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Optional decomposition for continuous semimartingales under arbitrary filtrations*

Ioannis Karatzas[†] Constantinos Kardaras[‡]

Abstract

We present an elementary treatment of the Optional Decomposition Theorem for continuous semimartingales and general filtrations. This treatment does not assume the existence of equivalent local martingale measure(s), only that of strictly positive local martingale deflator(s).

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Introduction

The *Optional Decomposition Theorem* (ODT) is an important result in the field of Stochastic Analysis, and more particularly in Mathematical Finance. In one of its most "classical" forms, following [Kra96], the ODT can be stated as follows. For some $d \in \mathbb{N}$, let X be a $\mathbb{R}^d$-valued locally bounded semimartingale on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbf{F} = \{\mathcal{F}(t)\}_{t \in \mathbb{R}_+}$, and assume that $\mathcal{Q}$, the collection of probability measures that are equivalent to $\mathbb{P}$ and bestow the local martingale property on X, is non-empty. Then, a given nonnegative process V is a supermartingale under *all* probabilities in $\mathcal{Q}$, if and only if it admits the "optional" decomposition

$$V = V(0) + \int_0^{\cdot} \langle H(t), \mathrm{d}X(t) \rangle - C\,; \qquad \text{(OD)}$$

here H is a predictable X-integrable process, and C is a nondecreasing right-continuous adapted process with $C(0) = 0$.

The representation (OD) is relevant in the setting of Mathematical Finance. Indeed, suppose the components of X represent returns of the (discounted) prices of assets in a financial market. If $H = (H_i)_{i \in \{1,\dots,d\}}$ is the investment strategy of an agent in the market, where H_i stands for the amount of currency in asset i held in the portfolio, for all $i \in \{1, \dots, d\}$, and C measures the agent's aggregate consumption, then V in (OD) corresponds to the wealth process generated by the investment/consumption

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strategy (H, C) starting with initial capital $V(0)$. The ODT offers an illuminating "dual" characterization of all such wealth processes, as supermartingales under all *equivalent local martingale measures* of X. Using this characterization, the ODT establishes the *superhedging duality* via use of dynamic programming techniques in non-Markovian settings; see, for instance, [KS98, Chapter 5] or [Kra96].

Stochastic controllability results, similar to the ODT and obtained via martingale methods, can be traced as far back as [DV73] and, in the context of Mathematical Finance, to [KLSX91]. A version of the ODT when X is driven by Brownian motion under quasi-left-continuous filtrations appears in [EKQ95]. The first paper to treat the ODT for general locally bounded semimartingales is [Kra96], where functional (convex) analytic methods and results from [DS94] were employed. In [FK97], the more general case of constraints on investment is considered, using essentially similar arguments. In [FK98], the assumption of local boundedness on the semimartingale integrator X is dropped and, more importantly, the authors avoid infinite-dimensional convex analysis by following an alternative approach via predictable characteristics; this involves Lagrange multipliers, separating hyperplane arguments in Euclidean space, and measurable selections. Although the treatments of the ODT in the aforementioned papers are quite general, they do require a significant level of sophistication; indeed, they involve either use of difficult functional-analytic results, or deep knowledge of the General Theory of Processes as presented, e.g., in [JS03, Chapters I and II].

The present paper offers a rather elementary proof of the ODT in its Stricker-Yan [SY98] formulation, for continuous-path semimartingale integrators X but *arbitrary* filtrations $\mathbb{F}$. Instead of assuming that the collection $\mathcal{Q}$ of equivalent local martingale measures is nonempty, we use the more "localized" assumption that the class $\mathcal{Y}$ of *strictly positive local martingale deflators* is non-empty. This assumption $\mathcal{Y} \neq \emptyset$ is both more general and more descriptive: it allows for an equivalent structural characterization of its validity by inspecting the local drift and local covariation processes of X, as mentioned in Theorem 1.1. By contrast, the more restrictive assumption $\mathcal{Q} \neq \emptyset$ does *not* admit such a descriptive characterization. (In fact, [SY98] treats the ODT using the condition $\mathcal{Y} \neq \emptyset$, by applying a localisation technique on the result assuming $\mathcal{Q} \neq \emptyset$. By contrast, the arguments provided here are direct.) The important pedagogical element of the paper is that it avoids use of functional analysis and predictable characteristics in order to obtain the ODT. Since arbitrary filtrations support local martingales with potential jumps at both accessible and totally inaccessible times, it is impossible to avoid entirely the use of certain results from the general theory of Stochastic Processes. However, we feel that the path taken here is as elementary as possible. Although some intersection with previous work exists (notably, [EKQ95], as well as [Jac12] which deals with continuous asset prices and continuous filtrations), we believe that the present treatment is more straightforward.

