Rudolf Rupp and Amol Sasane

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Rudolf Rupp and Amol Sasane*

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Abstract: A corona type theorem is given for the ring $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$ of periodic distributions in $\mathbb{R}^d$ in terms of the sequence of Fourier coefficients of these distributions, which have at most polynomial growth. It is also shown that the Bass stable rank and the topological stable rank of $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$ are both equal to 1.

Keywords: periodic distributions, Fourier series, Bass stable rank, topological stable rank, Bézout equation

Classification: MSC Primary 46F05; Secondary 42B05, 55N15, 18F25

1 Introduction

The aim of this short note is to study some algebraic and topological questions associated with the "Bézout equation"

$$\mathbf{b}_1\mathbf{a}_1 + \cdots + \mathbf{b}_N\mathbf{a}_N = \mathbf{e},$$

where $\mathbf{b}_i, \mathbf{a}_i$ $(1 \le i \le N)$ are elements of the commutative unital topological ring

$$(\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d), +, \star, \mathcal{T}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)}),$$

defined below, and $\mathbf{e}$ denotes the identity element (which will be the locally finite sum of Dirac distributions placed at a lattice formed by the periods in $\mathbb{A}$, as explained below). The Bézout equation in rings of distributions arises in problems of robust filtering, image processing, etc, see for example [2].

1.1 The spaces $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$ and $s'(\mathbb{Z}^d)$

For background on periodic distributions and its Fourier series theory, we refer the reader to [4, Chapter 16] and [10, p.527-529].

Consider the space $s'(\mathbb{Z}^d)$ of all complex valued maps on $\mathbb{Z}^d$ of at most polynomial growth, that is,

$$s'(\mathbb{Z}^d) := \{\mathbf{a} : \mathbb{Z}^d \to \mathbb{C} \mid \exists M > 0 : \exists k \in \mathbb{N} : \forall \mathbf{n} \in \mathbb{Z}^d : |\mathbf{a}(\mathbf{n})| \le M(1 + \lVert\mathbf{n}\rVert)^k\},$$

where $\lVert \cdot \rVert$ denotes the 1-norm in $\mathbb{R}^d$. Then this is a unital commutative ring $(s'(\mathbb{Z}^d), +, \cdot)$ with pointwise operations $+$ and $\cdot$, and the unit element being the constant function

$$\mathbf{n} \mapsto 1 \quad (\mathbf{n} \in \mathbb{Z}^d).$$

We now equip it with a topology $\mathcal{T}_{s'(\mathbb{Z}^d)}$, as follows. First, consider the locally convex topological vector space $s(\mathbb{Z}^d)$ of rapidly decreasing sequences:

$$s(\mathbb{Z}^d) := \left\{\mathbf{a} : \mathbb{Z}^d \to \mathbb{C} \;\middle|\; \forall k \in \mathbb{N},\ \sup_{\mathbf{n}\in\mathbb{Z}^d}(1 + \lVert\mathbf{n}\rVert)^k|\mathbf{a}(\mathbf{n})| < +\infty\right\},$$

***Corresponding Author: Amol Sasane:** Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom, E-mail: sasane@lse.ac.uk
Rudolf Rupp: Fakultät für Angewandte Mathematik, Physik und Allgemeinwissenschaften, TH-Nürnberg, Kesslerplatz 12, D-90489 Nürnberg, Germany, E-mail: Rudolf.Rupp@th-nuernberg.de

with pointwise operations, and with the topology given by the family of seminorms

$$p_k(\mathbf{b}) := \sup_{\mathbf{n}\in\mathbb{Z}^d}(1+|\mathbf{n}|)^k|\mathbf{b}(\mathbf{n})|, \quad \mathbf{b}\in s(\mathbb{Z}^d), \quad k\in\mathbb{N}.$$

(That is, the topology on $s(\mathbb{Z}^d)$ is the weakest one making all the seminorms continuous.) Note that there is a natural duality between $s'(\mathbb{Z}^d)$ and $s(\mathbb{Z}^d)$, namely a bilinear form on $s'(\mathbb{Z}^d)\times s(\mathbb{Z}^d)$ defined as follows: for $\mathbf{a}\in s'(\mathbb{Z}^d)$ and $\mathbf{b}\in s(\mathbb{Z}^d)$,

$$\langle \mathbf{a},\mathbf{b}\rangle_{s'(\mathbb{Z}^d)\times s(\mathbb{Z}^d)} := \sum_{\mathbf{n}\in\mathbb{Z}^d}\mathbf{a}(\mathbf{n})\mathbf{b}(\mathbf{n}).$$

