2009:Q4

	$\sigma\left(M_t^*\right)$	$Sk\left(M_t^*\right)$	$Kurt\left(M_t^*\right)$
A. HJD kernel			
25 FF	0.45	−0.01	3.12
Market	0.22	0.61	3.91
10 momentum	0.41	0.05	3.41
10 industry	0.32	0.54	4.21
B. Minimum entropy kernel			
25 FF	0.91/0.71	4.53/2.07	28.4/9.31
Market	0.26/0.24	3.14/1.84	19.1/8.59
10 momentum	0.69/0.57	3.82/1.78	22.0/7.22
10 industry	0.45/0.39	5.08/2.32	39.6/11.8

The table reports the moments of the SDF computed using the (1) the HJD minimum linear adjustment (panel A) and (2) the minimum relative entropy log-linear adjustment (panel B). The test assets used in the estimation of the minimum adjustment consist of the 25 size and book-to-market-equity-sorted portfolios (Row 1), the market portfolio (Row 2), the ten momentum-sorted portfolios (Row 3), and the ten industry-sorted portfolios (Row 4). The data are quarterly for 1947:Q1-2009:Q4.

A.4 Data Description

At the quarterly frequency, we use six different sets of assets: (1) the market portfolio, (2) the 25 Fama-French portfolios, (3) the 10 size-sorted portfolios, (4) the 10 book-to-market-equity-sorted portfolios, (5) the ten momentum-sorted portfolios, and (6) the ten industry-sorted portfolios. At the annual frequency, we use the same sets of assets, except the 25 Fama-French portfolios are replaced by the 6 portfolios formed by sorting stocks on the basis of size and book-to-market-equity because of the small time-series dimension available at the annual frequency.

Our proxy for the market return is the Center for Research in Security Prices (CRSP) value-weighted index of all stocks on the NYSE, AMEX, and NASDAQ. The proxy for the risk-free rate is the one-month Treasury-bill rate obtained from the CRSP files. The returns on all the portfolios are obtained from Kenneth French's data library. Quarterly (annual) returns for the above assets are computed by compounding monthly returns within each quarter (year), and converted to real using the personal consumption deflator. Excess returns on the assets are then computed by subtracting the risk-free rate.

Finally, for each dynamic asset pricing model, the information bounds and the nonparametrically extracted and model-implied time series of the SDF depend on consumption data. For the standard Consumption-CAPM of Breeden (1979) and Rubinstein (1976), the external habit models of Campbell and Cochrane (1999) and Menzly et al. (2004), and the long-run risks model of Bansal and Yaron (2004), we use per capita real personal consumption expenditures on nondurable goods from the National Income and Product Accounts (NIPA). We make the standard "end-of-period" timing assumption that consumption during quarter t takes place at the end of the quarter. For the housing model of Piazzesi et al. (2007), aggregate consumption is measured as expenditures on nondurables and services excluding housing services.

A.5 Extracting the Model-Implied SDF for the Menzly et al. (2004) Model

The SDF in this model is given by

$$M_t = \delta\left(C_t/C_{t-1}\right)^{-\gamma}\left(S_t/S_{t-1}\right)^{-\gamma},\qquad(\text{A10})$$

where δ is the subjective time discount factor, γ is the utility curvature parameter, $S_t = \frac{C_t - X_t}{C_t}$ denotes the surplus consumption ratio, and X_t is the habit component.

The inverse surplus, $Y_t = \frac{1}{S_t}$, follows a mean-reverting process:

$$dY_t = k\left(\overline{Y} - Y_t\right)dt - \alpha\left(Y_t - \lambda\right)\sigma_c dB_t.$$

Therefore, using Ito's lemma, $s_t \equiv \ln(S_t) = -\ln(Y_t)$ follows the process

$$
ds_t = -\frac{1}{Y_t} dY_t + \frac{1}{2Y_t^2} (dY_t)^2
$$

$$
= -\frac{1}{Y_t} k(\overline{Y} - Y_t) dt + \frac{1}{Y_t} \alpha (Y_t - \lambda) \sigma_c dB_t + \frac{1}{2Y_t^2} \alpha^2 (Y_t - \lambda)^2 \sigma_c^2 dt
$$

$$
= \left[k(1 - \overline{Y} S_t) + \frac{1}{2} \alpha^2 (1 - \lambda S_t)^2 \sigma_c^2 \right] dt + \alpha (1 - \lambda S_t) \sigma_c dB_t.
$$

Therefore, discretizing the process, we have

$$
\Delta s_{t+1} = k(1 - \overline{Y} S_t) + \frac{1}{2} \alpha^2 (1 - \lambda S_t)^2 \sigma_c^2 + \alpha (1 - \lambda S_t) \sigma_c \varepsilon_{t+1},
$$

where $\varepsilon_{t+1} \sim i.i.d. \ N(0,1)$.

