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Effros, Baire, Steinhaus and Non-Separability

By A. J. Ostaszewski

Abstract. We give a short proof of an improved version of the Effros Open Mapping Principle via a shift-compactness theorem (also with a short proof), involving 'sequential analysis' rather than separability, deducing it from the Baire property in a general Baire-space setting (rather than under topological completeness). It is applicable to absolutely-analytic normed groups (which include complete metrizable topological groups), and via a Steinhaus-type Sum-set Theorem (also a consequence of the shift-compactness theorem) includes the classical Open Mapping Theorem (separable or otherwise).

Keywords: Open Mapping Theorem, absolutely analytic sets, base-σ-discrete maps, demi-open maps, Baire spaces, Baire property, group-action shift-compactness.

Classification Numbers: 26A03; 04A15; 02K20.

1 Introduction

We generalize a classic theorem of Effros [Eff] beyond its usual separable context. Viewed, despite the separability, as a group-action counterpart of the Open Mapping Theorem OMT (that a surjective continuous linear map between Fréchet spaces is open – cf. [Rud]), it has come to be called the *Open Mapping Principle* – see [Anc, §1]. Our 'non-separable' approach is motivated by a sequential property related to the Steinhaus-type Sum-set Theorem (that 0 is an interior point of $A - A$, for non-meagre A with BP, the Baire property – [Pic]), because of the following argument (which goes back to Pettis [Pe]).

Consider $L : E \to F$, a linear, continuous surjection between Fréchet spaces, and U a neighbourhood (nhd) of the origin. Choose A an *open* nhd of the origin with $A - A \subseteq U$; as $L(A)$ is non-meagre (since $\{nL(A) : n \in \mathbb{N}\}$ covers F) and has BP (see Proposition 2 in §2.3), $L(A) - L(A)$ is a nhd of the origin by the Sum-set Theorem. But of course

$$L(U) \supseteq L(A) - L(A),$$

so $L(U)$ is a nhd of the origin. So L is an open mapping.[1]

Throughout this paper, without further comment, all spaces considered will be metrizable, but not necessarily separable. We recall the Birkhoff-Kakutani theorem (cf. [HewR, §II.8.3]), that a metrizable group G with neutral element e_G has a right-invariant metric d_R^G. Passage to $||g|| := d_R^G(g, e_G)$ yields a (group) norm (invariant under inversion, satisfying the triangle inequality), which justifies calling these *normed groups*; any Fréchet space qua additive group, equipped with an F-norm ([KalPR, Ch. 1 §2]), is a natural example (cf. *Auth* in §2.2). Recall that a *Baire space* is one in which Baire's theorem holds – see [AaL]. Below we need the following.

Definitions 1 (cf. [Pe]). For G a metrizable group, say that $\varphi : G \times X \to X$ is a *Nikodym* group action (or that it has the Nikodym property) if for every non-empty open neighbourhood U of e_G and every $x \in X$ the set $Ux = \varphi_x(U) := \varphi(x, U)$ contains a non-meagre *Baire set*. (Here Baire set, as opposed to Baire space as above, means 'set with the Baire property'.)
2. A^q denotes the *quasi-interior* of A – the largest open set U with $U \backslash A$ meagre (cf. [Ost1, §4]); other terms ('analytic', 'base-σ-discrete', 'group action') are recalled later.

Concerning when the above property holds see §2.3. Our main results are Theorems S and E below, with Corollaries in §2.3 including OMT; see below for commentary.

Theorem S (Shift-compactness Theorem). *For T a Baire non-meagre subset of a metric space X and G a group, Baire under a right-invariant metric, and with separately continuous and transitive Nikodym action on X:*

for every convergent sequence x_n with limit x and any Baire non-meagre $A \subseteq G$ with $e_G \in A^q$ and $A^q x \cap T^q \neq \emptyset$, there are $\alpha \in A$ and an integer N such that $\alpha x \in T$ and

$$\{\alpha(x_n) : n > N\} \subseteq T.$$

In particular, this is so if G is analytic and all point-evaluation maps φ_x are base-σ-discrete.

[1] This proof is presumably well-known – so simple and similar to that for the automatic continuity of homomorphisms – but we have no textbook reference; cf. [KalPR, Cor. 1.5].