Equip $s'(\mathbb{Z}^d)$ by its natural weak-⋆ topology $\mathcal{T}_{s'(\mathbb{Z}^d)}$ as a dual of $s(\mathbb{Z}^d)$. This topology $\mathcal{T}_{s'(\mathbb{Z}^d)}$ can be described in terms of convergence of nets as follows: a net $(\mathbf{a}_i)_{i\in I}$ in $s'(\mathbb{Z}^d)$ converges to $\mathbf{a}$ in $s'(\mathbb{Z}^d)$ if and only if for every $\mathbf{b}\in s(\mathbb{Z}^d)$, we have that

$$\lim_i\langle\mathbf{a}_i,\mathbf{b}\rangle = \langle\mathbf{a},\mathbf{b}\rangle.$$

Then $s'(\mathbb{Z}^d)$ equipped with pointwise operations, and the above topology $\mathcal{T}_{s'(\mathbb{Z}^d)}$, is a topological ring. Moreover, this is isomorphic as a topological ring to $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$ ($\simeq \mathcal{D}'(\mathbb{T}^d)$), where the latter is equipped with pointwise addition and multiplication taken as convolution, and its natural dual topology $\mathcal{T}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)}$, as elaborated below.

For $\mathbf{a}\in\mathbb{R}^d$, the *translation operation* $\mathbf{S_a}$ on distributions in $\mathcal{D}'(\mathbb{R}^d)$ is defined by

$$\langle \mathbf{S_a}(T),\varphi\rangle = \langle T,\varphi(\cdot+\mathbf{a})\rangle \text{ for all } \varphi\in\mathcal{D}(\mathbb{R}^d).$$

A distribution $T\in\mathcal{D}'(\mathbb{R}^d)$ is said to be *periodic with a period* $\mathbf{a}\in\mathbb{R}^d\setminus\{\mathbf{0}\}$ if $T=\mathbf{S_a}(T)$. Let

$$\mathbb{A} := \{\mathbf{a_1},\cdots,\mathbf{a_d}\}$$

be a linearly independent set of d vectors in $\mathbb{R}^d$. We define $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$ to be the set of all distributions T that satisfy

$$\mathbf{S}_{\mathbf{a_k}}(T) = T \quad (k=1,\cdots,d).$$

From [3, §34], T is a tempered distribution, and from the above it follows by taking Fourier transforms that

$$(1-e^{2\pi i\mathbf{a_k}\cdot\mathbf{y}})\widehat{T} = 0 \text{ for } k=1,\cdots,d.$$

It can be seen that

$$\widehat{T} = \sum_{\mathbf{v}\in A^{-1}\mathbb{Z}^d}\alpha_{\mathbf{v}}(T)\delta_{\mathbf{v}},$$

for some scalars $\alpha_{\mathbf{v}}(T)\in\mathbb{C}$, and where A is the matrix with its rows equal to the transposes of the column vectors $\mathbf{a_1},\cdots,\mathbf{a_d}$:

$$A := \begin{bmatrix}\mathbf{a_1}^\top\\ \vdots\\ \mathbf{a_d}^\top\end{bmatrix}.$$

Also, in the above, $\delta_{\mathbf{v}}$ denotes the usual Dirac measure with support in $\mathbf{v}$:

$$\langle\delta_{\mathbf{v}},\psi\rangle = \psi(\mathbf{v}) \text{ for } \psi\in\mathcal{D}(\mathbb{R}^d).$$

Then the Fourier coefficients $\alpha_{\mathbf{v}}(T)$ give rise to an element in $s'(\mathbb{Z}^d)$, and vice versa, every element in $s'(\mathbb{Z}^d)$ is the set of Fourier coefficients of some periodic distribution. In this manner the topological ring

$$(\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d),+,\star,\mathcal{T}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)})$$

of periodic distributions on $\mathbb{R}^d$ is isomorphic (as topological rings) to

$$(s'(\mathbb{Z}^d),+,\cdot,\mathcal{T}_{s'(\mathbb{Z}^d)}).$$

In the sequel we will mostly just consider the topological ring

$$(s'(\mathbb{Z}^d),+,\cdot,\mathcal{T}_{s'(\mathbb{Z}^d)})$$

while stating our results and demonstrating proofs, with the tacit understanding that analogous results also hold for $(\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d),+,\star,\mathcal{T}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)})$.

