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MARKOVIAN NASH EQUILIBRIUM IN FINANCIAL MARKETS WITH ASYMMETRIC INFORMATION AND RELATED FORWARD–BACKWARD SYSTEMS

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This paper develops a new methodology for studying continuous-time Nash equilibrium in a financial market with asymmetrically informed agents. This approach allows us to lift the restriction of risk neutrality imposed on market makers by the current literature. It turns out that, when the market makers are risk averse, the optimal strategies of the agents are solutions of a forward–backward system of partial and stochastic differential equations. In particular, the price set by the market makers solves a nonstandard "quadratic" backward stochastic differential equation. The main result of the paper is the existence of a Markovian solution to this forward–backward system on an arbitrary time interval, which is obtained via a fixed-point argument on the space of absolutely continuous distribution functions. Moreover, the equilibrium obtained in this paper is able to explain several stylized facts which are not captured by the current asymmetric information models.

1. Introduction. In this paper, we address the long-standing open problem[1] of existence of an equilibrium in a financial market with asymmetrically informed traders and risk averse market makers in continuous-time with finite horizon. In such a market, the price of the traded asset is an equilibrium outcome of a game between the market makers and an informed trader who possesses superior information. Both market makers and the informed trader choose their controls adapted to their filtrations. We assume that the market makers obtain their information through their interactions with the traders and have the obligation to absorb the total demand for the asset. Therefore, their filtration is the one generated by the total demand process, Y. The informed trader, on the other hand, has the filtration jointly generated by the market prices and her private information. In this game the market makers' control is the price, S, while the control of the informed trader is her trading strategy, X. Thus, the equilibrium price should satisfy the following conditions: (i) the informed trader's optimisation problem has a solution, and (ii) given this solution, the price S fulfils the market makers' objectives.[2]

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[1]This problem was posed by Subrahmanyam in [40].

[2]The focus of this paper is the equilibrium between the market makers and the strategic informed trader as well as the resulting price. We do not study in depth the interaction among the market

The study of this game goes back to [29], which is the canonical model in market microstructure theory for the analysis of strategic trading in the presence of private information (see [3, 14] and [37] for a review of Kyle's model as well as a discussion of its relationship with other market microstructure models). Various extensions of the original model have been studied in the literature; see, among others, [2, 4, 5, 8, 15, 17–19] and [28].

The original model and all these extensions assume that the market makers are risk-neutral and compete in a *Bertrand fashion* for the total demand (see Section 12.C of [36] for the definition). This means that, in equilibrium, the utility of any market maker is a martingale. Since the utility is linear, this in turn implies that the optimal strategy for the market makers is to set the price to be the conditional expectation of the fundamental value of the asset given their filtration. In particular, in these models there is always a unique price satisfying the objective of the market makers for any control of the informed trader. Furthermore, the martingale property of the price results in the optimal strategy of the informed trader being *inconspicuous* in the equilibrium; that is, the law of Y in its own filtration is the same as that of $Y - X$ in its own filtration.

Whereas the risk-neutrality of the market makers makes the model tractable, it is not consistent with the observed market behaviour. Indeed, there is a vast empirical evidence that the market makers are risk averse and exercise their control in a way that total demand mean reverts around a target level at a speed determined by their risk aversion (see [24] and [33] for New York Stock Exchange, [23] for London Stock Exchange, [12] for Foreign Exchange; for a survey of related literature and results, see Sections 1.2 and 1.3 in [10]).

Although relaxing the assumption of market makers' risk neutrality is natural and has been prompted by the empirical evidence, there has been only one attempt in the literature to investigate the effect of such an extension. Subrahmanyam in [40] considered a one-period model where market makers with identical exponential utilities set the price that makes their utilities martingales. This assumption is the direct analogue of the original Kyle model discussed above in the context of risk averse market makers. The tractability of the model considered in [40] relies on the fact that in a one-period setting there exists an optimal response for the market makers for any strategy of the insider. However, the existence of such responses is uncertain in a multi-period setting. Indeed, Subrahmanyam noted that an extension of his model to a multi-period setting is not possible due to the strategic behaviour of the agents.

The aforementioned difficulty with the existence of an optimal response for the market makers persists in continuous time. More precisely, given a trading strategy of the informed trader, the optimal response of the market makers is found via

makers faced with a given demand process. The reader is referred to the recent manuscript of Bank and Kramkov [6] for an in-depth analysis of this interaction.

solving the backward stochastic differential equation (BSDE)

$$dS_t = Z_t \, d\beta_t - \frac{c}{2} Y_t Z_t^2 \, dt, \tag{1.1}$$

$$\exp(cY_1 S_1) = \mathbb{E}\big[\exp(cY_1 V) | \mathcal{F}_1^Y\big], \tag{1.2}$$

where $c > 0$ is a constant, V is a bounded random variable representing the fundamental value of the asset, Y is a given total demand process and β is a Brownian motion with respect to $\mathcal{F}^Y$ – the filtration of the market makers generated by Y. A solution to this BSDE is a pair (Z, S) of $\mathcal{F}^Y$-adapted processes satisfying (1.1) and (1.2). When this BSDE admits a solution, S is the price that makes the utilities of the market makers martingales.

Although the terminal condition is unconventional, as Y and V are given, the right-hand side of (1.2) is a fixed $\mathcal{F}_1^Y$-measurable random variable. Thus, we can rewrite the terminal condition as $S_1 = \xi$, which is bounded due to the boundedness of V. The form of the driver, on the other hand, poses a real difficulty since the process Y multiplying Z^2 is in general unbounded. This renders the system (1.1)–(1.2) outside the realm of standard quadratic BSDEs.

The price response of the market makers is only one side of the equilibrium. To characterise an equilibrium, we also need to find the level of total demand, Y, implied by the informed agent's optimal trading strategy. Consistent with the literature, we assume that the total demand is driven by a Brownian motion and has a drift which is determined by the informed trader. Hence, an equilibrium consists of (α, S), where α is the optimal drift given S, and S satisfies the forward–backward stochastic differential equation (FBSDE)

$$dY_t = d\beta_t + \hat{\alpha}\big(t, (Y_s)_{s \le t}\big) \, dt, \tag{1.3}$$

$$dS_t = Z_t \, d\beta_t - \frac{c}{2} Y_t Z_t^2 \, dt, \tag{1.4}$$

$$\exp(cY_1 S_1) = \mathbb{E}\big[\exp(cY_1 V) | \mathcal{F}_1^Y\big], \tag{1.5}$$

where $\hat{\alpha}$ is the $\mathcal{F}^Y$-optional projection of α. It is well known that the existence of a solution for FBSDEs is quite delicate even when the driver is globally Lipschitz and satisfies a linear growth condition. Antonelli [1] showed the existence and uniqueness of a solution over a small time interval via a fixed-point algorithm on a Banach space of processes. This result has been extended by [22] and [20] to arbitrary time intervals by pasting solutions obtained for small time intervals. An alternative technique for solving FBSDEs is the so-called *four-step scheme* introduced by [31], which requires strong smoothness on the coefficients of the system and is based on the link between quasi-linear partial differential equations. When the driver is quadratic, the problem becomes more complicated and only few results are available. Moreover, since available results originate from the solvability of quadratic BSDEs, the standard assumption in the current literature is that the driver is bounded by $k(1 + z^2)$ for some constant k (see, e.g., [25]). However, as

(1.4) does not fit into the current paradigm of quadratic BSDEs, these results are not applicable to our setting.

Despite these difficulties, we obtain a solution to this system with $S_t = H(t, Y_t)$ for some smooth function H, when α is the optimal drift of the informed trader given S. This solution provides a Markovian equilibrium for the model that we consider. We show that in this case the system (1.3)–(1.5) transforms into

$$H_t + \frac{1}{2}H_{yy} = 0, \tag{1.6}$$

$$dY_t = d\beta_t - \frac{c}{2}Y_t H_y(t, Y_t)\,dt, \tag{1.7}$$

$$V \overset{d}{=} H(1, Y_1), \tag{1.8}$$

provided Y has a smooth transition density, where the last equality is an equality in distribution. This is still a forward–backward system of a forward SDE and a backward PDE such that the terminal condition of the PDE depends on the solution of the SDE, which in turn depends on H. This coupling between the SDE and the PDE suggests a use of a fixed-point algorithm.

Indeed, if we are given a continuous distribution for Y_1, (1.8) yields a function $H(1, y)$, which is increasing in y. This allows us to obtain $H(t, y)$ via (1.6), and Y via (1.7). Hence, this procedure defines a mapping from the space of distributions into itself. We show in Theorem 4.1, via Schauder's fixed-point theorem, that this mapping has a fixed point under the assumption that $V = f(\eta)$ for some increasing and bounded f satisfying some mild regularity conditions, and a standard normal random variable η.

The validity of Schauder's fixed-point theorem in our setting relies heavily on the properties of solutions of (1.7) for any given function H satisfying (1.6) with a bounded and increasing terminal condition. These properties are explored in Lemmata 4.1–4.3. In particular, we obtain a remarkable connection between the laws of Y_1 and that of Brownian motion. Namely, we prove that

$$\mathbb{E}\big[(Y_1 - x)^+\big] \geq \mathbb{E}\big[\big(e^{-cC}B_1 - x\big)^+\big] > 0,$$
$$\mathbb{E}\big[(-x - Y_1)^+\big] \geq \mathbb{E}\big[\big(-x - e^{-cC}B_1\big)^+\big] > 0,$$

for all $x > 0$, where C is a constant that depends only on the bound on H. We also show that Y has a smooth transition density.