1 The Setting

1.1 Preliminaries

We shall work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a right continuous filtration $\mathbf{F} = \{\mathcal{F}(t)\}_{t \in \mathbb{R}_+}$. We stress that no further assumption is made on the filtration. We do not even require the usual hypothesis of augmentation by null sets, as semimartingale integration theory can be developed without it; see, for example, [JS03, Chapter I].

Let $X = (X_i)_{i \in \{1,\dots,d\}}$ be a d-dimensional semimartingale with continuous paths. We write $X = A + M$ for the Doob-Meyer decomposition of X ; here A is a d-dimensional process with continuous paths of finite variation and $A(0) = 0$, and M is a d-dimensional local martingale with continuous paths.

For $i \in \{1, \dots, d\}$, we shall denote by $\check{A}_i$ the process of finite first variation associated with A_i. Upon defining $G := \sum_{i=1}^d \left(\check{A}_i + [M_i, M_i]\right)$, it follows that there exist a d-dimensional predictable process a, and a predictable process c taking values in the set of nonnegative-definite matrices, such that $A = \int_0^\cdot a(t)\mathrm{d}G(t)$ and $[M_i, M_j] = \int_0^\cdot c_{ij}(t)\mathrm{d}G(t)$ hold for $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, d\}$.

We shall denote by $\mathcal{P}$ the predictable σ-field on $\Omega \times \mathbb{R}_+$, and by $\mathbb{P} \otimes G$ the measure on the product measurable space $(\Omega \times \mathbb{R}_+, \mathcal{P})$ which satisfies $(\mathbb{P} \otimes G)[J] = \mathbb{E}\left[\int_0^\infty \mathbf{1}_J(t)\mathrm{d}G(t)\right]$ for all $J \in \mathcal{P}$.

Let $\mathcal{P}(X)$ denote the collection of all d-dimensional, predictable and X-integrable processes. A given d-dimensional predictable process H belongs to $\mathcal{P}(X)$ if, and only if, both processes $\int_0^\cdot |\langle H(t), a(t)\rangle| \, \mathrm{d}G(t)$ and $\int_0^\cdot \langle H(t), c(t)H(t)\rangle \, \mathrm{d}G(t)$ are finitely-valued.

We shall be using the notation

$$\mathcal{E}(Z) \;:=\; \exp\left(Z - \frac{1}{2}\,[Z,Z]^c\right) \cdot \prod_{t \le \cdot} \big(1 + \Delta Z(t)\big) \exp\left(-\Delta Z(t)\right)$$

for the *stochastic exponential* of a scalar semimartingale Z with $Z(0) = 0$; we note that this process satisfies the integral equation $\mathcal{E}(Z) \;=\; 1 + \int_0^\cdot \mathcal{E}(Z)(t-)\, \mathrm{d}Z(t)$.

1.2 Strictly positive local martingale deflators

We define $\mathcal{Y}$ as the collection of all *strictly positive* local martingales Y with $Y(0) = 1$, such that YX_i is a local martingale for all $i \in \{1, \dots, d\}$. The next result gives conditions on the drift and local covariance structure of X which are equivalent to the requirement of non-emptiness for $\mathcal{Y}$.

Theorem 1.1. In the above setup, the following two conditions are equivalent:

1. $\mathcal{Y} \neq \emptyset$.
2. There exists a d-dimensional, predictable process ρ, such that $a = c\rho$ holds $(\mathbb{P} \otimes G)$-a.e. and the process $\int_0^\cdot \langle \rho(t), c(t)\rho(t)\rangle \, \mathrm{d}G(t)$ is finitely-valued.

The structural conditions in statement (2) of Theorem 1.1 have appeared previously—see, for example, [Sch95] or [KS98, Theorem 4.2, page 12]. A proof of Theorem 1.1 can be found in [Kar10, Section 4]. We shall not repeat it here, but will provide some discussion in order to introduce important quantities that will be used later on.

