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The joint distribution of Parisian and Hitting times of the Brownian motion with Application to Parisian Option Pricing

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Abstract We study the joint law of Parisian time and hitting time of a drifted Brownian motion by using a three-state semi-Markov model, obtained through perturbation. We obtain a martingale, to which we can apply the optional sampling theorem and derive the double Laplace transform. This general result is applied to address problems in option pricing. We introduce a new option related to Parisian options, being triggered when the age of an excursion exceeds a certain time or/and a barrier is hit. We obtain an explicit expression for the Laplace transform of its fair price.

Keywords Parisian options · Excursion time · three state semi-Markov model · Laplace transform

Mathematics Subject Classification (2010) 91B25 · 60K15 · 60J27 · 60J65
JEL Classification G13

1 Introduction

Parisian options were introduced by Chesney, Jeanblanc-Picqué and Yor [8] in 1997. They are similar to path-dependent barrier options where the contract is defined in terms of staying above or below a certain level for a fixed period of time, instead of just touching the barrier. The so-called excursion time denotes for the time spent between two crossovers of the fixed barrier. On the other hand, one can also add up all excursion times and consider the so-called occupation time which leads to the examination of cumulative Parisian options. This has been studied by Chesney et al. [8] and Dassios and Wu [13], Cai et al. [6] and Zhang [22]. One motivation of introducing

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Parisian options lies in the insensitivity to influential agents; it is significantly more expensive to manipulate these kind of options. Variations of the Parisian option can be found in the double sided Parisian option by Anderluh and Weide [2] or the double barrier Parisian option by Dassios and Wu [12]. American-style Parisian options have been studied by Haber et al. [16] and Chesney and Gauthier [7]. Schröder [20], [21] studies Parisian excursions and finds a convolution representation for the Brownian minimum-length excursion law. Hedging of Parisian options are developed as consequences of these results.

Even though Parisian options are not exchange traded, they are used as building blocks in structured products, such as convertible bonds, which offer the holder the right but not the obligation to convert the bond at any time to a pre-specified number of shares. Most convertible bonds contain the call provision, allowing the issuer to buy back the bond at the so-called call price, in order to manage the company's debt-equity ratio. Upon issuer's call, the holder either redeems at call price or converts. Apart from the hard call constraint, which protects the conversion privilege to be called away too early, the soft call constraint requires the stock price to be higher than a certain price level. This is sensitive to market manipulation by the issuer, which can be counteracted with the Parisian feature. The Parisian feature requires the stock price to stay above a level for a certain time. These callable convertible bonds with Parisian feature are commonly traded in the OTC market in Hong Kong , see [3], [18].

We introduce a new type of option, the so-called ParisianHit option, which in contrast to the Parisian option takes both the excursion time and the hitting time of a pre-specified barrier into account. One version of this modification, called MinParisianHit option, is triggered if either the age of an excursion above a level reaches a certain time or another barrier is hit before maturity. The MaxParisianHit on the other hand gets activated when both excursion age exceeds a certain time and a barrier is hit. The key for pricing these kind of options lies in deriving the joint law of excursion and hitting time. Here, we study excursion and hitting time using a three state semi-Markov model indicating whether the process is in a positive or negative excursion and above or below a fixed barrier. This will allow us to compute the double Laplace transform of these two stopping times, which can be inverted numerically using techniques as in Labart and Lelong [17]. Gauthier [14], [15] studies the first instant when a standard Brownian motion either spends consecutively more than a certain time above a certain level, or reaches another level, i.e. the minimum of Parisian and hitting time. Gauthier's result are presented as Laplace transforms and coincide with our Lemma 4.1 and Lemma 4.2 by setting $\mu = 0$ and $\tilde{h} \equiv 0$. In this paper we generalise these results and the concept of the Parisian time by deriving the joint probability of the Parisian and hitting time. This allows us to also find the distribution of the maximum of Parisian and hitting time.

The paper is structured as follows. In section 2 we motivate this paper with the financial application of pricing ParisianHit options. The pricing problem reduces to finding the joint distribution of Parisian and hitting time. We use the approach of a three state semi-Markov model on a perturbed Brownian motion with drift, which has been introduced by Dassios and Wu [10] and present it in section 3. This perturbed Brownian motion has the same behaviour as a drifted Brownian motion, except it moves toward the other side of the barrier by a jump of size ε each time it hits zero, disposing of the difficulty of the origin being regular. The semi-Markov process allows us to define an infinitesimal generator where the solution of the martingale problem provides us with the single Laplace transform of excursion and hitting time in section 4. Dividing up into the two possible cases in section 4.1 and 4.2 we derive an explicit form of the double Laplace transform of hitting and Parisian time for drifted Brownian motion. Section 5 is devoted to the application to option pricing and explains the MinParisianHit and MaxParisianHit option in detail. Using results about the double Laplace transform we are now able to price ParisianHit options.

2 Motivation

Following the Black-Scholes framework, let $(S_t)_{t\geq 0}$ be the stock price process following a geometric Brownian motion, i.e. solving the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

and call L the level. We define the times

$$g_{L,t}(S) = \sup\{s \leq t : S_s = L\},$$
$$d_{L,t}(S) = \inf\{s \geq t : S_s = L\}.$$

The trajectory of S between $g_{L,t}(S)$ and $d_{L,t}(S)$ is the excursion of S at level L, which straddles time t. The variables $g_{L,t}(S)$ and $d_{L,t}(S)$ are called the left and right ends of the excursion. Assuming that the interest rate r is constant, the process representing the risk neutral asset price is given by

$$S_t = S_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W_t},$$

solving the stochastic differential equation $dS_t = rS_t dt + \sigma S_t dW_t$. We denote the equivalent martingale measure by $\bar{\mathbb{Q}}$.

We define $\tau^+_{L,d}(S)$ as the first time the age of an excursion above L for the price process is greater or equal to d and $H_B(S)$ as the first hitting time of a barrier $B > L$, i.e.

$$\tau^+_{L,d}(S) = \inf\{t \geq 0 | \mathbf{1}_{S_t > L}(t - g^S_{L,t}) \geq d\},$$
$$H_B(S) = \inf\{t \geq 0 | S_t = B\}.$$

We introduce the notation

$$m = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right),$$
$$l = \frac{1}{\sigma}\ln\frac{L}{S_0},$$
$$b = \frac{1}{\sigma}\ln\frac{B}{S_0}$$

and define the process $(Z_t)_{t\geq 0} = (W_t + mt)_{t\geq 0}$. This process Z contains a drift making it impossible for us to calculate the probability exactly. Our strategy is now to tilt the sloped line back to a horizontal line. We write $S_t = S_0 e^{\sigma Z_t}$ with $Z_t = W_t + mt$. The condition $S_t \leq L$ becomes $Z_t \leq l$. Using Girsanov's theorem we introduce a new probability measure $\mathbb{Q}$, which makes Z a $\mathbb{Q}$-Brownian motion. The Radon-Nikodym derivative is given by

$$\left.\frac{d\mathbb{Q}}{d\bar{\mathbb{Q}}}\right|_{\mathscr{F}_T} = e^{mZ_T - \frac{m^2}{2}T}. \tag{2.1}$$

We define the first time at which the age of an excursion above the level l for the process $(Z_t)_{t\geq 0}$ is greater than or equal to d:

$$\tau^+_{l,d}(Z) = \inf\{t \geq 0 | \mathbf{1}_{Z_t > l}(t - g_{l,t}) \geq d\}$$
$$g_{l,t}(Z) = \sup\{u \leq t | Z_u = l\}$$

In the case where $l = 0$, we shall use the shortcut $\tau^+_d(Z)$ and $g_t(Z)$.

Our so-called MinParisianHit Option is triggered either when the age of an excursion above L reaches time d or a barrier $B > L$ is hit by the underlying price process S. More precisely, a MinParisianHit Up-and-In is activated at the minimum of both stopping times, i.e. $\min\{\tau^+_{L,d}(S), H_B(S)\}$.

The MinParisianHit Up-and-In Call option has payoff

$$(S_T - K)^+ \mathbf{1}_{\min\{\tau^+_{L,d}(S), H_B(S)\} \leq T},$$

where K denotes the strike price.
Using risk-neutral valuation and Girsanov's change of measure (2.1), the price of this option can be written in the following way.

$$\begin{aligned} &minPHC^u_i(S_0, T, K, L, d, r) \\ &= e^{-(r+\frac{1}{2}m^2)T} \int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty} (S_0 e^{\sigma z} - K) e^{mz} \mathbb{Q}_0\left(Z_T \in dz, \min\{\tau^+_d(Z), H_b(Z)\} \leq T\right) \end{aligned} \tag{2.2}$$

Hence, finding the fair price for a MinParisianHit option reduces to finding the joint probability of position at maturity and minimum of Parisian and hitting times.

Our so-called MaxParisianHit Option on the other hand is triggered, when both the barrier B is hit and the excursion age exceeds duration d above L, the payoff becomes

$$(S_T - K)^+ \mathbf{1}_{\{\max\{\tau^+_{L,d}(S), H_B(S)\} \leq T\}},$$

and the option pricing problem can be reduced in a similar way, i.e.

$$\begin{aligned} & maxPHC_i^u(S_0, T, K, L, d, r) = \\ & \quad = e^{-(r+\frac{1}{2}m^2)T} \int\limits_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty} (S_0 e^{\sigma z} - K) e^{mz} \mathbb{Q}_0(Z_T \in dz, \max\{\tau^+_d(Z), H_b(Z)\} \leq T). \end{aligned} \tag{2.3}$$

This will be discussed in further detail in section 5.2.

We can see from equation (2.2) and in further detail in section 5 that both pricing problems can be solved by determining the joint distribution of hitting and Parisian time of a drifted Brownian motion. This is our focus for the next sections 3 and 4, where our main results are presented in Propositions 4.2 and 4.3.

Instead of finding a closed form solution for the joint density of hitting and Parisian time, we focus on deriving the double Laplace transform which uniquely determines the probability distribution.

3 Perturbed Brownian motion and the Martingale problem

This section is the most technical one and we give a brief outline of the steps that we follow: The property of the sample path of Brownian motions of being regular around the origin zero results in the occurrence of infinitely many small excursions. In order to counteract this problem we perturb the Brownian motion by a small jump at the origin. The construction can be found in equations (3.5) - (3.8) and follows Dassios and Wu [10]. Next, we construct a continuous time finite state Markov process in equation (3.9), which distinguishes between whether the process is below 0 or above 0 or the barrier b. This Markov process has an associated infinitesimal generator and we can formulate the martingale problem in equation (3.16). We construct a martingale of the form $f_i(U_t(X), t) = e^{-\beta t} h_i(U_t(X))$. This function f looks arbitrary at first sight, however it is chosen in such a way that after applying Doob's optional sampling theorem in equation (3.19) it yields the Laplace transform of the desired stopping times.

What is important to note is that this outlined procedure is not limited to ParisianHit option pricing within the Black-Scholes framework but can be used to solve similar problems where the stochastic process does not follow a Brownian motion.

3.1 Definition

For any stochastic process Y we define for fixed $t > 0$ the times

$$
\begin{aligned}
g_t(Y) &= \sup\{s \le t | \mathrm{sgn}(Y_s) \ne \mathrm{sgn}(Y_t)\}, && (3.1)\\
d_t(Y) &= \inf\{s \ge t | \mathrm{sgn}(Y_s) \ne \mathrm{sgn}(Y_t)\}, && (3.2)\\
\tau_d^+(Y) &= \inf\{t > 0 | (t - g_t(Y))\mathbf{1}_{Y_t > 0} \ge d\}, && (3.3)\\
H_b(Y) &= \inf\{t \ge 0 | Y_t = b\}. && (3.4)
\end{aligned}
$$

The time interval $(d_t(Y), g_t(Y))$ is the excursion interval straddling time t and the time $g_t(Y) - d_t(Y)$ is called excursion time. $\tau_d^+(Y)$ denotes the first time the process Y spends time d above zero, the so-called Parisian time above zero.