The existence of solution to the system (1.6)–(1.8) ensures the existence of a Markovian solution for the price process which makes the utilities of market makers martingales once the drift of total demand, Y has the form given in equation (1.7). However, in order for such a drift to appear in equilibrium, it should be optimal for the insider to choose a drift whose $\mathcal{F}^Y$-optional projection has this form.

To this end, we establish in Proposition 3.1 that the sole criterion of optimality for the insider is that the strategy fulfils the bridge condition $H(1, Y_1) = V$. Thus, if Markovian equilibrium exists, the equilibrium pair (H, Y) solves the system (1.6)–(1.8) *and* satisfies $H(1, Y_1) = V$. The existence of such a pair is precisely the result of Theorem 5.1, which allows us to establish the existence of the equilibrium in Theorem 5.2.

The paper is structured as follows. Section 2 describes the model we consider while Section 3 is devoted to the (formal) derivation of the system (1.6)–(1.8) and characterisation of the optimal strategy of the informed trader. Section 4 establishes the existence of solution to the system (1.6)–(1.8) and Section 5 proves the existence of the equilibrium. In Section 6, we discuss the impact of risk aversion on the market behaviour in the equilibrium and explore the connections to the empirical literature.

2. Market structure. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right continuity and $\mathbb{P}$-completeness. We suppose that $\mathcal{F}_0$ is not trivial and there exists an $\mathcal{F}_0$-measurable standard normal random variable, η. Moreover, the filtered probability space also supports a standard Brownian motion, B, with $B_0 = 0$, and thus, B is independent of η. We define $V := f(\eta)$ for some bounded and strictly increasing function f with a continuous derivative.

As all the randomness in our model will depend only on V and B, we shall take $\mathcal{F} = \sigma(\tilde{\mathcal{N}}, \tilde{\mathcal{F}})$, where $\tilde{\mathcal{F}}$ is the minimal σ-field with respect to which V and $(B_t)_{t\in[0,1]}$ are measurable and $\tilde{\mathcal{N}} = \{E : E \subset F \text{ for some } F \in \tilde{\mathcal{F}} \text{ with } \mathbb{P}(F) = 0\}$. Moreover, in view of the independence of V and B, we may assume the existence of a family of probability measures, $(\mathbb{P}^v)$ such that the disintegration formula

$$\mathbb{P}(E) = \int_{f(\mathbb{R})} \mathbb{P}^v(E)\mathbb{P}(V \in dv)$$

holds for all $E \in \mathcal{F}$, and for all $v \in f(\mathbb{R})$ the measure $\mathbb{P}^v$ satisfies $\mathbb{P}^v(E) = \mathbb{P}(E|V = v)$. The existence of such a family is easily justified when we consider $\Omega = f(\mathbb{R}) \times C([0, 1], \mathbb{R})$, where $C([0, 1], \mathbb{R})$ is the space of real valued continuous functions on $[0, 1]$.

We consider a market in which the risk free interest rate is set to 0 and a single risky asset is traded. The fundamental value of this asset equals V, which will be announced at time $t = 1$.

There are three types of agents that interact in this market:

(i) Liquidity traders who trade for reasons exogenous to the model and whose cumulative demand at time t is given by σB_t for some constant $\sigma > 0$.

(ii) A single informed trader, who knows V from time $t = 0$ onward, and is risk neutral. We will call the informed trader *insider* in what follows and denote her cumulative demand at time t by X_t. The filtration of the insider, $\mathcal{F}^I$, is generated by observing the price of the risky asset and V. Thus, an insider who has the information that $V = v$ possesses the minimal right continuous filtration generated by V and the price process, and completed with the null sets of $\mathbb{P}^v$.

(iii) Market makers observe only the net demand of the risky asset, $Y = X + \sigma B$, thus, their filtration, $\mathcal{F}^M$, is the minimal right-continuous filtration generated by Y and completed with $\mathbb{P}$-null sets. The number of market makers is assumed to be $N \geq 2$.

We also assume that the market makers have identical preferences described by the common utility function, $U(x) = -e^{-\rho x}$, and compete in a Bertrand fashion for the net demand of the risky asset. In case of several market makers quoting the same winning price, we adopt the convention that the total order is equally split among them.

Similar to [2], we assume that the market makers set the price of risky security, S, as $S_t = H(t, Y_t)$ for some function H.

To understand the subtlety of the equilibrium derived later, it is important to observe that an insider who is given the information that $V = v$ has the probability measure $\mathbb{P}^v$ on $(\Omega, \mathcal{F})$ while the probability measure of the market makers is given by $\mathbb{P}$, and these measures are singular with respect to each other as $\mathbb{P}^v(V = v) = 1$, whereas $\mathbb{P}(V = v) = 0$ in our settings.

We now define admissibility of functions H for the market makers (which will be called *pricing rule* in what follows) and admissibility of the trading strategy of the insider. The conditions we impose are standard in the literature and were first introduced in [2]. The integrability conditions (2.1) and (2.2) prevent the insider from following doubling strategies (see [2] for the discussion). The absolute continuity of insider's strategies is without any loss of generality since strategies with a martingale component and/or jumps are strictly suboptimal as shown in [2].

DEFINITION 2.1. A function $H : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ is a *pricing rule* if $H \in C^{1,2}$, strictly increasing in y and satisfies

$$\mathbb{E}H^2(1, \sigma B_1) < \infty \quad \text{and} \quad \mathbb{E}\int_0^1 H^2(t, \sigma B_t)\, dt < \infty. \tag{2.1}$$

The class of such functions is denoted with $\mathcal{H}$.

Note that since any pricing rule is strictly monotone, B is adapted to $\mathcal{F}^I$. The admissible strategies for the insider is defined in the following.

DEFINITION 2.2. An insider strategy, X, is *admissible* for a given pricing rule, H, if $X_t = \int_0^t \alpha_s\, ds$ for some $\mathcal{F}^I$-progressively measurable α such that, for all $v \in f(\mathbb{R})$, we have $\mathbb{P}^v(\int_0^1 |\alpha_s|\, ds < \infty) = 1$,

$$\mathbb{E}^v \int_0^1 H^2(t, X_t + \sigma B_t)\, dt < \infty, \tag{2.2}$$

and $\mathbb{E}^v[\min\{0, W_1^X\}] > -\infty$, where W_1^X is the terminal wealth of insider given by

$$W_1^X := \int_0^1 X_s\, dH(s, Y_s) + X_1\big(V - H(1, Y_1)\big) = \int_0^1 \big(V - H(s, Y_s)\big)\, dX_s. \tag{2.3}$$

The class of admissible strategies for a given pricing rule H will be denoted by $\mathcal{A}(H)$.

Observe that for any X of finite variation W_1^X is well defined since $V - H(s, X_s)$ is a continuous process for any pricing rule, $\mathbb{P}^v$-a.s.

The first term in (2.3) corresponds to continuous trading in the risky asset, while the second term exists due to a potential discontinuity in the asset price when the value becomes public knowledge at time $t = 1$. The second expression for the wealth follows from integration by parts.

Given the definition of a pricing rule and admissible trading strategies, we can now define an equilibrium as follows.

DEFINITION 2.3. A pair (H^*, X^*) is an equilibrium if $H^* \in \mathcal{H}$, $X^* \in \mathcal{A}(H^*)$, and

(i) given H^*, the insider's strategy X^* solves her optimisation problem:

$$\mathbb{E}^v[W_1^{X^*}] = \sup_{X \in \mathcal{A}(H^*)} \mathbb{E}^v[W_1^X] \qquad \forall v \in f(\mathbb{R}).$$

(ii) Given X^*, the pricing rule H^* is such that the market makers' wealth satisfies zero-utility gain condition, that is, $U(G)$ is a $(\mathcal{F}^M, \mathbb{P})$-martingale, where

$$G_t := -\frac{1}{N} \int_0^t Y_s^* \, dH^*(s, Y_s^*) + \mathbf{1}_{t=1} \frac{Y_1^*}{N} \big(H^*(Y_1^*, 1) - V\big). \tag{2.4}$$

The above is the formulation of a Markovian Nash equilibrium in our model. The condition for the optimality of insider's strategy is a straightforward description of the best response of the insider for a given pricing rule. The market makers' optimality condition follows the tradition of Kyle models where each market makers' utility remains a martingale due to the Bertrand competition among them. Indeed, suppose that one of the market makers, say MM_i, decides to deviate at some time t from this pricing rule by, for example, selling at a higher price than H would suggest in order to achieve a positive utility gain. However, the other market makers could then offer to sell at a slightly lower price which would still allow them to make a positive utility gain. Moreover, as this lower price is more favourable to the traders, no one will trade with MM_i eliminating any opportunities for a utility gain. Deviation from the zero-utility gain condition by buying at a lower price is also suboptimal for a similar reason. Clearly, buying (resp., selling) at a higher (resp., lower) price is suboptimal since it leads to a loss in the utility. Thus, a pricing rule satisfying the zero-utility gain condition is the best response of the market makers. The zero-utility gain condition is also a direct continuous-time analogue of the concept of *autarky utility* defining the equilibrium in the one-period Kyle model of [40] studying the effects of the risk aversion of market makers on equilibrium. Recall that the market makers are identical by assumption and, therefore, they offer the same price quotes in equilibrium and the order is split equally among them due to our order splitting convention when there are more than one winning quote.

3. Characterisation of equilibrium. In this section, we show that a Markovian equilibrium of this game is described by a forward–backward system of stochastic and partial differential equations given by (1.6)–(1.8) by first studying the optimal response of the market makers for a given strategy of the insider, and then characterising the profit maximising strategies for the insider. The heuristic arguments below which are used to characterise the equilibrium will be made rigorous in subsequent sections.