This theorem has wide-ranging consequences, including Steinhaus' Sumset Theorem – see the survey article [Ost4], and the recent [BinO3].

Theorem E (Effros Theorem – Baire version). *If*
(i) *the normed group G has separately continuous and transitive Nikodym action on X;*
(ii) *G is Baire under the norm topology and X is non-meagre*
– then for any open neighbourhood U of e_G and any $x \in X$ the set $Ux := \{u(x) : u \in U\}$ is a neighbourhood of x, so that in particular the point-evaluation maps $g \to g(x)$ are open for each x. That is, the action of G is micro-transitive.

In particular, this holds if G is analytic and Baire, and all point-evaluation maps φ_x are base-σ-discrete.

By Proposition B2 (§2.3) X, being non-meagre here, is also a Baire space.

The classical counterpart of Theorem E has G a Polish group; van Mill's version [vMil1] requires the group G to be analytic (i.e. the continuous image of some Polish space, cf. [JayR], [Kec2]). The Baire version above improves the version given in [Ost3], where the group is almost complete. (The two cited sources taken together cover the literature.)

A result due to Loy [Loy] and to Hoffmann-Jørgensen [HofJ, Th. 2.3.6 p. 355] asserts that a Baire, separable, analytic *topological group* is Polish (as a consequence of an analytic group being metrizable – for which see again [HofJ, Th. 2.3.6]), so in the analytic separable case Theorem E reduces to its classical version.

Unlike the proof of the Effros Theorem attributed to Becker in [Kec1, Th. 3.1], the one offered here does not employ the Kuratowski-Ulam Theorem (the Category version of the Fubini Theorem), a result known to fail beyond the separable context (as shown in [Pol], cf. [vMilP], but see [FreNR]).

For further commentary (connections between convexity and the Baire property, relation to van Mill's separation property in [vMil2], certain specializations) see the extended version of this paper on arXiv.

2 Analyticity, micro-action, shift-compactness

We recall some definitions from general topology, before turning to ones that are group-related. We refer to [Eng] for general topological usage (but prefer 'meagre' to 'of first category').

2.1 Analyticity

We say that a subspace S of a metric space X has a *Souslin-$\mathcal{H}$ representation* if there is a *determining system* $\langle H(i|n)\rangle := \langle H(i|n) : i \in \mathbb{N}^{\mathbb{N}}\rangle$ of sets in $\mathcal{H}$ with ([Rog], [Han2])

$$S = \bigcup\nolimits_{i\in I} \bigcap\nolimits_{n\in\mathbb{N}} H(i|n),\ \ (I := \mathbb{N}^{\mathbb{N}},\ \ i|n := (i_1, ..., i_n)).$$

A topological space is an (absolutely) *analytic* space if it is embeddable as a Souslin-$\mathcal{F}$ set in its own metric completion (with $\mathcal{F}$ the closed sets); in particular, in a complete metric space $\mathcal{G}_\delta$-subsets (being $\mathcal{F}_{\sigma\delta}$) are analytic. For more recent generalizations see e.g. [NamP]. According to Nikodym's theorem, if $\mathcal{H}$ above comprises Baire sets, then also S is Baire (the Baire property is preserved by the Souslin operation): so analytic subspaces are Baire sets. For background – see [Kec2] Th. 21.6 (the Lusin-Sierpiński Theorem) and the closely related Cor. 29.14 (Nikodym Theorem), cf. the treatment in [Kur] Cor. 1 p. 482, or [JayR] pp. 42-43. For the extended Souslin operation of non-separable descriptive theory see also [Ost2]. This motivates our interest in analyticity as a carrier of the Baire property, especially as continuous images of separable analytic sets are separable, hence Baire.