1.3 Discussion of Theorem 1.1

Let us start by assuming that condition (2) of Theorem 1.1 holds. Since $a = c\rho$ implies that $\langle \rho, a\rangle = \langle \rho, c\rho\rangle = |\langle \rho, a\rangle|$ holds $(\mathbb{P} \otimes G)$-a.e., it follows that ρ is X-integrable, i.e., $\rho \in \mathcal{P}(X)$. Then, the continuous-path semimartingale

$$\widehat{V} \;:=\; \mathcal{E}\left(\int_0^\cdot \langle \rho(t), \mathrm{d}X(t)\rangle\right) \tag{1.1}$$

is well-defined and satisfies the integral equation

$$\widehat{V} \;=\; 1 + \int_0^\cdot \widehat{V}(t)\, \langle \rho(t), \mathrm{d}X(t)\rangle\,. \tag{1.2}$$

Straightforward computations show now that $(1/\widehat{V})$ is a local martingale, as is $(X_i/\widehat{V})$ for all $i \in \{1, \dots, d\}$; consequently, $(1/\widehat{V}) \in \mathcal{Y}$. In fact, whenever L is a local martingale with $L(0) = 0$, $\Delta L > -1$ and $[L, M] = 0$, the product $\big(1/\widehat{V}\big)\, \mathcal{E}(L)$ is an element of $\mathcal{Y}$. This multiplicative structure of $\mathcal{Y}$ under the path-continuity of X is crucial in the proof of the

ODT presented here; if jumps are present in X, the present approach does not appear to generalize in any straightforward way. Although we shall not make direct use of this fact, let us also note that every element of $\mathcal{Y}$ can be written as $\big(1/\widehat{V}\big)\,\mathcal{E}(L)$, where L is a local martingale with $L(0)=0$, $\Delta L > -1$ and $[L,M]=0$; see, for instance, the argument in [LŽ07, Proposition 3.2].

The argument of the preceding paragraph establishes the implication $(2) \Rightarrow (1)$ in Theorem 1.1. For completeness we discuss now briefly, how the failure of condition (2) implies the failure of condition (1) in Theorem 1.1. The arguments outlined below highlight the further equivalence of the conditions in Theorem 1.1 with the notion of *no arbitrage of the first kind,* and with the existence of a so-called "numéraire portfolio" in financial markets. This is the minimal condition that allows problems like hedging of contingent claims and utility maximization to have meaningful solutions. Detailed discussion and complete arguments are in [Kar10, especially Section 4].

Two contingencies need to be considered:

(i) The vector a fails to be in the range of the matrix c, on a predictable set E of strictly positive $(\mathbb{P}\otimes G)$-measure. In this case one can find $\zeta \in \mathcal{P}(X)$, such that $c\,\zeta = 0$ and the process $\int_0^\cdot \langle \zeta(t), \mathrm{d}X(t)\rangle = \int_0^\cdot \langle \zeta(t), a(t)\rangle\,\mathrm{d}G(t)$ is nondecreasing everywhere and eventually strictly positive on E. This implies in a straightforward way that $\mathcal{Y}=\emptyset$.

(ii) A d-dimensional predictable process ρ exists, so that $a = c\rho$ holds $(\mathbb{P}\otimes G)$-a.e.; but the event $\big\{\int_0^T \langle \rho(t), c(t)\rho(t)\rangle\,\mathrm{d}G(t)=\infty\big\}$ has positive probability for some $T>0$. Then, upon noting that $\rho\mathbf{1}_{\{|\rho|\le n\}} \in \mathcal{P}(X)$ holds for all $n\in\mathbb{N}$, and defining

$$V_n := \mathcal{E}\left(\int_0^\cdot \big\langle \rho(t)\mathbf{1}_{\{|\rho(t)|\le n\}}, \mathrm{d}X(t)\big\rangle\right),$$

one can show that the collection $\{V_n(T)\mid n\in\mathbb{N}\}$ is unbounded in probability. This again implies $\mathcal{Y}=\emptyset$.

Indeed, if $\mathcal{Y}$ were not empty, then for any $Y\in\mathcal{Y}$ it would be straightforward to check that YV_n is a nonnegative local martingale—thus, a supermartingale—for all $n\in\mathbb{N}$. By Doob's maximal inequality, this would imply that $\{Y(T)V_n(T)\mid n\in\mathbb{N}\}$ is bounded in probability; and since Y is strictly positive, the set $\{V_n(T)\mid n\in\mathbb{N}\}$ would then be bounded in probability. But we have already seen that the opposite is true.