1.2 Main results

We will prove three results (Theorem 1.1, 1.4 and 1.8), listed below:

Theorem 1.1. *Let* $\mathbf{a}_1, \cdots, \mathbf{a}_N \in s'(\mathbb{Z}^d)$*. Then the following are equivalent:*

1. *There exist* $\mathbf{b}_1, \cdots, \mathbf{b}_N \in s'(\mathbb{Z}^d)$ *such that*

$$\mathbf{b}_1\mathbf{a}_1 + \cdots + \mathbf{b}_N\mathbf{a}_N = 1$$

2. *There exists a* $\delta > 0$ *and a* $K \in \mathbb{N}$ *such that*

$$\forall \mathbf{n} \in \mathbb{Z}^d, \ |\mathbf{a}_1(\mathbf{n})| + \cdots + |\mathbf{a}_N(\mathbf{n})| \geq \delta(1 + \|\mathbf{n}\|)^{-K}.$$

In light of the isomorphism between $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$ and $s'(\mathbb{Z}^d)$, this result can be viewed as an analogue of a result for the solvability of the Bézout equation in the ring $\mathcal{E}'(\mathbb{R}^d)$ of compactly supported distributions recalled below (see for instance [9, Corollary 3.1]):

Proposition 1.2. *Let* $f_1, \cdots, f_N \in \mathcal{E}'(\mathbb{R}^d)$*. Then the following are equivalent:*

1. *There exist* $g_1, \cdots, g_N \in \mathcal{E}'(\mathbb{R}^d)$ *such that*

$$f_1 * g_1 + \cdots + f_N * g_N = \delta.$$

2. *There are positive constants* δ, K, M *such that for all* $\mathbf{z} \in \mathbb{C}^d$,

$$|\widehat{f}_1(\mathbf{z})| + \cdots + |\widehat{f}_N(\mathbf{z})| \geq \delta(1 + \|\mathbf{z}\|)^{-K} e^{-M\|\mathrm{Im}(\mathbf{z})\|}.$$

Our second main result is the determination of the Bass stable rank of $s'(\mathbb{Z}^d)$. This notion of stable rank was introduced by Hyman Bass in order to facilitate some computations in algebraic K-theory where it plays a role analogous to dimension; see [1],

Definition 1.3 (Bass stable rank)**.** Let R be a commutative unital ring with identity element 1. We assume that $1 \neq 0$, that is R is not the trivial ring $\{0\}$.

1. An N-tuple $(\mathbf{a}_1, \cdots, \mathbf{a}_N) \in R^N$ is said to be *invertible* (or *unimodular*), if there exists $(\mathbf{b}_1, \cdots, \mathbf{b}_N) \in R^N$ such that the Bézout equation
$$\mathbf{b}_1\mathbf{a}_1 + \cdots + \mathbf{b}_N\mathbf{a}_N = 1$$
is satisfied. The set of all invertible N-tuples is denoted by $U_N(R)$.
2. An $(N+1)$-tuple $(\mathbf{a}_1, \cdots, \mathbf{a}_N, \boldsymbol{\alpha}) \in U_{N+1}(R)$ is called *reducible* if there exists $(\mathbf{h}_1, \cdots, \mathbf{h}_N) \in R^N$ such that
$$(\mathbf{a}_1 + \mathbf{h}_1\boldsymbol{\alpha}, \cdots, \mathbf{a}_N + \mathbf{h}_N\boldsymbol{\alpha}) \in U_N(R).$$
3. The *Bass stable rank* of R, denoted by bsr R, is the smallest integer N such that every element in $U_{N+1}(R)$ is reducible. If no such N exists, then bsr $R := \infty$.

Here is the reason for taking the smallest such number N: if every $(N+1)$-tuple is reducible, then also every $(N+k)$-tuple, $k \geq 1$, is reducible, see [1]. With this terminology, our second main result is the following:

Theorem 1.4. *The Bass stable rank of* $s'(\mathbb{Z}^d)$ *is* 1.

As an immediate consequence of this result, we also have that $s'(\mathbb{Z}^d)$ is a Hermite ring, as elaborated below.