Suppose that X is an admissible trading strategy of the insider so that Y in its own filtration satisfies

$$dY_t = \sigma\, dB_t^Y + \hat{\alpha}_t\, dt,$$

where B^Y is an $\mathcal{F}^M$-Brownian motion and $\hat{\alpha}$ is the $\mathcal{F}^M$-optional projection of α. The best response of the market makers is to choose a price, S, that will satisfy the zero-utility gain condition. Let price S follow

$$dS_t = Z_t\, dB_t^Y + \mu_t\, dt,$$

for some predictable process Z and an optional process μ that are to be determined by the market makers. As there is a potential discrepancy between S_1 and V, there is a possibility of a jump in the market makers' wealth at time 1. More precisely,

$$\Delta G_1 = \frac{Y_1}{N}(S_1 - V).$$

However, the zero-utility gain condition implies

$$1 = \mathbb{E}\left[\exp\left(-\frac{\rho Y_1}{N}(S_1 - V)\right)\Big|\mathcal{F}_1^M\right],$$

which is equivalent to

$$\mathbb{E}\left[\exp\left(\frac{\rho Y_1}{N} V\right)\Big|\mathcal{F}_1^M\right] = \exp\left(\frac{\rho Y_1}{N} S_1\right). \tag{3.1}$$

On the other hand, if we compute the dynamics of $U(G)$ for $t < 1$ by Itô's formula, we obtain

$$dU(G_t) = U(G_t)\frac{\rho}{N} Y_t \left\{\sigma_t\, dB_t^Y + \left(\mu_t + \frac{\rho}{2N} Y_t \sigma_t^2\right) dt\right\}.$$

Reiterating the zero-utility gain condition for $t < 1$ shows that we must have

$$\mu_t = -\frac{\rho}{2N} Y_t Z_t^2.$$

Therefore, the zero-utility gain condition stipulates that the price S follows

$$dS_t = Z_t\,dB_t^Y - \frac{\rho}{2N}Y_t Z_t^2\,dt, \tag{3.2}$$

and the market makers' problem is to find (Z, S) to solve (3.2) with the terminal condition (3.1) given the total demand process Y.

The BSDE in (3.2) is reminiscent of the quadratic BSDEs, which have been studied extensively, and the connection of which to problems arising in mathematical finance is well established (see, e.g., [7, 13, 27] and the references therein). The essential deviation of (3.2) from the BSDEs considered in these papers is that the coefficient of Z_t^2 in (3.2) is $\frac{\rho}{2N}Y_t$, which is in general unbounded. This makes the direct application of the results contained in the current literature for quadratic BSDEs to (3.2) impossible.

However, if we turn to a Markovian equilibrium, that is, consider $S_t = H(t, Y_t)$, it is natural to expect that in equilibrium $\hat{\alpha}_t = \hat{\alpha}_t(t, Y_t, S_t, Z_t)$ for some deterministic function $\hat{\alpha}$ so that

$$dY_t = \sigma\,dB_t^Y + \hat{\alpha}(t, Y_t, S_t, Z_t)\,dt. \tag{3.3}$$

Thus, if a Markovian equilibrium can be attained it will provide a Markovian solution to the FBSDE defined by (3.1)–(3.3), where $\hat{\alpha}$ is the optimal drift chosen by the insider.

We now turn to the optimisation problem for the insider when $S_t = H(t, Y_t)$ for an admissible pricing rule H. Observe that from the point of view of the insider the total demand process follows

$$dY_t = \sigma\,dB_t + \alpha_t\,dt,$$

for a given insider's strategy $X_t = \int_0^t \alpha_s\,ds$. And the value function, Ψ, can be defined as

$$\Psi(t, y) = \sup_{X\in\mathcal{A}(H)} \mathbb{E}^v\left[\int_t^1 \big(V - H(s, Y_s)\big)\alpha_s\,ds \,\middle|\, Y_t = y\right].$$

Then, a formal application of the dynamic programming principle leads to the HJB equation

$$\Psi_t + \frac{\sigma^2}{2}\Psi_{yy} + \sup_{\alpha}\{\alpha(\Psi_y + V - H)\} = 0.$$

Since the term to be maximised is linear in α, the only way to ensure the finiteness of solution is to set

$$\Psi_y = H - V,$$

which yields $\Psi_t + \frac{\sigma^2}{2}\Psi_{yy} = 0$. Then, by straightforward calculations we see that H must satisfy a backward heat equation

$$H_t + \frac{\sigma^2}{2}H_{yy} = 0,$$

and, therefore, Itô's formula will yield that S should satisfy

$$dS_t = \sigma H_y(t, Y_t)\, dY_t.$$

Combining this with (3.2) and (3.3) implies

$$\frac{z}{\sigma}\hat{\alpha}(t, y, s, z) = -\frac{\rho}{2N} yz^2,$$

that is,

$$\hat{\alpha}(t, y, s, z) = -\frac{\rho\sigma}{2N} yz \tag{3.4}$$

as soon as we note that $z = \sigma H_y(t, y)$ by the choice of S.

The above form of $\hat{\alpha}$ is necessary in order for the market makers to quote a Markovian pricing rule. However, in order for such $\hat{\alpha}$ to appear in equilibrium, it should be optimal for the insider to choose a drift whose $\mathcal{F}^M$-optional projection has this form. In Proposition 3.1, we will show that the sole criterion of optimality for the insider is that the strategy fulfils the bridge condition $H(1, Y_1) = V$. Thus, if a Markovian equilibrium exists,

$$dY_t = \sigma\, dB_t^Y - \frac{\sigma^2\rho}{2N} Y_t H_y(t, Y_t), \tag{3.5}$$

and H solves the backward heat equation above and satisfies $H(1, Y_1) = V$.

As we show in Sections 4 and 3 a pair (H, Y) satisfying the above conditions exists for some admissible insider trading strategy and that it indeed constitutes an equilibrium. In order to see that this equilibrium is indeed feasible, suppose that we have a pair (H, Y) which solves the following system of equations:

$$H_t + \frac{1}{2}\sigma^2 H_{yy} = 0, \tag{3.6}$$

$$dY_t = \sigma\, d\beta_t - \frac{\sigma^2\rho}{2N} Y_t H_y(t, Y_t)\, dt, \tag{3.7}$$

$$V \overset{d}{=} H(1, Y_1), \tag{3.8}$$

with $Y_0 = 0$ where β is a Brownian motion on some given probability space and Y is understood to be a strong solution of the forward SDE. Further assume that the transition probability of Y possesses a smooth density, p. Then the theory of filtration enlargements gives us (see Theorem 1.6 in [34]) that Y solves the SDE

$$dY_t = \sigma\, d\tilde{\beta}_t + \left\{\sigma^2 \frac{p_y}{p}(t, Y_t; 1, Y_1) - \frac{\sigma^2\rho}{2N} Y_t H_y(t, Y_t)\right\} dt, \tag{3.9}$$

where $\tilde{\beta}$ is a Brownian motion with respect to the natural filtration of Y initially enlarged with the random variable Y_1 and, in particular, independent of Y_1. Thus, if $\tilde{V}$ is a random variable with the same distribution as V and independent of $\tilde{\beta}$, we can replace Y_1 with $H^{-1}(1, \tilde{V})$ in (3.9) and obtain the SDE

$$d\tilde{Y}_t = \sigma\, d\tilde{\beta}_t + \left\{\sigma^2 \frac{p_y}{p}\big(t, \tilde{Y}_t; 1, H^{-1}(1, \tilde{V})\big) - \frac{\sigma^2 \rho}{2N} \tilde{Y}_t H_y(t, \tilde{Y}_t)\right\} dt.$$

Now, suppose that the solutions of this SDE are unique in law. Then $\tilde{Y}$ will have the same law as Y, which yields in particular that $\tilde{Y}_1 = H^{-1}(1, \tilde{V})$ and in its own filtration $\tilde{Y}$ follows

$$d\tilde{Y}_t = \sigma\, dB_t^{\tilde{Y}} - \frac{\sigma^2 \rho}{2N} \tilde{Y}_t H_y(t, \tilde{Y}_t)\, dt,$$

for some Brownian motion $B^{\tilde{Y}}$.

The above discussion makes it clear what the optimal strategy of the insider should be given H. Since V is independent of B, the optimal number of shares of the risky asset held by the insider at time t equals

$$\int_0^t \left\{\sigma^2 \frac{p_y}{p}\big(s, Y_s; 1, H^{-1}(1, V)\big) - \frac{\sigma^2 \rho}{2N} Y_s H_y(s, Y_s)\right\} ds.$$

This ensures that Y follows (3.5) in its own filtration and $H(1, Y_1) = V$ achieving the optimality conditions for the insider as well as those for the market makers.

These considerations imply that the question of existence of the equilibrium can be reduced to the problem of existence of a solution to the system (3.6)–(3.8) with process Y admitting a smooth transition density. Despite the apparent simplicity, the existence of a solution to this system is far from being a trivial matter. Indeed, in order to determine H via the basic PDE in (3.6), we first need to know its boundary condition. However, the boundary condition for H, (3.8), requires the knowledge of the distribution of Y_1 which can only be determined if we know H. Thus, this problem is appropriate for the employment of a fixed-point theorem which indeed yields the existence of the solution as demonstrated in the next section.

We end this section by proving the optimality criteria for the insider that we used in order to establish the above system.