However, the continuous image of an analytic space is not in general analytic – for an example of failure see [Han3] Ex. 3.12. But this does happen when, additionally, the continuous map is base-σ-discrete, as defined below (*Hansell's Theorem*, [Han3] Cor. 4.2)**.** This technical condition is the standard assumption for preservation of analyticity and holds automatically in the separable realm. Special cases include *closed surjective* maps and *open-to-analytic injective* maps (taking open sets to analytic sets). To define the key concept just mentioned, recall that for an (indexed) family $\mathcal{B} := \{B_t : t \in T\}$:

(i) $\mathcal{B}$ is *index-discrete* in the space X (or just *discrete* when the index set T is understood) if every point in X has a nhd meeting the sets B_t for at most one $t \in T$,

(ii) $\mathcal{B}$ is *σ-discrete* if $\mathcal{B} = \bigcup_n \mathcal{B}_n$ where each set $\mathcal{B}_n$ is discrete as in (i), and

(iii) $\mathcal{B}$ is a *base for* $\mathcal{A}$ if every member of $\mathcal{A}$ is the union of a subfamily of $\mathcal{B}$. For $\mathcal{T}$ a topology (the family of all open sets) with $\mathcal{B} \subseteq \mathcal{T}$ a base for $\mathcal{T}$, this reduces to $\mathcal{B}$ being simply a (topological) *base.*

Definitions. 1. ([Mic1], Def. 2.1) Call $f : X \to Y$ *base-σ-discrete* (or *co-σ-discrete,* [Han3, §3]) if the image under f of any discrete family in X

has a σ-discrete base in Y.
2 ([Han3, §2]). An indexed family $\mathcal{A} := \{A_t : t \in T\}$ is *σ-discretely decomposable* if there are discrete families $\mathcal{A}_n := \{A_{tn} : t \in T\}$ such that $A_t = \bigcup_n A_{tn}$ for each t.
3 ([Mic1], Def. 3.3). Call $f : X \to Y$ *index-σ-discrete* if the image under f of any discrete family $\mathcal{E}$ in X is σ-discretely decomposable in Y. (Note $f(\mathcal{E})$ is regarded as indexed by $\mathcal{E}$, so could be discrete without being index-discrete.)

2.2 Action, micro-action, shift-compactness

Recall that a normed group G *acts continuously* on X if there is a continuous mapping $\varphi : G \times X \to X$ such that $\varphi(e_G, x) = x$ and $\varphi(gh, x) = \varphi(g, \varphi(h, x))$ $(x \in X, g, h \in G)$.The action φ is *separately continuous* if $g : x \mapsto \varphi(g, x)$ is continuous for each g, and $\varphi_x : g \mapsto \varphi(g, x)$ is continuous for each x; in such circumstances:
(i) the elements $g \in G$ yield autohomeomorphisms of X via $g : x \mapsto g(x) := \varphi(g, x)$ (as g^{-1} is continuous), and
(ii) point-evaluation of these homeomorphisms, $\varphi_x(g) = g(x)$, is continuous.
In certain situations joint continuity of action is implied by separate continuity (see [Bou] and literature cited in [Ost2]).

The action is *transitive* if for any x, y in X there is $g \in G$ such that $g(x) = y$. For later purposes (§2.3 and 3), say that the action of G on X is *weakly micro-transitive* if for $x \in X$ and each nhd A of e_G the set

$$\mathrm{cl}(Ax) = \mathrm{cl}\{ax : a \in A\}$$

has x as an interior point (in X). The action is *micro-transitive* ('transitive in the small' – for details see [vMil1]) if for $x \in X$ and each nhd A of e_G the set

$$Ax = \{ax : a \in A\}$$

is a nhd of x. This (norm) property implies that Ux is open for U open in G (i.e. that here each φ_x is an open mapping). We refer to Ax as an *x orbit* (the A-orbit of x). The following group action connects the Open Mapping Theorem to the present context.

Example (Induced homomorphic action). A surjective, continuous homomorphism $\lambda : G \to H$ between normed groups induces a transitive action of G on H via $\varphi^\lambda(g, h) := \lambda(g)h$ (cf. [Ost2] Th. 5.1), specializing for G, H Fréchet spaces (regarded as normed, additive groups) and $\lambda = L : G \to H$ linear (Ancel [Anc] and van Mill [vMil1]) to

$$\varphi^L(a, b) := L(a) + b.$$

Of course for Fréchet spaces, by the Open Mapping Theorem itself, φ^L has the Nikodym property.

Definitions. 1. $Auth(X)$ denotes the autohomeomorphisms of a metric space (X, d^X); this is a group under composition. $\mathcal{H}(X)$ comprises those $h \in Auth(X)$ of bounded norm:

$$||h|| := \sup_{x \in X} d^X(h(x), x) < \infty.$$

2. For a normed group G acting on X, say that X has the *crimping property* (property C for short) w.r.t. G if, for each $x \in X$ and each sequence $\{x_n\} \to x$, there exists in G a sequence $\{g_n\} \to e_G$ with $g_n(x) = x_n$. (This and a variant occurs in [Ban, Ch. III; Th.4]; and [ChCh]; for the term see [BinO2].)