Let W^μ, with $W_t^\mu = W_t + \mu t$, be a Brownian motion with drift $\mu \ge 0$ and $W_0^\mu = 0$, where W is a standard Brownian motion under the probability measure $\mathbb{Q}$. We notice that the origin zero is a regular point of the process, resulting in the occurrence of infinitely many small excursions. In order to counteract this problem, the perturbed Brownian motion $W^{\varepsilon,\mu}$ has been introduced by Dassios and Wu [10] as follows. Define the sequence of stopping times for $\varepsilon > 0$ and $n \in \mathbb{N}_0$,

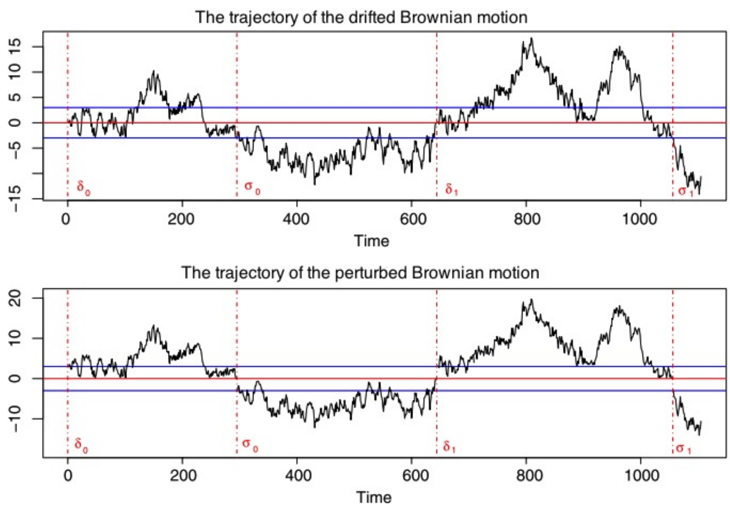
$$
\begin{aligned}
\delta_0 &= 0, && (3.5)\\
\sigma_n &= \inf\{t > \delta_n | W_t^\mu = -\varepsilon\}, && (3.6)\\
\delta_{n+1} &= \inf\{t > \sigma_n | W_t^\mu = 0\}. && (3.7)
\end{aligned}
$$

Define the perturbed drifted Brownian motion

$$
W_t^{\varepsilon,\mu} = \begin{cases} W_t^\mu + \varepsilon & \text{, if } \delta_n \le t < \sigma_n \\ W_t^\mu & \text{, if } \sigma_n \le t < \delta_{n+1} \end{cases} \tag{3.8}
$$

By introducing the jumps of size ε towards the other side of zero whenever zero is hit by W^μ we get a process $W^{\varepsilon,\mu}$ with a very clear structure of excursions above and below zero, making zero an irregular point. This construction has been introduced by Dassios and Wu [10]. See Figure 3.1 for illustration. With the superscript ε we denote quantities based on the perturbed process $W^{\varepsilon,\mu}$, e.g. $H_b(W^{\varepsilon,\mu}) = \inf\{t \ge 0 | W_t^{\varepsilon,\mu} = b\}$. By construction we have $W_t^{\varepsilon,\mu} \xrightarrow{a.s.} W_t^\mu$ for all $t \ge 0$, as ε approaches zero. The quantities defined based on $W_t^{\varepsilon,\mu}$ also converge to those of the drifted Brownian motion W_t^μ. This has been proven in Dassios and Wu [10], [11] and Lim [19].

Fig. 3.1 Sample paths of W^{μ} and $W^{\varepsilon,\mu}$, see Dassios and Wu [10]



3.2 Markov Process construction

It is clear from the definition above that we are actually considering two states, namely the state when the stochastic process $W^{\varepsilon,\mu}$ is below zero and the state when it is above zero. Our final goal is to find the joint density of the Parisian time above or below zero and the hitting time of a specified barrier b, $H_b(W^{\mu})$. Hence, we construct an artificial absorbing state for the time the process $W^{\varepsilon,\mu}$ spends above barrier $b > 0$. For each state above and below zero we are now interested in the time it spends in it. We introduce a new process based on $W^{\varepsilon,\mu}$ by

$$X_t = \begin{cases} 2 & \text{, if } W_t^{\varepsilon,\mu} \geq b \\ 1 & \text{, if } 0 < W_t^{\varepsilon,\mu} < b \\ -1 & \text{, if } W_t^{\varepsilon,\mu} \leq 0. \end{cases} \tag{3.9}$$

Clearly, definitions (3.1), (3.2), (3.3), and (3.4) hold similarly for the process X. We define state 2 to be an absorbing state, i.e. once b is hit, the process does not return to state 1 anymore.

Define $U_t(X) := t - g_t(X)$ to be the time elapsed in the current state, either state -1 or state 1 and 2 combined. Note, that $U_t(X)$ only distinguishes between above or below zero and converges to $U_t(W^{\mu}) = t - g_t(W^{\mu})$, the time elapsed above or below zero in the current excursion of the drifted Brownian motion W^{μ}. If the notation is unambiguous, we will abbreviate the definition of the time elapsed for the Brownian motion,

$U_t = U_t(W^\mu)$. $(X_t, U_t(X))$ becomes a Markov process. Hence, X is a three-state semi-Markov process with state space $\{2,1,-1\}$. The transition intensities $\lambda_{i,j}(u)$ for X satisfy

$$\mathbb{Q}\left(X_{t+\Delta t} = j, i \neq j | X_t = i, U_t(X) = u\right) = \lambda_{i,j}(u)\Delta t + o(\Delta t) \tag{3.10}$$

$$\mathbb{Q}\left(X_{t+\Delta t} = i | X_t = i, U_t(X) = u\right) = 1 - \sum_{j\neq i} \lambda_{i,j}(u)\Delta t + o(\Delta t) \tag{3.11}$$

for $i, j = 2, 1, -1$. Define the survival probability and transition density by

$$\bar{Q}_i(t) = e^{-\int_0^t \sum\limits_{j\neq i} \lambda_{i,j}(v)dv}, \tag{3.12}$$

$$q_{i,j}(t) = \lambda_{i,j}(t)\bar{Q}_i(t). \tag{3.13}$$

In order to simplify notations we define $\hat{Q}_{i,j}(\beta)$ and $\tilde{Q}_{i,j}(\beta)$ to be

$$\hat{Q}_{i,j}(\beta) = \int_0^d e^{-\beta s} q_{i,j}(s)ds, \tag{3.14}$$

$$\tilde{Q}_{i,j}(\beta) = \int_0^\infty e^{-\beta s} q_{i,j}(s)ds. \tag{3.15}$$

3.3 Martingale problem

Having constructed the process X and its time elapsed in the current state, we now consider a bounded function $f : \{2,1,-1\} \times \mathbb{R}^2 \to \mathbb{R}$. The infinitesimal generator $\mathscr{A}$ is an operator making

$$f(X_t, U_t(X), t) - \int_0^t \mathscr{A} f(X_s, U_s(X), s)ds \tag{3.16}$$

a martingale. We shall use the shortcut $f_i(z,u) = f(i,z,u)$ and $\mathscr{A} f_{X_t}(U_t(X),t) = \mathscr{A} f(X_t, U_t(X), t)$.

Hence, solving $\mathscr{A} f = 0$, subject to certain conditions, will provide us with martingales of the form $f_{X_t}(U_t(X),t)$, to which we can apply the optional sampling theorem to obtain the Laplace transforms of interest. We have for the generator

$$\begin{aligned}\mathscr{A} f_1(u,t) = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial u} + \lambda_{1,1}(u)\left(f_{-1}(0,t) - f_1(u,t)\right) + \\ + \lambda_{1,2}(u)\left(f_2(u,t) - f_1(u,t)\right), \\ \mathscr{A} f_{-1}(u,t) = \frac{\partial f_{-1}}{\partial t} + \frac{\partial f_{-1}}{\partial u} + \lambda_{-1,1}(u)\left(f_1(0,t) - f_{-1}(u,t)\right).\end{aligned}$$

Since we are not interested in what happens after the absorbing state 2 has been reached, we do not define $\mathscr{A} f_2$, the generator starting from state 2.

We assume the function f having the form $f_i(u,t) = e^{-\beta t} h_i(u)$, where $\beta \in \mathbb{R}^+$ is a positive constant, and solve $\mathscr{A} f \equiv 0$ with the constraints $h_1(d) = B$ and $h_{-1}(\infty) = 0$ with constant B. Since state 2 is an absorbing state, we may assign any bounded function at will. We choose $h_2(u) = A\tilde{h}(u)$, where A is an arbitrary constants. The function $\tilde{h}$ will be motivated and defined in the proof of Proposition 4.2. The intuition behind choosing the constraint $h_{-1}(\infty) = 0$ is, that in we are not concerned with the time elapsed below zero, hence, we let the excursion window below zero approach infinity. A and B on the other hand are constants, indicating different scenarios and clarified in Lemma 3.2.

The reason for choosing this form for the function f is our objective to derive the Laplace transform of stopping times.

Lemma 3.1. *Using the conditions above, the initial value of the function* $f_1(0,0) = h_1(0)$ *is given by*

$$h_1(0) = \frac{Be^{-\beta d}\bar{Q}_1(d) + A\int_0^d e^{-\beta w}\tilde{h}(w) q_{1,2}(w)dw}{1 - \tilde{Q}_{-1,1}(\beta)\hat{Q}_{1,-1}(\beta)}. \tag{3.17}$$

Proof $\mathscr{A} f \equiv 0$ transforms into

$$\frac{dh_1(u)}{du} - (\beta + \lambda_{1,-1}(u) + \lambda_{1,2}(u))h_1(u) + \lambda_{1,-1}(u)h_{-1}(0) + A\lambda_{1,2}(u)\tilde{h}(u) = 0,$$
$$\frac{dh_{-1}(u)}{du} - (\beta + \lambda_{-1,1}(u))h_{-1}(u) + \lambda_{-1,1}(u)h_1(0) = 0.$$

Using the integrating factor method for ordinary differential equations and the constraints we find

$$\begin{aligned} h_1(u) &= Be^{-\int_u^d \beta \lambda_{1,-1}(v) + \lambda_{1,2}(v)dv} + \int_u^d \big(\lambda_{1,-1}(w)h_{-1}(0) + \\ &\qquad + A\lambda_{1,2}(w)\tilde{h}(w)\big)e^{-\int_u^w \beta\lambda_{1,-1}(v)+\lambda_{1,2}(v)dv}dw\,, \quad 0 \le u \le d \\ h_{-1}(u) &= h_1(0)\int_u^\infty \lambda_{-1,1}(w)e^{-\int_u^w \beta + \lambda_{-1,1}(v)dv}dw\,, \qquad\qquad u \ge 0. \end{aligned}$$

Setting $u = 0$ and solving the system of equations gives us

$$\begin{aligned} h_1(0) &= \frac{Be^{-\int_0^d \beta + \lambda_{1,-1}(v) + \lambda_{1,2}(v)dv} + A\int_0^d \lambda_{1,2}(w)\tilde{h}(w)e^{-\int_0^w \beta+\lambda_{1,-1}(v)+\lambda_{1,2}(v)dv}dw}{1 - \int_0^\infty \lambda_{-1,1}(w)e^{-\int_0^w \beta+\lambda_{-1,1}(v)dv}dw \int_0^d \lambda_{1,-1}(w)e^{-\int_0^w \beta+\lambda_{1,-1}(v)+\lambda_{1,2}(v)dv}dw} \\ &= \frac{Be^{-\beta d}\bar{Q}_1(d) + A\int_0^d e^{-\beta w}\tilde{h}(w)q_{1,2}(w)dw}{1 - \tilde{Q}_{-1,1}(\beta)\hat{Q}_{1,-1}(\beta)}, \end{aligned}$$

where $\bar{Q}_i(t)$, $q_{1,2}(t)$, $\lambda_{i,j}(u)$, $\hat{Q}_{i,j}(\beta)$ and $\tilde{Q}_{i,j}(\beta)$ have been defined in (3.12), (3.13), (3.10), (3.14) and (3.15).

For the transition densities we use results from Borodin and Salminen [5] (formula (2.0.2) and formulae (3.0.2), (3.0.6)). Without loss of generality we assume $b > \varepsilon > 0$. Therefore, it is not possible to go straight from state -1 to state 2 and vice versa, i.e. $q_{-1,2}(t) = q_{2,-1}(t) = 0$.

