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SOME CURIOSITIES OF THE ALGEBRA OF BOUNDED DIRICHLET SERIES

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Abstract. It is shown that the algebra $\mathscr{H}^\infty$ of bounded Dirichlet series is not a coherent ring, and has infinite Bass stable rank. As corollaries of the latter result, it is derived that $\mathscr{H}^\infty$ has infinite topological stable rank and infinite Krull dimension.

1. Introduction

The aim of this short note is to make explicit two observations about algebraic properties of the ring $\mathscr{H}^\infty$ of bounded Dirichlet series. In particular we will show that

(1) $\mathscr{H}^\infty$ is not a coherent ring. (This is essentially an immediate consequence of Eric Amar's proof of the noncoherence of the Hardy algebra $H^\infty(\mathbb{D}^n)$ of the polydisk $\mathbb{D}^n$ for $n \geq 3$ [1].)
(2) $\mathscr{H}^\infty$ has infinite Bass stable rank. (This is a straightforward adaptation of the first author's proof of the fact that the stable rank of the infinite polydisk algebra is infinite [12]). As corollaries, we obtain that $\mathscr{H}^\infty$ has infinite topological stable rank, and infinite Krull dimension.

Before giving the relevant definitions, we briefly mention that $\mathscr{H}^\infty$ is a closed Banach subalgebra of the classical Hardy algebra $H^\infty(\mathbb{C}_{>0})$ consisting of all bounded and holomorphic functions in the open right half plane

$$\mathbb{C}_{>0} := \{s \in \mathbb{C} : \text{Re}(s) > 0\},$$

and it is striking to compare our findings with the corresponding results for $H^\infty(\mathbb{C}_{>0})$:

	$H^\infty(\mathbb{C}_{>0})$	$\mathscr{H}^\infty$
Coherent?	Yes (See [11])	No
Bass stable rank	1 (See [17])	∞
Topological stable rank	2 (See [16])	∞
Krull dimension	∞ (See [13])	∞

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Nevertheless the above results for $\mathscr{H}^\infty$ lend support to Harald Bohr's idea of interpreting Dirichlet series as functions of infinitely many complex variables, a key theme used in the proofs of the main results in this note.

We recall the pertinent definitions below.

1.1. **The algebra $\mathscr{H}^\infty$ of bounded Dirichlet series.** $\mathscr{H}^\infty$ denotes the set of Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{1.1}$$

where $(a_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers, such that f is holomorphic and bounded in $\mathbb{C}_{>0}$. Equipped with pointwise operations and the supremum norm,

$$\|f\|_\infty := \sup_{s\in\mathbb{C}_{>0}} |f(s)|, \quad f \in \mathscr{H}^\infty,$$

$\mathscr{H}^\infty$ is a unital commutative Banach algebra. In [8, Theorem 3.1], it was shown that the Banach algebra $\mathscr{H}^\infty$ is precisely the multiplier space of the Hilbert space $\mathcal{H}$ of Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

for which

$$\|f\|_{\mathcal{H}}^2 := \sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

The importance of the Hilbert space $\mathcal{H}$ stems from the fact that its kernel function $K_{\mathcal{H}}(z, w)$ is related to the Riemann zeta function ζ:

$$K_{\mathcal{H}}(z, w) = \zeta(z + \overline{w}).$$

For $m \in \mathbb{N}$, let $\mathscr{H}_m^\infty$ be the closed subalgebra of $\mathscr{H}^\infty$ consisting of Dirichlet series of the form (1.1) involving only integers n generated by the first m primes $2, 3, \cdots, p_m$.

1.2. **$\mathscr{H}^\infty = H^\infty(\mathbb{D}^\infty)$.** In [8, Lemma 2.3 and the proof of Theorem 3.1], it was established that $\mathscr{H}^\infty$ is isometrically (Banach algebra) isomorphic to $H^\infty(\mathbb{D}^\infty)$, a certain algebra of functions analytic in the infinite dimensional polydisk, defined below. As this plays a central role in what follows, we give an outline of this based on [8], [15] and [10].