Definition 1.5 (Hermite ring)**.** A commutative unital ring R is called a *Hermite ring* if for every $N \in \mathbb{N}$, and every $(\mathbf{a}_1, \cdots, \mathbf{a}_N) \in U_N(R)$, there exists a $N \times N$ matrix $\mathbf{A} \in R^{N\times N}$, which is invertible as an element of $R^{N\times N}$, and such that its first column entries $\mathbf{A}_{i1}$ coincide with $\mathbf{a}_i$:

$$\mathbf{A}_{i1} = \mathbf{a}_i \quad (i = 1, \cdots, N).$$

Corollary 1.6. *$s'(\mathbb{Z}^d)$ is a Hermite ring.*

Proof. This follows from the known result that commutative unital rings having stable rank at most 2 are Hermite (see for example [7, Theorem 20.13, p. 301]). ☐

Finally, we will also determine the topological stable rank of the topological ring $s'(\mathbb{Z}^d)$. Marc Rieffel, in the seminal paper [8], introduced a notion of topological stable rank, analogous to the concept of Bass stable rank, for C^*-algebras. Motivated by his definition, one can more generally consider topological rings instead of only C^*-algebras, and consider the following generalization of his notion of topological stable rank.

Definition 1.7 (Topological stable rank)**.** Let R be a commutative unital topological ring with topology $\mathcal{T}$. The *topological stable rank*, tsr$R$, of $(R, \mathcal{T})$ is the least integer N for which $U_N(R)$ is dense in R^N. If no such N exists, then tsr $R := \infty$.

Our third and final main result is:

Theorem 1.8. *The topological stable rank of $s'(\mathbb{Z}^d)$ is* 1.

The organization of the paper is as follows: In Sections 2, 3, 4, we give the proofs of Theorems 1.1, 1.4, 1.8, respectively.

2 Corona type theorem

Proof of Theorem 1.1.
(1)⇒(2): Suppose there exist $\mathbf{b}_1, \cdots, \mathbf{b}_N \in s'(\mathbb{Z}^d)$ such that

$$\mathbf{b}_1\mathbf{a}_1 + \cdots + \mathbf{b}_N\mathbf{a}_N = 1.$$

Then we can choose a large enough $K \in \mathbb{N}$ and $M > 0$ such that for all $\mathbf{n} \in \mathbb{Z}^d$ and $i = 1, \cdots, N$,

$$|\mathbf{b}_i(\mathbf{n})| \le M(1 + |\mathbf{n}|)^K.$$

Then for all $\mathbf{n} \in \mathbb{Z}^d$, we have

$$\begin{aligned} 1 &= |1| = |\mathbf{b}_1(\mathbf{n})\mathbf{a}_1(\mathbf{n}) + \cdots + \mathbf{b}_N(\mathbf{n})\mathbf{a}_N(\mathbf{n})| \\ &\le |\mathbf{b}_1(\mathbf{n})||\mathbf{a}_1(\mathbf{n})| + \cdots + |\mathbf{b}_N(\mathbf{n})||\mathbf{a}_N(\mathbf{n})| \\ &\le M(1 + |\mathbf{n}|)^K(|\mathbf{a}_1(\mathbf{n})| + \cdots + |\mathbf{a}_N(\mathbf{n})|), \end{aligned}$$

and so the corona type condition follows by a rearrangement (and with $\delta := 1/M$).

(2)⇒(1): From the corona type condition, it follows that

$$\mathbf{n} \mapsto \frac{1}{|\mathbf{a}_1(\mathbf{n})| + \cdots + |\mathbf{a}_N(\mathbf{n})|}$$

is an element of $s'(\mathbb{Z}^d)$. Next, for $i = 1, \cdots, N$ and $\mathbf{n} \in \mathbb{Z}^d$, define

$$\mathbf{b}_i(\mathbf{n}) := \frac{e^{-i\mathrm{Arg}(\mathbf{a}_i(\mathbf{n}))}}{|\mathbf{a}_1(\mathbf{n})| + \cdots + |\mathbf{a}_N(\mathbf{n})|},$$

where Arg denotes the principal argument of a complex number, living in $(-\pi, \pi]$, and is taken as 0 for the complex number 0. Then we see that $\mathbf{b}_1, \cdots, \mathbf{b}_N \in s'(\mathbb{Z}^d)$ and

$$\mathbf{b}_1\mathbf{a}_1 + \cdots + \mathbf{b}_N\mathbf{a}_N = 1$$

This completes the proof. ☐

3 Bass stable rank of $s'(\mathbb{Z}^d)$

Proof of Theorem 1.4. Suppose that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2$ are elements of $s'(\mathbb{Z}^d)$ such that