For a subgroup $\mathcal{G} \subseteq \mathcal{H}(X)$, say that X has the *crimping property* w.r.t. $\mathcal{G}$ if X has the crimping property w.r.t. to the natural action $(g, x) \to g(x)$ from $\mathcal{G} \times X \to X$. (This action is continuous relative to the left or right norm topology on $\mathcal{G}$ – cf. [Dug] XII.8.3, p. 271.)

3. As a matter of convenience, say that the *Effros property* (or *property E*) holds for the group G acting on X if the action is micro-transitive, as above.

4. For a subgroup $\mathcal{G} \subseteq Auth(X)$ say that X is *$\mathcal{G}$-shift-compact* (or, shift-compact under $\mathcal{G}$) if for any convergent sequence $x_n \to x_0$, any open subset U in X and any Baire set T co-meagre in U, there is $g \in \mathcal{G}$ with $g(x_n) \in T \cap U$ along a subsequence. Call the space *shift-compact* if it is $\mathcal{H}(X)$-shift-compact (cf. [MilO], [Ost5]).

In such a space, any Baire non-meagre set is locally co-meagre (co-meagre on open sets) in view of Prop. B2 below.

We shall prove in § 3.1 equivalence between the Effros and Crimping properties:

Theorem EC. *The Effros property holds for a group G acting on X iff X has the Crimping property w.r.t. G.*

We now clarify the role of shift-compactness.

Proposition B1. *For any subgroup $\mathcal{G} \subseteq \mathcal{H}(X)$, if X is $\mathcal{G}$-shift-compact, then X is a Baire space.*

Proof. We argue as in [vMil2] Prop 3.1 (1). Suppose otherwise; then X contains a non-empty meagre open set. By Banach's Category Theorem (or localization principle, for which see [JayR] p. 42, or [Kel] Th. 6.35), the union of all such sets is a largest open meagre set M, and is non-empty. Thus $X\backslash M$ is a co-meagre Baire set. For any $x \in M$ the constant sequence $x_n \equiv x$ is convergent and, since $X\backslash M$ is co-meagre in X, there is $g \in G$ with $g(x) \in X\backslash M$. But, as g is a homeomorphism, $g(M)$ is a non-empty open meagre set, so is contained in M, implying $g(x) \in M$, a contradiction. □

A similar argument gives the following and clarifies an assumption in Theorem E.

Proposition B2 (cf. [vMil2]; [HofJ, Prop. 2.2.3]). *If X is non-meagre and G acts transitively on X, then X is a Baire space.*

Proof. As above, refer again to M, the union of all meagre open sets, which, being meagre, has non-empty complement. For x_0 in this complement and any non-empty open U pick $u \in U$ and $g \in G$ such that $g(x_0) = u$. Now, as g is continuous, $g^{-1}(U)$ is a nhd of x_0, so is non-meagre, since every nhd of x_0 is non-meagre. But g is a homeomorphism, so $U = g(g^{-1}(U)$ is non-meagre. So X is Baire, as every non-empty open set is non-meagre. □

2.3 Nikodym actions

The following result generalizes one that, for separable groups G, is usually a first step in proving the weakly micro-transitive variant of the classical Effros Theorem (cf. Ancel [Anc] Lemma 3, [Ost3] Th. 2). Indeed, one may think of it as giving a form of 'very weak micro-transitivity'.

Proposition 1. *If G is a normed group, acting transitively on a non-meagre space X with each point evaluation map $\varphi_x : g \mapsto g(x)$ base-σ-discrete relative – then for each non-empty open U in G and each $x \in X$ the set Ux is non-meagre in X.*

In particular, if G is analytic, then G is a Nikodym action.