Remark 1.2. *A Reduction:* Under condition (2) of Theorem 1.1, a given d-dimensional and predictable process H belongs to $\mathcal{P}(X)$ if and only if $\int_0^\cdot \langle H(t), c(t)H(t)\rangle\,\mathrm{d}G(t)$ is finitely-valued. Indeed, the Cauchy-Schwartz inequality (see also [KS91, Proposition 3.2.14]) and the $(\mathbb{P}\otimes G)$-identity $a=c\rho$ imply then

$$\int_0^\cdot |\langle H(t), a(t)\rangle|\,\mathrm{d}\,G(t) \le \left(\int_0^\cdot \langle H(t), c(t)H(t)\rangle\,\mathrm{d}G(t)\right)^{1/2}\left(\int_0^\cdot \langle \rho(t), c(t)\rho(t)\rangle\,\mathrm{d}G(t)\right)^{1/2},$$

and show that $\int_0^\cdot |\langle H(t), a(t)\rangle|\,\mathrm{d}G(t)$ is *a fortiori* finitely-valued.

In obvious notation, we have $\mathcal{P}(X)=\mathcal{P}(M)$ under the condition (2) of Theorem 1.1.

Remark 1.3. *An Interpretation:* It follows from (1.2) that the process $\widehat{V}$ can be interpreted as the strictly positive wealth generated by the predictable process ρ viewed as a "portfolio", starting with a unit of capital. The components of X represent then the *returns* of the various assets in an equity market; the strictly positive processes $S_i=\mathcal{E}(X_i)$ are the *prices* of these assets; the components of ρ stand for the *proportions* of current wealth invested in each one of these assets; whereas the scalar processes $\vartheta_i = (\widehat{V}/S_i)\rho_i$ (respectively, $H_i=\widehat{V}\rho_i$) keep track of the numbers of *shares* (resp., amounts of currency) invested in the various assets.

1.4 Optional Decomposition Theorem

The following is the main result of this work. It is proved in Section 2.

Theorem 1.4. Assume that $\mathcal{Y} \neq \emptyset$. Let V be an adapted càdlàg process, locally bounded from below. Then, the following statements are equivalent:

1. The product YV is a local supermartinagle, for all $Y \in \mathcal{Y}$.
2. The process V is of the form

$$V \;=\; V(0) + \int_0^{\cdot} \langle H(t), \mathrm{d}X(t)\rangle - C\,, \tag{1.3}$$

   where $H \in \mathcal{P}(X)$ and C is a nondecreasing, right-continuous and adapted process with $C(0) = 0$, which is locally bounded from above.

The properties of right continuity and adaptedness, imply that the non-decreasing process C in (1.3) is *optional* – whence the terminology “Optional Decomposition Theorem". This process may, however, fail to be predictable, as it is in the classical Doob-Meyer decomposition.

Remark 1.5. *On Uniqueness:* In the present setting, the decomposition (1.3) of V is unique in the following sense: If in addition to (1.3) we have also

$$V \;=\; V(0) + \int_0^{\cdot} \langle H'(t), \mathrm{d}X(t)\rangle - C'$$

for some $H' \in \mathcal{P}(X)$ and some nondecreasing, right-continuous and adapted process C' with $C'(0) = 0$, then $C = C'$ and $\int_0^{\cdot} \langle H(t), \mathrm{d}X(t)\rangle = \int_0^{\cdot} \langle H'(t), \mathrm{d}X(t)\rangle$ hold modulo evanescence.

Indeed, let $\widehat{Y} := 1/\widehat{V}$, $D = C' - C$, $F := H' - H$, and note that

$$\widehat{Y} D = \widehat{Y} \int_0^{\cdot} \langle F(t), \mathrm{d}X(t)\rangle$$

is a continuous-path local martingale. This follows from integration-by-parts on the right-hand side of the last equality, the equation (1.2) for $\widehat{V}$, and property (2) in Theorem 1.1 for ρ. (For the latter local martingale property, see also the proof of the implication $(2) \Rightarrow (1)$ in Theorem 1.4 in the beginning of §2.2.) Integrating by parts again, we see that

$$\int_0^{\cdot} \widehat{Y}(t)\, \mathrm{d}D(t) = \widehat{Y} D - \int_0^{\cdot} D(t-)\, \mathrm{d}\widehat{Y}(t)$$

is both a continuous-path local martingale and a finite-variation process, which implies $\int_0^{\cdot} \widehat{Y}(t)\, \mathrm{d}D(t) = 0$ modulo evanescence. Since $\widehat{Y}$ is strictly positive, this last fact implies $D = 0$ modulo evanescence, completing the argument.