$$\mathbf{b}_1\mathbf{a}_1 + \mathbf{b}_2\mathbf{a}_2 = 1.$$

Define $\mathbf{u}_1 \in s'(\mathbb{Z}^d)$ by $\mathbf{u}_1(\mathbf{n}) = 1 + |\mathbf{a}_1(\mathbf{n})|$, $\mathbf{n} \in \mathbb{Z}^d$. Then by using Theorem 1.1, we can see that $\mathbf{u}_1$ is invertible in $s'(\mathbb{Z}^d)$. Set $\mathbf{A}_1 := \mathbf{a}_1\mathbf{u}_1^{-1}$, that is,

$$\mathbf{A}_1(\mathbf{n}) = \frac{\mathbf{a}_1(\mathbf{n})}{1 + |\mathbf{a}_1(\mathbf{n})|} \quad (\mathbf{n} \in \mathbb{Z}^d).$$

We note that $|\mathbf{A}_1(\mathbf{n})| \le 1$ for all $\mathbf{n} \in \mathbb{Z}^d$. Define $\mathbf{B}_1 \in s'(\mathbb{Z}^d)$ by $\mathbf{B}_1 = \mathbf{b}_1\mathbf{u}_1$. Then $\mathbf{b}_1\mathbf{a}_1 = \mathbf{B}_1\mathbf{A}_1$, and so

$$1 = \mathbf{b}_1\mathbf{a}_1 + \mathbf{b}_2\mathbf{a}_2 = \mathbf{B}_1\mathbf{A}_1 + \mathbf{b}_2\mathbf{a}_2.$$

Define $\widetilde{\mathbf{B}}_1$ by

$$\widetilde{\mathbf{B}}_1(\mathbf{n}) = \begin{cases} \epsilon & \text{if } |\mathbf{B}_1(\mathbf{n})| \le \epsilon, \\ \mathbf{B}_1(\mathbf{n}) & \text{if } |\mathbf{B}_1(\mathbf{n})| > \epsilon, \end{cases}$$

where the $\epsilon > 0$ will be determined later. Then clearly $\widetilde{\mathbf{B}}_1 \in s'(\mathbb{Z}^d)$, and in fact is invertible in $s'(\mathbb{Z}^d)$ because it is bounded below by ϵ. Moreover, $\widetilde{\mathbf{B}}_1$ "approximates" $\mathbf{B}_1$ pointwise:

$$|\widetilde{\mathbf{B}}_1(\mathbf{n}) - \mathbf{B}_1(\mathbf{n})| \le 2\epsilon \quad (\mathbf{n} \in \mathbb{Z}^d).$$

We also note that

$$\widetilde{\mathbf{B}}_1\mathbf{A}_1 + \mathbf{b}_2\mathbf{a}_2 = \widetilde{\mathbf{B}}_1\mathbf{A}_1 + (1 - \mathbf{B}_1\mathbf{A}_1) = 1 + (\widetilde{\mathbf{B}}_1 - \mathbf{B}_1)\mathbf{A}_1.$$

Since for all $\mathbf{n} \in \mathbb{Z}^d$ we have that

$$|(1 + (\widetilde{\mathbf{B}}_1 - \mathbf{B}_1)\mathbf{A}_1)(\mathbf{n})| \ge 1 - 2\epsilon \cdot 1 = 1 - 2\epsilon \overset{(\epsilon:=1/4)}{=} \frac{1}{2},$$

for the choice $\epsilon = 1/4$, it follows by Theorem 1.1, that $1 + (\widetilde{\mathbf{B}}_1 - \mathbf{B}_1)\mathbf{A}_1$ is an invertible element in $s'(\mathbb{Z}^d)$.

Now with $\mathbf{h} \in s'(\mathbb{Z}^d)$ defined by $\mathbf{h} := \widetilde{\mathbf{B}}_1^{-1}\mathbf{u}_1\mathbf{b}_2$, we have

$$\begin{aligned} \mathbf{a}_1 + \mathbf{h}\mathbf{a}_2 &= \widetilde{\mathbf{B}}_1^{-1}\mathbf{u}_1(\widetilde{\mathbf{B}}_1\mathbf{u}_1^{-1}\mathbf{a}_1 + \mathbf{b}_2\mathbf{a}_2) \\ &= \widetilde{\mathbf{B}}_1^{-1}\mathbf{u}_1(\widetilde{\mathbf{B}}_1\mathbf{A}_1 + \mathbf{b}_2\mathbf{a}_2) \\ &= \widetilde{\mathbf{B}}_1^{-1}\mathbf{u}_1(1 + (\widetilde{\mathbf{B}}_1 - \mathbf{B}_1)\mathbf{A}_1). \end{aligned}$$