PROPOSITION 3.1. *Suppose H is a pricing rule satisfying*

$$H_t + \tfrac{1}{2}\sigma^2 H_{yy} = 0. \tag{3.10}$$

If $X_t = \int_0^t \alpha_s\, ds$ for some $\mathcal{F}^I$-progressively measurable α such that, for all $v \in f(\mathbb{R})$, we have $\mathbb{P}^v(\int_0^1 |\alpha_s|\, ds < \infty) = 1$,

$$\mathbb{E}^v\left(\int_0^1 H^2(t, X_t + \sigma B_t)\, dt\right) < \infty \tag{3.11}$$

and

$$H(1, X_1 + Z_1) = V, \qquad \mathbb{P}^v\text{-a.s.}, \tag{3.12}$$

then $X \in \mathcal{A}(H)$ and it is an optimal strategy for the insider.

PROOF. We adapt the arguments in [2] and [41] to our case. Consider the function

$$\Psi(t,y) := \int_{\xi(t)}^{y} \{H(t,u) - V\}\,du + \frac{1}{2}\sigma^2 \int_t^1 H_y\big(s,\xi(s)\big)\,ds, \tag{3.13}$$

where $\xi(t)$ is the unique solution of $H(t,\xi(t)) = V$. Direct calculations show

$$\Psi_y(t,y) = H(t,y) - V \tag{3.14}$$

and

$$\Psi_t + \frac{\sigma^2}{2}\Psi_{yy} = 0.$$

Therefore, from (3.14) and Itô's formula it follows that

$$\begin{aligned} \Psi(1,Y_1) - \Psi(0,0) &= \int_0^1 \{H(t,Y_t) - V\}\,dY_t \\ &= -W_1^X + \int_0^1 \{H(t,Y_t) - V\}\sigma\,dB_t \end{aligned} \tag{3.15}$$

for any X such that $X_t = \int_0^t \alpha_s\,ds$ with $\mathbb{P}^v(\int_0^1 |\alpha_s|\,ds < \infty) = 1$. Using (3.15) and admissibility properties of X (see Definition 2.2), insider's optimisation problem becomes

$$\sup_{X\in\mathcal{A}(H)} \mathbb{E}^v[W_1^X] = \sup_{X\in\mathcal{A}(H)} \mathbb{E}^v\left[\int_0^1 \big(V - H(t,Y_t)\big)\,dX_t\right] \tag{3.16}$$

$$= \mathbb{E}^v[\Psi(0,0)] - \inf_{X\in\mathcal{A}(H)} \mathbb{E}^v[\Psi(1,Y_1)], \tag{3.17}$$

where the last equality is due to (2.2).

Since $\Psi(1,Y_1) = \int_{\xi(1)}^{Y_1}\{H(1,u) - V\}\,du$ is strictly positive unless $Y_1 = \xi(1)$ as $H(1,y)$ is strictly increasing, the conclusion will follow as soon as X is shown to be admissible. In view of (3.15),

$$W_1^X = \Psi(0,0) + \int_0^1 \{H(t,Y_t) - V\}\sigma\,dB_t,$$

and, therefore, the admissibility of X follows from (3.11). □

4. The main result and its proof.

In this section, we state and prove the main result of this paper that establishes the existence of a solution to the system given by (3.6)–(3.8).

THEOREM 4.1. *There is a pair (H,Y) that solves the system of equations* (3.6)–(3.8). *Moreover,* $0 < H_y(t,y) \le C\frac{1}{\sqrt{1-t}}$ *for all* $(t,y) \in [0,1) \times \mathbb{R}$ *and for some constant C. Furthermore, Y is the unique strong solution of* (3.7) *and admits* *a regular transition density,*[3] $p(s, y; t, z)$, *for all* $0 \le s \le t \le 1$ *and* $(y, z) \in \mathbb{R}^2$ *such that, for any fixed* (t, z), $p(s, y; t, z) > 0$ *on* $[0, t) \times \mathbb{R}$ *and is* $C^{1,2}([0, t) \times \mathbb{R})$.

We will prove this theorem by an application of Schauder's fixed-point theorem. Observe that if we start with an absolutely continuous probability measure on $\mathbb{R}$ with full support, (3.8) yields an increasing function $H(1, \cdot)$, which defines an H solving (3.6). If we then plug this function into the SDE of (3.7), we arrive at a new probability measure on $\mathbb{R}$ associated with the distribution of Y_1. This procedure defines a transformation from the space of probability measures on $\mathbb{R}$ into itself. Application of Schauder's fixed-point theorem requires a suitable choice of a closed and convex subset, D, of probability measures on $\mathbb{R}$ such that the above transformation maps D into itself and satisfies the conditions of Schauder's fixed-point theorem.

Before we present the proof of the fixed-point result, we collect some useful facts on the behaviour of the solutions of (3.7) in the following lemmata. The first lemma observes a striking relationship between the time 1 laws of the solutions of (3.7) and that of B_{σ^2}. An immediate consequence of this lemma is that the law of Y_1, where Y is the solution of (3.7) for a given H, has a full support on $\mathbb{R}$. This property allows us to compute the law of Y_1 via a Girsanov transform using the law of B_1, which is achieved in the second lemma.

LEMMA 4.1. *Suppose* $H \in C^{1,2}([0, 1) \times \mathbb{R})$ *satisfies* $0 \le H_y(t, y) \le C \frac{1}{\sqrt{1-t}}$ *for all* $(t, y) \in [0, 1) \times \mathbb{R}$, *and some constant* C. *Let* $c \ge 0$ *be a constant, then the stochastic differential equation*

$$dY_t = \sigma\, dB_t - cY_t H_y(t, Y_t)\, dt \tag{4.1}$$

has a unique strong solution on $[0, 1]$. *Moreover, for any* $x > 0$,

$$\mathbb{E}[(Y_1 - x)^+] \ge \mathbb{E}[(e^{-2cC} B_{\sigma^2} - x)^+] > 0, \tag{4.2}$$

$$\mathbb{E}[(-x - Y_1)^+] \ge \mathbb{E}[(-x - e^{-2cC} B_{\sigma^2})^+] > 0, \tag{4.3}$$

and, in particular, $\mathbb{P}(Y_1 \le y) \in (0, 1)$ *for all* $y \in \mathbb{R}$.

PROOF. Since $yH_y(t, y)$ is locally Lipschitz on $[0, T] \times \mathbb{R}$ for any $T < 1$, the above equation has a unique strong solution on $[0, T]$ upto an explosion time τ. Since T is arbitrary this implies the existence of a unique continuous strong solution on $[0, 1 \wedge \tau)$. Let $\tau_n := \inf\{t \in [0, 1) : |Y_t| > n\}$ and observe that $\tau_n \uparrow \tau$, a.s. Moreover, for any $t \in [0, 1]$

$$\begin{aligned} Y^2_{t\wedge\tau_n} &= 2\int_0^{t\wedge\tau_n} Y_s \sigma\, dB_s - 2c\int_0^{t\wedge\tau_n} Y_s^2 H_y(s, Y_s)\, ds + \sigma^2(t \wedge \tau_n) \\ &\le 2\int_0^{t\wedge\tau_n} Y_s\sigma\, dB_s + \sigma^2(t\wedge\tau_n). \end{aligned}$$

[3] See the last paragraph on page 76 of [35] for a definition.

Thus, by Itô's isometry and the elementary inequality $x \leq 1 + x^2$,

$$\mathbb{E}[Y_{t\wedge\tau_n}^2] \leq 1 + \sigma^2 + 4\sigma^2 \int_0^t \mathbb{E}[Y_s^2 \mathbf{1}_{[s<\tau_n]}]\,ds \leq 1 + \sigma^2 + 4\sigma^2 \int_0^t \mathbb{E}[Y_{s\wedge\tau_n}^2]\,ds.$$

Therefore, Gronwall's inequality yields $\mathbb{E}[Y_{t\wedge\tau_n}^2] \leq (1+\sigma^2)e^{4\sigma^2}$ for all $t \in [0,1]$ and $n \geq 1$. Thus, $(Y_{t\wedge\tau_n})_{n\geq 1}$ is uniformly integrable, and consequently, $\mathbb{P}(\tau < t) = 0$ and $\mathbb{E}[Y_t^2] \leq (1+\sigma^2)e^{4\sigma^2}$ for all $t \in [0,1]$, that is, Y never explodes and the SDE has a nonexploding strong solution.

To obtain the estimates (4.2) and (4.3), let

$$\tilde{Y}_t = Y_t \exp\biggl(c\int_0^t H_y(s, Y_s)\,ds\biggr)$$

and observe that

$$\tilde{Y}_t = \int_0^t \exp\biggl(c\int_0^s H_y(r, Y_r)\,dr\biggr)\sigma\,dB_s.$$

Thus, $\tilde{Y}_t = W_{T_t}$ for some Brownian motion W and the time change T_t satisfying

$$\sigma^2 t \leq T_t = \sigma^2 \int_0^t \exp\biggl(2c\int_0^s H_y(r, Y_r)\,dr\biggr)ds \leq \sigma^2 \exp(4cC)t.$$