Now, the Euler equation for the return on the aggregate consumption claim is

$$
E_t \left(e^{m_{t+1} + r_{c,tt+1}} \right) = 1, \tag{A11}
$$

where $r_{c,t+1}$ denotes the continuously compounded return on the consumption claim. We rely on log-linear approximations for $r_{c,t+1}$, as in Campbell and Shiller (1988):

$$
r_{c,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}, \tag{A12}
$$

where z_t is the log price-consumption ratio. In Equation (A12), $\kappa_1 = \frac{e^{\overline{z}}}{1 + e^{\overline{z}}}$ and $\kappa_0 = \log(1 + e^{\overline{z}}) - \kappa_1 e^{\overline{z}}$, where $\overline{z}$ denotes the long-run mean of the log price-consumption ratio. We conjecture that z_t is affine in the single state variable s_t:

$$
z_t = A_0 + A_1 s_t. \tag{A13}
$$

To verify the conjecture and also solve for A_0 and A_1, we substitute the expressions for $r_{c,t+1}$ and z_t from Equations (A12) and (A13), respectively, into the Euler equation (A11):

$$
E_t \left(\exp\{ \ln\delta - \gamma \Delta c_{t+1} - \gamma \Delta s_{t+1} + \kappa_0 + \kappa_1 z_{c,t+1} - z_t + \Delta c_{t+1} \} \right) = 1,
$$

$$
\Rightarrow E_t \left(\exp \left\{ \begin{array}{l} \ln\delta - \gamma \mu_c - \gamma \sigma_c \varepsilon_{t+1} - \gamma k(1 - \overline{Y} S_t) - \frac{1}{2} \gamma \alpha^2 (1 - \lambda S_t)^2 \sigma_c^2 - \gamma \alpha (1 - \lambda S_t) \sigma_c \varepsilon_{t+1} \\ + \kappa_0 + \kappa_1 A_0 + \kappa_1 A_1 \left[k(1 - \overline{Y} S_t) + \frac{1}{2} \alpha^2 (1 - \lambda S_t)^2 \sigma_c^2 + \alpha (1 - \lambda S_t) \sigma_c \varepsilon_{t+1} + s_t \right] \\ - A_0 - A_1 s_t + \mu_c + \sigma_c \varepsilon_{t+1} \end{array} \right\} \right) = 1.
$$

Using the properties of conditionally lognormal random variables, we have

$$
0 = \ln\delta - \gamma \mu_c - \gamma k + \gamma k \overline{Y} S_t - \frac{1}{2} \gamma \alpha^2 \sigma_c^2 - \frac{1}{2} \gamma \alpha^2 \lambda^2 \sigma_c^2 S_t^2 + \gamma \alpha^2 \sigma_c^2 \lambda S_t + \kappa_0 + \kappa_1 A_0
$$

$$
+ \kappa_1 A_1 k - \kappa_1 A_1 k \overline{Y} S_t + \frac{1}{2} \kappa_1 A_1 \alpha^2 \sigma_c^2 + \frac{1}{2} \kappa_1 A_1 \alpha^2 \sigma_c^2 \lambda^2 S_t^2 - \kappa_1 A_1 \alpha^2 \sigma_c^2 \lambda S_t + \kappa_1 A_1 s_t
$$

$$
- A_0 - A_1 s_t + \mu_c + \frac{1}{2} [-\gamma - \gamma \alpha (1 - \lambda S_t) + \kappa_1 A_1 \alpha (1 - \lambda S_t) + 1]^2 \sigma_c^2,
$$

which implies

$$
0 = \begin{pmatrix} \ln\delta - \gamma \mu_c - \gamma k - \frac{1}{2} \gamma \alpha^2 \sigma_c^2 + \kappa_0 + \kappa_1 A_0 + \kappa_1 A_1 k + \frac{1}{2} \kappa_1 A_1 \alpha^2 \sigma_c^2 \\ -A_0 + \mu_c + \frac{1}{2} [-\gamma - \gamma \alpha + \kappa_1 A_1 \alpha + 1]^2 \sigma_c^2 \end{pmatrix}
$$

$$
+ \begin{pmatrix} \gamma k \overline{Y} + \gamma \alpha^2 \sigma_c^2 \lambda - \kappa_1 A_1 k \overline{Y} - \kappa_1 A_1 \alpha^2 \sigma_c^2 \lambda \\ + [\gamma \alpha \lambda - \kappa_1 A_1 \alpha \lambda][-\gamma - \gamma \alpha + \kappa_1 A_1 \alpha + 1] \sigma_c^2 \end{pmatrix} S_t
$$

$$
+ (\kappa_1 A_1 - A_1) s_t
$$

$$
+ \left(-\frac{1}{2} \gamma \alpha^2 \lambda^2 \sigma_c^2 + \frac{1}{2} \kappa_1 A_1 \alpha^2 \sigma_c^2 \lambda^2 + \frac{1}{2} (\gamma \alpha \lambda - \kappa_1 A_1 \alpha \lambda)^2 \sigma_c^2 \right) S_t^2.
$$