With the definition $H_{a,b}(Y) = \inf\{t \geq 0 | Y_t = a \text{ or } Y_t = b\}$ for the first exit time of interval (a,b) with $a,b \in \mathbb{R}$ and $a < b$ by a general stochastic process Y, and the function

$$ss_t(x,y) = \sum_{k=-\infty}^{\infty} \frac{(2k+1)y - x}{\sqrt{2\pi t^3}} e^{-\frac{((2k+1)y-x)^2}{2t}},$$

(see e.g. Borodin and Salminen [5], Appendix 2, 9. Theta functions of imaginary argument and related functions), the quantities $q_{i,j}(t)$, $\hat{Q}_{i,j}(\beta)$, $\tilde{Q}_{i,j}(\beta)$ and $\bar{Q}_i(d)$ can be calculated:

$$\begin{aligned}
q_{1,-1}(t) &= \frac{1}{dt}\mathbb{P}_\varepsilon(H_{0,b}(W^{\varepsilon,\mu}) \in dt, W^{\varepsilon,\mu}_{H_{0,b}} = 0) = e^{-\mu\varepsilon - \frac{\mu^2 t}{2}} ss_t(b-\varepsilon, b) \\
&= e^{-\mu\varepsilon - \frac{\mu^2 t}{2}} \sum_{k=-\infty}^{\infty} \frac{\varepsilon + 2kb}{\sqrt{2\pi t^3}} e^{-\frac{(\varepsilon+2kb)^2}{2t}} \\
&= e^{-\mu\varepsilon - \frac{\mu^2 t}{2}} \sum_{k=0}^{\infty} \left[\frac{2kb+\varepsilon}{\sqrt{2\pi t^3}} e^{-\frac{(2kb+\varepsilon)^2}{2t}} - \frac{2kb-\varepsilon}{\sqrt{2\pi t^3}} e^{-\frac{(2kb-\varepsilon)^2}{2t}} \right] - \frac{\varepsilon}{\sqrt{2\pi t^3}} e^{-\frac{(\varepsilon+\mu t)^2}{2t}}
\end{aligned}$$

$$q_{-1,1}(t) = \frac{\varepsilon}{\sqrt{2\pi t^3}} e^{-\frac{(\varepsilon-\mu t)^2}{2t}}$$

$$\begin{aligned}
q_{1,2}(t) &= \frac{1}{dt}\mathbb{P}_\varepsilon(H_{0,b}(W^{\varepsilon,\mu}) \in dt, W^{\varepsilon,\mu}_{H_{0,b}} = b) = e^{\mu(b-\varepsilon) - \frac{\mu^2 t}{2}} ss_t(\varepsilon, b) \\
&= e^{\mu(b-\varepsilon) - \frac{\mu^2 t}{2}} \sum_{k=-\infty}^{\infty} \frac{b - \varepsilon + 2kb}{\sqrt{2\pi t^3}} e^{-\frac{(b-\varepsilon+2kb)^2}{2t}} \\
&= e^{\mu(b-\varepsilon) - \frac{\mu^2 t}{2}} \sum_{k=0}^{\infty} \frac{(2k+1)b - \varepsilon}{\sqrt{2\pi t^3}} e^{-\frac{((2k+1)b-\varepsilon)^2}{2t}} - \frac{(2k+1)b+\varepsilon}{\sqrt{2\pi t^3}} e^{-\frac{((2k+1)b+\varepsilon)^2}{2t}}
\end{aligned}$$

$$\begin{aligned}
\hat{Q}_{1,2}(\beta) = \sum_{k=0}^{\infty} e^{(\mu-(2k+1)\sqrt{2\beta+\mu^2})b} &\left[e^{\varepsilon(\sqrt{2\beta+\mu^2}-\mu)} \mathcal{N}\left(-\frac{(2k+1)b-\varepsilon}{\sqrt{d}} + \sqrt{(2\beta+\mu^2)d} \right) - \right. \\
&\left. - e^{-\varepsilon(\sqrt{2\beta+\mu^2}+\mu)} \mathcal{N}\left(-\frac{(2k+1)b+\varepsilon}{\sqrt{d}} + \sqrt{(2\beta+\mu^2)d} \right) \right] + \\
+ e^{(\mu+(2k+1)\sqrt{2\beta+\mu^2})b} &\left[e^{-\varepsilon(\sqrt{2\beta+\mu^2}+\mu)} \mathcal{N}\left(-\frac{(2k+1)b-\varepsilon}{\sqrt{d}} - \sqrt{(2\beta+\mu^2)d} \right) - \right. \\
&\left. - e^{\varepsilon(\sqrt{2\beta+\mu^2}-\mu)} \mathcal{N}\left(-\frac{(2k+1)b+\varepsilon}{\sqrt{d}} - \sqrt{(2\beta+\mu^2)d} \right) \right]
\end{aligned}$$

$$\tilde{Q}_{-1,1}(\beta) = e^{(\mu-\sqrt{2\beta+\mu^2})\varepsilon}$$

$$\hat{Q}_{1,-1}(\beta) = \int_{s=0}^{d} e^{-\beta s} e^{-\mu\varepsilon-\frac{\mu^2 s}{2}} \sum_{k=0}^{\infty} \left[\frac{2kb+\varepsilon}{\sqrt{2\pi s^3}} e^{-\frac{(2kb+\varepsilon)^2}{2s}} - \frac{2kb-\varepsilon}{\sqrt{2\pi s^3}} e^{-\frac{(2kb-\varepsilon)^2}{2s}} \right] -$$

$$- e^{-\beta s} \frac{\varepsilon}{\sqrt{2\pi s^3}} e^{-\frac{(\varepsilon+\mu t)^2}{2s}} ds$$

$$= e^{-\mu\varepsilon} \Bigg\{ \sum_{k=0}^{\infty} \Bigg[e^{-\sqrt{2\beta+\mu^2}(2kb+\varepsilon)} \mathcal{N}\left(-\frac{2kb+\varepsilon}{\sqrt{d}} + \sqrt{(2\beta+\mu^2)d} \right) +$$

$$+ e^{\sqrt{2\beta+\mu^2}(2kb+\varepsilon)} \mathcal{N}\left(-\frac{2kb+\varepsilon}{\sqrt{d}} - \sqrt{(2\beta+\mu^2)d} \right) - e^{-\sqrt{2\beta+\mu^2}(2kb-\varepsilon)} \times$$

$$\times \mathcal{N}\left(-\frac{2kb-\varepsilon}{\sqrt{d}} + \sqrt{(2\beta+\mu^2)d} \right) - e^{\sqrt{2\beta+\mu^2}(2kb-\varepsilon)} \mathcal{N}\left(-\frac{2kb-\varepsilon}{\sqrt{d}} - \sqrt{(2\beta+\mu^2)d} \right) \Bigg] -$$

$$- e^{-\sqrt{2\beta+\mu^2}\varepsilon} \mathcal{N}\left(-\frac{\varepsilon}{\sqrt{d}} + \sqrt{(2\beta+\mu^2)d} \right) - e^{\sqrt{2\beta+\mu^2}\varepsilon} \mathcal{N}\left(-\frac{\varepsilon}{\sqrt{d}} - \sqrt{(2\beta+\mu^2)d} \right) \Bigg\}$$

$$\bar{Q}_1(d) = \mathbb{P}_\varepsilon(H_0(W^{\varepsilon,\mu}) > d, H_b(W^{\varepsilon,\mu}) > d)$$

$$= \int_d^\infty e^{-\frac{\mu^2 t}{2}} \left(e^{-\mu\varepsilon} ss_t(b-\varepsilon, b) + e^{\mu(b-\varepsilon)} ss_t(\varepsilon, b) \right) dt$$

$$= \sum_{k=0}^{\infty} \Bigg\{ e^{-\mu(2kb+2\varepsilon)} \mathcal{N}\left(\frac{2kb+\varepsilon}{\sqrt{d}} - \mu\sqrt{d} \right) - e^{2kb\mu} \mathcal{N}\left(-\frac{2kb+\varepsilon}{\sqrt{d}} - \mu\sqrt{d} \right) -$$

$$- e^{-2kb\mu} \mathcal{N}\left(\frac{2kb-\varepsilon}{\sqrt{d}} - \mu\sqrt{d} \right) + e^{\mu(2kb-2\varepsilon)} \mathcal{N}\left(-\frac{2kb-\varepsilon}{\sqrt{d}} - \mu\sqrt{d} \right) +$$

$$+ e^{-2kb\mu} \mathcal{N}\left(\frac{(2k+1)b-\varepsilon}{\sqrt{d}} - \mu\sqrt{d} \right) - e^{2kb\mu+2\mu(b-\varepsilon)} \mathcal{N}\left(-\frac{(2k+1)b-\varepsilon}{\sqrt{d}} - \mu\sqrt{d} \right) -$$

$$- e^{-2kb\mu-2\mu\varepsilon} \mathcal{N}\left(\frac{(2k+1)b+\varepsilon}{\sqrt{d}} - \mu\sqrt{d} \right) + e^{2kb\mu+2\mu b} \mathcal{N}\left(-\frac{(2k+1)b+\varepsilon}{\sqrt{d}} - \mu\sqrt{d} \right) \Bigg\} -$$

$$- e^{-2\mu\varepsilon} \mathcal{N}\left(\frac{\varepsilon}{\sqrt{d}} - \mu\sqrt{d} \right) + \mathcal{N}\left(-\frac{\varepsilon}{\sqrt{d}} - \mu\sqrt{d} \right)$$

Remark 3.1. With the subscript behind the expected value we denote the starting position of any stochastic process Y, i.e. for any function f

$$\mathbb{E}_x(f(Y)) = \mathbb{E}(f(Y); Y_0 = x)$$

In the case of no subscript we assume the process to start at zero. The superscript announces under which probability measure we take the expectation, i.e.

$$\mathbb{E}^{\mathbb{P}}(f(Y)) = \int_{-\infty}^{\infty} f(x)\mathbb{P}(Y \in dx).$$

If not specified the notation should be clear.

3.4 An important Lemma

In the following we present an important lemma which is the main building block in pricing ParisianHit options.

Lemma 3.2. *For the perturbed Brownian motion with drift, we find the Laplace transform to be*

$$A\mathbb{E}_{\varepsilon}^{\mathbb{Q}}\left(e^{-\beta H_b(W^{\varepsilon,\mu})}\tilde{h}(U_{H_b}(W^{\varepsilon,\mu}))\mathbf{1}_{H_b(W^{\varepsilon,\mu})<\tau_d^+(W^{\varepsilon,\mu})}\right) + B\mathbb{E}_{\varepsilon}^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^{\varepsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\varepsilon,\mu})<H_b(W^{\varepsilon,\mu})}\right)$$
$$= \frac{Be^{-\beta d}\bar{Q}_1(d) + A\int_0^d e^{-\beta w}\tilde{h}(w)q_{1,2}(w)dw}{1-\tilde{Q}_{-1,1}(\beta)\hat{Q}_{1,-1}(\beta)}, \quad (3.18)$$

where A and B are arbitrary constants.

Proof Solving $\mathscr{A}f \equiv 0$ with constraints $h_1(d) = B$ and $h_{-1}(\infty) = 0$, provides us with a martingale of the form $\hat{M}_t := f_{X_t}(U_t(X),t) = e^{-\beta t}h_{X_t}(U_t(X))$. Recall that state 2, denoting for the perturbed Brownian motion above barrier b, is an absorbing state. Hence, we may choose h_2 to be any arbitrary bounded function. We assign h_2 to be $h_2(u) = A\tilde{h}(u)$, where A is a constant and $\tilde{h}$ is a bounded function, which will be specified in the proof of Proposition 4.2.
Let $\tau(W^{\varepsilon,\mu}) = \min\{H_b(W^{\varepsilon,\mu}), \tau_d^+(W^{\varepsilon,\mu})\}$, then optional sampling theorem on martingale $\hat{M}$ with stopping time $\tau(W^{\varepsilon,\mu}) \wedge t$ yields

$$\mathbb{E}_{\varepsilon}^{\mathbb{Q}}\left(\hat{M}_{\tau(W^{\varepsilon,\mu})\wedge t}\right) = \mathbb{E}_{\varepsilon}^{\mathbb{Q}}(\hat{M}_0). \quad (3.19)$$

$h_1(u)$ is a continuous function and therefore bounded on the compact interval $[0,d]$. Hence, there exists a constant K, such that $|h_1(U_t(X))| \leq K$ for all $U_t(X) \in [0,d]$. Furthermore, we have assumed that $h_2(u)$ is a bounded function. Therefore Lebesgue's Dominated Convergence Theorem applies, yielding for the l.h.s. of (3.19):

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}_{\varepsilon}^{\mathbb{Q}}\left(\hat{M}_{\tau(W^{\varepsilon,\mu})\wedge t}\right) &= \mathbb{E}_{\varepsilon}^{\mathbb{Q}}\left(\hat{M}_{\tau(W^{\varepsilon,\mu})}\right)\\
&= \mathbb{E}_{\varepsilon}^{\mathbb{Q}}\left(e^{-\beta H_b(W^{\varepsilon,\mu})}h_2(U_{H_b(W^{\varepsilon,\mu})}(W^{\varepsilon,\mu}))\mathbf{1}_{H_b(W^{\varepsilon,\mu})<\tau_d^+(W^{\varepsilon,\mu})}\right)+\\
&\quad+\mathbb{E}_{\varepsilon}^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^{\varepsilon,\mu})}h_1(U_{\tau_d^+(W^{\varepsilon,\mu})}(W^{\varepsilon,\mu}))\mathbf{1}_{\tau_d^+(W^{\varepsilon,\mu})<H_b(W^{\varepsilon,\mu})}\right)\\
&= A\mathbb{E}_{\varepsilon}^{\mathbb{Q}}\left(e^{-\beta H_b(W^{\varepsilon,\mu})}\tilde{h}(U_{H_b(W^{\varepsilon,\mu})}(W^{\varepsilon,\mu}))\mathbf{1}_{H_b(W^{\varepsilon,\mu})<\tau_d^+(W^{\varepsilon,\mu})}\right)+\\
&\quad+B\mathbb{E}_{\varepsilon}^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^{\varepsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\varepsilon,\mu})<H_b(W^{\varepsilon,\mu})}\right).
\end{aligned}$$

For the r.h.s. of (3.19) we have $\mathbb{E}_{\varepsilon}^{\mathbb{Q}}(\hat{M}_0)=h_1(0)$ and the claim follows from Lemma 3.1.

4 Double Laplace transform of Parisian and Hitting times

This section is the main part of the paper and devoted to finding the double Laplace transform of Parisian and hitting times. We firstly derive the limiting Laplace transform through results on the perturbed process and distinguish between the two possible scenarios $H_b(W^\mu)<\tau_d^+(W^\mu)$ and $\tau_d^+(W^\mu)<H_b(W^\mu)$.