Thus, by the optional sampling theorem, for any $K \in \mathbb{R}$ we have

$$\mathbb{E}[(\tilde{Y}_1 - K)^+] = \mathbb{E}[(W_{T_1} - K)^+] \geq \mathbb{E}[(W_{\sigma^2} - K)^+] > 0,$$
$$\mathbb{E}[(K - \tilde{Y}_1)^+] = \mathbb{E}[(K - W_{T_1})^+] \geq \mathbb{E}[(K - W_{\sigma^2})^+] > 0,$$

which implies (4.2) and (4.3). $\square$

LEMMA 4.2. *Let h be a bounded, nondecreasing, and absolutely continuous function, which is not constant. Consider the solution, H, of (3.6) with the terminal condition h. Then $|H(t,\cdot)| \leq \|h\|_\infty$ for $t \leq 1$, and $0 < H_y(t,\cdot) \leq C\frac{1}{\sqrt{1-t}}$ for $t < 1$, where $C = \sqrt{\frac{2}{\sigma^2\pi}}\|h\|_\infty$. Consequently, there exists a unique, strong solution, Y, of (4.1) and, for any bounded and continuous function g and $T \leq 1$, we have*

$$\mathbb{E}[g(Y_T)] = \mathbb{E}^{\mathbb{Q}}[g(\sigma W_T)M_T],$$

where W is a Brownian motion on a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\in[0,1]}, \mathbb{Q})$ and $(M_t)_{t\in[0,1]}$ is a strictly positive $((\tilde{\mathcal{F}}_t), \mathbb{Q})$-martingale given by

$$M_t := \exp\biggl(-c\int_0^t W_s H_y(s, \sigma W_s)\,dW_s - \frac{c^2}{2}\int_0^t W_s^2 H_y^2(s, \sigma W_s)\,ds\biggr), \tag{4.4}$$

with c being the constant from Lemma 4.1. Furthermore, $\mathbb{Q}$-a.s., $M_1 \leq e^{2cC}$, and

$$\biggl|\int_0^\tau W_s H_y(s, \sigma W_s)\,dW_s\biggr| \leq K(1 + |W_\tau|) \leq K(1 + W_1^*), \tag{4.5}$$

where τ is any stopping time with respect to the natural filtration of W such that $\tau \le 1$, $\mathbb{Q}$-a.s., $W_t^* = \sup_{s\le t} |W_s|$, and K is some constant that depends only on σ and $\|h\|_\infty$.

PROOF. Observe that

$$H(t,y) = \int_{\mathbb{R}} h(z) q\big(\sigma^2(1-t), z-y\big)\,dz,$$

where $q(t,x)$ is the probability density of a normal random variable with mean 0 and variance t. Then, clearly, $|H(t,y)| \le \int_{\mathbb{R}} |h(z)| q(\sigma^2(1-t), z-y)\,dz \le \|h\|_\infty$. Moreover, $H_y(t,y)$ is strictly positive whenever $t<1$. Indeed, differentiating above, we have

$$H_y(t,y) = \int_{\mathbb{R}} h(z) q_y\big(\sigma^2(1-t), z-y\big)\,dz = \int_{\mathbb{R}} q\big(\sigma^2(1-t), z-y\big)\,dh(z) > 0.$$

On the other hand,

$$\begin{aligned} H_y(t,y) &= \int_{\mathbb{R}} h(z) \frac{z-y}{\sigma^2(1-t)} q\big(\sigma^2(1-t), z-y\big)\,dz \\ &\le \sup_{z\in\mathbb{R}} h(z) \int_{\mathbb{R}} \frac{|z-y|}{\sigma^2(1-t)} q\big(\sigma^2(1-t), z-y\big)\,dz \le C\frac{1}{\sqrt{1-t}}, \end{aligned} \tag{4.6}$$

where $C = \|h(z)\|_\infty \sqrt{\frac{2}{\sigma^2\pi}}$. Hence, Lemma 4.1 implies the existence and the uniqueness of a strong solution to (4.1).

Next, we will characterise the distribution of Y on $[0,T]$ for $T<1$ by constructing a weak solution to (4.1) via a Girsanov transform. To this end, let W be a Brownian motion on some filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\in[0,1]}, \mathbb{Q})$. Then M is a martingale on $[0,T]$ by Corollary 3.5.16 in [26]. Thus, if we define $\tilde{\mathbb{P}}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ by $d\tilde{\mathbb{P}}/d\mathbb{Q} = M_T$, σW solves (4.1) under $\tilde{\mathbb{P}}$ on $[0,T]$. Due to the uniqueness in law of the solutions of (4.1), for any continuous and bounded function g we therefore have

$$\mathbb{E}^{\tilde{\mathbb{P}}}[g(Y_T)] = E^{\mathbb{Q}}[g(\sigma W_T) M_T].$$

We next aim to extend the above equality to $T=1$, which would follow from the dominated convergence theorem once we demonstrate that M is a bounded martingale. Direct calculations lead to

$$\begin{aligned} M_T = \exp\bigg(& B(T, \sigma W_T) \\ & + c\int_0^T \Big\{ H_y(s,\sigma W_s) - \frac{1}{2} H_y(s,0) - \frac{c}{2} W_s^2 H_y^2(s,\sigma W_s)\Big\}\,ds\bigg), \end{aligned}$$

where

$$B(t,y) = -\frac{c}{\sigma^2}\int_0^y x H_y(t,x)\,dx \le 0 \tag{4.7}$$

since H_y is positive. Thus, for any $t \le T$,

$$M_t \le \exp\left(c \int_0^1 H_y(s, \sigma W_s)\, ds\right) \le e^{2cC} \tag{4.8}$$

implying

$$\mathbb{E}^{\tilde{\mathbb{P}}}[g(Y_1)] = \mathbb{E}^{\mathbb{Q}}[g(\sigma W_1) M_1],$$

where $M_1 := \lim_{T \to 1} M_T$.

Our next goal is to prove the estimate in (4.5) which will, in turn, imply that M_1 is strictly positive. Let τ be a stopping time with respect to the natural filtration of W and bounded by 1. Then

$$\begin{aligned} c\left|\int_0^\tau W_s H_y(s, \sigma W_s)\, dW_s\right| &\le \left|B(\tau, \sigma W_\tau)\right| + c\int_0^\tau \left|H_y(s, \sigma W_s) - \frac{1}{2} H_y(s, 0)\right| ds \\ &\le \left|B(\tau, \sigma W_\tau)\right| + 3cC, \end{aligned}$$

where $B(t, y)$ is given by (4.7). A simple application of integration by parts on $B(t, y)$ yields that $|B(t, y)| \le \frac{2c}{\sigma^2}|y|\|h\|_\infty$. Hence,

$$\left|\int_0^\tau W_s H_y(s, \sigma W_s)\, dW_s\right| \le K\left(1 + |W_\tau|\right) \le K\left(1 + W_1^*\right),$$

for some K that depends on σ and $\|h\|_\infty$ only.

The above estimate also shows that $c\int_0^t W_s H_y(s, \sigma W_s)\, dW_s$ is a square integrable martingale on [0, 1] with

$$c^2 \int_0^1 E^{\mathbb{Q}}[W_s^2 (H_y(s, \sigma W_s))^2]\, ds \le 2c^2 K^2 (1 + E^{\mathbb{Q}}(W_1^*)^2) < \infty.$$

As $\{\omega : M_1(\omega) = 0\} \subseteq \{\omega : \int_0^1 W_s^2(\omega)(H_y(s, \sigma W_s(\omega)))^2\, ds = \infty\}$, this yields that M is strictly positive on [0, 1], $\mathbb{Q}$-a.s. and

$$M_1 = \exp\left(-c\int_0^1 W_s H_y(s, \sigma W_s)\, dW_s - \frac{c^2}{2}\int_0^1 W_s^2 H_y^2(s, \sigma W_s)\, ds\right). \qquad \square$$

Next lemma is not needed for the fixed-point algorithm that we will consider in order to show the existence of a solution to the system (3.6)–(3.8). On the other hand, it shows that any solution to (3.7) has a smooth transition density, which is necessary to construct the equilibrium in our model.

LEMMA 4.3. *Let h be a nonconstant, bounded, nondecreasing, absolutely continuous function, the derivative of which is bounded on compacts. Consider the solution, H, of* (3.6) *with the terminal condition h. Then the unique strong solution, Y, of* (4.1) *admits a regular transition density $p(s, y; t, z)$ for all $0 \le s \le t \le 1$ for $(y, z) \in \mathbb{R}^2$. Moreover, for any fixed (t, z), $p(s, y; t, z) > 0$ on $[0, t) \times \mathbb{R}$ and is $C^{1,2}([0, t) \times \mathbb{R})$.*

PROOF. Due to the Lemma 4.2, we have $0 < H_y(t, y) \le C\frac{1}{\sqrt{1-t}}$ for $t < 1$ and $y \in \mathbb{R}$, where H is the solution of (3.6) with the terminal condition h, and $C = \sqrt{\frac{2}{\sigma^2\pi}}\|h\|_\infty$. Furthermore, there exists a unique the solution, Y, to (3.7) and for any bounded function g and $0 \le t < u \le 1$,

$$\begin{aligned}
&\mathbb{E}\big[g(Y_u)|Y_t = y\big]\\
&\quad= \mathbb{E}^{\mathbb{Q}}\biggl[g(\sigma W_u)\exp\biggl(-c\int_t^u W_s H_y(s, \sigma W_s)\, dW_s\\
&\qquad - \frac{c^2}{2}\int_0^t W_s^2 H_y^2(s, \sigma W_s)\, ds\biggr)\Big| W_t = \frac{y}{\sigma}\biggr].
\end{aligned}$$

Thus, a regular transition density of Y can be defined as

$$p(t, y; u, z) = q\bigl(\sigma^2(u-t), z-y\bigr)r(t, y; u, z), \qquad 0 \le t < u \le 1, \tag{4.9}$$

where

$$\begin{aligned}
&r(t, y; u, z)\\
&\quad:= \mathbb{E}^{\mathbb{Q}_{t\to u}^{y\to z}}\biggl[\exp\biggl(-\frac{c}{\sigma^2}\int_t^u Y_s H_y(s, Y_s)\, dY_s\\
&\qquad - \frac{c^2}{2\sigma^2}\int_t^u Y_s^2 H_y^2(s, Y_s)\, ds\biggr)\biggr],
\end{aligned}\tag{4.10}$$

with $\frac{Y}{\sigma}$ being a Brownian bridge from $\frac{y}{\sigma}$ to $\frac{z}{\sigma}$ on the interval $[t, u]$ under measure $\mathbb{Q}_{t\to u}^{y\to z}$. Indeed, the representation (4.9) holds once we show that r is a measurable function and Chapman–Kolmogorov equations hold. In fact, as we show below r is continuous with respect to all its parameters, hence, measurable (the easy task of validating Chapman–Kolmogorov equation is left to the reader).