Table A2
Bounds for RRA when all parameters are allowed to vary

	HJ-bound	$Q1/Q2$-bounds	$M1/M2$-bounds	$\Psi1/\Psi2$-bounds
A. Quarterly data, 1947:Q1-2009:Q4				
CC	2.2	4.0/3.8	4.0/3.8	4.3/4.2
MSV	29.0	36.2/35.9	38.0/38.1	50.9/52.5
BY	3.0	4.0/4.0	4.0/4.0	4.0/4.0
PST	19.1	25.2/24.0	25.4/24.1	24.1/23.1
B. Annual data, 1929-2009				
CC	0.1	0.1/0.1	0.1/0.1	0.1/0.1
MSV	11.3	18.6/16.2	19.3/17.2	28.6/27.2
BY	4.0	4.0/4.0	4.0/4.0	4.0/4.0
PST	4.3	6.8/5.8	6.8/5.8	6.3/5.4

The table reports the minimum values of the utility curvature parameter γ at which the model-implied SDF satisfies the HJ (Column 1), Q (Column 2), M (Column 3), and Ψ (Column 4) bounds using quarterly data for 1947:Q1-2009:Q4 (panel A) and annual data for 1929-2009 (panel B). Columns 2-4 has two entries in each cell that correspond to whether the filtered ψ^*-component of the SDF and, therefore, the filtered SDF are estimated using Equation (6), reported on the left, or Equation (4), reported on the right. CC, MSV, BY, and PST denote, respectively, the models of Campbell and Cochrane (1999), Menzly et al. (2004), Bansal and Yaron (2004), and Piazzesi et al. (2007).

Using the approximations $s_t \approx S_t - 1$ and $S_t^2 \approx -\overline{S}^2 + 2\overline{S}S_t$, we have

$$0 = \begin{pmatrix} \ln\delta - \gamma\mu_c - \gamma k - \frac{1}{2}\gamma\alpha^2\sigma_c^2 + \kappa_0 + \kappa_1 A_0 + \kappa_1 A_1 k + \frac{1}{2}\kappa_1 A_1\alpha^2\sigma_c^2 \\ -A_0 + \mu_c + \frac{1}{2}[-\gamma - \gamma\alpha + \kappa_1 A_1\alpha + 1]^2\sigma_c^2 \end{pmatrix}$$

$$+ \begin{pmatrix} \gamma k\overline{Y} + \gamma\alpha^2\sigma_c^2\lambda - \kappa_1 A_1 k\overline{Y} - \kappa_1 A_1\alpha^2\sigma_c^2\lambda \\ +[\gamma\alpha\lambda - \kappa_1 A_1\alpha\lambda][-\gamma - \gamma\alpha + \kappa_1 A_1\alpha + 1]\sigma_c^2 \end{pmatrix} S_t$$

$$+ (\kappa_1 A_1 - A_1)(S_t - 1)$$

$$+ \left(-\frac{1}{2}\gamma\alpha^2\lambda^2\sigma_c^2 + \frac{1}{2}\kappa_1 A_1\alpha^2\sigma_c^2\lambda^2 + \frac{1}{2}(\gamma\alpha\lambda - \kappa_1 A_1\alpha\lambda)^2\sigma_c^2 \right)\left(-\overline{S}^2 + 2\overline{S}S_t \right).$$

We use the method of undetermined coefficients and set to zero the constant term and the coefficient of S_t to obtain two equations in the two unknowns A_0 and A_1:

$$0 = \begin{pmatrix} \ln\delta - \gamma\mu_c - \gamma k - \frac{1}{2}\gamma\alpha^2\sigma_c^2 + \kappa_0 + \kappa_1 A_0 + \kappa_1 A_1 k + \frac{1}{2}\kappa_1 A_1\alpha^2\sigma_c^2 \\ -A_0 + \mu_c + \frac{1}{2}[-\gamma - \gamma\alpha + \kappa_1 A_1\alpha + 1]^2\sigma_c^2 \end{pmatrix}$$

$$- (\kappa_1 A_1 - A_1)$$

$$- \left(-\frac{1}{2}\gamma\alpha^2\lambda^2\sigma_c^2 + \frac{1}{2}\kappa_1 A_1\alpha^2\sigma_c^2\lambda^2 + \frac{1}{2}(\gamma\alpha\lambda - \kappa_1 A_1\alpha\lambda)^2\sigma_c^2 \right)\overline{S}^2. \tag{A14}$$