Proof. We first work in the right norm topology, i.e. derived from the assumed right-invariant metric $d_R^G(s,t) = ||st^{-1}||$. Suppose that $u \in U$, and so without loss of generality assume that $U = B_\varepsilon(u) = B_\varepsilon(e_G)u$ (open balls of radius some $\varepsilon > 0$); then put $y := ux$ and $W = B_\varepsilon(e_G)$. Then $Ux = Wy$. Next work in the left norm topology, derived from $d_L^G(s,t) = ||s^{-1}t|| = d_R^G(s^{-1}, t^{-1})$ (for which $W = B_\varepsilon(e_G)$ is still a nhd of e_G). As each set hW for $h \in G$ is now open (since now the left shift $g \to hg$ is a homeomorphism), the open family $\mathcal{W} = \{gW : g \in G\}$ covers G. As G is metrizable (and so has a σ-discrete base), the cover $\mathcal{W}$ has a σ-discrete refinement, say $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$, with each $\mathcal{V}_n$ discrete. Put $X_n := \bigcup\{Vy : V \in \mathcal{V}_n\}$; then $X = \bigcup_{n\in\mathbb{N}} X_n$, as $X = Gy$, and so X_n is non-meagre for some n, for $n = N$ say. Since φ_y is base-σ-discrete, $\{Vy : V \in \mathcal{V}_N\}$ has a σ-discrete base, say $\mathcal{B} = \bigcup_{m\in\mathbb{N}} \mathcal{B}_m$, with each $\mathcal{B}_m$ discrete. Then, as $\mathcal{B}$ is a base for $\{Vy : V \in \mathcal{V}_N\}$,

$$X_N = \bigcup_{m\in\mathbb{N}} \left(\bigcup\{B \in \mathcal{B}_m : (\exists V \in \mathcal{V}_N) B \subseteq Vy\} \right).$$

So for some m, say for $m = M$,

$$\bigcup\{B \in \mathcal{B}_m : (\exists V \in \mathcal{V}_N) B \subseteq Vy\}$$

is non-meagre. But as $\mathcal{B}_M$ is discrete, by Banach's Category Theorem (cf. Prop. B1), there are $\hat{B} \in \mathcal{B}_M$ and $\hat{V} \in \mathcal{V}_N$ with $\hat{B} \subseteq \hat{V}y$ such that $\hat{B}$ is non-meagre. As $\mathcal{V}$ refines $\mathcal{W}$, there is some $\hat{g} \in G$ with $\hat{V} \subseteq \hat{g}W$, so $\hat{B} \subseteq \hat{V}y \subseteq \hat{g}Wy$, and so $\hat{g}Wy$ is non-meagre. As $\hat{g}^{-1}$ is a homeomorphism of X, $Wy = Ux$ is also non-meagre in X.

If G is analytic, then as U is open, it is also analytic (since open sets are $\mathcal{F}_\sigma$ and Souslin-$\mathcal{F}$ subsets of analytic sets are analytic, cf. [JayR]), and hence so is $\varphi_x(U)$. Indeed, since φ_x is continuous and base-σ-discrete, Ax is analytic (Hansell's Theorem, §2.1), so Souslin-$\mathcal{F}$, and so Baire by Nikodym's Theorem (§2.1). □

Definition. (Ancel [Anc]). Call the map φ_x *countably-covered* if there exist self-homeomorphisms h_n^x of X for $n \in \mathbb{N}$ such that for any open nhd U in G the sets $\{h_n^x(\varphi_x(U)) : n \in \mathbb{N}\}$ cover X.

Proposition 1′ (cf. Ancell [Anc]) *For the action $\varphi : G \times X \to X$ with X non-meagre, if each map φ_x is countably-covered and takes open sets to sets with the Baire property, then the action has the Nikodym property.*

Proof. If φ_x is countably-covered, then there exist self-homeomorphisms h_n^x of X for $n \in \mathbb{N}$ such that for any open nhd U in G the sets $\{h_n^x(\varphi_x(U)) : n \in \mathbb{N}\}$ cover X. Then for X non-meagre, there is $n \in \mathbb{N}$ with $h_n^x(\varphi_x(U))$ non-meagre, so $Ux = \varphi_x(U)$ is itself non-meagre, being a homeomorphic copy of $h_n^x(\varphi_x(U))$. As Ux is assumed Baire, the action has the Nikodym property. □

For E separable, an immediate consequence of *continuous* maps taking open sets to analytic sets (which are Baire sets) and of Prop. $1'$ is that φ^L is a Nikodym action.