As the last expression on the right hand side is a product of invertibles from $s'(\mathbb{Z}^d)$, it follows that $\mathbf{a}_1 + \mathbf{h}\mathbf{a}_2$ is an invertible element in $s'(\mathbb{Z}^d)$. This shows that the Bass stable rank of $s'(\mathbb{Z}^d)$ is 1. □

Remark 3.1. In fact, a modification of the proof gives a more general result, as outlined below.

Let $c(\mathbb{Z}^d)$ denote the ring of all complex valued maps on $\mathbb{Z}^d$ with the usual pointwise operations, and $\ell^\infty(\mathbb{Z}^d)$ denote the subring of $c(\mathbb{Z}^d)$ consisting of all maps with a bounded range. Suppose that R is a subring of $c(\mathbb{Z}^d)$ possessing the following properties (P1), (P3) and either (P2) or (P2′):

(P1) $\ell^\infty(\mathbb{Z}^d) \subset R$,
(P2) $f \in R \;\Rightarrow\; \overline{f} \in R$. (Here $\overline{\cdot}$ indicates pointwise complex conjugation.)
(P2′) $f \in R \;\Rightarrow\; |f| \in R$. (Here $|\cdot|$ denotes taking pointwise complex absolute value.)
(P3) $|f| \ge \delta > 0 \;\Rightarrow\; f \in U_1(R)$.

Note that once we have (P1), the properties (P2) and (P2′) are equivalent, thanks to the identity $\overline{z}e^{i\mathrm{Arg}(z)} = |z|$ for any complex number z, where Arg denotes the principal argument of a complex number, living in $(-\pi, \pi]$, and we set $\mathrm{Arg}(0) := 0$.

Claim: $\mathrm{bsr}(R) = 1$.

We carry out the same proof as above, and note the following key changes in the arguments:

1. $\mathbf{u}_1 \in R$ because $|\mathbf{a}_1| \in R$, using (P2) or (P2$'$). Moreover (P3) shows also that $\mathbf{u}_1$ is invertible in R. Also, then $\mathbf{A}_1 \in R$ and $\mathbf{B}_1 \in R$.
2. Let's show that $\widetilde{\mathbf{B}}_1 \in R$. To this end, write additively $\widetilde{\mathbf{B}}_1 = \mathbf{x} + \mathbf{B}_1 \cdot \mathbf{y}$, where

$$\mathbf{x}(\mathbf{n}) := \left\{ \begin{array}{ll} \epsilon & \text{if } |\mathbf{B}_1(\mathbf{n})| \le \epsilon, \\ 0 & \text{if } |\mathbf{B}_1(\mathbf{n})| > \epsilon \end{array} \right\} \text{ and } \mathbf{y}(\mathbf{n}) := \left\{ \begin{array}{ll} 0 & \text{if } |\mathbf{B}_1(\mathbf{n})| \le \epsilon, \\ 1 & \text{if } |\mathbf{B}_1(\mathbf{n})| > \epsilon \end{array} \right\}.$$

   As $\mathbf{x}, \mathbf{y}$ are bounded, they belong to R by (P1). Thus, since R is a ring, so does $\widetilde{\mathbf{B}}_1 = \mathbf{x} + \mathbf{B}_1 \cdot \mathbf{y}$.
3. We prove that $\widetilde{\mathbf{B}}_1$ is invertible in R. To this end, factor multiplicatively $\widetilde{\mathbf{B}}_1 = (1 + |\widetilde{\mathbf{B}}_1|) \cdot \mathbf{C}_1$, where

$$\mathbf{C}_1 = \frac{\widetilde{\mathbf{B}}_1}{1 + |\widetilde{\mathbf{B}}_1|}.$$

   The first factor, $1 + |\widetilde{\mathbf{B}}_1|$, belongs to R, using (P2) and the fact that $\widetilde{\mathbf{B}}_1 \in R$ (shown above). Moreover, $1 + |\widetilde{\mathbf{B}}_1|$ is invertible in R by (P3).
   Now the function $x \mapsto \dfrac{x}{1+x} : (0, \infty) \to (0, \infty)$ is increasing, and so

$$\frac{\epsilon}{1+\epsilon} \le |\mathbf{C}_1| \le 1.$$

   The rightmost inequality shows that $\mathbf{C}_1 \in R$ (as it is bounded), while the leftmost inequality then gives the invertibility of $\mathbf{C}_1$ in R (using (P3)). The rest of the proof is the same, mutatis mutandis, as the proof of Theorem 1.4.