A seminal observation made by H. Bohr [3], is that if we put

$$z_1 = \frac{1}{2^s}, \quad z_2 = \frac{1}{3^s}, \quad z_3 = \frac{1}{5^s}, \cdots, z_n = \frac{1}{p_n^s}, \cdots,$$

where p_n denotes the nth prime, then, in view of the Fundamental Theorem of Arithmetic, formally a Dirichlet series in $\mathscr{H}_n^\infty$ or $\mathscr{H}^\infty$ can be considered as a power series of infinitely many variables. Indeed, each n has a unique expansion

$$n = p_1^{\alpha_1(n)} \cdots p_{r(n)}^{\alpha_{r(n)}(n)},$$

with nonnegative $\alpha_j(n)$s, and so, from (1.1), we obtain the formal power series

$$F(\mathbf{z}) = \sum_{n=1}^{\infty} a_n z_1^{\alpha_1(n)} \cdots z_{r(n)}^{\alpha_{r(n)}(n)}, \tag{1.2}$$

where $\mathbf{z} = (z_1, \cdots, z_m)$ or $\mathbf{z} = (z_1, z_2, z_3, \cdots)$ depending on whether f is a function in $\mathscr{H}_m^\infty$ or in $\mathscr{H}^\infty$. Let us recall Kronecker's Theorem on diophantine approximation [7, Chapter XXIII]:

Proposition 1.1. *For each $m \in \mathbb{N}$, the map*

$$t \mapsto (2^{-it}, 3^{-it}, \cdots, p_m^{-it}) : (0, \infty) \to \mathbb{T}^m$$

has dense range in $\mathbb{T}^m$, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

Using the above and the Maximum Principle, it can be shown that for $f \in \mathscr{H}_m^\infty$,

$$\|f\|_\infty = \|F\|_\infty, \tag{1.3}$$

where the norm on the right hand side is the $H^\infty(\mathbb{D}^m)$ norm. Here $H^\infty(\mathbb{D}^m)$ denotes the usual Hardy algebra of bounded holomorphic functions on the polydisk $\mathbb{D}^m$, endowed with the supremum norm:

$$\|F\|_\infty := \sup_{\mathbf{z} \in \mathbb{D}^m} |F(\mathbf{z})|, \quad F \in H^\infty(\mathbb{D}^m).$$

In [8], it was shown that this result also holds in the infinite dimensional case. In order to describe this result, we introduce some notation. Let c_0 be the Banach space of complex sequences tending to 0 at infinity, with the induced norm from ℓ^∞, and let B be the open unit ball of that Banach space. Thus with $\mathbb{N} := \{1, 2, 3, \cdots\}$ and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$,

$$B = c_0 \cap \mathbb{D}^{\mathbb{N}}.$$

For a point $\mathbf{z} = (z_1, \cdots, z_m, \cdots) \in B$, we set $\mathbf{z}^{(m)} := (z_1, \cdots, z_m, 0, \cdots)$, that is, $z_k = 0$ for $k > m$. Substituting $\mathbf{z}^{(m)}$ in the argument of F given formally by (1.2), we obtain a function

$$(z_1, \cdots, z_m) \mapsto F(\mathbf{z}^{(m)}),$$

which we call the mth-section F_m (after Bohr's terminology "$m$te abschnitt"). F is said to be in $H^\infty(\mathbb{D}^\infty)$ if the H^∞ norm of these functions F_m are uniformly bounded, and denote the supremum of these norms to be $\|F\|_\infty$. Using Schwarz's Lemma for the polydisk, it can be seen that for $m < \ell$,

$$|F(\mathbf{z}^{(m)}) - F(\mathbf{z}^{(\ell)})| \le 2\|f\|_\infty \cdot \max\{|z_j| : m < j \le \ell\},$$

and so we may define

$$F(\mathbf{z}) = \lim_{m \to \infty} F(\mathbf{z}^{(m)}).$$

It was shown in [8] that (1.3) remains true in the infinite dimensional case, and so we may associate $\mathscr{H}^\infty$ with $H^\infty(\mathbb{D}^\infty)$.

Proposition 1.2 ([8])**.** *There exists a Banach algebra isometric isomorphism $\iota : \mathscr{H}^\infty \to H^\infty(\mathbb{D}^\infty)$.*

1.3. **Coherence.**

Definition 1.3. Let R be a unital commutative ring, and for $n \in \mathbb{N}$, let $R^n = R \times \cdots \times R$ (n times).
For $\mathbf{f} = (f_1, \cdots, f_n) \in R^n$, a *relation* $\mathbf{g}$ on $\mathbf{f}$ is an n-tuple $\mathbf{g} = (g_1, \cdots, g_n)$ in R^n such that

$$g_1 f_1 + \cdots + g_n f_n = 0.$$

The set of all relations on $\mathbf{f}$ is denoted by $\mathbf{f}^\perp$.
The ring R is said to be *coherent* if for each n and each $\mathbf{f} \in R^n$, the R-module $\mathbf{f}^\perp$ is finitely generated.