First, observe that the Itô formula and the PDE (3.6) satisfied by H yield [recall that $B(t, y)$ is given by (4.7)]

$$\begin{aligned}
&e^{B(t,y)-B(u,z)}r(t, y; u, z)\\
&\quad= \mathbb{E}^{\mathbb{Q}_{t\to u}^{y\to z}}\biggl[\exp\biggl(c\int_t^u\biggl\{H_y(s, Y_s) - \frac{1}{2}H_y(s, 0) - \frac{c}{2\sigma^2}Y_s^2 H_y^2(s, Y_s)\biggr\}\, ds\biggr)\biggr]\\
&\quad= \exp\biggl(-\frac{c}{2}\int_t^u H_y(s, 0)\, ds\biggr)\\
&\qquad\times \mathbb{E}^{\mathbb{Q}_{t\to u}^{y\to z}}\biggl[\exp\biggl(c\int_t^u\biggl\{H_y(s, Y_s) - \frac{c}{2\sigma^2}Y_s^2 H_y^2(s, Y_s)\biggr\}\, ds\biggr)\biggr].
\end{aligned}\tag{4.11}$$

Moreover, in view of the SDE representation of Brownian bridges (see Section 5.6.B in [26]), the law of Y under $\mathbb{Q}_{t\to u}^{y\to z}$ is the same as that of $\tilde{Y}$ under $\mathbb{Q}$, where

$$\tilde{Y}_s := y\frac{u-s}{u-t} + z\frac{s-t}{u-t} + \sigma(u-s)\int_t^s \frac{dW_r}{u-r}, \qquad s \in [t, u].$$

Therefore,

$$\mathbb{E}^{\mathbb{Q}^{y\to z}_{t\to u}}\left[\exp\left(c\int_t^u\left\{H_y(s,Y_s)-\frac{c}{2\sigma^2}Y_s^2H_y^2(s,Y_s)\right\}ds\right)\right]$$
$$=\mathbb{E}^{\mathbb{Q}}\left[\exp\left(c\int_t^u\left\{H_y(s,\tilde{Y}_s)-\frac{c}{2\sigma^2}\tilde{Y}_s^2H_y^2(s,\tilde{Y}_s)\right\}ds\right)\right],$$

and the desired continuity follows from the continuity of $\tilde{Y}$ with respect to (t,y,u,z) and the dominated convergence theorem that applies due to the bounds on H_y.

In order to show that $r(t,y;u,z)>0$ for all $u\le 1$, it suffices to show that

$$\mathbb{Q}^{y\to z}_{t\to u}\left(\int_t^u Y_s^2H_y^2(s,Y_s)\,ds<\infty\right)=1. \tag{4.12}$$

Indeed, due the uniform bounds on H_y, the nonnegative random variable inside the conditional expectation in (4.11) is zero only if $\int_t^u Y_s^2H_y^2(s,Y_s)\,ds=\infty$. To this end, fix an ω and observe that $K_{t,u}:=\sup_{t\le s\le u}Y_s^2$ satisfies $\mathbb{Q}^{y\to z}_{t\to u}(0<K_{t,u}<\infty)=1$. Therefore,

$$\begin{aligned}\frac{1}{4K_{t,u}}\int_t^u Y_s^2H_y^2(s,Y_s)\,ds&\le\frac14\int_t^u H_y^2(s,Y_s)\,ds\\&=\frac14\int_t^u\left(\int_{\mathbb{R}}h'(z)q\big(\sigma^2(1-s),z-Y_s\big)\,dz\right)^2ds\\&\le\int_t^u\left(\int_{-1}^1h'(z+Y_s)q\big(\sigma^2(1-s),z\big)\,dz\right)^2ds\\&\quad+\int_t^u\left(\int_1^\infty h'(z+Y_s)q\big(\sigma^2(1-s),z\big)\,dz\right)^2ds\\&\quad+\int_t^u\left(\int_{-\infty}^{-1}h'(z+Y_s)q\big(\sigma^2(1-s),z\big)\,dz\right)^2ds.\end{aligned}$$

Observe that, for the fixed ω, Y is a continuous function of time and, therefore, takes values in a compact set, which implies that $h'(z+Y_s)$ is bounded for $z\in[-1,1]$ and all $s\in[t,u]$. This implies that the first integral is finite since $\int_{-1}^1 q(\sigma^2(1-s),z)\,dz<1$.

To see the finiteness of the second integral, apply integration by parts to get

$$\int_t^u\left(-\tilde{h}(1+Y_s)q\big(\sigma^2(1-s),1\big)+\int_1^\infty\tilde{h}(z+Y_s)\frac{z}{\sigma^2(1-s)}q\big(\sigma^2(1-s),z\big)\,dz\right)^2ds,$$

where $\tilde{h}=h+\|h\|_\infty$. Note that $\tilde{h}$ is positive, therefore, the above integral is bounded from above by

$$8\|h\|_\infty^2\int_t^u q^2\big(\sigma^2(1-s),1\big)\,ds<\infty.$$

The third integral can be shown to be finite in the same way.

In order to show that $p(t, y; u, z) \in C^{1,2}([0, u) \times \mathbb{R})$ for fixed (u, z), where $u < 1$, we will show that it is the fundamental solution of a parabolic differential equation (see page 3 of [21] for the definition of fundamental solutions). In view of the relationship between the fundamental solutions of PDEs and transition densities of diffusion processes (see the discussion following Definition 5.7.9 in [26]), let us consider the PDE

$$u_t + \frac{1}{2}\sigma^2 u_{yy} - cyH_y u_y = 0 \tag{4.13}$$

on the interval $[0, T]$ where $T < 1$. The existence of a fundamental solution to this PDE will follow from Theorem 1 in [9] once we show that conditions (i)–(iii) on page 28 of [9] are satisfied. Condition (i) is trivially satisfied for σ being a constant. Moreover, since

$$\left|\frac{\partial}{\partial y} yH_y\right| = |H_y + yH_{yy}|$$

and $H_y(t, y) \le C\frac{1}{\sqrt{1-T}}$ for all $(t, y) \in [0, T] \times \mathbb{R}$, we can conclude that the function $\frac{\partial}{\partial y} yH_y$ is locally bounded if H_{yy} can be shown to be bounded in (t, y) when y belongs to a bounded interval. Indeed, by directly differentiating H we obtain

$$\begin{aligned}|H_{yy}| &\le \frac{1}{\sigma^2(1-t)}\biggl(\int_{\mathbb{R}} |H(1, z)| q\bigl(\sigma^2(1-t), z-y\bigr)\, dz \\ &\qquad + \int_{\mathbb{R}} |H(1, z)| \frac{(z-y)^2}{\sigma^2(1-t)} q\bigl(\sigma^2(1-t), z-y\bigr)\, dz\biggr) \\ &\le 2\frac{\|h(z)\|_\infty}{\sigma^2(1-T)},\end{aligned}$$

that is, H_{yy} is uniformly bounded on $[0, T] \times \mathbb{R}$. Thus, we have shown that condition (ii) was satisfied. Since the constant functions solve the (4.13), condition (iii) is satisfied as well; thus, a fundamental solution, $\Gamma(t, y; s, z)$ to (4.13) exists. In particular, if one considers this PDE with the boundary condition $u(T, y) = g(y)$ for some bounded g, the solution is given by

$$u(t, y) = \int_{\mathbb{R}} g(z)\Gamma(t, y; T, z)\, dz.$$

On the other hand, since the SDE (4.1) satisfies the hypotheses of Theorem 5.7.6 in [26] on the time interval $[0, T]$, u has the following stochastic representation by this theorem:

$$u(t, y) = \mathbb{E}\bigl[g(Y_T) | Y_t = y\bigr] = \int_{\mathbb{R}} g(z)p(t, y; T, z)\, dz.$$

Thus,

$$\int_{\mathbb{R}} g(z)p(t, y; T, z)\, dz = \int_{\mathbb{R}} g(z)\Gamma(t, y; T, z)\, dz,$$

and since g is arbitrary and both Γ and p are continuous in their parameters, we deduce $p(t, y; T, z) = \Gamma(t, y; T, z)$ for all $0 \le t < T < 1$, and thus, it is $C^{1,2}$ on $[0, T) \times \mathbb{R}$ for $T < 1$.

To show that $p(t, y; 1, z)$ is $C^{1,2}$ on $[0, 1) \times \mathbb{R}$ for each z consider (4.13) with the boundary condition $u(T, y) = p(T, y; 1, z)$. Note that $u(T, y)$ is bounded in y since, due to (4.11), we have

$$
\begin{aligned}
r(t, y; u, z) &\le e^{2cC} e^{-B(t,y)} = e^{2cC} e^{(c/\sigma^2)(yH(t,y) - \int_0^y H(t,x)\,dx)} \\
&\le e^{2cC} e^{(c\|h\|_\infty/\sigma^2)|y|},
\end{aligned}
\tag{4.14}
$$

where the first inequality is due to bounds on $H_y(t, y)$ and the last one due to the bounds on $H(t, y)$ obtained in Lemma 4.2. Thus, there exists a unique classical solution, $u(t, y)$, to (4.13), with the boundary condition $u(T, y) = p(T, y; 1, z)$, given by

$$
u(t, y) = \int_{\mathbb{R}} p(t, y; T, x) p(T, x; 1, z)\,dx
$$

by the definition of fundamental solutions. However, by Chapman–Kolmogorov equations,

$$
\int_{\mathbb{R}} p(t, y; T, x) p(T, x; 1, z)\,dx = p(t, y; 1, z),
\tag{4.15}
$$

which in turn yields that $p(t, y; 1, z) \in C^{1,2}([0, T) \times \mathbb{R})$. Since T is arbitrary, we have $p(t, y; 1, z) \in C^{1,2}([0, 1) \times \mathbb{R})$. □

Having collected all the prerequisites, we can now prove our main theorem.