For the general context, one needs *demi-open* continuous maps, which preserve almost completeness (absolute $\mathcal{G}_\delta$ sets modulo meagre sets – see [Mic2] and its antecedent [Nol]), as it is not known which linear maps are base-σ-discrete – a delicate matter to determine, since the former include continuous linear surjections (by Lemma 1 below) and preserve almost analyticity as opposed to analyticity.

For present purposes, however, the *monotonicity property* below suffices. We omit the proof of the following observation (for which see the opening step in [Rud, 2.11], or [Con, Ch. 3 §12.3], or the Appendix in the arXiv version of this paper). For the underlying translation-invariant metric of a Fréchet space denote below by $B(a, r)$ the open r-ball with centre a.

Lemma 1. *For a continuous linear map $L : X \to Y$ from a Fréchet space X to a normed space Y, for $s < t < r$*

$$\mathrm{int}(\mathrm{cl}L(B(0, s))) \subseteq L(B(0, t)) \subseteq L(B(0, r)).$$

Hence for $L(a, r)$ convex, either $L(B(a, r))$ is meagre or differs from $\mathrm{int}L(B(a, r))$ *by a meagre set.*

Proposition 2. *For L a continuous linear surjection from a Fréchet space E to a non-meagre normed space F, the action φ^L has the Nikodym property.*

Proof. As in Prop $1'$ for $L : E \to F$ a continuous linear surjection, $\{\varphi_x^L : x \in F\}$ are countably-covered. Indeed, fixing $x \in F$

$$h_n^x(z) := n(z - x) \qquad (n \in \mathbb{N} \text{ and } z \in F)$$

is on the one hand a self-homeomorphism satisfying $h_n^x(\varphi_x(L(V))) = L(nV)$, since $n[(L(v) + x) - x] = nL(v) = L(nv)$; on the other hand the family

$$\{h_n^x(L(V) + x) : n \geq 1\}$$

covers F, as $\{nV : n \in \mathbb{N}\}$ covers E for V any open nhd of the origin in E (by the 'absorbing' property, cf. [Con, 4.1.13], [Rud, 1.33]). In particular, $nL(B(0,1))$ is non-meagre for some n, and so $L(B(0,s))$ is non-meagre for any s. By Lemma 1, $L(B(0,t))$ for any $t > s$ contains the non-meagre Baire set $\mathrm{cl}L(B(0,s))$. □

Corollary 1 below is now immediate; it is used in [Ost2, Th. 5.1] to prove the 'Semi-Completeness Theorem', an Ellis-type theorem [Ell, Cor. 2] (cf. [Ost6]) giving a one-sided continuity condition which implies that a right-topological group generated by a right-invariant metric is a topological group.

Corollary 1 (cf. [Ost2, Th. 5.1], 'Open Homomorphism Theorem'). *If the continuous surjective homomorphism λ between normed groups G and H, with G analytic and H a Baire space, is base-σ-discrete, then λ is open; in particular, for λ bijective, λ^{-1} is continuous.*

Corollary 2. *For $L : E \to F$ a continuous surjective linear map between Fréchet spaces, the point evaluations φ_b^L for $b \in F$ are open, and so L is an open mapping.*

Proof. By surjectivity of L, the action is transitive, and by Prop 2 the action φ^L has the Nikodym property. So by Theorem E above the point-evaluations maps φ_b^L are open. Hence so also is L. □

3 Proofs

3.1 Proof that E $\Longleftrightarrow$ C

In [BinO1] Th. 3.15 we showed that if the Effros property holds for the action of a group G on X, then X has the crimping property w.r.t. G. We recall the argument, as it is short. Suppose that $x = \lim x_n$. For each n, take $U = B^G_{1/n}(e_G)$; then $Ux := \{u(x) : u \in U\}$ is an open nhd of x, and so there exists $h_{n,m} \in U$ with $h_{n,m}(x) = x_m$ for all m large enough, say for all $m > m(n)$. Without loss of generality we may assume that $m(1) < m(2) < ....$. Put $h_m := e_G$ for $m < m(1)$, and for $m(k) \leq m < m(k+1)$ take $h_m := h_{k,m}$. Then $h_m \in B^G_{1/k}(e_G)$, so h_m converges to e_G and $h_m(e_G) = x_m$.