and

$$0 = \begin{pmatrix} \gamma k\overline{Y} + \gamma\alpha^2\sigma_c^2\lambda - \kappa_1 A_1 k\overline{Y} - \kappa_1 A_1\alpha^2\sigma_c^2\lambda \\ +[\gamma\alpha\lambda - \kappa_1 A_1\alpha\lambda][-\gamma - \gamma\alpha + \kappa_1 A_1\alpha + 1]\sigma_c^2 \end{pmatrix}$$

$$+ (\kappa_1 A_1 - A_1)$$

$$+ 2\overline{S}\left(-\frac{1}{2}\gamma\alpha^2\lambda^2\sigma_c^2 + \frac{1}{2}\kappa_1 A_1\alpha^2\sigma_c^2\lambda^2 + \frac{1}{2}(\gamma\alpha\lambda - \kappa_1 A_1\alpha\lambda)^2\sigma_c^2 \right). \tag{A15}$$

Solving the equations for A_0 and A_1 gives the equilibrium solution for the log price-consumption ratio in Equation (A13). Note that Equation (A15) implies a quadratic equation for A_1:

$$0 = \left(-\kappa_1^2 \alpha^2 \lambda \sigma_c^2 + \overline{S} \kappa_1^2 \alpha^2 \lambda^2 \sigma_c^2 \right) A_1^2$$

$$+ \left(\begin{array}{c} -\kappa_1 k \overline{Y} - \kappa_1 \alpha^2 \sigma_c^2 \lambda + \gamma \alpha \lambda \kappa_1 \alpha \sigma_c^2 - \kappa_1 \alpha \lambda [-\gamma - \gamma \alpha + 1] \sigma_c^2 \\ + \kappa_1 - 1 + \overline{S} \kappa_1 \alpha^2 \sigma_c^2 \lambda^2 - 2 \overline{S} \gamma \alpha^2 \lambda \sigma_c^2 \kappa_1 \end{array} \right) A_1$$

$$+ \left(\gamma k \overline{Y} + \gamma \alpha^2 \sigma_c^2 \lambda + \gamma \alpha \lambda [-\gamma - \gamma \alpha + 1] \sigma_c^2 - \overline{S} \gamma \alpha^2 \lambda^2 \sigma_c^2 + \overline{S} \gamma^2 \alpha^2 \lambda^2 \sigma_c^2 \right).$$

We choose the smaller root of the quadratic equation as the economically meaningful solution because it implies a positive relation between the log price-consumption ratio and the surplus consumption ratio, unlike the bigger root that implies a negative relation between the variables.

We proxy the log price-consumption ratio using the observable log price-dividend ratio and use Equation (A13) to extract the time series of the state variable s_t. This extracted time series can then be used to obtain the time series of the model-implied SDF and its missing component.

Note that the model is calibrated at the quarterly frequency. Since we evaluate the empirical plausibility of models at the quarterly, as well as annual, frequencies, we obtain the annual estimates of the model parameters as follows. As a first step, we simulate a long sample (five million observations) of the state variable Y from

$$\Delta Y_{q,\tau+1} = k_q \left(\overline{Y}_q - Y_{q,\tau} \right) - \alpha_q \left(Y_{q,\tau} - \lambda_q \right) \sigma_{q,c} \varepsilon_{\tau+1}, \quad \varepsilon_{\tau+1} \sim i.i.d. N(0,1),$$

treating the calibrated quarterly parameter values as the truth. The subscript q in the above equation denotes quarterly. As a second step, we aggregate the simulated data into annual nonoverlapping observations:

$$Y_{a,t} = Y_{q,\tau} + Y_{q,\tau-1} + Y_{q,\tau-2} + Y_{q,\tau-3}, \text{ for } \tau = 1, 2, 3, \ldots$$

$$\Delta Y_{a,t+1} = Y_{a,t+1} - Y_{a,t},$$

where τ denotes quarter t and t denotes year t. As a final step, we estimate the model parameters at the annual frequency from the equation

$$\Delta Y_{a,t+1} = k_a \left(\overline{Y}_a - Y_{a,t} \right) - \alpha_a \left(Y_{a,t} - \lambda_a \right) \sigma_{a,c} e_{t+1}, \quad \varepsilon_{t+1} \sim i.i.d. N(0,1),$$

treating the state variable $Y_{a,t}$ as observed and using the method of moments approach. This step produces the following annual estimates of the parameters: $\overline{Y}_a = 33.99531$, $k_a = .8689003$, $\alpha_a = 3.49499$, $\lambda_a = 29.843719$. The mean , $\mu_{a,c}$, and volatility , $\sigma_{a,c}$, of aggregate consumption growth are set equal to their sample values.

A.6 Extracting the Model-Implied SDF for the Menzly et al. (2004) Model
The SDF in this model is given by

$$M_{t+1} = \delta^\theta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\rho}} R_{c,t+1}^{\theta-1},$$

where $R_{c,t+1}$ is the unobservable gross return on an asset that delivers aggregate consumption as its dividend each period.