Proposition 4.1. *The Laplace transform of the hitting and Parisian times for drifted Brownian motion W^μ is given by*

$$\begin{aligned}
&A\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)}\tilde{h}(U_{H_b})\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right)+B\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^\mu)}\mathbf{1}_{\tau_d^+(W^\mu)<H_b(W^\mu)}\right)=\\
&\quad=\left\{Be^{-\beta d}\left(\sum_{k=0}^{\infty}2\left[z(k,0,\mu)-e^{\mu b}z(k+\frac{1}{2},0,\mu)\right]-z(0,0,\mu)\right)+\right.\\
&\quad\left.+A\int_0^d e^{-\beta w}\tilde{h}(w)\sqrt{\frac{2}{\pi w^3}}e^{\mu b-\frac{\mu^2 w}{2}}\sum_{k=0}^{\infty}\left(\frac{(2k+1)^2b^2}{w}-1\right)e^{-\frac{(2k+1)^2b^2}{2w}}dw\right\}\times\\
&\quad\times\left\{\sum_{k=0}^{\infty}2\left[z(k,\beta,\mu)+\sqrt{2\beta+\mu^2}e^{-\sqrt{2\beta+\mu^2}2kb}\right]-z(0,\beta,\mu)-2\sqrt{2\beta+\mu^2}\right\}^{-1},
\end{aligned}$$

where the function z is defined as

$$\begin{aligned}
&z(k,\beta,\mu)=\\
&\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\beta+\mu^2)d}{2}-\frac{2(kb)^2}{d}}-\sqrt{2\beta+\mu^2}\left(e^{\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(-\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)+\right.\\
&\qquad\left.+e^{-\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)\right). \quad (4.1)
\end{aligned}$$

Proof In order to find the Laplace transform for the drifted Brownian motion, we take the limit from results about $W^{\varepsilon,\mu}$ and therefore we let ε approach zero in equation (3.18). In particular, notice that by construction we have $W_t^{\varepsilon,\mu} \xrightarrow{a.s.} W_t^{\mu}$ for all $t \geq 0$ as ε approaches zero. The quantities defined based on $W_t^{\varepsilon,\mu}$ also converge to those of the drifted Brownian motion W_t^{μ}. Furthermore, $e^{-\beta H_b(W^{\mu})}\tilde{h}(U_{H_b})$ and $e^{-\beta \tau_d^+(W^{\mu})}$ are both bounded functions. Recall, that U_{H_b} is the abbreviation for $U_{H_b(W^{\mu})}(W^{\mu})$. Thus dominated convergence applies to get the result for W_t^{μ},

$$\begin{aligned}
&A\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^{\mu})}\tilde{h}(U_{H_b})\mathbf{1}_{H_b(W^{\mu})<\tau_d^+(W^{\mu})}\right)+B\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^{\mu})}\mathbf{1}_{\tau_d^+(W^{\mu})<H_b(W^{\mu})}\right)\\
&\quad=\lim_{\varepsilon\to 0}A\mathbb{E}_{\varepsilon}^{\mathbb{Q}}\left(e^{-\beta H_b(W^{\varepsilon,\mu})}\tilde{h}(U_{H_b(W^{\varepsilon,\mu})}(W^{\varepsilon,\mu}))\mathbf{1}_{H_b(W^{\varepsilon,\mu})<\tau_d^+(W^{\varepsilon,\mu})}\right)+\\
&\qquad+B\mathbb{E}_{\varepsilon}^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^{\varepsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\varepsilon,\mu})<H_b(W^{\varepsilon,\mu})}\right)\\
&\quad=\lim_{\varepsilon\to 0}\frac{Be^{-\beta d}\bar{Q}_1(d)+A\int_0^d e^{-\beta w}\tilde{h}(w)q_{1,2}(w)dw}{1-\tilde{Q}_{-1,1}(\beta)\hat{Q}_{1,-1}(\beta)}
\end{aligned}\tag{4.2}$$

We refer to Dassios and Wu [10], [11] and Lim [19] for further details. Therefore, letting ε go to zero in the result of Lemma 3.2 will provide us with the Laplace transform for the drifted Brownian motion. In order to apply L'Hôpital's rule, we take the derivative with respect to ε and find for the denominator of (3.18):

$$\begin{aligned}
&\frac{\partial}{\partial\varepsilon}\left(1-\tilde{Q}_{-1,1}(\beta)\hat{Q}_{1,-1}(\beta)\right)\underset{\varepsilon\to 0}{\longrightarrow}\\
&\quad\sum_{k=0}^{\infty}\Bigg(2\sqrt{2\beta+\mu^2}\Bigg[e^{-\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(-\frac{2kb}{\sqrt{d}}+\sqrt{(2\beta+\mu^2)d}\right)-\\
&\quad-e^{\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(-\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)\Bigg]+2\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\beta+\mu^2)d}{2}-\frac{2(kb)^2}{d}}\Bigg)-\\
&\qquad-2\sqrt{2\beta+\mu^2}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)d}\right)-\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\beta+\mu^2)d}{2}}\\
&=\sum_{k=0}^{\infty}\Bigg(2\sqrt{2\beta+\mu^2}\Bigg[e^{-\sqrt{2\beta+\mu^2}2kb}-e^{-\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)-\\
&\quad-e^{\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(-\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)\Bigg]+2\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\beta+\mu^2)d}{2}-\frac{2(kb)^2}{d}}\Bigg)-\\
&\qquad-2\sqrt{2\beta+\mu^2}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)d}\right)-\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\beta+\mu^2)d}{2}}
\end{aligned}\tag{4.3}$$

For the numerator we find

$$\frac{\partial}{\partial\varepsilon}\bar{Q}_1(d)\underset{\varepsilon\to 0}{\longrightarrow}\sum_{k=0}^{\infty}\Bigg\{2\sqrt{\frac{2}{\pi d}}e^{-\frac{(2kb)^2}{2d}-\frac{\mu^2 d}{2}}-2\sqrt{\frac{2}{\pi d}}e^{-\frac{(2k+1)^2b^2}{2d}-\frac{\mu^2 d}{2}+\mu b}+$$
$$+2\mu\Bigg[e^{(2k+1)\mu b+\mu b}\mathcal{N}\left(-\frac{(2k+1)b}{\sqrt{d}}-\mu\sqrt{d}\right)+e^{-(2k+1)\mu b+\mu b}\mathcal{N}\left(\frac{(2k+1)b}{\sqrt{d}}-\mu\sqrt{d}\right)-$$
$$-e^{2k\mu b}\mathcal{N}\left(-\frac{2kb}{\sqrt{d}}-\mu\sqrt{d}\right)-e^{-2k\mu b}\mathcal{N}\left(\frac{2kb}{\sqrt{d}}-\mu\sqrt{d}\right)\Bigg]\Bigg\}-\sqrt{\frac{2}{\pi d}}e^{-\frac{\mu^2 d}{d}}+2\mu\mathcal{N}\left(-\mu\sqrt{d}\right) \tag{4.4}$$

and

$$\frac{\partial}{\partial\varepsilon}q_{1,2}(t)\underset{\varepsilon\to 0}{\longrightarrow}\sqrt{\frac{2}{\pi t^3}}e^{\mu b-\frac{\mu^2 t}{2}}\sum_{k=0}^{\infty}\left(\frac{(2k+1)^2b^2}{t}-1\right)e^{-\frac{(2k+1)^2b^2}{2t}} \tag{4.5}$$

Inserting calculations (4.3), (4.4) and (4.5) into equation (4.2) yields the proposition.

4.1 Case $H_b(W^\mu) < \tau_d^+(W^\mu)$

In the case where the barrier b is hit before the excursion above zero of length d is completed, we have found the single Laplace transform of the hitting time of the drifted Brownian motion in Proposition 4.1.

Lemma 4.1.

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)}\tilde{h}(U_{H_b})\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right)$$
$$=\frac{\int_0^d e^{-\beta w}\tilde{h}(w)\sqrt{\frac{2}{\pi w^3}}e^{\mu b-\frac{\mu^2 w}{2}}\sum_{k=0}^{\infty}\left(\frac{(2k+1)^2b^2}{w}-1\right)e^{-\frac{(2k+1)^2b^2}{2w}}dw}{\sum_{k=0}^{\infty}2\left[z(k,\beta,\mu)+\sqrt{2\beta+\mu^2}e^{-\sqrt{2\beta+\mu^2}2kb}\right]-z(0,\beta,\mu)-2\sqrt{2\beta+\mu^2}},$$

where z is defined as in (4.1)

$$z(k,\beta,\mu)=$$
$$\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\beta+\mu^2)d}{2}-\frac{2(kb)^2}{d}}-\sqrt{2\beta+\mu^2}\Bigg(e^{\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(-\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)+$$
$$+e^{-\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)\Bigg). \tag{4.6}$$

We are now interested in finding the double Laplace transform of hitting and Parisian times in the case that b is hit before excursion exceeds d. We will now make an appropriate choice of the bounded function $\tilde{h}$, where the intuition will become clear in the proof of the following Proposition.

Proposition 4.2. *The double Laplace transform of hitting and Parisian times of a drifted Brownian motion W^μ, where $H_b(W^\mu) < \tau_d^+(W^\mu)$, is*

$$\begin{aligned}
&\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)-\gamma\tau_d^+(W^\mu)}\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right) = \\
&\int_0^d e^{-\beta w}\left[e^{-\gamma d}\left(1-e^{-2\mu b}\mathcal{N}\left(\frac{\mu(d-w)-b}{\sqrt{d-w}}\right)-\mathcal{N}\left(\frac{-\mu(d-w)-b}{\sqrt{d-w}}\right)\right)+\right.\\
&\quad+\mathbb{E}_0^{\mathbb{Q}}(e^{-\gamma\hat{\tau}_d^+})\left(e^{-(\sqrt{2\gamma+\mu^2}+\mu)b}\mathcal{N}\left(\sqrt{(2\gamma+\mu^2)(d-w)}-\frac{b}{\sqrt{d-w}}\right)+\right.\\
&\quad\quad\left.\left.+e^{\sqrt{2\gamma+\mu^2}-\mu)b}\mathcal{N}\left(-\sqrt{(2\gamma+\mu^2)(d-w)}-\frac{b}{\sqrt{d-w}}\right)\right)\right]\times\\
&\quad\quad\times\sqrt{\frac{2}{\pi w^3}}e^{\mu b-\frac{\mu^2 w}{2}}\sum_{k=0}^{\infty}\left(\frac{(2k+1)^2b^2}{w}-1\right)e^{-\frac{(2k+1)^2b^2}{2w}}dw\times\\
&\times\left\{\sum_{k=0}^{\infty}2\left[z(k,\beta,\mu)+\sqrt{2\beta+\mu^2}e^{-\sqrt{2\beta+\mu^2}2kb}\right]-z(0,\beta,\mu)-2\sqrt{2\beta+\mu^2}\right\}^{-1},
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}_0^{\mathbb{Q}}(e^{-\gamma\hat{\tau}_d^+(W^\mu)}) &= \frac{2\mu e^{-\gamma d}\mathcal{N}\left(\mu\sqrt{d}\right)+\sqrt{\frac{2}{\pi d}}e^{-\gamma d-\frac{\mu^2 d}{2}}}{2\sqrt{2\gamma+\mu^2}\mathcal{N}\left(\sqrt{(2\gamma+\mu^2)d}\right)+\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\gamma+\mu^2)d}{2}}}\\
&= \frac{e^{-\gamma d}(z(0,0,\mu)+2\mu)}{z(0,\gamma,\mu)+2\sqrt{2\gamma+\mu^2}},
\end{aligned}$$

and the function z is defined in equation (4.1).