PROOF OF THEOREM 4.1. In the setting of Lemma 4.2, M defines an equivalent change of measure between the laws of Y and σW. Thus, if we define $r(y)$ by [see (4.10)]

$$
r(y) := r(0, 0; 1, y) = \mathbb{E}^{\mathbb{Q}}\left[M_1 \middle| W_1 = \frac{y}{\sigma}\right],
\tag{4.16}
$$

then

$$
\mathbb{E}[g(Y_1)] = \int_{\mathbb{R}} g(y) q(\sigma^2, y) r(y)\,dy
$$

and, therefore, the probability density of Y_1 under $\mathbb{P}$ is given by

$$
q(\sigma^2, y) r(y) \le q(\sigma^2, y) e^{2cC}.
\tag{4.17}
$$

The existence of a solution to the system of equations (3.6)–(3.8) will be shown via a fixed-point argument applied to a certain operator mapping a class of distribution functions on $\mathbb{R}$ into itself.

Schauder's fixed-point theorem (see Theorem 7.1.2 in [21]) states that if D is a closed convex subset of a Banach space and $T : D \mapsto D$ is a continuous operator, then it has a fixed point if the space TD is precompact, that is, every sequence in TD has a subsequence which converges to some element of the Banach space. In order to apply this theorem, we first need to find a suitable Banach space which contains a class of probability distribution functions on $\mathbb{R}$ that is large enough to contain the distribution of Y_1 where Y is one of the components of the solution to the system of equations (3.6)–(3.8). In view of the above discussion, the distribution of Y_1 will be continuous, in fact it will admit a density. Thus, we may take $C_b(\mathbb{R})$, *the space of bounded continuous functions on* $\mathbb{R}$, equipped with the sup norm as our underlying Banach space and set $\mathcal{P}$ as the space of absolutely continuous distribution functions on $\mathbb{R}$, that is, $P \in \mathcal{P}$ if P is increasing, $P(-\infty) = 0$, $P(\infty) = 1$, and there exists a measurable function P' such that $P(y) = \int_{-\infty}^{y} P'(z)\,dz$. Then we can define the set

$$\begin{aligned} D = \biggl\{ & P \in \mathcal{P} : P'(z) \le C^* q(\sigma^2, z), \forall z \in \mathbb{R}, \\ & \int_x^\infty (y-x) P'(y)\,dy \ge \mathbb{E}\biggl[\biggl(\frac{1}{C^*} W_{\sigma^2} - x\biggr)^+\biggr], \forall x > 0, \\ & -\int_{-\infty}^{-x} (y+x) P'(y)\,dy \ge \mathbb{E}\biggl[\biggl(-x - \frac{1}{C^*} W_{\sigma^2}\biggr)^+\biggr], \forall x > 0 \biggr\}, \end{aligned}$$

where

$$C^* := \exp\biggl(\frac{\rho \| f(z) \|_\infty}{N} \sqrt{\frac{2}{\pi}}\sigma\biggr).$$

The reason for this judicious choice of C^* will become apparent when we define the operator T. We will prove the existence of a fixed point in four steps.

Step 1: *D is a closed convex set.* It is clear that D is convex. To see it is also closed, suppose that P_n is a sequence of elements in D converging to some element, P, of the Banach space in the sup norm. Clearly, P is increasing with $P(-\infty) = 0$ and $P(\infty) = 1$. Moreover, for any $x \le y$ in $\mathbb{R}$, it follows from Fatou's lemma that

$$0 \le P(y) - P(x) = \lim_{n\to\infty} \int_x^y P_n'(z)\,dz \le \int_x^y \limsup_{n\to\infty} P_n'(z)\,dz,$$

since each P_n' is bounded from above by the same integrable function, which in turn is an upper bound to the positive function $\limsup_{n\to\infty} P_n'$. However, this implies that P is absolutely continuous and, in particular, there exists a function P' with $0 \le P'(z) \le \limsup_{n\to\infty} P_n'(z) \le C^* q(\sigma^2, z)$ for all $z \in \mathbb{R}$.

To complete the proof that D is closed, we need to show

$$\int_x^\infty (y-x) P'(y)\,dy \ge \mathbb{E}^{\mathbb{Q}}\biggl[\biggl(\frac{1}{C^*} W_{\sigma^2} - x\biggr)^+\biggr] \qquad \forall x > 0.$$

Since P_n converges P weakly, there exists a probability space supporting random variables $(Y_n)_{n\geq 0}$ and Y such that $Y_n \to Y$, a.s., Y_n has distribution P_n, and Y has distribution P. Note that one can directly verify that

$$\int_{\mathbb{R}} (y-x)^2 P_n(dy) \leq C^* \mathbb{E}\big[(W_{\sigma^2} - x)^2\big],$$

which shows the uniform integrability of the sequence $(Y_n - x)^+$. Therefore,

$$\int_x^\infty (y-x)P(dy) = \lim_{n\to\infty} \int_x^\infty (y-x)P_n(dy) \geq \mathbb{E}\left[\left(\frac{1}{C^*} W_{\sigma^2} - x\right)^+\right].$$

Similar arguments show the other inequality. Thus, D is closed.

Step 2: *Defining the operator* T. For any $P \in D$, let $H : [0,1] \times \mathbb{R} \mapsto \mathbb{R}$ be the unique function which solves the following boundary value problem:

$$\begin{aligned} &H_t + \frac{\sigma^2}{2} H_{yy} = 0, \\ &H(1, y) = f\big(\Phi^{-1}\big(P(y)\big)\big), \end{aligned} \tag{4.18}$$

where Φ is the cumulative distribution function of a standard normal random variable. Observe that $h(z) := f(\Phi^{-1}(P(z)))$ is a bounded, increasing function. Moreover, its derivative given by $f'(\Phi^{-1}(P(y)))(\Phi^{-1})'(P(y))P'(y)$ is well defined for all $y \in \mathbb{R}$ as $P(y) \in (0,1)$ for all $P \in D$ and $y \in \mathbb{R}$ and, therefore, h is also absolutely continuous. Thus, by Lemma 4.2, for all $t < 1$, $0 < H_y(t,y) \leq C\frac{1}{\sqrt{1-t}}$, where $C = \sqrt{\frac{2}{\sigma^2\pi}}\|f\|_\infty$ is independent of the choice of P.

To this H one can associate a unique process Y which solves (4.1) for $c = \frac{\sigma^2\rho}{2N}$ and Y_1 is a continuous random variable with the probability density $q(\sigma^2, y)r(y)$, where r is defined in (4.16). Thus, we can define

$$TP(y) = \int_{-\infty}^y q\big(\sigma^2, z\big) r(z)\, dz.$$

Note that TP belongs to D due to (4.2), (4.3) and (4.17).

Step 3: T *is precompact.* Since TD is an equicontinuous family of functions, by a version of the Ascoli–Arzela theorem (see Corollary III.3.3 in [30]), if P_n is a sequence in TD then it admits a subsequence which converges pointwise to $P \in C_b(\mathbb{R})$. Moreover, this convergence is uniform on every compact subset of $\mathbb{R}$. This would mean that TD is precompact once we show that the convergence is uniform over all $\mathbb{R}$.

To do so, let us assume without loss of generality that P_n itself is the convergent subsequence and consider any $\varepsilon > 0$. Due to the definition of D, there exist x^* and x_* such that

$$P_n(x) \leq C^* \int_{-\infty}^x q\big(\sigma^2, y\big)\, dy \leq C^* \int_{-\infty}^{x_*} q\big(\sigma^2, y\big)\, dy$$

$$
\begin{aligned}
&= C^*\Phi(x_*) \le \frac{\varepsilon}{6} \qquad \forall x \le x_*;\\
1 - P_n(x) &\le C^* \int_x^\infty q(\sigma^2, y)\,dy \le C^* \int_{x^*}^\infty q(\sigma^2, y)\,dy\\
&= C^*\Phi(-x^*) \le \frac{\varepsilon}{6} \qquad \forall x \ge x^*.
\end{aligned}
$$

Since P_n converges to P pointwise, we also have with the same x^* and x_* that

$$
P(x) \le \frac{\varepsilon}{4} \qquad \forall x \le x_*; \qquad 1 - P(x) \le \frac{\varepsilon}{4} \qquad \forall x \ge x^*.
$$

Since the convergence is uniform on the compact $[x_*, x^*]$, there exist a K such that for all $n \ge K$

$$
\sup_{x \in [x_*, x^*]} \bigl|P_n(x) - P(x)\bigr| \le \frac{\varepsilon}{3}.
$$

Thus, for any $n \ge K$ we have

$$
\begin{aligned}
\sup_{x\in\mathbb{R}} \bigl|P_n(x) - P(x)\bigr| \le{}& \sup_{x \in [x_*, x^*]} \bigl|P_n(x) - P(x)\bigr| + \sup_{x \in (-\infty, x_*]} \bigl(P_n(x) + P(x)\bigr)\\
&+ \sup_{x \in [x^*, \infty)} \bigl(1 - P_n(x) + 1 - P(x)\bigr) \le \varepsilon.
\end{aligned}
$$

Thus, we have shown that the convergence of P_n to P is uniform on $\mathbb{R}$, that is, TD is precompact in $C_b(\mathbb{R})$ equipped with the sup norm. Hence, Schauder's fixed-point theorem yields T has a fixed point provided T is a continuous operator, which we show next.