Finally, let us note that for the integrands H and H' we may only conclude that $H = H'$ holds in the $(\mathbb{P} \otimes G)$-a.e. sense.

2 Proof of Theorem 1.4

We shall denote by $\mathcal{L}^{\mathrm{c}}$ the collection of all local martingales L with continuous paths and $L(0) = 0$; furthermore, we shall denote by $\mathcal{L}^{\mathrm{d}}$ the collection of all local martingales L with $\Delta L > -1$ and $L(0) = 0$ which are purely discontinuous, i.e., satisfy $[L, \Lambda] \equiv 0$ for all $\Lambda \in \mathcal{L}^{\mathrm{c}}$.

2.1 An intermediate, but crucial, result

In order to prove Theorem 1.4, we first isolate an auxiliary result that will enable us eventually to deal only with continuous-path local martingales. In a sense, the result that follows constitutes the main technical argument in this proof. In [Jac12], the author treats the ODT for continuous semimartingales under the additional assumption that the underlying filtration supports only continuous martingales; in this case, it is completely straightforward to check that the process B appearing in Lemma 2.1 is automatically nondecreasing, making the proof of the ODT considerably simpler.

Lemma 2.1. Let B be a semimartingale with the following properties:

1. B be a locally bounded from above, and $B(0) = 0$.
2. $B + [B, L]$ is a local submartingale, for every $L \in \mathcal{L}^{\mathrm{d}}$.
3. $[B, L] = 0$, for every $L \in \mathcal{L}^{\mathrm{c}}$.

Then, B is actually non-decreasing.

Proof. Property (2) implies that B itself is a local submartingale (just take $L \equiv 0$ there). Standard localization arguments imply that we may take B to be bounded from above; therefore, we shall assume the existence of $b \in \mathbb{R}_+$ such that $B \leq b$. In view of the submartingale convergence theorem, B is an actual submartingale with last element $B(\infty)$, and $\mathbb{E}\,[B(T)] \geq \mathbb{E}\,[B(0)] = 0$ holds for all stopping times T.

Consider now a countable collection $(\tau_n)_{n\in\mathbb{N}}$ of *predictable* stopping times that exhaust the accessible jump-times of B; see [Del72, Theorem T30, page 84]. Defining also the predictable set

$$J := \bigcup_{n\in\mathbb{N}} [\![\tau_n, \tau_n]\!]\,,$$

we note that the process

$$B^J := \int_{(0,\cdot\,]} \mathbf{1}_J(t)\,\mathrm{d}B(t) = \sum_{n\in\mathbb{N}} \Delta B(\tau_n)\,\mathbf{1}_{\{\tau_n \leq \cdot\}}$$

is a local submartingale.

• We shall first show that B^J is nondecreasing, which amounts to showing that $\Delta B(\tau_n) \geq 0$ holds on $\{\tau_n < \infty\}$ for all $n \in \mathbb{N}$. To this end, we define for each $n \in \mathbb{N}$ the $\mathcal{F}(\tau_n-)$-measurable random variable

$$p_n := \mathbb{P}\,[\,\Delta B(\tau_n) < 0 \mid \mathcal{F}(\tau_n-)\,]\,\mathbf{1}_{\{\tau_n<\infty\}}\,.$$

On the event $\{\tau_n < \infty,\, p_n = 0\}$, we have $\Delta B(\tau_n) \geq 0$. For $n \in \mathbb{N}$ and $k \in \mathbb{N}$, we define $L_{n,k} \in \mathcal{L}^{\mathrm{d}}$ to be the local martingale with $L_{n,k}(0) = 0$ and a single jump at τ_n, such that

$$\Delta L_{n,k}(\tau_n) = \big(1 - (1/k)\big)\,\big((1/p_n)\,\mathbf{1}_{\{\Delta B(\tau_n)<0\}} - 1\big)\,\mathbf{1}_{\{p_n>0\}}\,.$$