Proof In order to find the double Laplace transform

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)}e^{-\gamma\tau_d^+(W^\mu)}\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right)$$

in the case where $H_b(W^\mu) < \tau_d^+(W^\mu)$, we define our previously generic function $\tilde{h}$ to be

$$\tilde{h}(U_{H_b}) = \mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma\tau_d^+(W^\mu)}\middle|\mathscr{F}_{H_b(W^\mu)}\right),$$

where $\{\mathscr{F}_t\}_{t\geq 0}$ denotes the standard filtration associated with the Brownian motion. Hence, the l.h.s. of Lemma 4.1 becomes

$$\begin{aligned}
&\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)}\tilde{h}(U_{H_b})\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right)\\
&\quad=\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)}\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma\tau_d^+(W^\mu)}\middle|\mathscr{F}_{H_b(W^\mu)}\right)\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right)\\
&\quad=\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)}e^{-\gamma\tau_d^+(W^\mu)}\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right)
\end{aligned}$$

with our choice of $\tilde{h}$. On the other hand, we have

$$\begin{aligned}
\tilde{h}(U_{H_b})&=\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma(H_b(W^\mu)+d-U_{H_b})}\mathbf{1}_{\tilde{H}_0(W^\mu)>d-U_{H_b}}\middle|\mathscr{F}_{H_b(W^\mu)}\right)+\\
&\quad+\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma(H_b(W^\mu)+\tilde{H}_0(W^\mu)+\hat{\tau}_d^+(W^\mu))}\mathbf{1}_{\tilde{H}_0<d-U_{H_b}}\middle|\mathscr{F}_{H_b(W^\mu)}\right)\\
&=e^{-\gamma H_b(W^\mu)}\Bigg[e^{-\gamma(d-U_{H_b})}\mathbb{P}_b(\tilde{H}_0(W^\mu)>d-U_{H_b})+\\
&\quad+\mathbb{E}_b^{\mathbb{Q}}\left(e^{-\gamma\tilde{H}_0(W^\mu)}\mathbf{1}_{\tilde{H}_0(W^\mu)<d-U_{H_b}}\right)\mathbb{E}_0^{\mathbb{Q}}(e^{-\gamma\hat{\tau}_d^+(W^\mu)})\Bigg],
\end{aligned}$$

where $\tilde{H}_0(W^\mu)$ is the first hitting time of zero restarted at time $H_b(W^\mu)$ and hence independent of $H_b(W^\mu)$ and $\hat{\tau}_d^+(W^\mu)$ is the first time the excursion lasts time d above zero restarted at time $\tilde{H}_0(W^\mu)$ and therefore also independent of $H_b(W^\mu)$. For the derivation of the Laplace transform of $\hat{\tau}_d^+(W^\mu)$, we set $A=0$, $B=1$ and let b approach infinity in Proposition 4.1. Notice that $\hat{\tau}_d^+(W^\mu)$ and $\tau_d^+(W^\mu)$ are identically distributed, due to the strong Markov property of the Brownian motion. It immediately yields

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma\tau_d^+(W^\mu)}\right)=\frac{e^{-\gamma d}(z(0,0,\mu)+2\mu)}{z(0,\gamma,\mu)+2\sqrt{2\gamma+\mu^2}},$$

where the 2μ in the numerator comes in from the odd case in equation (4.4).
For the other quantities, straightforward calculation yields

$$\begin{aligned}
\mathbb{P}_b(\tilde{H}_0(W^\mu)>d-U_{H_b})&=\int_{d-U_{H_b}}^{\infty}\frac{b}{\sqrt{2\pi t^3}}e^{-\frac{(b+\mu t)^2}{2t}}dt\\
&=1-e^{-2\mu b}\mathcal{N}\left(\frac{\mu(d-U_{H_b})-b}{\sqrt{d-U_{H_b}}}\right)-\mathcal{N}\left(\frac{-\mu(d-U_{H_b})-b}{\sqrt{d-U_{H_b}}}\right)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_b^{\mathbb{Q}}\left(e^{-\gamma\tilde{H}_0(W^\mu)}\mathbf{1}_{\tilde{H}_0<d-U_{H_b}}\right)&=e^{-(\sqrt{2\gamma+\mu^2}+\mu)b}\mathcal{N}\left(\sqrt{(2\gamma+\mu^2)(d-U_{H_b})}-\frac{b}{\sqrt{d-U_{H_b}}}\right)\\
&\quad+e^{\sqrt{2\gamma+\mu^2}-\mu)b}\mathcal{N}\left(-\sqrt{(2\gamma+\mu^2)(d-U_{H_b})}-\frac{b}{\sqrt{d-U_{H_b}}}\right)
\end{aligned}$$

Inserting these calculations into Lemma 4.1 yields the proposition.

4.2 Case $\tau_d^+(W^\mu) < H_b(W^\mu)$

In the case where the excursion has exceeded length d before hitting the barrier $b > 0$, we conclude from Proposition 4.1

Lemma 4.2.

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^\mu)}\mathbf{1}_{\tau_d^+(W^\mu)<H_b(W^\mu)}\right)$$
$$= \frac{e^{-\beta d}\left\{\sum_{k=0}^{\infty} 2\left[z(k,0,\mu) - e^{\mu b}z(k+\frac{1}{2},0,\mu)\right] - z(0,0,\mu)\right\}}{\sum_{k=0}^{\infty} 2\left[z(k,\beta,\mu) + \sqrt{2\beta+\mu^2}e^{-\sqrt{2\beta+\mu^2}2kb}\right] - z(0,\beta,\mu) - 2\sqrt{2\beta+\mu^2}}$$

where the function z is defined in equation (4.1).

This lemma allows us to compute the probability, that the Parisian time happens before the hitting time of b by setting $\beta = \mu = 0$, as outlined in the following corollary.

Corollary 4.1. *For the standard Brownian motion W the probability that the excursion exceeds time d before hitting barrier b is given by*

$$\mathbb{Q}\left(\tau_d^+(W) < H_b(W)\right) = 1 - \frac{2\sum_{k=0}^{\infty} e^{-\frac{(2k+1)^2b^2}{2d}} - 1}{2\sum_{k=0}^{\infty} e^{-\frac{(2kb)^2}{2d}} - 1}$$

Now, the double Laplace transform of hitting and Parisian times in the case where the excursion has exceeded length d before hitting b, can be derived.

Proposition 4.3. *The double Laplace transform of hitting and Parisian times for the drifted Brownian motion W^μ in the case where $\tau_d^+(W^\mu) < H_b(W^\mu)$ is given by*

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^\mu)-\gamma H_b(W^\mu)}\mathbf{1}_{\tau_d^+(W^\mu)<H_b(W^\mu)}\right) =$$
$$\Bigg\{e^{-\beta d}\left[e^{-b(\sqrt{2\gamma+\mu^2}-\mu)}\mathcal{N}\left(\frac{b}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d}\right) - e^{b(\sqrt{2\gamma+\mu^2}-\mu)}\times\right.$$
$$\left.\times\mathcal{N}\left(-\frac{b}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d}\right)\right]\sum_{k=0}^{\infty} 2\left[z(k,0,\mu) - e^{\mu b}z(k+\frac{1}{2},0,\mu)\right] - z(0,0,\mu)\Bigg\}\times$$
$$\times\Bigg\{\left[\sum_{k=0}^{\infty} 2\left[z(k,\beta+\gamma,\mu) + \sqrt{2(\beta+\gamma)+\mu^2}e^{-\sqrt{2(\beta+\gamma)+\mu^2}2kb}\right] -\right.$$
$$\left.-z(0,\beta+\gamma,\mu) - 2\sqrt{2(\beta+\gamma)+\mu^2}\right]\left[1 - \mathcal{N}\left(\frac{\mu d - b}{\sqrt{d}}\right) - e^{2\mu b}\mathcal{N}\left(\frac{-\mu d - b}{\sqrt{d}}\right)\right]\Bigg\}^{-1},$$

where the function z is defined by (4.1).

Proof In order to find the double Laplace transform in this case, we define a new infinitesimal generator for the perturbed Brownian motion $W^{\varepsilon,\mu}$ starting at time $\tau_d^+(W^{\varepsilon,\mu})$. We can do this due to the strong Markov property of the Brownian motion. State 2, which stands for $W^{\varepsilon,\mu}$ above barrier b, is an absorbing state, hence nothing comes back from there. Also, we are not concerned with state -1, denoting for $W^{\varepsilon,\mu}$ below zero, because our excursion has already exceeded time d and we are now only interested in hitting b. With this motivation the generator becomes

$$\mathscr{A} f_1(u,t) = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial u} + \lambda_{1,2}(u)\left(f_2(u,t) - f_1(u,t)\right),$$

where we choose f_2 to be $f_2(u,t) = e^{-\gamma t}$. Since state 2 is absorbing, the function f_2 can be assigned arbitrarily. Note, that our choice of f_2 is a bounded function.

Furthermore, at time $\tau_d^+(W^{\varepsilon,\mu})$ we are in state 1. Similar to the proof of Lemma 3.2, we solve $\mathscr{A} f \equiv 0$ in order to derive a martingale of the form $\hat{M}_t := f_{X_t}(U_t(X),t) = e^{-\beta t} h_{X_t}(U_t(X))$. However, notice that we have $f_1(d,0) = h_1(d)$, because per definitionem our time elapsed at starting time $\tau_d^+(W^{\varepsilon,\mu})$ is d. Since we have already achieved an excursion above zero of length d, we are not concerned about any excursions any longer, hence we choose the constraint $h_1(\infty) = 0$. Solving $\mathscr{A} f \equiv 0$ yields

$$h_1(u) = \int_u^\infty \lambda_{1,2}(w) e^{-\int_u^w \gamma + \lambda_{1,2}(v)dv} dw\,, \quad 0 \leq u \leq \infty,$$

where

$$\lambda_{1,2}(t) e^{-\int_0^t \lambda_{1,2}(v)dv} = p_{12}(t) = \mathbb{P}_\varepsilon(H_b(W^\mu) \in dt) = \frac{b-\varepsilon}{\sqrt{2\pi t^3}} e^{-\frac{(b-\varepsilon-\mu t)^2}{2t}}.$$

Hence,

$$
\begin{aligned}
h_1(d) &= \frac{e^{\gamma d}\int_d^\infty e^{-\gamma w} p_{12}(w)dw}{1-\int_0^d p_{12}(s)ds} \\
&= \left\{ e^{\gamma d}\left[e^{-(b-\varepsilon)(\sqrt{2\gamma+\mu^2}-\mu)} \mathcal{N}\left(\frac{b-\varepsilon}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d}\right) - e^{(b-\varepsilon)(\sqrt{2\gamma+\mu^2}-\mu)} \times \right.\right. \\
&\quad \left.\left. \times \mathcal{N}\left(-\frac{b-\varepsilon}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d}\right)\right]\right\} \times \left\{ 1 - \mathcal{N}\left(\frac{\mu d-(b-\varepsilon)}{\sqrt{d}}\right) - \right. \\
&\quad \left. - e^{2\mu(b-\varepsilon)}\mathcal{N}\left(\frac{-\mu d-(b-\varepsilon)}{\sqrt{d}}\right)\right\}^{-1} \\
&\xrightarrow{\varepsilon\to 0} \left\{ e^{\gamma d}\left[e^{-b(\sqrt{2\gamma+\mu^2}-\mu)} \mathcal{N}\left(\frac{b}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d}\right) - e^{b(\sqrt{2\gamma+\mu^2}-\mu)} \times \right.\right. \\
&\quad \left.\left. \times \mathcal{N}\left(-\frac{b}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d}\right)\right]\right\} \times \left\{ 1 - \mathcal{N}\left(\frac{\mu d-b}{\sqrt{d}}\right) - \right. \\
&\quad \left. - e^{2\mu b}\mathcal{N}\left(\frac{-\mu d-b}{\sqrt{d}}\right)\right\}^{-1}.
\end{aligned}
$$

As a result, we have found a martingale $\hat{M}_t := f_{X_t}(U_t(X),t)$ with $\hat{M}_0 = f_1(d,0) = h_1(d)$. Also, with $\hat{\hat{H}}_b(W^{\varepsilon,\mu})$ being the first hitting time of b of our process restarted at $\tau_d^+(W^{\varepsilon,\mu})$ and hence $H_b(W^{\varepsilon,\mu}) = \tau_d^+(W^{\varepsilon,\mu}) + \hat{\hat{H}}_b(W^{\varepsilon,\mu})$. Furthermore, note the following:

$$
\hat{M}_{\hat{\hat{H}}_b(W^{\varepsilon,\mu})} = f_2(U_{\hat{\hat{H}}_b(W^{\varepsilon,\mu})}(X), \hat{\hat{H}}_b(W^{\varepsilon,\mu})) = e^{-\gamma \hat{\hat{H}}_b(W^{\varepsilon,\mu})}
$$

Notice that at hitting time of b, the process $W^{\varepsilon,\mu}$ is in state 2.
Hence, the optional sampling theorem on martingale $\hat{M}_t$ with stopping time $\hat{\hat{H}}_b(W^{\varepsilon,\mu}) \wedge t$ yields

$$
\mathbb{E}_\varepsilon^{\mathbb{Q}}\left(\hat{M}_{\hat{\hat{H}}_b(W^{\varepsilon,\mu})\wedge t}\right) = \mathbb{E}_\varepsilon^{\mathbb{Q}}(\hat{M}_0).
$$

Notice, that by construction

$$
\mathbb{E}_\varepsilon^{\mathbb{Q}}(\hat{M}_0) = h_1(d).
$$

Furthermore, $h_1(u)$ is continuous and decreasing due to the integral limit. Hence, there exists a constant K, such that $|h_1(U_t(X))| \leq K$ for all $U_t(X)$. Therefore, Lebesgue's Dominated Convergence Theorem applies and we derive

$$
\lim_{t\to\infty} \mathbb{E}_\varepsilon^{\mathbb{Q}}\left(\hat{M}_{\hat{\hat{H}}_b(W^{\varepsilon,\mu})\wedge t}\right) = \mathbb{E}_\varepsilon^{\mathbb{Q}}\left(\hat{M}_{\hat{\hat{H}}_b(W^{\varepsilon,\mu})}\right) = \mathbb{E}_\varepsilon^{\mathbb{Q}}\left(e^{-\gamma \hat{\hat{H}}_b(W^{\varepsilon,\mu})}\right).
$$

Hence, $h_1(d) = \mathbb{E}_\varepsilon^{\mathbb{Q}}(e^{-\gamma \hat{H}_b(W^{\varepsilon,\mu})})$ and the double Laplace becomes

$$
\begin{aligned}
&\mathbb{E}_\varepsilon^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^{\varepsilon,\mu})}e^{-\gamma H_b(W^{\varepsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\varepsilon,\mu})<H_b(W^{\varepsilon,\mu})}\right)\\
&\quad = \mathbb{E}_\varepsilon^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^{\varepsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\varepsilon,\mu})<H_b(W^{\varepsilon,\mu})}\mathbb{E}_\varepsilon^{\mathbb{Q}}(e^{-\gamma H_b(W^{\varepsilon,\mu})}\big|\tau_d^+(W^{\varepsilon,\mu}))\right)\\
&\quad = \mathbb{E}_\varepsilon^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^{\varepsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\varepsilon,\mu})<H_b(W^{\varepsilon,\mu})}\mathbb{E}_\varepsilon^{\mathbb{Q}}(e^{-\gamma(\tau_d^+(W^{\varepsilon,\mu})+\hat{H}_b(W^{\varepsilon,\mu}))}\big|\tau_d^+(W^{\varepsilon,\mu}))\right)\\
&\quad = h_1(d)\mathbb{E}_\varepsilon^{\mathbb{Q}}\left(e^{-(\beta+\gamma)\tau_d^+(W^{\varepsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\varepsilon,\mu})<H_b(W^{\varepsilon,\mu})}\right).
\end{aligned}
$$

Together with Lemma 4.2 we conclude the proposition.