A property which is equivalent to coherence is that the intersection of any two finitely generated ideals in R is finitely generated, and the annihilator of any element is finitely generated [4]. We refer the reader to the article [5] and the monograph [6] for the relevance of the property of coherence in commutative algebra. All Noetherian rings are coherent, but not all coherent rings are Noetherian. (For example, the polynomial ring $\mathbb{C}[x_1, x_2, x_3, \cdots]$ is not Noetherian because the sequence of ideals $\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \langle x_1, x_2, x_3 \rangle \subset \cdots$ is ascending and not stationary, but $\mathbb{C}[x_1, x_2, x_3, \cdots]$ is coherent [6, Corollary 2.3.4].)

In the context of algebras of holomorphic functions in the unit disk $\mathbb{D}$, we mention [11], where it was shown that the Hardy algebra $H^\infty(\mathbb{D})$ is coherent, while the disk algebra $A(\mathbb{D})$ isn't. For $n \geq 3$, Amar [1] showed that the Hardy algebra $H^\infty(\mathbb{D}^n)$ is not coherent. (It is worth mentioning that whether the Hardy algebra $H^\infty(\mathbb{D}^2)$ of the bidisk is coherent or not seems to be an open problem.) Using Amar's result, we will prove the following result:

Theorem 1.4. *$\mathscr{H}^\infty$ is not coherent.*

1.4. **Stable rank.** In algebraic K-theory, the notion of (Bass) stable rank of a ring was introduced in order to facilitate K-theoretic computations [2].

Definition 1.5. Let R be a commutative ring with an identity element (denoted by 1).
An element $(a_1, \cdots, a_n) \in R^n$ is called *unimodular* if there exist elements $b_1, \cdots, b_n$ in R such that

$$b_1 a_1 + \cdots + b_n a_n = 1.$$

The set of all unimodular elements of R^n is denoted by $U_n(R)$.
We say that $a = (a_1, \cdots, a_{n+1}) \in U_{n+1}(R)$ is *reducible* if there exists an element $(x_1, \cdots, x_n) \in R^n$ such that

$$(a_1 + x_1 a_{n+1},\ \cdots,\ a_n + x_n a_{n+1}) \in U_n(R).$$

The *Bass stable rank of* R is the least integer $n \in \mathbb{N}$ for which every $a \in U_{n+1}(R)$ is reducible. If there is no such integer n, we say that R *has infinite stable rank.*

Using the same idea as in [12, Proposition 1] (that the infinite polydisk algebra $A(\mathbb{D}^\infty)$ has infinite Bass stable rank), we show the following.

Theorem 1.6. *The Bass stable rank of $\mathscr{H}^\infty$ is infinite.*

For Banach algebras, an analogue of the Bass stable rank, called the topological stable rank, was introduced by Marc Rieffel in [14].

Definition 1.7. Let R be a commutative complex Banach algebra with unit element 1. The least integer n for which $U_n(R)$ is dense in R^n is called the *topological stable rank of* R. We say R *has infinite topological stable rank* if no such integer n exists.

Corollary 1.8. *The topological stable rank of $\mathscr{H}^\infty$ is infinite.*

Proof. This follows from the inequality that the Bass stable rank of a commutative unital semisimple complex Banach algebra is at most equal to its topological stable rank; see [14, Corollary 2.4]. □

Definition 1.9. The Krull dimension of a commutative ring R is the supremum of the lengths of chains of distinct proper prime ideals of R.

Corollary 1.10. *The Krull dimension of $\mathscr{H}^\infty$ is infinite.*

Proof. This follows from the fact that if a ring has Krull dimension d, then its Bass stable rank is at most $d+2$; see [9]. □

2. Noncoherence of $\mathscr{H}^\infty$

We will use the following fact due to Amar [1, Proof of Theorem 1.(ii)].