We note that $B + [B, L_{n,k}] = B \leq b$ holds on $[\![0, \tau_n[\![$, while on the event $\{\tau_n < \infty\,,\, p_n > 0\}$ we have

$$\begin{aligned}\Delta\left(B + [B, L_{n,k}]\right)(\tau_n) &= \Delta B(\tau_n)\big(1 + \Delta L_{n,k}(\tau_n)\big)\\ &= \frac{\Delta B(\tau_n)}{k} - \frac{k-1}{kp_n}\,(\Delta B(\tau_n))^- \leq (\Delta B(\tau_n))^+\,.\end{aligned}$$

Property (2) imposed on B and the above facts imply that $B+[B, L_{n,k}]$ is a local submartingale bounded from above by b on $[0, \tau_n]$. It follows that on the event $\{\tau_n < \infty,\, p_n > 0\}$ we have $\mathbb{E}\,[\,\Delta B(\tau_n)\,(1 + \Delta L_{n,k}(\tau_n))\;\mid \mathcal{F}(\tau_n-)\,] \geq 0$, which translates into

$$\mathbb{E}\left[(\Delta B(\tau_n))^-\;\mid \mathcal{F}(\tau_n-)\right] \leq \frac{\mathbb{E}\left[(\Delta B(\tau_n))^+\;\mid \mathcal{F}(\tau_n-)\right]}{1 + \big((k-1)/p_n\big)},\;\text{ on the event }\{\tau_n < \infty\,,\; p_n > 0\}\,,$$

for all $k \in \mathbb{N}$. Sending $k \to \infty$, it follows that $\Delta B(\tau_n) \geq 0$ holds on $\{\tau_n < \infty,\, p_n > 0\}$. We conclude that we have indeed $\Delta B(\tau_n) \geq 0$ on the event $\{\tau_n < \infty\}$.

- It remains to show that

$$B' := B - B^J = \int_{(0,\cdot]} \mathbf{1}_{(\Omega\times\mathbb{R}_+)\setminus J}(t)\mathrm{d}B(t)$$

is also a nondecreasing process. We note that B' inherits some of the properties of B: in particular, we have $B'(0) = 0$ and for every $L \in \mathcal{L}^{\mathsf{d}}$, the process

$$B' + [B', L] = \int_0^{\cdot} \mathbf{1}_{(\Omega\times\mathbb{R}_+)\setminus J}(t) \big(\mathrm{d}B(t) + \mathrm{d}[B, L](t)\big)$$

is a local submartingale. We also have $B' \leq B \leq b$, and additionally B' has jumps only at totally inaccessible stopping times. To ease the notation we write B instead of B' for the remainder of this proof, assuming from now onwards that B jumps only at totally inaccessible stopping times. For each $n \in \mathbb{N}$, we define the local martingale

$$L_n := n \sum_{t\leq\cdot} \mathbf{1}_{\{\Delta B(t)\leq -1/n\}} - D_n\,,$$

where D_n is a suitable nondecreasing process with continuous paths (since the jumps of B are totally inaccessible—see [Del72, Theorem T40, page 111].). Observe that $L_n \in \mathcal{L}^{\mathsf{d}}$, which implies that $B + [B, L_n]$ is a local submartingale for all $n \in \mathbb{N}$. Furthermore, we note that

$$B + [B, L_n] = B + n \sum_{t\leq\cdot} \Delta B(t)\, \mathbf{1}_{\{\Delta B(t)\leq -1/n\}} \leq b,$$

which implies that $B + [B, L_n]$ is a true submartingale. It follows that

$$n\, \mathbb{E}\left[-\sum_{t\leq n} \Delta B(t)\, \mathbf{1}_{\{\Delta B(t)\leq -1/n\}}\right] \leq \mathbb{E}\left[B(n) - B(0)\right] = \mathbb{E}\left[B(n)\right] \leq b\,, \quad \forall\, n \in \mathbb{N}\,,$$

and the monotone convergence theorem gives $\mathbb{E}\big[\sum_{t\in\mathbb{R}_+} (\Delta B(t))^-\big] = 0$, which implies $\Delta B \geq 0$. Continuing, we define for each $n \in \mathbb{N}$ a new local martingale