For the converse, suppose that the Effros property fails for G acting on X. Then for some open nhd U of e_G and some $x \in X$, $Ux := \{u(x) : u \in U\}$ is not an open nhd of x. So for each n there is a point $x_n \in B_{1/n}(x) \backslash Ux$. As x_n converges to x there are homeomorphisms h_n converging to the identity e_G with $h_n(x) = x_n$. As U is an open nhd of e_G and since h_n converges to e_G, there is N such that $h_n \in U$ for $n > N$. In particular, for any $n > N$, $h_n(x) = x_n \in Ux$, a contradiction.

3.2 Weak S

We view Th. S as having 'two tasks': to find a 'translator of the sequence' τ, and to locate it in a given Baire non-meagre subset of the group – provided that subset satisfies a consistency condition (a necessary condition).

For clarity we break the tasks the into two steps – the first delivering a weaker version of S in Proposition 3 below. The arguments are based on the following lemma. We note a corollary, observed earlier by van Mill in the case of metric topological groups ([vMil2, Prop. 3.4]), which concerns a co-meagre set, but we need its refinement to a localized version for a non-meagre set.

Separation Lemma. *Let G be a normed group, with separately continuous and transitive Nikodym action on a non-meagre space X. Then for any point x and any F closed nowhere dense, $W_{x,F} := \{\alpha \in G : \alpha(x) \notin F\}$ is dense open in G. In particular, G separates points from nowhere dense closed sets.*

Proof. The set $W_{x,F}$ is open, being of the form $\varphi_x^{-1}(X \backslash F)$ with φ_x continuous (by assumption). By the Nikodym property, for U any non-empty open set in G, the set Ux is non-meagre, and so $Ux \backslash F$ is non-empty, as F is meagre. But then for some $u \in U$ we have $u(x) \notin F$. $\square$

Corollary 2. *If G is a normed group, Baire in the norm topology with transitive and separately continuously Nikodym action on a non-meagre space X space, and T is co-meagre in X– then for countable $D \subseteq X$, the set $\{g : g(D) \subseteq T\}$ is a dense $\mathcal{G}_\delta$.*

In particular, this holds if G is analytic and each point-evaluation map $\varphi_x : g \to g(x)$ is base-σ-discrete.

Proof. Without loss of generality, the co-meagre set is of the form $T = U \backslash \bigcup_{n \in \omega} F_n$ with each F_n closed and nowhere dense, and U open. Then, by the Separation Lemma and as G is Baire,

$$\{g \in G : g(D) \subseteq T\} = \bigcap_{n\in\omega} \{g : g(D) \cap F_n = \emptyset\} = \bigcap_{d\in D, n\in\omega} \{g : g(d) \notin F_n\}$$

is a dense $\mathcal{G}_\delta$. □

Proposition 3. *If T is a Baire non-meagre subset of a metric space X and G a normed group, Baire in its norm topology, acting separately continuously and transitively on X, with the Nikodym property – then, for every convergent sequence x_n with limit x_0 there is $\tau \in G$ and an integer N with $\tau x_0 \in T$ and*

$$\{\tau(x_n) : n > N\} \subseteq T.$$

Proof. Write $T := M \cup (U \backslash \bigcup_{n\in\omega} F_n)$ with U open, M meagre and each F_n closed and nowhere dense in X. Let $u_0 \in T \cap U$. By transitivity there is $\sigma \in G$ with $\sigma x_0 = u_0$. Put $u_n := \sigma x_n$. Then $u_n \to u_0$. Put

$$C := \bigcap_{m,n\in\omega} \{\alpha \in G : \alpha(u_m) \notin F_n\},$$

a dense $\mathcal{G}_\delta$ in G; then, by the Separation Lemma above, as G is Baire,

$$\{\alpha \in G : \alpha(u_0) \in U\} \cap C$$

is non-empty. For α in this set we have $\alpha(u_0) \in U \backslash \bigcup_{n\in\omega} F_n$. Now $\alpha(u_n) \to \alpha(u_0)$, by continuity of α, and U is open. So for some N we have for $n > N$ that $\alpha(u_n) \in U$. Since $\{\alpha(u_m) : m = 1, 2, ..\} \in X \backslash \bigcup_{n\in\omega} F_n$, we have for $n > N$ that $\alpha(u_n) \in U \backslash \bigcup_{n\in\omega} F_n \subseteq T$.