Proposition 2.1. *$(z_1 - z_2, z_2 - z_3)^\perp$ is not a finitely generated $H^\infty(\mathbb{D}^3)$-module.*

Proof of Theorem 1.4. The main idea of the proof is that, using the isomorphism ι, essentially we boil the problem down to working with $H^\infty(\mathbb{D}^\infty)$. Let

$$\begin{aligned} f_1 &:= \frac{1}{2^s} - \frac{1}{3^s}, \\ f_2 &:= \frac{1}{3^s} - \frac{1}{5^s}. \end{aligned}$$

Then $\iota(f_1) = z_1 - z_2$ and $\iota(f_2) = z_2 - z_3$. Suppose that $(f_1, f_2)^\perp$ is a finitely generated $\mathscr{H}^\infty$-module, say by

$$\begin{bmatrix} g_1^{(1)} \\ g_1^{(2)} \end{bmatrix}, \dots, \begin{bmatrix} g_r^{(1)} \\ g_r^{(2)} \end{bmatrix} \in (\mathscr{H}^\infty)^2.$$

We will show that the 3rd section of the image under ι of the above elements generate $(z_1 - z_2, z_2 - z_3)^\perp$ in $H^\infty(\mathbb{D}^3)$, contradicting Proposition 2.1. If

$$\begin{bmatrix} G^{(1)} \\ G^{(2)} \end{bmatrix} \in (H^\infty(\mathbb{D}^3))^2 \cap (F_1, F_2)^\perp,$$

then $F_1G^{(1)} + F_2G^{(2)} = 0$, and by applying ι^{-1}, we see that

$$\begin{bmatrix} \iota^{-1}G^{(1)} \\ \iota^{-1}G^{(2)} \end{bmatrix} \in (f_1, f_2)^\perp.$$

So there exist $\alpha^{(1)}, \cdots, \alpha^{(r)} \in \mathscr{H}^\infty$ such that

$$\begin{bmatrix} \iota^{-1}G^{(1)} \\ \iota^{-1}G^{(2)} \end{bmatrix} = \alpha^{(1)} \begin{bmatrix} g_1^{(1)} \\ g_1^{(2)} \end{bmatrix} + \cdots + \alpha^{(r)} \begin{bmatrix} g_r^{(1)} \\ g_r^{(2)} \end{bmatrix}.$$

Applying ι, we obtain

$$\begin{bmatrix} G^{(1)} \\ G^{(2)} \end{bmatrix} = \iota(\alpha^{(1)}) \begin{bmatrix} \iota(g_1^{(1)}) \\ \iota(g_1^{(2)}) \end{bmatrix} + \cdots + \iota(\alpha^{(r)}) \begin{bmatrix} \iota(g_r^{(1)}) \\ \iota(g_r^{(2)}) \end{bmatrix}.$$

Finally taking the 3rd section, we obtain

$$\begin{bmatrix} G^{(1)}(z_1, z_2, z_3) \\ G^{(2)}(z_1, z_2, z_3) \end{bmatrix} = \sum_{j=1}^{r} (\iota(\alpha^{(j)}))(\mathbf{z}^{(3)}) \begin{bmatrix} (\iota(g_j^{(1)}))(\mathbf{z}^{(3)}) \\ (\iota(g_j^{(2)}))(\mathbf{z}^{(3)}) \end{bmatrix}.$$

So it follows that

$$\begin{bmatrix} (\iota(g_1^{(1)}))(\mathbf{z}^{(3)}) \\ (\iota(g_1^{(2)}))(\mathbf{z}^{(3)}) \end{bmatrix}, \cdots, \begin{bmatrix} (\iota(g_r^{(1)}))(\mathbf{z}^{(3)}) \\ (\iota(g_r^{(2)}))(\mathbf{z}^{(3)}) \end{bmatrix}$$

generate $(z_1 - z_2, z_2 - z_3)^\perp$, a contradiction to Amar's result, Proposition 2.1. □

3. Stable rank of $\mathscr{H}^\infty$

The proof of Theorem 1.6 is a straightforward adaptation of the first author's proof of the fact that the Bass stable rank of the infinite polydisk algebra is infinite [12, Proposition 1]. In [12], the infinite polydisk algebra $A(\mathbb{D}^\infty)$ is the uniform closure of the algebra generated by the coordinate functions $z_1, z_2, z_3, \cdots$ on the countably infinite polydisk $\overline{\mathbb{D}} \times \overline{\mathbb{D}} \times \overline{\mathbb{D}} \times \cdots$.