$$\widetilde{L}_n := \widetilde{D}_n - \big(1 - (1/n)\big) \sum_{t\leq\cdot} \mathbf{1}_{\{\Delta B(t)\geq 1/n\}}\,,$$

where $\widetilde{D}_n$ is an appropriate continuous and nondecreasing process. Note that we have again $\widetilde{L}_n \in \mathcal{L}^{\mathsf{d}}$ for all $n \in \mathbb{N}$, which implies that the processes

$$B + [B, \widetilde{L}_n] = B - \big(1 - 1/n\big) \sum_{t\leq\cdot} \Delta B(t)\, \mathbf{1}_{\{\Delta B(t)\geq 1/n\}}\,, \qquad n \in \mathbb{N}$$

are local submartingales, uniformly bounded from above by b. Thus, it follows that $\sum_{t\leq\cdot} \Delta B(t)$ is finitely-valued; and that $\widetilde{B} := B - \sum_{t\leq\cdot} \Delta B(t)$ is a local submartingale. Recalling that the jumps of B occur only at totally inaccessible stopping times, this last process $\widetilde{B}$ has continuous paths (again, see [Del72, Theorem T40, page 111]) and is strongly orthogonal to all continuous-path local martingales, which means that it is of finite variation. Since it is a local submartingale, this process has to be nondecreasing. It follows from this reasoning that the process

$$B = \left(B - \sum_{t\leq\cdot} \Delta B(t)\right) + \sum_{t\leq\cdot} \Delta B(t)$$

is nondecreasing, and this concludes the argument. □

Let us note that the proof of Lemma 2.1 would be slightly simpler, were we to make the assumption that the underlying filtration supports only quasi-left-continuous martingales; this is because in this case the process B^J is predictable.

2.2 Proof of Theorem 1.4

The implication $(2) \Rightarrow (1)$ is straightforward. Indeed, we fix $Y \in \mathcal{Y}$ and note that

$$Z := \int_0^{\cdot} Y(t-)\,\mathrm{d}X(t) + [Y, X] = YX - X(0) - \int_0^{\cdot} X(t)\,\mathrm{d}Y(t)$$

is a d-dimensional local martingale. Then, if $V = V(0) + \int_0^{\cdot} \langle H(t), \mathrm{d}X(t)\rangle - C$ is a process as in (1.3), one computes

$$YV = V(0) + \int_0^{\cdot} V(t-)\,\mathrm{d}Y(t) + \int_0^{\cdot} \langle H(t), \mathrm{d}Z(t)\rangle - \int_0^{\cdot} Y(t)\,\mathrm{d}C(t),$$

which shows that YV is a local supermartingale.

• For the implication $(1) \Rightarrow (2)$, let us assume that the process V as in the statement of the theorem is such that YV is a local supermartingale for all $Y \in \mathcal{Y}$. In particular, recalling the notation of (1.1), we note that $U := (V/\widehat{V})$ is a local supermartingale. We apply to the continuous local martingale part of this process the Kunita-Watanabe decomposition with respect to the continuous local martingale M, and obtain the Doob-Meyer decomposition

$$U := V/\widehat{V} = V(0) + \int_0^{\cdot} \langle \theta(t), \mathrm{d}M(t)\rangle + N - B. \tag{2.1}$$

Here $\theta \in \mathcal{P}(X)$ (recall Remark 1.2), and $N \in \mathcal{L}^{\mathrm{c}}$ satisfies $[N, M] = 0$, whereas B is a local submartingale with $B(0) = 0$ and "purely discontinuous", in the sense that

$$[B, L] = 0 \quad \text{holds for every } L \in \mathcal{L}^{\mathrm{c}}\,. \tag{2.2}$$

In particular, we note that $[B, N] = 0 = [B, M]$.

(i): The first item of business is to show that the process B in (2.1) is actually *nondecreasing;* for this we shall use Lemma 2.1. Since $N + \int_0^{\cdot} \langle \theta(t), \mathrm{d}M(t)\rangle$ is continuous and U locally bounded from below, the process $B = V(0) + N + \int_0^{\cdot} \langle \theta(t), \mathrm{d}M(t)\rangle - U$ is locally bounded from above.