Using the Campbell-Shiller log-linearization for $r_{c,t+1} \equiv \ln \left(R_{c,t+1} \right)$:

$$r_{c,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1},$$

where z_t is the log price-consumption ratio. Noting that the model implies that the equilibrium $z_t = A_0 + A_1 x_t + A_2 \sigma_t^2$, we have

$$\ln M_t = [\theta \ln \delta + (\theta-1)(\kappa_0 + \kappa_1 A_0 - A_0)] - \gamma \Delta c_{t+1}$$

$$+ (\theta-1)\kappa_1 A_1 x_{t+1} + (\theta-1)\kappa_1 A_2 \sigma_{t+1}^2 - (\theta-1)A_1 x_t - (\theta-1)A_2 \sigma_t^2. \quad (A16)$$

This is Equation (24) in the text. To obtain the time series of the SDF and its ψ component, we extract the state variables, x_t and σ_t^2, from observed data using two different procedures.

First, we extract the state variables from consumption data. To do so, we assume the same time-series specification for the aggregate consumption growth process as in Bansal and Yaron (2004), with a square-root process for the variance (as in Hansen et al. 2007) as the only exception:

$$\Delta c_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}, \tag{A17}$$

$$x_{t+1} = \rho x_t + \phi_e \sigma_t e_{t+1}, \tag{A18}$$

$$\sigma_{t+1}^2 = \sigma^2 (1 - \nu_1) + \nu_1 \sigma_t^2 + \sigma_w \sigma_t w_{t+1}. \tag{A19}$$

Note that the Bansal and Yaron (2004) model is calibrated at the monthly frequency with the *monthly* parameter values being $\mu = .0015$, $\rho = .979$, $\phi_e = .044$, $\sigma = .0078$, $\nu_1 = .987$, *and* $\sigma_w = .00029487$. We need to extract the *quarterly* state variables $x_{t,q}$ and $\sigma_{t,q}^2$. As a first step, we simulate a long sample (five million observations) from the above system, treating the given parameter values as the truth and retaining the simulated state variables. As a second step, we aggregate the simulated data into quarterly nonoverlapping observations:

$$\Delta c_{t,q} = \Delta c_t + \Delta c_{t-1} + \Delta c_{t-2}, \text{ for } t = 3, 6, 9, \ldots$$

$$x_{t,q} = x_t + x_{t-1} + x_{t-2}$$

$$\sigma_{t,q}^2 = \sigma_t^2 + \sigma_{t-1}^2 + \sigma_{t-2}^2.$$

As a third step, we estimate the model parameters in Equations (A17)–(A19) using these quarterly observations and treating the state variables as observed. This step produces the following quarterly estimates of the parameters:

$$\rho_q = \rho_m^3 = .9383137,$$

$$\nu_{1,q} = \nu_{1,m}^3 = .9615048,$$

$$\mu_q = 3 \times \mu_m = .0045,$$

$$\sigma_q^2 = Mean\left(\sigma_{t,q}^2\right) = .0001822490,$$

$$\phi_{e,q} = \sqrt{\frac{Var\left(x_{t+1,q} - \rho_q x_{t,q}\right)}{\sigma_q^2}} = .1084845,$$

$$\sigma_{w,q} = \sqrt{\frac{Var\left(\sigma_{t+1,q}^2 - \sigma_q^2 \left(1 - \nu_{1,q}\right) - \nu_{1,q} \sigma_{t,q}^2\right)}{\sigma_q^2}} = 0.0007328592,$$

where the variables with subscript m are the monthly calibrated values, and the means and variances are the ones obtained in the simulated sample. As a fourth step, we run a Bayesian smoother through the historical quarterly consumption growth treating the quarterly parameters as being known with certainty. The smoother produces estimates of the quarterly state variables $\hat{x}_{t,q}$ and $\hat{\sigma}_{t,q}^2$.

The same steps can be applied to obtain the parameter estimates and, therefore, the time series of the state variables at the annual frequency. In this case, we have $\rho_a = .7751617$, $\nu_{1,a} = .8546845$, $\mu_a = .018$, $\sigma_a^2 = .0007299038$, $\phi_{e,a} = .3853643$, and $\sigma_{w,a} = .00270020$.

Using the point estimates of the parameters and the extracted time series of the state variables at the relevant frequency, the SDF and its missing ψ component are obtained from Equation (24).