4 Topological stable rank of $s'(\mathbb{Z}^d)$

Proof of Theorem 1.8. Let $\mathbf{a} \in s'(\mathbb{Z}^d)$. We will construct a net $(\mathbf{a}_\epsilon)_{\epsilon>0}$ with index set as the directed set $(0, \infty)$ and the usual order of real numbers, of invertible elements $\mathbf{a}_\epsilon$ in $s'(\mathbb{Z}^d)$ such that $(\mathbf{a}_\epsilon)_{\epsilon>0}$ converges to $\mathbf{a}$. Define for $\epsilon > 0$ and $\mathbf{n} \in \mathbb{Z}^d$

$$\mathbf{a}_\epsilon(\mathbf{n}) = \left\{ \begin{array}{ll} \epsilon & \text{if } |\mathbf{a}(n)| \le \epsilon, \\ \mathbf{a}(\mathbf{n}) & \text{if } |\mathbf{a}(n)| > \epsilon. \end{array} \right.$$

Then $|\mathbf{a}_\epsilon(\mathbf{n})| \ge \epsilon$ for all $\mathbf{n} \in \mathbb{Z}^d$, and so $\mathbf{a}_\epsilon$ is invertible in $s'(\mathbb{Z}^d)$ for all $\epsilon > 0$. Moreover, for every $\mathbf{b} \in s(\mathbb{Z}^d)$, we have

$$\begin{aligned} \left| \langle (\mathbf{a}_\epsilon - \mathbf{a}), \mathbf{b} \rangle_{s'(\mathbb{Z}^d) \times s(\mathbb{Z}^d)} \right| &= \left| \sum_{\mathbf{n} \in \mathbb{Z}^d} (\mathbf{a}_\epsilon - \mathbf{a})(\mathbf{n}) \mathbf{b}(\mathbf{n}) \right| \\ &\le \sum_{\mathbf{n} \in \mathbb{Z}^d} |\mathbf{a}_\epsilon(\mathbf{n}) - \mathbf{a}(\mathbf{n})| \cdot |\mathbf{b}(\mathbf{n})| \\ &\le \sum_{\mathbf{n} \in \mathbb{Z}^d} 2\epsilon \cdot |\mathbf{b}(\mathbf{n})| \\ &\le 2\epsilon \sum_{\mathbf{n} \in \mathbb{Z}^d} \frac{K}{(1 + |\mathbf{n}|)^2} \le K' \epsilon, \end{aligned}$$

where K (and K') is a constant (depending on $\mathbf{b}$). So the net $(\mathbf{a}_\epsilon)_{\epsilon>0}$ of invertible elements $\mathbf{a}_\epsilon$ in $s'(\mathbb{Z}^d)$ converges in the weak-* topology of $s'(\mathbb{Z}^d)$ to $\mathbf{a}$. Hence $U_1(s'(\mathbb{Z}^d))$ is dense in $s'(\mathbb{Z}^d)$, that is, the topological stable rank of $s'(\mathbb{Z}^d)$ is 1. □

Remark 4.1. It is known that for a commutative Q-algebra (namely a unital topological algebra in which the set of units forms an open set) R, one has $\text{bsr}(R) \le \text{tsr}(R)$ (see for example the proof of [8, Theorem 2.3]). However, it can be seen that $s'(\mathbb{Z}^d)$ is *not* a Q-algebra (see below), and so the result in this section does not render the result in the previous section superfluous.

Let us show that the set of units in $s'(\mathbb{Z}^d)$ does not form an open set in the weak-$\star$ topology of $s'(\mathbb{Z}^d)$. Consider the net $(\mathbf{a}_\epsilon)_{\epsilon>0}$, where

$$\mathbf{a}_\epsilon := e^{-\epsilon|\mathbf{n}|} \quad (\mathbf{n} \in \mathbb{Z}^d).$$

Then each $\mathbf{a}_\epsilon$, $\epsilon > 0$, is not invertible in $s'(\mathbb{Z}^d)$ because it is easily seen that

$$\neg\Big(\exists \delta > 0 : \exists K \in \mathbb{N} : |\mathbf{a}_\epsilon(\mathbf{n})| \geq \delta(1 + |\mathbf{n}|)^{-K}\Big).$$