Let us fix $L \in \mathcal{L}^{\mathrm{d}}$. From $(1/\widehat{V})\,\mathcal{E}(L) \in \mathcal{Y}$ we observe – e.g., using the product rule – that the process

$$V \cdot \big(1/\widehat{V}\big)\,\mathcal{E}(L) = U\,\mathcal{E}(L) = \left(V(0) + N + \int_0^{\cdot} \langle \theta(t), \mathrm{d}M(t)\rangle - B\right)\mathcal{E}(L)$$

is a local supermartingale. Furthermore, the process $\big(N + \int_0^{\cdot} \langle \theta(t), \mathrm{d}M(t)\rangle\big)\,\mathcal{E}(L)$ is a local martingale, and it follows from these two observations that

$$B\,\mathcal{E}(L) = \int_0^{\cdot} \mathcal{E}(L)(t-)\,\mathrm{d}\,(B + [B, L])\,(t) + \int_0^{\cdot} (B\mathcal{E}(L))\,(t-)\,\mathrm{d}L(t)$$

is a local submartingale. This, in turn, implies that

$$B + [B, L] = \int_0^{\cdot} \frac{1}{\mathcal{E}(L)(t-)}\,\mathrm{d}\big(B \cdot \mathcal{E}(L)\big)(t) - \int_0^{\cdot} B(t-)\,\mathrm{d}L(t)$$

is also a local submartingale. Recalling the property (2.2) and invoking Lemma 2.1, we conclude from this observation that the local submartingale B in the decomposition (2.1) is indeed non-decreasing.

(ii): The second item of business is to show that $N \equiv 0$ holds in (2.1). The crucial observation here is that, because of $[N, M] = 0$, the product $(1/\widehat{V})\,\mathcal{E}(nN)$ is an element of $\mathcal{Y}$ for all $n \in \mathbb{N}$. As a consequence, $U\mathcal{E}(nN) = V \cdot (1/\widehat{V})\,\mathcal{E}(nN)$ is a local supermartingale for all $n \in \mathbb{N}$.

Since $[\mathcal{E}(nN), B] = 0$ and $[\mathcal{E}(nN), M] = 0$, it follows that $\mathcal{E}(nN)(B - N)$ is a local submartingale for all $n \in \mathbb{N}$. Now we observe

$$\mathcal{E}(nN)(B-N) = \int_0^\cdot (B-N)(t-)\,\mathrm{d}\mathcal{E}(nN)(t) + \int_0^\cdot \mathcal{E}(nN)(t-)\,\mathrm{d}\big(B - N - [nN, N]\big)(t),$$

from which it follows that $B - n[N, N]$ is a local submartingale for all $n \in \mathbb{N}$. This is only possible if $[N, N] = 0$ which, since $N(0) = 0$, implies $N \equiv 0$; as a result, (2.1) becomes

$$U \;=\; (V/\widehat{V}) \;=\; V(0) + \int_0^\cdot \langle \theta(t), \mathrm{d}M(t)\rangle - B. \tag{2.3}$$

(iii): We are now in a position to conclude. Yet another application of the integration-by-parts formula on $V = \widehat{V}\,U$ gives, in conjunction with (1.2), (2.3) and Theorem 1.1, the decomposition

$$\begin{aligned} V &= V(0) + \int_0^\cdot U(t-)\,\mathrm{d}\widehat{V}(t) + \int_0^\cdot \widehat{V}(t)\,\mathrm{d}U(t) + [\widehat{V}, U] \\ &= V(0) + \int_0^\cdot U(t-)\,\mathrm{d}\widehat{V}(t) + \int_0^\cdot \widehat{V}(t)\,\langle \theta(t),\, \mathrm{d}X(t)\rangle - \int_0^\cdot \widehat{V}(t)\,\mathrm{d}B(t) \\ &= V(0) + \int_0^\cdot \widehat{V}(t)\,\langle U(t-)\rho(t) + \theta(t), \mathrm{d}X(t)\rangle - \int_0^\cdot \widehat{V}(t)\,\mathrm{d}B(t). \end{aligned}$$

Defining $U_-(t) := U(t-)$ for $t > 0$, as well as

$$H \;:=\; \widehat{V}\big(U_-\,\rho + \theta\big) \;\in\; \mathcal{P}(X) \quad \text{ and } \quad C \;:=\; \int_0^\cdot \widehat{V}(t)\,\mathrm{d}B(t)\,,$$

we obtain the decomposition (1.3) as claimed. □

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