Our second procedure for extracting the state variables relies on asset market data. For the log-linearized version of the model, the observable log market-wide price-dividend ratio, $z_{m,t}$, and the log gross risk-free rate, $r_{f,t}$, are affine functions of the state variables, x_t and σ_t^2. Therefore, Constantinides and Ghosh (2011) argue that these affine functions may be inverted to express

Table A3
Bounds for RRA, quarterly data, 1947:Q1-2009:Q4

	HJ-bound	$Q1/Q2$-bounds	$M1/M2$-bounds	$\Psi 1/\Psi 2$-bounds
A. State variables extracted from consumption				
CC	9	16/14	14/14	19/21
MSV	31	41/38	41/42	60/61
BY	>100	$>100/>100$	$>100/>100$	$>100/>100$
PST	69	93/86	112/106	86/85
B. State variables extracted from asset prices				
CC	18	39/43	33/46	47/48
MSV	69	90/84	$>100/>100$	$>100/>100$
BY	4	5/5	5/5	5/5

The table reports the minimum values of the utility curvature parameter γ at which the model-implied SDF satisfies the HJ (Column 1), Q (Column 2), M (Column 3), and Ψ (Column 4) bounds using quarterly data for 1947:Q1-2009:Q4. Columns 2-4 has two entries in each cell that correspond to whether the filtered ψ^*-component of the SDF and, therefore, the filtered SDF are estimated using equation (6), reported on the left, or Equation (4), reported on the right. Panels A and B present results when the models' state variables are extracted from consumption data and asset market data, respectively. CC, MSV, BY, and PST denote, respectively, the models of Campbell and Cochrane (1999), Menzly et al. (2004), Bansal and Yaron (2004), and Piazzesi et al. (2007).

the unobservable state variables, x_t and σ_t^2, in terms of the observables, $z_{m,t}$ and $r_{f,t}$. Following this approach, the pricing kernel in Equation (A16) can be expressed as a function of observable variables:

$$\ln M_t = c_1' - \gamma \Delta c_t + c_3' \left(r_{f,t} - \frac{1}{\kappa_1} r_{f,t-1} \right) + c_4' \left(z_{m,t} - \frac{1}{\kappa_1} z_{m,t-1} \right), \qquad (A20)$$

where the parameters (c_1', c_3', c_4') are functions of the underlying time-series and preference parameters.

Since the model is calibrated at the monthly frequency, we obtain the pricing kernels at the quarterly and annual frequencies by aggregating the monthly kernels. For instance, the quarterly pricing kernel, M^q, is obtained as

$$\ln M_t^q = -\gamma \Delta^q c_t + \ln \psi_t^q,$$

where $\Delta^q c_t$ denotes the quarterly log-consumption difference and $\ln \psi_t^q$ is given by

$$\ln \psi_t^q = 3c_1' + \sum_{i=0}^{2} \left[c_3' \left(r_{f,t-i} - \kappa_1 r_{f,t-i-1} \right) + c_4' \left(z_{m,t-i} - \kappa_1 z_{m,t-i-1} \right) \right].$$

Therefore, using the monthly calibrated parameter values from Bansal and Yaron (2004) and the historical monthly time series of the market-wide price-dividend ratio and risk-free rate, we obtain the time series of the SDF and its missing component at the quarterly and annual frequencies from the above two equations.

A.7 Additional Robustness Checks
A.7.1 Entropy bounds when all model parameters are simultaneously allowed to vary. In the empirical analysis on the entropy bounds, we have focused on one-dimensional bounds as a function of the risk-aversion parameter, γ, while fixing the other parameters at the authors' calibrated values. In other words, we have computed the minimum values of γ at which the model-implied SDFs satisfy the HJ, Q, M, and Ψ bounds, while holding the remaining model parameters fixed at their calibrated values. As a robustness check, in this section, we compute the minimum values of γ at which the model-implied SDFs satisfy the bounds, while allowing the remaining

model parameters to simultaneously vary over two standard error intervals around their calibrated values.

For the external habit models of Campbell and Cochrane (1999) and Menzly et al. (2004), the model-implied SDFs are obtained by extracting the surplus consumption ratio from aggregate consumption data. While the state variable also may be extracted from the price-dividend ratio, the Menzly et al. (2004) model admits a closed-form solution for the price-dividend ratio only for $\gamma = 1$, and this motivates our choice for the extraction of the state variable in the external habit models. For the Bansal and Yaron (2004) long-run risks model, on the other hand, we extract the two state variables by inverting the closed-form solutions for the price-dividend ratio and risk-free rate. While the state variables also can be extracted from aggregate consumption data using Bayesian smoothing procedures, the computing time makes it prohibitively expensive to do this, while allowing all the parameters to vary simultaneously (since the Bayesian smoothing would have to be computed for each of set of parameter values considered). Finally, for the Piazzesi et al. (2007) model, the state variable is directly observable from the BEA tables and, therefore, does not need to be extracted from either consumption or asset market data.

The results are presented in Table A2. The table shows that, for each model, the *HJ*, *Q*, *M*, and Ψ bounds are satisfied for smaller values of γ when the other parameters are allowed to vary simultaneously compared to Tables 6 and 7 where the other parameters are held fixed. However, as in the latter tables, the *CC*, *MSV*, and *PST* models still require much larger values of risk aversion to satisfy the bounds compared with the authors' calibrated values at the quarterly frequency.