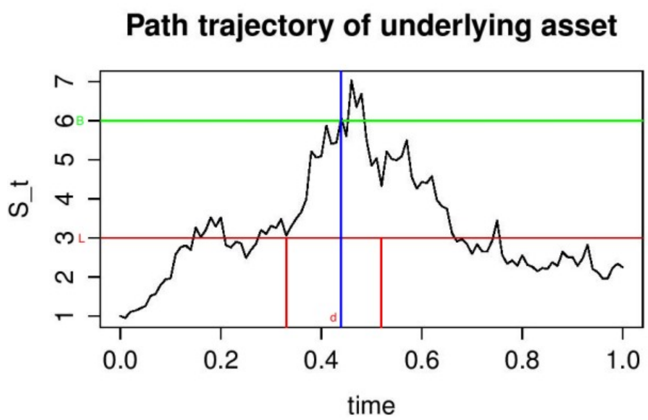
5 Pricing ParisianHit Options

Let $(S_t)_{t\geq 0}$ be the stock price process following a geometric Brownian motion and we recall all definitions from section 2.

5.1 Option triggered at Minimum of Parisian and Hitting times

Our so-called MinParisianHit Option is triggered either when the age of an excursion above L reaches time d or a barrier $B > L$ is hit by the underlying price process S. More precisely, a MinParisianHit Up-and-In is activated at the minimum of both stopping times, i.e. $\min\{\tau_{L,d}^+(S), H_B(S)\}$. This time is illustrated by the blue line in Figure 5.1.

Fig. 5.1 Minimum of Parisian and hitting times



To simplify calculations we assume from now on that the underlying process starts at the barrier, i.e. $S_0 = L$ or equivalently $l = 0$, hence we can use results from our three states Semi-Markov model. The more general case, where $S_0 \neq L$ and the strong Markov property of the Brownian motion applies, will be discussed in the Appendix.

The MinParisianHit Up-and-In Call option has payoff

$$(S_T - K)^+ \mathbf{1}_{\min\{\tau^+_{L,d}(S), H_B(S)\} \leq T},$$

where K denotes the strike price.
Using risk-neutral valuation and Girsanov's change of measure (2.1), the price of this option can be written in the following way.

$$\begin{aligned} minPHC^u_i(S_0,T,K,L,d,r) &= e^{-rT}\mathbb{E}^{\bar{\mathbb{Q}}}_{S_0}\left((S_T-K)^+\mathbf{1}_{\min\{\tau^+_{L,d}(S),H_B(S)\}\leq T}\right) \\ &= e^{-(r+\frac{1}{2}m^2)T}\mathbb{E}^{\mathbb{Q}}_0\left((S_0e^{\sigma Z_T}-K)^+e^{mZ_T}\mathbf{1}_{\min\{\tau^+_d(Z),H_b(Z)\}\leq T}\right) \\ &= e^{-(r+\frac{1}{2}m^2)T}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty}(S_0e^{\sigma z}-K)e^{mz}\mathbb{Q}_0\left(Z_T\in dz, \min\{\tau^+_d(Z),H_b(Z)\}\leq T\right) \end{aligned} \tag{5.1}$$

Hence, finding the fair price for a MinParisianHit option reduces to finding the joint probability of position at maturity and minimum of Parisian and hitting times.

Remark 5.1. We fix the notation for inverse Laplace transforms. Given a function $F(\beta)$, the inverse Laplace transform of F, denoted by $\mathscr{L}^{-1}\{F(\beta)\}$, is the function f whose Laplace transform is F, i.e.

$$f(t) = \mathscr{L}^{-1}_{\beta}\{F(\beta)\}|_t \iff \mathscr{L}_t\{f(t)\}(\beta) := \int_0^\infty e^{-\beta t} f(t)dt = F(\beta).$$

Note, that we consider the inverse Laplace transform with respect to the transformation variable β at the evaluation point t. If not otherwise stated we take from now on $\mathscr{L}^{-1}_{\beta}\{F(\beta)\}|_t$ as a function of the time variable t.

Proposition 5.1. *The joint density of position at maturity and minimum of hitting and Parisian times for standard Brownian motion is*

$$\begin{aligned} &\mathbb{Q}_0(Z_T\in dz, \min\{\tau^+_d(Z),H_b(Z)\}\leq T) = \int\limits_{t=0}^{T}\int\limits_{w=-\infty}^{b}\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(z-w)^2}{2(T-t)}}\times \\ &\times\left[\frac{\sum\limits_{k=-\infty}^{\infty}\frac{w+2kb}{d}e^{-\frac{(w+2kb)^2}{2d}}}{2\sum\limits_{k=0}^{\infty}\left(e^{-\frac{(2kb)^2}{2d}}-e^{-\frac{(2k+1)^2b^2}{2d}}\right)}\mathscr{L}^{-1}_{\beta}\{H_1(\beta)\}|_t+\delta_{(w-b)}\mathscr{L}^{-1}_{\beta}\{H_2(\beta)\}|_t\right]dw\,dt \end{aligned}$$

with

$$H_1(\beta) = \frac{e^{-\beta d}\left(2\sum_{k=0}^{\infty}\left[z(k,0,0) - z(k+\frac{1}{2},0,0)\right] - z(0,0,0)\right)}{2\sum_{k=0}^{\infty}\left[z(k,\beta,0) + \sqrt{2\beta}e^{-\sqrt{2\beta}2kb}\right] - z(0,\beta,0) - 2\sqrt{2\beta}}$$

$$H_2(\beta) = \frac{2\sum_{k=0}^{\infty} z(k+\frac{1}{2},\beta,0) + \sqrt{2\beta}e^{-(2k+1)\sqrt{2\beta}b}}{2\sum_{k=0}^{\infty}\left[z(k,\beta,0) + \sqrt{2\beta}e^{-\sqrt{2\beta}2kb}\right] - z(0,\beta,0) - 2\sqrt{2\beta}}$$

and z defined by (4.1) and δ_x being the Dirac delta function.

Proof Let Z denote a standard Brownian motion and $\tau(Z) := \min\{\tau_d^+(Z), H_b(Z)\}$. The joint probability of position at maturity and minimum of Parisian and hitting times can be decomposed in the following way:

$$\begin{aligned}
&\mathbb{Q}_0(Z_T \in dz, \min\{\tau_d^+(Z), H_b(Z)\} \leq T) = \int_{t=0}^{T}\int_{w=-\infty}^{b} \mathbb{Q}_0(Z_T \in dz, \tau(Z) \in dt, Z_\tau \in dw) \\
&= \int_{t=0}^{T}\int_{w=-\infty}^{b} \mathbb{Q}_0(Z_T \in dz | \tau(Z) = t, Z_\tau \in dw)\mathbb{Q}_0(\tau(Z) \in dt, Z_\tau \in dw) \\
&= \int_{t=0}^{T}\int_{w=-\infty}^{b} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(z-w)^2}{2(T-t)}} dz \mathbb{Q}_0(\tau(Z) \in dt, Z_\tau \in dw) \\
&= \int_{t=0}^{T}\int_{w=-\infty}^{b} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(z-w)^2}{2(T-t)}} dz \Big[\mathbb{Q}_0(\tau(Z) \in dt, Z_\tau \in dw | H_b(Z) < \tau_d^+(Z)) \times \\
&\times \mathbb{Q}_0(H_b(Z) < \tau_d^+(Z)) + \mathbb{Q}_0(\tau(Z) \in dt, Z_\tau \in dw | \tau_d^+(Z) < H_b(Z)) \times \\
&\times \mathbb{Q}_0(\tau_d^+(Z) < H_b(Z))\Big]
\end{aligned}$$

We find

$$\begin{aligned}
&\mathbb{Q}_0(\tau(Z) \in dt, Z_\tau \in dw | \tau_d^+(Z) < H_b(Z))\mathbb{Q}_0(\tau_d^+(Z) < H_b(Z)) \\
&\quad = \mathbb{Q}_0(Z_{\tau_d^+} \in dw | \tau(Z) = t, \tau_d^+(Z) < H_b(Z))\mathbb{Q}_0(\tau(Z) \in dt | \tau_d^+(Z) < H_b(Z)) \times \\
&\quad \times \mathbb{Q}_0(\tau_d^+(Z) < H_b(Z)) \\
&\quad = \mathbb{Q}_0(Z_{\tau_d^+} \in dw | \tau(Z) = t, \tau_d^+(Z) < H_b(Z))\mathbb{Q}_0(\tau(Z) \in dt, \tau_d^+(Z) < H_b(Z)).
\end{aligned} \tag{5.2}$$

For the first term on the r.h.s. we notice

$$
\begin{aligned}
&\mathbb{Q}_0(Z_{\tau_d^+} \in dw | \tau(Z) = t, \tau_d^+(Z) < H_b(Z)) \\
&= \lim_{\varepsilon \to 0} \mathbb{Q}_\varepsilon(Z_d \in dw | \inf_{0<s<d} Z_s > 0, \sup_{0<s<d} Z_s < b) \\
&= \lim_{\varepsilon \to 0} \frac{\mathbb{Q}_\varepsilon(Z_d \in dw, \inf_{0<s<d} Z_s > 0, \sup_{0<s<d} Z_s < b)}{\mathbb{Q}_\varepsilon(\inf_{0<s<d} Z_s > 0, \sup_{0<s<d} Z_s < b)} \\
&= \lim_{\varepsilon \to 0} \frac{\sum_{k=-\infty}^{\infty} e^{-\frac{(w-\varepsilon+2kb)^2}{2d}} - e^{-\frac{(w+\varepsilon+2kb)^2}{2d}}}{\sum_{k=-\infty}^{\infty} \int_0^b e^{-\frac{(z-\varepsilon+2kb)^2}{2d}} - e^{-\frac{(z+\varepsilon+2kb)^2}{2d}} dz} dw \\
&= \frac{\sum_{k=-\infty}^{\infty} \frac{w+2kb}{d} e^{-\frac{(w+2kb)^2}{2d}}}{2 \sum_{k=0}^{\infty} \left(e^{-\frac{(2kb)^2}{2d}} - e^{-\frac{(2k+1)^2 b^2}{2d}} \right)} dw. \qquad (5.3)
\end{aligned}
$$

Notice that the first equality results from the position at Parisian time, $Z_{\tau_d^+}$, being independent of time $\tau_d^+(Z) = t$. See Chesney et al. [8], section 8.3.1, for further details. Formulae for the third line can be found in Borodin and Salminen [5], Chapter 1. Brownian motion, formulae (1.15.4) and (1.15.8). The second term on the r.h.s. of equation (5.2) can be calculated via inverting the Laplace transform of the minimum of hitting and Parisian times. The Laplace transform has been found in Lemma 4.2. With $\mu = 0$ we derive

$$
\begin{aligned}
&\mathbb{Q}_0(\tau(Z) \in dt, \tau_d^+(Z) < H_b(Z)) = \mathscr{L}_\beta^{-1} \left\{ \mathbb{E}_0^{\mathbb{Q}} \left(e^{-\beta \tau_d^+(Z)} \mathbf{1}_{\tau_d^+(Z) < H_b(Z)} \right) \right\} \Big|_t \, dt \\
&= \mathscr{L}_\beta^{-1} \left\{ \frac{e^{-\beta d} \left(\sum_{k=0}^{\infty} 2 \left[z(k,0,0) - z(k+\frac{1}{2},0,0) \right] - z(0,0,0) \right)}{\sum_{k=0}^{\infty} 2 \left[z(k,\beta,0) + \sqrt{2\beta} e^{-\sqrt{2\beta} 2kb} \right] - z(0,\beta,0) - 2\sqrt{2\beta}} \right\} \Bigg|_t \, dt,
\end{aligned}
$$

where $z(k,\beta,\mu)$ is defined as in (4.1) to be

$$
\begin{aligned}
&z(k,\beta,\mu) = \\
&\sqrt{\frac{2}{\pi d}} e^{-\frac{(2\beta+\mu^2)d}{2} - \frac{2(kb)^2}{d}} - \sqrt{2\beta+\mu^2} \Bigg(e^{\sqrt{2\beta+\mu^2} 2kb} \mathscr{N} \left(-\frac{2kb}{\sqrt{d}} - \sqrt{(2\beta+\mu^2)d} \right) + \\
&\qquad + e^{-\sqrt{2\beta+\mu^2} 2kb} \mathscr{N} \left(\frac{2kb}{\sqrt{d}} - \sqrt{(2\beta+\mu^2)d} \right) \Bigg).
\end{aligned}
$$