But the net $(\mathbf{a}_\epsilon)_{\epsilon>0}$ converges to the invertible element $(1)_{\mathbf{n}\in\mathbb{Z}^d}$: indeed for every $\mathbf{b} \in s(\mathbb{Z}^d)$, we have

$$\begin{aligned} |\langle(\mathbf{a}_\epsilon - \mathbf{a}), \mathbf{b}\rangle_{s'(\mathbb{Z}^d)\times s(\mathbb{Z}^d)}| &= \Big|\sum_{\mathbf{n}\in\mathbb{Z}^d}(1 - e^{-\epsilon|\mathbf{n}|})\mathbf{b}(\mathbf{n})\Big| \\ &\leq \sum_{\mathbf{n}\in\mathbb{Z}^d} \epsilon|n||\mathbf{b}(\mathbf{n})| \\ &\leq \epsilon \sum_{\mathbf{n}\in\mathbb{Z}^d} |\mathbf{n}| \cdot \frac{K}{(1 + |\mathbf{n}|)^3} = K' \cdot \epsilon, \end{aligned}$$

for some constant K (and K') depending on $\mathbf{b}$. In the above, we have used the inequality $0 < 1 - e^{-x} < x$ for $x > 0$, which follows from the Mean Value Theorem.

5 An open problem

While we have computed the stable ranks of $s'(\mathbb{Z}^d) \simeq \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$, the stable ranks of $\mathcal{E}'(\mathbb{R}^d)$ do not seem to be known. Via Fourier transformation, and by the Payley-Wiener-Schwartz Theorem [6, Theorem 7.3.1, p.181], $(\mathcal{E}'(\mathbb{R}^d), +, \star)$ is isomorphic to the ring $E_{\exp}$ of entire functions of exponential type:

$$E_{\exp}(\mathbb{C}^d) := \Big\{f : \mathbb{C}^d \to \mathbb{C} : \exists C > 0 : \exists N \in \mathbb{N} : \exists M > 0 : |f(\mathbf{z})| \leq Ce^{M|\mathrm{Im}(\mathbf{z})|}\Big\},$$

equipped with pointwise operations. Is $\mathrm{bsr}(E_{\exp}(\mathbb{C}^d)) < \infty$? If so what is it? The Bass stable rank of $E_{\exp}(\mathbb{C})$ can be shown to be at least 2 (see below).

First, we note that every unit in $E_{\exp}(\mathbb{C})$ has the form $u(z) = e^{a+bz}$ for some constants a, b. This follows directly from the Weierstrass factorisation of such functions, but here is a direct proof: writing $u(z) = e^{h(z)}$, where h denotes an entire function, we conclude that $|\mathrm{Re}(h(z))| \leq A + B|z|$, but then Schwarz's formula for the disc $|\zeta| = 2r$ and $|z| = r$ gives

$$h(z) = \frac{1}{2\pi i} \int_{|\zeta|=2r} \frac{\zeta + z}{\zeta - z} \mathrm{Re}(h(\zeta)) \frac{d\zeta}{\zeta} + i\mathrm{Im}(h(0))$$

and so,

$$|h(z) - i\mathrm{Im}(h(0))| \leq \frac{3r}{r} \sup_{|\zeta|=2r} |\mathrm{Re}(h(\zeta))| \leq 3(A + 2rB).$$

Now Cauchy's estimate for the coefficients of a Taylor series shows that h must be a polynomial of degree 1, that is $h(z) = a + bz$ for some a, b.

Finally, let us show that $\mathrm{bsr}(E_{\exp}(\mathbb{C})) > 1$. Note that $(z - 1, z^3)$ is a unimodular pair in $E_{\exp}(\mathbb{C})$ (with a polynomial solution to the Bézout equation), but is not reducible. Assuming the contrary, there are $h \in E_{\exp}(\mathbb{C})$ and constants a, b such that $z - 1 + h(z)z^3 = e^{a+bz}$. Differentiating twice gives:

$$h''(z)z^3 + 6h'(z)z^2 + h(z)6z = b^2 e^{a+bz},$$

and putting $z = 0$ gives $b = 0$. So $z - 1 + h(z)z^3 = e^a$ for all z. Differentiating this gives $1 + h'(z)z^3 + h(z)3z^2 = 0$, and putting $z = 0$ now gives the contradiction that $1 = 0$.

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