A.7.2 Entropy bounds when the risk-free rate is included as an additional test asset.
In the empirical analysis, we have used the *excess* returns (in excess of the risk-free rate) on a broad cross-section of risky assets to extract the SDF and obtain entropy bounds for the SDF and its components. As a robustness check, we repeat the empirical exercise using as test assets the gross returns (instead of excess returns) on the cross section of size- and book-to-market-equity-sorted, momentum-sorted, and industry-sorted portfolios, and the return on the risk-free asset.

In this case, the relevant Euler equation is

$$\mathbf{1}_N = \mathbb{E}[m(\theta, t)\psi_t \mathbf{R}_t],$$

where $\mathbf{R}_t \in \mathbb{R}^N$ is a vector of gross returns and $\mathbf{1}_N$ is an N-dimensional vector of ones. Under weak regularity conditions, the above pricing restrictions for the SDF can be rewritten as

$$\bar{\psi}^{-1}\mathbf{1}_N = \mathbb{E}^\Psi[m(\theta, t)\mathbf{R}_t]$$

or as

$$\bar{M}^{-1}\mathbf{1}_N = \mathbb{E}^Q[\mathbf{R}_t],$$

where $\bar{x} \equiv \mathbb{E}[x_t]$, $\frac{\psi_t}{\bar{\psi}} = \frac{d\Psi}{dP}$, and $\frac{M_t}{\bar{M}} = \frac{dQ}{dP}$. Therefore, Equations (4)–(7) can be reformulated, respectively, as Equations (A21)–(A24) below:

$$\hat{\Psi} \equiv \underset{\Psi}{\arg\min} \, D(\Psi \parallel P) \equiv \underset{\Psi}{\arg\min} \int \frac{d\Psi}{dP} \ln \frac{d\Psi}{dP} dP \text{ s.t. } \bar{\psi}^{-1}\mathbf{1}_N = \mathbb{E}^\Psi[m(\theta, t)\mathbf{R}_t], \qquad (A21)$$

with its dual solution given (up to a positive scale constant) by

$$\hat{\psi}_t = \frac{e^{\lambda(\theta)'\left[m(\theta, t)\mathbf{R}_t - \bar{\psi}^{-1}\mathbf{1}_N\right]}}{\sum\limits_{t=1}^{T} e^{\lambda(\theta)'\left[m(\theta, t)\mathbf{R}_t - \bar{\psi}^{-1}\mathbf{1}_N\right]}} = \frac{e^{\lambda(\theta)'m(\theta, t)\mathbf{R}_t}}{\sum\limits_{t=1}^{T} e^{\lambda(\theta)'m(\theta, t)\mathbf{R}_t}}, \quad \forall t$$

where $\lambda(\theta) \in \mathbb{R}^N$ is the solution to the following unconstrained convex problem

$$\lambda(\theta) \equiv \underset{\lambda}{\arg\min} \, \frac{1}{T} \sum_{t=1}^{T} e^{\lambda'\left[m(\theta, t)\mathbf{R}_t - \bar{\psi}^{-1}\mathbf{1}_N\right]};$$

$$\hat{Q} \equiv \operatorname*{arg\,min}_{Q} D(Q \parallel P) \equiv \operatorname*{arg\,min}_{Q} \int \frac{dQ}{dP} \ln \frac{dQ}{dP} dP \quad \text{s.t.} \quad \bar{M}^{-1}\mathbf{1}_N = \mathbb{E}^Q[\mathbf{R}_t], \tag{A22}$$

with its dual solution given (up to a positive scale constant) by

$$\hat{M}_t = \frac{e^{\lambda'\mathbf{R}_t}}{\displaystyle\sum_{t=1}^{T} e^{\lambda'\mathbf{R}_t}}, \quad \forall t,$$

where $\lambda \in \mathbb{R}^N$ is the solution to

$$\lambda(\theta) \equiv \operatorname*{arg\,min}_{\lambda} \frac{1}{T} \sum_{t=1}^{T} e^{\lambda'\left[\mathbf{R}_t - \bar{M}^{-1}\mathbf{1}_N\right]};$$

$$\hat{\Psi} \equiv \operatorname*{arg\,min}_{\Psi} D(P \parallel \Psi) \equiv \operatorname*{arg\,min}_{\Psi} \int \ln \frac{dP}{d\Psi} dP \quad \text{s.t.} \quad \bar{\psi}^{-1}\mathbf{1}_N = \mathbb{E}^\Psi[m(\theta,t)\mathbf{R}_t], \tag{A23}$$

with its dual solution given (up to a positive scale constant) by

$$\hat{\psi}_t = \frac{1}{T\left[1 + \lambda(\theta)'\left(m(\theta,t)\mathbf{R}_t - \bar{\psi}^{-1}\mathbf{1}_N\right)\right]}, \quad \forall t,$$