We also have in the case that $H_b(Z) < \tau_d^+(Z)$,

$$\begin{aligned}
&\mathbb{Q}_0(\tau(Z) \in dt, Z_\tau \in dw | H_b(Z) < \tau_d^+(Z))\mathbb{Q}_0(H_b(Z) < \tau_d^+(Z)) \\
&\quad = \mathbb{Q}_0(Z_{H_b} \in dw | \tau(Z) = t, H_b(Z) < \tau_d^+(Z))\mathbb{Q}_0(\tau(Z) \in dt, H_b(Z) < \tau_d^+(Z)).
\end{aligned}$$

Since Z_{H_b} conditionally on $H_b(Z)$ is deterministic the probability becomes the Dirac delta function at point b, hence

$$\mathbb{Q}_0(Z_{H_b} \in dw | \tau(Z) = t, H_b(Z) < \tau_d^+(Z)) = \delta_{(w-b)} dw,$$

where the Dirac delta function is defined for all $x \in \mathbb{R}$ as

$$\delta_x = \begin{cases} 0 & , \text{if } x \neq 0 \\ \infty & , \text{if } x = 0, \end{cases}$$

and also satisfying the identity

$$\int_{-\infty}^{\infty} \delta_x \, dx = 1.$$

By inversion of the Laplace transform in Lemma 4.1 with $h \equiv 1$, we firstly derive for the numerator

$$\begin{aligned}
&\frac{\partial}{\partial \varepsilon}\hat{Q}_{1,2}(\beta) \longrightarrow \sum_{k=0}^{\infty} 2\sqrt{\frac{2}{\pi d}} e^{\mu b - \frac{(2k+1)^2 b^2}{2d} - \frac{(2\beta+\mu^2)d}{2}} + 2\sqrt{2\beta+\mu^2} e^{\mu b} \Bigg[e^{-(2k+1)\sqrt{2\beta+\mu^2} b} \times \\
&\times \mathcal{N}\left(-\frac{(2k+1)b}{\sqrt{d}} + \sqrt{(2\beta+\mu^2)d}\right) - e^{(2k+1)\sqrt{2\beta+\mu^2} b} \mathcal{N}\left(-\frac{(2k+1)b}{\sqrt{d}} - \sqrt{(2\beta+\mu^2)d}\right)\Bigg] \\
&= 2e^{\mu b} \sum_{k=0}^{\infty} z(k+\frac{1}{2}, \beta, \mu) + \sqrt{2\beta+\mu^2} e^{-(2k+1)\sqrt{2\beta+\mu^2} b}.
\end{aligned}$$

Setting $\mu = 0$, we yield

$$\begin{aligned}
&\mathbb{Q}_0(\tau(Z) \in dt, H_b(Z) < \tau_d^+(Z)) = \mathscr{L}_\beta^{-1}\left\{\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(Z)} \mathbf{1}_{H_b(Z) < \tau_d^+(Z)}\right)\right\}\Big|_t \, dt \\
&\quad = \mathscr{L}_\beta^{-1}\left\{\frac{2\sum_{k=0}^{\infty} z(k+\frac{1}{2}, \beta, 0) + \sqrt{2\beta} e^{-(2k+1)\sqrt{2\beta} b}}{2\sum_{k=0}^{\infty}\left[z(k,\beta,0) + \sqrt{2\beta} e^{-\sqrt{2\beta} 2kb}\right] - z(0,\beta,0) - 2\sqrt{2\beta}}\right\}\Bigg|_t \, dt.
\end{aligned}$$

Putting things together the proposition follows.

We are now able to price a MinParisianHit option by combining Proposition 5.1 and equation (5.1), in particular the fair price of a MinParisianHit Up-and-In Call option can be calculated via evaluating the integral

$$
\begin{aligned}
&minPHC_i^u(S_0,T,K,L,d,r) \\
&= e^{-(r+\frac{1}{2}m^2)T}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty}(S_0e^{\sigma z}-K)e^{mz}\mathbb{Q}_0\left(Z_T\in dz,\min\{\tau_d^+(Z),H_b(Z)\}\le T\right), \quad (5.4)
\end{aligned}
$$

where the joint probability has been derived in Proposition 5.1.

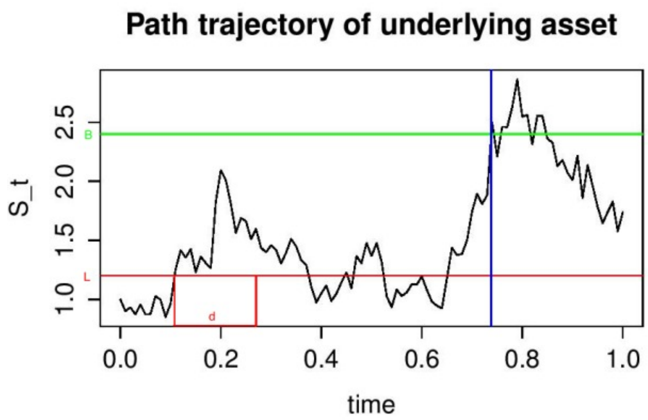
5.2 Option triggered at Maximum of Parisian and Hitting times

Our so-called MaxParisianHit Option is triggered, when both the barrier B is hit and the excursion age exceeds duration d above L. Hence, the payoff of a Call option with strike K becomes

$$
(S_T-K)^+\mathbf{1}_{\{\tau_{L,d}^+(S)\le T, H_B(S)\le T\}} = (S_T-K)^+\mathbf{1}_{\{\max\{\tau_{L,d}^+(S),H_B(S)\}\le T\}}.
$$

The maximum of Parisian and hitting times is illustrated by the blue line in Figure 5.2.

Fig. 5.2 Maximum of Parisian and hitting times



As in the previous case the problem reduces to finding the joint density of hitting and Parisian times and position for a drifted Brownian motion which then can be related to the joint density of hitting and Parisian time for standard Brownian motion due to Girsanov. We also assume $S_0 = L$, thus $\tau_{l,d}^+(Z) = \tau_d^+(Z)$, and discuss the more general case $S_0 \neq L$ in the Appendix. The fair price becomes

$$
\begin{aligned}
maxPHC_i^u(S_0,T,K,L,d,r) &= e^{-rT}\mathbb{E}_{S_0}^{\bar{\mathbb{Q}}}\left((S_T-K)^+\mathbf{1}_{\{\tau^+_{L,d}(S)\le T, H_B(S)\le T\}}\right)\\
&= e^{-(r+\frac{1}{2}m^2)T}\mathbb{E}_0^{\mathbb{Q}}\left((S_0e^{\sigma Z_T}-K)^+e^{mZ_T}\mathbf{1}_{\{\tau^+_d(Z)\le T, H_b(Z)\le T\}}\right)\\
&= e^{-(r+\frac{1}{2}m^2)T}\int\limits_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty}(S_0e^{\sigma z}-K)e^{mz}\mathbb{Q}_0(Z_T\in dz, \max\{\tau^+_d(Z),H_b(Z)\}\le T).
\end{aligned}
\tag{5.5}
$$

Hence, finding the fair price of a MaxParisianHit option reduces to finding the joint probability of position at maturity and maximum of Parisian and hitting times.

Proposition 5.2. *The joint probability of position at maturity and maximum of hitting and Parisian times of standard Brownian motion is*

$$
\begin{aligned}
&\mathbb{Q}_0(Z_T\in dz, \max\{\tau^+_d(Z),H_b(Z)\}\le T)\\
&\qquad = \int\limits_{t=0}^{T}\int\limits_{w=-\infty}^{\infty}\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(z-w)^2}{2(T-t)}}\left\{\frac{|w|}{\pi\sqrt{(t-d)d^3}}e^{-\frac{w^2}{2d}}-\right.\\
&\left.-\frac{\sum\limits_{k=-\infty}^{\infty}\frac{w+2kb}{d}e^{-\frac{(w+2kb)^2}{2d}}}{\sum\limits_{k=-\infty}^{\infty}\left(e^{-\frac{(2kb)^2}{2d}}-e^{-\frac{(2k+1)^2b^2}{2d}}\right)}\mathscr{L}^{-1}_{\beta}\{H_1(\beta)\}|_t\,dt+\delta_{(w-b)}\mathscr{L}^{-1}_{\gamma}\{H_3(\gamma)\}|_t\right\}dw\,dt\,dz,
\end{aligned}
$$

where

$$
H_1(\beta)=\frac{e^{-\beta d}\left(2\sum\limits_{k=0}^{\infty}\left[z(k,0,0)-z(k+\frac{1}{2},0,0)\right]-z(0,0,0)\right)}{2\sum\limits_{k=0}^{\infty}\left[z(k,\beta,0)+\sqrt{2\beta}e^{-\sqrt{2\beta}2kb}\right]-z(0,\beta,0)-2\sqrt{2\beta}},
$$

$$
\begin{aligned}
H_3(\gamma)=&\left\{\left[e^{-\sqrt{2\gamma}b}\mathscr{N}\left(\frac{b}{\sqrt{d}}-\sqrt{2\gamma d}\right)-e^{\sqrt{2\gamma}b}\mathscr{N}\left(-\frac{b}{\sqrt{d}}-\sqrt{2\gamma d}\right)\right]\times\right.\\
&\left.\times\sum_{k=0}^{\infty}2\left[z(k,0,0)-z(k+\frac{1}{2},0,0)\right]-z(0,0,0)\right\}\times\\
&\times\left\{\left[\sum_{k=0}^{\infty}2\left[z(k,\gamma,0)+\sqrt{2\gamma}e^{-\sqrt{2\gamma}2kb}\right]-z(0,\gamma,0)-2\sqrt{2\gamma}\right]\times\left[1-2\mathscr{N}\left(-\frac{b}{\sqrt{d}}\right)\right]\right\}^{-1},
\end{aligned}
$$

with z defined by (4.1) and δ_x denoting the Dirac delta function.

Proof Let $\bar{\tau}(Z) = \max\{\tau_d^+(Z), H_b(Z)\}$, we again have the following decomposition:

$$\begin{aligned}
&\mathbb{Q}_0(Z_T \in dz, \max\{\tau_d^+(Z), H_b(Z)\} \le T) \qquad (5.6)\\
&\quad = \int_{t=0}^{T}\int_{w=-\infty}^{\infty} \mathbb{Q}_0(Z_T \in dz, \bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw)\\
&\quad = \int_{t=0}^{T}\int_{w=-\infty}^{\infty} \mathbb{Q}_0(Z_T \in dz | \bar{\tau}(Z) = t, Z_{\bar{\tau}} \in dw)\mathbb{Q}_0(\bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw)\\
&\quad = \int_{t=0}^{T}\int_{w=-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(z-w)^2}{2(T-t)}} \mathbb{Q}_0(\bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw)dz\\
&\quad = \int_{t=0}^{T}\int_{w=-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(z-w)^2}{2(T-t)}} \Big[\mathbb{Q}_0(\bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw, H_b(Z) < \tau_d^+(Z)) +\\
&\qquad + \mathbb{Q}_0(\bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw, \tau_d^+(Z) < H_b(Z))\Big] dz. \qquad (5.7)
\end{aligned}$$

For the second part of the r.h.s. of equation (5.7) we have

$$\begin{aligned}
&\mathbb{Q}_0(\bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw, \tau_d^+(Z) < H_b(Z))\\
&\quad = \mathbb{Q}_0(Z_{H_b} \in dw | H_b(Z) = t, \tau_d^+(Z) < H_b(Z))\mathbb{Q}_0(H_b(Z) \in dt, \tau_d^+(Z) < H_b(Z))\\
&\quad = \delta_{(w-b)}\, \mathscr{L}_\gamma^{-1}\{H_3(\gamma)\}|_t dw,
\end{aligned}$$

where we know from Proposition 4.3 with $\mu = 0$ and $\beta = 0$

$$\begin{aligned}
&H_3(\gamma) = \mathbb{E}(e^{-\gamma H_b(Z)}\mathbf{1}_{\tau_d^+(Z) < H_b(Z)}) = \Bigg\{\left[e^{-\sqrt{2\gamma}b}\mathscr{N}\left(\frac{b}{\sqrt{d}} - \sqrt{2\gamma d}\right) - \right.\\
&\left. - e^{\sqrt{2\gamma}b}\mathscr{N}\left(-\frac{b}{\sqrt{d}} - \sqrt{2\gamma d}\right)\right]\sum_{k=0}^{\infty} 2\left[z(k,0,0) - z(k+\frac{1}{2},0,0)\right] - z(0,0,0)\Bigg\}\times\\
&\times\left\{\left[\sum_{k=0}^{\infty} 2\left[z(k,\gamma,0) + \sqrt{2\gamma}e^{-\sqrt{2\gamma}2kb}\right] - z(0,\gamma,0) - 2\sqrt{2\gamma}\right] \times \left[1 - 2\mathscr{N}\left(-\frac{b}{\sqrt{d}}\right)\right]\right\}^{-1}.
\end{aligned}$$

Notice the Dirac delta function which is motivated by the deterministic behaviour of Z_{H_b} conditioned on $H_b(Z) = t$.