Proof of Theorem 1.6: Fix $n \in \mathbb{N}$. Let $g \in \mathscr{H}^\infty$ be given by

$$g(s) := \prod_{j=1}^{n} \Big(1 - \frac{1}{(p_j p_{n+j})^s}\Big) \in \mathscr{H}^\infty. \tag{3.1}$$

Set

$$\mathbf{f} := \Big(\frac{1}{2^s}, \cdots, \frac{1}{p_n^s}, g\Big) \in (\mathscr{H}^\infty)^{n+1}.$$

We will show that $\mathbf{f} \in U_{n+1}(\mathscr{H}^\infty)$ is not reducible. First let us note that $\mathbf{f}$ is unimodular. Indeed, by expanding the product on the right hand side of (3.1), we obtain

$$g = 1 + \frac{1}{2^s} \cdot g_1 + \cdots + \frac{1}{p_n^s} \cdot g_n,$$

for some appropriate $g_1, \cdots, g_n \in \mathscr{H}^\infty$. Now suppose that $\mathbf{f}$ is reducible, and that there exist $h_1, \cdots, h_n \in \mathscr{H}^\infty$ such that

$$\Big(\frac{1}{2^s} + gh_1, \cdots, \frac{1}{p_n^s} + gh_n\Big) \in U_n(\mathscr{H}^\infty).$$

Let $y_1, \cdots, y_n \in \mathscr{H}^\infty$ be such that

$$\Big(\frac{1}{2^s} + gh_1\Big)y_1 + \cdots + \Big(\frac{1}{p_n^s} + gh_n\Big)y_n = 1.$$

Applying ι, we obtain

$$(z_1 + \iota(g)\iota(h_1))\iota(y_1) + \cdots + (z_n + \iota(g)\iota(h_n))\iota(y_n) = 1. \tag{3.2}$$

Let $\mathbf{h} := (\iota(h_1), \cdots, \iota(h_n))$. For $\mathbf{z} = (z_1, \cdots, z_n) \in \mathbb{C}^n$, we define

$$\mathbf{\Phi}(\mathbf{z}) = \begin{cases} -\mathbf{h}(z_1, \cdots, z_n, \overline{z_1}, \cdots, \overline{z_n}, 0, \cdots) \displaystyle\prod_{j=1}^n (1 - |z_j|^2) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } |z_j| < 1,\ j = 1, \cdots, n, \\ 0 \ \text{ otherwise.} \end{cases}$$

Then $\mathbf{\Phi}$ is a continuous map from $\mathbb{C}^n$ into $\mathbb{C}^n$. But $\mathbf{\Phi}$ vanishes outside $\mathbb{D}^n$, and so

$$\max_{\mathbf{z} \in \mathbb{D}^n} \|\mathbf{\Phi}(\mathbf{z})\|_2 = \sup_{\mathbf{z} \in \mathbb{C}^n} \|\mathbf{\Phi}(\mathbf{z})\|_2.$$

This implies that there must exist an $r \geq 1$ such that $\mathbf{\Phi}$ maps $K := r\overline{\mathbb{D}}^n$ into K. As K is compact and convex, by Brouwer's Fixed Point Theorem it follows that there exists a $\mathbf{z}_* \in K$ such that

$$\mathbf{\Phi}(\mathbf{z}_*) = \mathbf{z}_*.$$

Since $\mathbf{\Phi}$ is zero outside $\mathbb{D}^n$, we see that $\mathbf{z}_* \in \mathbb{D}^n$. Let $\mathbf{z}_* = (\zeta_1, \cdots, \zeta_n)$. Then for each $j \in \{1, \cdots, n\}$, we obtain

$$\begin{aligned} 0 &= \zeta_j + (\iota(h_j))(\zeta_1, \cdots, \zeta_n, \overline{\zeta_1}, \cdots, \overline{\zeta_n}, 0, \cdots) \prod_{k=1}^n (1 - |\zeta_k|^2) \\ &= \zeta_j + (\iota(h_j)\iota(g))(\zeta_1, \cdots, \zeta_n, \overline{\zeta_1}, \cdots, \overline{\zeta_n}, 0, \cdots). \end{aligned} \tag{3.3}$$

But from (3.2), we know that

$$\sum_{j=1}^n (z_j + \iota(h_j)\iota(g))\iota(y_j) = 1,$$

and this contradicts (3.3). As the choice of $n \in \mathbb{N}$ was arbitrary, it follows that the Bass stable rank of $\mathscr{H}^\infty$ is infinite. □

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