where $\lambda(\theta) \in \mathbb{R}^N$ is the solution to

$$\lambda(\theta) \equiv \operatorname*{arg\,min}_{\lambda} -\sum_{t=1}^{T} \log\left(1 + \lambda'\left(m(\theta,t)\mathbf{R}_t - \bar{\psi}^{-1}\mathbf{1}_N\right)\right);$$

$$\hat{Q} \equiv \operatorname*{arg\,min}_{Q} D(P \parallel Q) \equiv \operatorname*{arg\,min}_{Q} \int \ln \frac{dP}{dQ} dP \quad \text{s.t.} \quad \bar{M}^{-1}\mathbf{1}_N = \mathbb{E}^Q[\mathbf{R}_t] \tag{A24}$$

with its dual solution given (up to a positive scale constant) by

$$\hat{M}_t = \frac{1}{T\left[1 + \lambda(\theta)'\left(\mathbf{R}_t - \bar{M}^{-1}\mathbf{1}_N\right)\right]}, \quad \forall t$$

where $\lambda(\theta) \in \mathbb{R}^N$ is the solution to

$$\lambda(\theta) \equiv \operatorname*{arg\,min}_{\lambda} -\sum_{t=1}^{T} \log\left(1 + \lambda'\left(\mathbf{R}_t - \bar{M}^{-1}\mathbf{1}_N\right)\right).$$

Two observations are in order about the above results. First, looking at the dual optimizations, it is clear that different $\bar{M}$ and $\bar{\psi}$ will now matter in determining the solution; that is, changes in the means will change the estimated SDF, and not simply as a scaling. Second, $\bar{M}$ can be calibrated easily, since from the Euler equation we have

$$\bar{M} \equiv \mathbb{E}[m(\theta,t)\psi_t] = \mathbb{E}\left[1/R_t^f\right],$$

and, therefore, can be estimated using a sample analog. $\bar{\psi}$, on the other hand, can be recovered from

$$\bar{M} \equiv \mathbb{E}[m(\theta,t)\psi_t] = Cov(m(\theta,t);\psi_t) + \bar{m}\bar{\psi},$$

$$\therefore \bar{\psi} = \frac{\bar{M} - Cov(m(\theta,t);\psi_t)}{\bar{m}}.$$

Therefore, to calibrate $\bar{\psi}$, we can follow the following iterative procedure:

1. Set $\bar{\psi} = \frac{\bar{M}}{\bar{m}} = \frac{\frac{1}{T}\sum_{t=1}^{T} 1/R_t^f}{\frac{1}{T}\sum_t^T m(\theta,t)}$ as a starting guess.

2. Given $\bar{\psi}$, use the above entropy minimization procedures to estimate $\left\{\hat{\psi}_t\right\}_{t=1}^T$ (up to a positive constant κ).

3. Identify the scaling constant κ using the fact that, from the Euler equation for the risk-free rate, we have (as $T \to \infty$)

$$\kappa \frac{1}{T}\sum_{t=1}^{T} m(\theta,t)\hat{\psi}_t = \frac{1}{T}\sum_{t=1}^{T} 1/R_t^f \Rightarrow \kappa = \frac{\sum_{t=1}^{T} 1/R_t^f}{\sum_{t=1}^{T} m(\theta,t)\hat{\psi}_t}.$$

4. Compute an updated $\bar{\psi}$ using

$$\bar{\psi} = \frac{\bar{M} - \kappa\widehat{Cov}\left(m(\theta,t);\hat{\psi}_t\right)}{\bar{m}} = \frac{\frac{1}{T}\sum_{t=1}^{T} 1/R_t^f - \kappa\widehat{Cov}\left(m(\theta,t);\hat{\psi}_t\right)}{\frac{1}{T}\sum_t^T m(\theta,t),}$$

   where $\widehat{Cov}(.)$ is the sample analogue based covariance estimator.

5. With the new $\bar{\psi}$ at hand, go back to Step 2 and repeat until convergence of $\bar{\psi}$ is achieved. Once converge is achieved, the exact estimate (no more up to a constant) of ψ_t is given by $\kappa \times \hat{\psi}_t$.

   Table A3 repeats the analysis in Table 6 when the set of assets consists of the gross returns (instead of excess returns) on the 6 size and book-to-market-equity-sorted portfolios of Fama-French, 10 industry-sorted portfolios, 10 momentum-sorted portfolios, and the risk-free asset. The table shows that the inclusion of the risk-free rate as an additional asset in the estimation leaves the HJ, Q, M, and Ψ bounds on the SDF and its components virtually unchanged for all the asset pricing models considered.

   The results in the other tables also remain largely similar on inclusion of the risk-free rate and are omitted for the sake of brevity.

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