For the first part of the r.h.s of equation (5.7) we have

$$
\begin{aligned}
&\mathbb{Q}_0(\bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw, H_b(Z) < \tau_d^+(Z)) \\
&\quad = \mathbb{Q}_0(\tau_d^+(Z) \in dt, Z_{\tau_d^+} \in dw, H_b(Z) < \tau_d^+(Z)) \\
&\quad = \mathbb{Q}_0(Z_{\tau_d^+} \in dw, \tau_d^+(Z) \in dt) - \mathbb{Q}_0(Z_{\tau_d^+} \in dw, \tau_d^+(Z) \in dt, \tau_d^+(Z) < H_b(Z)).
\end{aligned}
$$

We have found in section 5.1, that with equation (5.2) and (5.3) combined we derive

$$
\begin{aligned}
&\mathbb{Q}_0(Z_{\tau_d^+} \in dw, \tau_d^+(Z) \in dt, \tau_d^+(Z) < H_b(Z)) \\
&\qquad = \frac{\sum_{k=-\infty}^{\infty} \frac{w+2kb}{d} e^{-\frac{(w+2kb)^2}{2d}}}{2\sum_{k=0}^{\infty}\left(e^{-\frac{(2kb)^2}{2d}} - e^{-\frac{(2k+1)^2 b^2}{2d}}\right)} \mathscr{L}_\beta^{-1}\{H_1(\beta)\}|_t \, dw \, dt.
\end{aligned}
$$

Also, [9] provides us with

$$
\mathbb{Q}_0(Z_{\tau_d^+} \in dw, \tau_d^+(Z) \in dt) = \frac{|w|}{\pi\sqrt{(t-d)d^3}} e^{-\frac{w^2}{2d}} dw \, dt.
$$

Hence, putting terms together we derive the proposition.

Proposition 5.2 allows us to derive the price of a MaxParisianHit option, in particular with equation (5.5) we find the fair price of a MaxParisianHit Up-and-In Call option

$$
\begin{aligned}
&maxPHC_i^u(S_0, T, K, L, d, r) \\
&= e^{-(r+\frac{1}{2}m^2)T} \int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty} (S_0 e^{\sigma z} - K) e^{mz} \mathbb{Q}_0(Z_T \in dz, \max\{\tau_d^+(Z), H_b(Z)\} \leq T),
\end{aligned}
\tag{5.8}
$$

where the joint probability has been found in Proposition 5.2.

In Proposition 4.2 and 4.3 we have derived the double Laplace transform of hitting and Parisian times for drifted Brownian motion. This main result leads to finding the joint distribution of the final position of Brownian motion and the minimum or maximum of hitting and Parisian time. We have established pricing formulae for MinParisianHit and MaxParisianHit options. These fair prices contain single Laplace transforms which need to be inverted numerically using techniques as in Labart and Lelong [17], Abate and Whitt [1] and Bernard et al. [4].

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A Appendix

In the case where the underlying asset does not start at the level L, i.e. $S_0 \neq L$, we want to make use of the strong Markov property of the Brownian motion. We distinguish between two possible scenarios, $S_0 < L$ and $S_0 > L$. From a financial point of view, we are only concerned with $L < B$, and therefore $l < b$.

The price of the MinParisianHit Up-and-In Call option (5.4) can be rewritten in the following form,

$$
\begin{aligned}
&minPHC_i^u(S_0,T,K,L,d,r) \\
&\qquad = e^{-(r+\frac{1}{2}m^2)T} \int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty} (S_0 e^{\sigma z} - K) e^{mz} \mathbb{Q}_0 \left(Z_T \in dz, \min\{\tau_{l,d}^+(Z), H_b(Z)\} \leq T \right),
\end{aligned}
$$

whereas the MaxParisianHit Up-and-In Call option (5.8) becomes

$$
\begin{aligned}
&maxPHC_i^u(S_0,T,K,L,d,r) \\
&\qquad = e^{-(r+\frac{1}{2}m^2)T} \int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty} (S_0 e^{\sigma z} - K) e^{mz} \mathbb{Q}_0 (Z_T \in dz, \max\{\tau_{l,d}^+(Z), H_b(Z)\} \leq T).
\end{aligned}
$$

The proofs of Propositions 5.2 and 5.2 suggest, that the pricing reduces to finding the Laplace transforms of hitting and Parisian time. This can be achieved by decomposing the stopping times and using known results for $S_0 = L$.

We look at the case $S_0 < L$ first. By definition it follows $l > 0$. Define the first hitting time of l for the $\mathbb{Q}$ - Brownian motion Z, with $Z_0 = 0$, to be $H_l(Z) = \inf\{t \geq 0 | Z_t = l\}$. By definition, we have

$$
\tau_{l,d}^+(Z) = H_l(Z) + \tau_{l,d}^+(\tilde{Z}),
$$

where $\tilde{Z}$ stands for a restarted Brownian motion at time $H_l(Z)$, i.e. $\tilde{Z}_0 = l$. Hence, we have equality in distribution of $\tau_{l,d}^+(\tilde{Z})$ and $\tau_d^+(Z)$. By the strong Markov property of the Brownian motion, we therefore have

$$
\mathbb{E}_0^{\mathbb{Q}} \left(e^{-\beta \tau_{l,d}^+(Z)} 1_{\tau_{l,d}^+(Z) < H_b(Z)} \right) = \mathbb{E}_0^{\mathbb{Q}} \left(e^{-\beta H_l(Z)} \right) \mathbb{E}_l^{\mathbb{Q}} \left(e^{-\beta \tau_{l,d}^+(\tilde{Z})} \mathbf{1}_{\tau_{l,d}^+(\tilde{Z}) < H_b(\tilde{Z})} \right).
$$

Clearly, $\mathbb{Q}_0 (H_l(Z) < H_b(Z)) = 1$ due to $l < b$. Notice, that $\mathbb{Q}_0 \left(\tau_{l,d}^+(Z) < H_b(Z) \right) = \mathbb{Q}_l \left(\tau_{l,d}^+(\tilde{Z}) < H_b(\tilde{Z}) \right)$, since $l < b$ and $\tau_{l,d}^+$ is concerned with the Parisian time above l. It is not difficult to see that

$$
\mathbb{E}_l^{\mathbb{Q}} \left(e^{-\beta \tau_{l,d}^+(\tilde{Z})} \mathbf{1}_{\tau_{l,d}^+(\tilde{Z}) < H_b(\tilde{Z})} \right) = \mathbb{E}_0^{\mathbb{Q}} \left(e^{-\beta \tau_d^+(Z)} \mathbf{1}_{\tau_d^+(Z) < H_b(Z)} \right),
$$

which has been calculated in Lemma 4.2 with $\mu = 0$. Also, according to [5], Chapter 1. Brownian motion, formula (2.0.1), we have

$$
\mathbb{E}_0^{\mathbb{Q}} \left(e^{-\beta H_l(Z)} \right) = e^{-l\sqrt{2\beta}},
$$

yielding

$$
\mathbb{E}_0^{\mathbb{Q}} \left(e^{-\beta \tau_{l,d}^+(Z)} 1_{\tau_{l,d}^+(Z) < H_b(Z)} \right) = \frac{e^{-l\sqrt{2\beta}-\beta d} \left\{ \sum_{k=0}^{\infty} 2 \left[z(k,0,0) - z(k+\frac{1}{2},0,0) \right] - z(0,0,0) \right\}}{\sum_{k=0}^{\infty} 2 \left[z(k,\beta,0) + \sqrt{2\beta} e^{-\sqrt{2\beta} 2kb} \right] - z(0,\beta,0) - 2\sqrt{2\beta}}.
$$

In the second case where $S_0 > L$, we have by definition $l < 0 < b$. Then $\tau^+_{l,d}(Z)$ can be decomposed into

$$\tau^+_{l,d}(Z) = \begin{cases} d & \text{, if } H_l(Z) \geq d \\ H_l(Z) + \tau^+_{l,d}(\tilde{Z}) & \text{, if } H_l(Z) < d \end{cases}$$

where $\tilde{Z}$ is a restarted Brownian motion at l. Hence,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}_0\left(e^{-\beta\tau^+_{l,d}(Z)}1_{\tau^+_{l,d}(Z)<H_b(Z)}\right)\\
&= \mathbb{E}^{\mathbb{Q}}_0\left(e^{-\beta d}1_{\tau^+_{l,d}(Z)<H_b(Z)}1_{H_l(Z)>d}\right) + \mathbb{E}^{\mathbb{Q}}_0\left(e^{-\beta H_l(Z)-\beta\tau^+_{l,d}(\tilde{Z})}1_{\tau^+_{l,d}(Z)<H_b(Z)}1_{H_l(Z)<d}\right)\\
&= e^{-\beta d}\mathbb{Q}_0\left(H_b(Z)>d, H_l(Z)>d\right) + \mathbb{E}^{\mathbb{Q}}_0\left(e^{-\beta H_l(Z)}1_{H_l(Z)<d}\right)\mathbb{E}^{\mathbb{Q}}_l\left(e^{-\beta\tau^+_{l,d}(\tilde{Z})}\mathbf{1}_{\tau^+_{l,d}(\tilde{Z})<H_b(\tilde{Z})}\right)
\end{aligned}$$

According to [5], Chapter 1. Brownian motion, formula (1.15.4),

$$\begin{aligned}
\mathbb{Q}_0\left(H_b(Z)>d, H_l(Z)>d\right) &= \mathbb{Q}_0\left(l < \inf_{0\leq s\leq d} Z_s, \sup_{0\leq s\leq d} Z_s < b\right)\\
&= \frac{1}{\sqrt{2\pi d}}\sum_{k=-\infty}^{\infty}\int_a^b\left(e^{-\frac{(z+2k(b-l))^2}{2d}} - e^{-\frac{(z-2l+2k(b-l))^2}{2d}}\right)dz.
\end{aligned}$$

Also, we can calculate

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_0\left(e^{-\beta H_l(Z)}1_{H_l(Z)<d}\right) &= \int_0^d e^{-\beta t}\frac{|l|}{\sqrt{2\pi t^3}}e^{-\frac{l^2}{2t}}\,dt\\
&= e^{-\sqrt{2\beta}|l|}\mathcal{N}\left(\sqrt{2\beta d} - \frac{|l|}{\sqrt{d}}\right) + e^{\sqrt{2\beta}|l|}\mathcal{N}\left(-\sqrt{2\beta d} - \frac{|l|}{\sqrt{d}}\right).
\end{aligned}$$

Again, we have the equality in distribution

$$\mathbb{E}^{\mathbb{Q}}_l\left(e^{-\beta\tau^+_{l,d}(\tilde{Z})}\mathbf{1}_{\tau^+_{l,d}(\tilde{Z})<H_b(\tilde{Z})}\right) = \mathbb{E}^{\mathbb{Q}}_0\left(e^{-\beta\tau^+_d(Z)}\mathbf{1}_{\tau^+_d(Z)<H_b(Z)}\right),$$

which has been calculated in Lemma 4.2 with $\mu = 0$. Altogether, it becomes

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}_0\left(e^{-\beta\tau^+_{l,d}(Z)}1_{\tau^+_{l,d}(Z)<H_b(Z)}\right)\\
&= \frac{e^{-\beta d}}{\sqrt{2\pi d}}\sum_{k=-\infty}^{\infty}\int_a^b\left(e^{-\frac{(z+2k(b-l))^2}{2d}} - e^{-\frac{(z-2l+2k(b-l))^2}{2d}}\right)dz+\\
&+\left[e^{-\sqrt{2\beta}|l|}\mathcal{N}\left(\sqrt{2\beta d} - \frac{|l|}{\sqrt{d}}\right) + e^{\sqrt{2\beta}|l|}\mathcal{N}\left(-\sqrt{2\beta d} - \frac{|l|}{\sqrt{d}}\right)\right]\times\\
&\times\frac{e^{-\beta d}\left(\sum\limits_{k=0}^{\infty}2\left[z(k,0,0) - z(k+\frac{1}{2},0,0)\right] - z(0,0,0)\right)}{\sum\limits_{k=0}^{\infty}2\left[z(k,\beta,0) + \sqrt{2\beta}e^{-\sqrt{2\beta}2kb}\right] - z(0,\beta,0) - 2\sqrt{2\beta}}.
\end{aligned}$$

Analogously, similar results when $H_b(Z) < \tau^+_{l,d}(Z)$, $l < b$, can be achieved.