Finally put $\tau := \alpha\sigma$; then $\tau(x_0) = \alpha\sigma(x_0) \in T$ and $\{\tau(x_n) : n > N\} \subseteq T$. □

3.3 Proof of S

We work in the right norm topology and use the notation of the preceding proof (of Proposition 3), so that U here is the quasi-interior of T and $\sigma x_0 = u_0$. As $e_G \in A^q$ and A is a non-meagre Baire set, we may without loss of generality write $A = B_\varepsilon(e_G) \backslash \bigcup_n G_n$, where each G_n is closed nowhere dense with $e_G \notin G_n$ and $B_\varepsilon(e_G)$ is the quasi-interior of A.

As $A^q x_0 \cap T^q$ is non-empty, there is $\alpha_0 \in B_\varepsilon(e_G)$ with $\alpha_0 x_0 \in U$ (but, we want a better α so that $\alpha x_0 \in T$ and $\alpha \in A$). Put $\beta_0 = \alpha_0 \sigma^{-1}$; then

$$\begin{aligned} \beta_0 &= \alpha_0\sigma^{-1} \in B_\varepsilon(e_G)\sigma^{-1} \cap \{\alpha : \alpha(x_0) \in U\}\sigma^{-1} \\ &= B_\varepsilon(e_G)\sigma^{-1} \cap \{\beta : \beta(\sigma x_0) \in U\} = B_\varepsilon(e_G)\sigma^{-1} \cap \{\beta : \beta(u_0) \in U\}, \end{aligned}$$

i.e. the open set $\{\beta : \beta(u_0) \in U\} \cap B_\varepsilon(e_G)\sigma^{-1}$ is non-empty. So

$$(C\backslash \bigcup_n G_n\sigma^{-1}) \cap \{\beta : \beta(u_0) \in U\} \cap B_\varepsilon(e_G)\sigma^{-1} \neq \emptyset,$$

since G is a Baire space and each $G_n\sigma^{-1}$ is closed and nowhere dense in G (as the right shift $g \to g\sigma^{-1}$ is a homeomorphism).

So there is β with $\beta(u_0) \in U$ such that $\alpha := \beta\sigma \in B_\varepsilon(e_G)\backslash \bigcup_n G_n = A$. That is, $\alpha x_0 = \beta u_0 \in U$; so $\beta(u_n) \in U$ for large n, for $n > N$ say, as $\alpha x_0 = \lim \alpha x_n = \lim \beta\sigma x_n = \lim \beta u_n$. But $\{\beta(u_m) : m = 1, 2, ..\} \in X\backslash \bigcup_n F_n$, as $\beta \in C$; so $\beta(u_n) \in U\backslash \bigcup_n F_n \subseteq T$ for $n > N$.

Finally, $\alpha(x_0) = \beta\sigma(x_0) \in T$ and $\{\alpha(x_n) : n > N\} \subseteq T$. $\square$

3.4 Proof that S $\implies$ E

Assume G acts transitively on X and that X is non-meagre. Let $B := B_\varepsilon(e_G)$ and suppose that for some x the set Bx is not a nhd of x. Then there is $x_n \to x$ with $x_n \notin Bx$ for each n. Take $A := B_{\varepsilon/2}(e_G)$ and note first that A is a symmetric open set ($A^{-1} = A$, since $||g|| = ||g^{-1}||$), and secondly that by the Nikodym property Ax contains a non-meagre, Baire subset T. So by Theorem S, as Ax meets T^q, there are $a \in A$ (which being open has the Baire property) and a co-finite $\mathbb{M}_a$ such that $ax_m \in Ax$ for $m \in \mathbb{M}_a$. For any such m, choose $b_m \in A$ with $ax_m = b_m x$. Then $x_m = a^{-1}b_m x \in A^2 x \subseteq Bx$, a contradiction (note that $a^{-1} \in A$, by symmetry).

As earlier, in the special case that G is (metrizable and) analytic, A is analytic, since open sets are $\mathcal{F}_\sigma$ and Souslin-$\mathcal{F}$ subsets of analytic sets are analytic, cf. [JayR, Th. 2.5.3], by Prop. 3 Ax is Baire non-meagre, as φ_x is base-σ-discrete.

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