Step 4: *T is continuous.* To this end, let $(P_n)_{n\ge 1} \subset D$ converge to $P \in D$ in the sup norm. As TP_n and TP belong to D, in view of Problem 14.8(c) in [11], pointwise convergence of TP_n to TP will imply uniform convergence since TP is continuous. To each P_n and P, we can associate functions H^n and H, B^n and B [see (4.7)], and the processes M^n and M from Lemma 4.2.

Pointwise convergence of TP_n to TP will follow immediately once we can show that for any continuous and bounded function g

$$
\lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}}\bigl[g(\sigma W_1) M_1^n\bigr] = \mathbb{E}^{\mathbb{Q}}\bigl[g(\sigma W_1) M_1\bigr].
$$

In view of the uniform bound on M^n and M due to (4.8), the above convergence will hold if we can show that M_1^n converges to M_1 in Q-probability.

In order to get the estimates to prove this convergence, first note that, due to Lemma 4.2 for any stopping time, τ, bounded by 1, we have

$$
c\left|\int_0^\tau W_s H_y^n(s, \sigma W_s)\,dW_s\right| \le K\bigl(1 + |W_\tau|\bigr) \le K\bigl(1 + W_1^*\bigr),
$$

for some K independent of n. This shows that $c\int_0^t W_s H_y^n(s,\sigma W_s)\,dW_s$ is a square integrable martingale on [0, 1] with

$$c^2\int_0^1 \mathbb{E}^{\mathbb{Q}}\big[W_s^2\big(H_y^n(s,\sigma W_s)\big)^2\big]\,ds \le K\big(1+\mathbb{E}^{\mathbb{Q}}(W_1^*)^2\big),$$

where K is a constant independent of n. Let

$$N_t^n := \int_0^t W_s\{H_y^n(s,\sigma W_s) - H_y(s,\sigma W_s)\}\,dW_s.$$

Since [recall that $B(t,y)$ is given by (4.7)]

$$-c\int_0^1 W_s H_y(s,\sigma W_s)\,dW_s = B(1,\sigma W_1) + c\int_0^1 \left\{H_y(s,\sigma W_s) - \frac{1}{2}H_y(s,0)\right\}ds,$$

integrating B^n and B by parts we obtain

$$\begin{aligned} cN_1^n &= \frac{c}{\sigma}W_1\big(H^n(1,\sigma W_1) - H(1,\sigma W_1)\big) - \frac{c}{\sigma^2}\int_0^{\sigma W_1}\{H^n(1,y) - H(1,y)\}\,dy \\ &\quad + c\int_0^1\{H_y(s,\sigma W_s) - H_y^n(s,\sigma W_s)\}\,ds - \frac{c}{2}\int_0^1\{H_y(s,0) - H_y^n(s,0)\}\,ds. \end{aligned}$$

As H^n are uniformly bounded and $H_y^n \le C\frac{1}{\sqrt{1-t}}$ for $t<1$, if we can show that $H^n(1,y)\to H(1,y)$ and $H_y^n(t,y)\to H_y(t,y)$ for all $y\in\mathbb{R}$ and $t\in[0,1)$, the above will immediately imply that N_1^n converges to 0, $\mathbb{Q}$-a.s. Moreover, it will also imply convergence in $L^p(\mathbb{Q})$ for all $p\in[1,\infty)$ in view of the bound obtained in (4.5). In particular, we will have

$$\lim_{n\to\infty}\mathbb{E}^{\mathbb{Q}}\left[\int_0^1 W_s^2\{H_y^n(s,\sigma W_s) - H_y(s,\sigma W_s)\}^2\,ds\right] = 0. \tag{4.19}$$

Thus,

$$\lim_{n\to\infty}\int_0^1 W_s^2\{H_y^n(s,\sigma W_s) - H_y(s,\sigma W_s)\}^2\,ds = 0 \qquad \text{in } \mathbb{Q}\text{-probability}. \tag{4.20}$$

Moreover,

$$\begin{aligned} &\int_0^1 W_s^2\big|H_y^n(s,\sigma W_s) - H_y(s,\sigma W_s)\big|H_y(s,\sigma W_s)\,ds \\ &\qquad \le \int_0^1 W_s^2\{H_y^n(s,\sigma W_s) - H_y(s,\sigma W_s)\}^2\,ds \int_0^1 W_s^2\big(H_y(s,\sigma W_s)\big)^2\,ds, \end{aligned}$$

which in turn [since due to (4.5), $\int_0^1 W_s^2(H_y(s,\sigma W_s))^2\,ds < \infty$ $\mathbb{Q}$-a.s.] implies

$$\lim_{n\to\infty}\int_0^1 W_s^2\{H_y^n(s,\sigma W_s) - H_y(s,\sigma W_s)\}H_y(s,\sigma W_s)\,ds = 0 \qquad \text{in } \mathbb{Q}\text{-probability}. \tag{4.21}$$

Combining (4.20) and (4.21), we can deduce that

$$\lim_{n\to\infty}\int_0^1 W_s^2\big(H_y^n(s,\sigma W_s)\big)^2\,ds=\int_0^1 W_s^2\big(H_y(s,\sigma W_s)\big)^2\,ds \qquad \text{in } \mathbb{Q}\text{-probability}.$$

Together with $N_1^n\to 0$, $\mathbb{Q}$-a.s. the above implies $M_1^n\to M_1$, $\mathbb{Q}$-probability.

Thus, it remains to show that $H^n(1,y)\to H(1,y)$ and $H_y^n(t,y)\to H_y(t,y)$ for all $y\in\mathbb{R}$ and $t\in[0,1)$. Indeed, $\lim_{n\to\infty}H^n(1,y)=\lim_{n\to\infty}f(\Phi^{-1}(P_n(y)))=f(\Phi^{-1}(P(y)))$ in view of the continuity of $f\circ\Phi^{-1}$ on $(0,1)$ and the fact that the sequence $(P_n(y))$ converges to a limit $P(y)\in(0,1)$ for any $y\in\mathbb{R}$ due to the definition of D.

Next, observe that for $t<1$

$$\begin{aligned}&\big|H_y^n(t,y)-H_y(t,y)\big|\\ &\quad\le\int_{\mathbb{R}}\big|H^n(1,z+y)-H(1,z+y)\big|\frac{|z|}{\sigma^2(1-t)}q\big(\sigma^2(1-t),z\big)\,dz.\end{aligned}$$

As H^n and H are bounded by $\|f\|_\infty$, the convergence to 0 follows from the dominated convergence theorem and that $H^n(1,y)\to H(1,y)$ as $n\to\infty$.

Thus, we have verified that T is continuous operator, D is a closed and convex subset of a Banach space and TD is precompact. Therefore, by Schauder's fixed-point theorem, T has a fixed point P, that is, $TP=P$. For this P, define H as the solution to (4.18) and Y as the corresponding unique solution to (3.7). Then (H,Y) is the solution to the system of equations (3.6)–(3.8).

To complete the proof of the theorem, we need to show that the solution to (3.7) has a transition density with the required smoothness and positivity properties. This follows from the Lemma 4.3 once we observe that $h(z):=f(\Phi^{-1}(P(z)))$ satisfies the required conditions. It is obvious that h is bounded (since f is), nonconstant and nondecreasing (as f, Φ and P are). Moreover, $h'(y)=f'(\Phi^{-1}(P(y)))(\Phi^{-1})'(P(y))P'(y)$, is well defined for all $y\in\mathbb{R}$ as $P(y)\in(0,1)$ for all $P\in D$ and $y\in\mathbb{R}$ and, therefore, h is absolutely continuous. Let $K\subset\mathbb{R}$ be a compact, then since P is continuous, $P(z)\in K_1$ for all $z\in K$, where $K_1\subset(0,1)$ is also a compact. As $\Phi^{-1}\in C^1((0,1))$, this implies that $(\Phi^{-1})'(P(y))$ is bounded for all $y\in K$. Similarly, $f'(\Phi^{-1}(P(y)))$ is bounded for all $y\in K$. As boundedness of P' follows from the fact that $P\in D$, this yields that h' is bounded on compacts and, therefore, satisfies the conditions of Lemma 4.3. □

5. Construction of the equilibrium. Suppose H is the function determined in Theorem 4.1. As briefly discussed in Section 3, if we can identify an admissible strategy X such that: (i) $\widehat{\alpha}$ is given by $-\frac{\sigma^2\rho}{2N}Y_tH_y(t,Y_t)$, and (ii) X_1 satisfies (3.12), then (H,X) will be a candidate equilibrium in view of Proposition 3.1 once we show that $U(G)$ is a true $\mathcal{F}^M$-martingale. The following theorem gives such an X.

THEOREM 5.1. *Let H and p be the functions defined in Theorem 4.1. Then there exists a unique process $(Y_t)_{t\in[0,1)}$ which solves*

$$
dY_t = \sigma\, dB_t + \left\{-\frac{\sigma^2\rho}{2N} Y_t H_y(t, Y_t) + \sigma^2 \frac{p_y}{p}\big(t, Y_t; 1, H^{-1}(1, V)\big)\right\} dt, \qquad t \in [0, T], \tag{5.1}
$$

for all $T < 1$. Moreover, Y is a $(\mathbb{P}^v, \mathcal{F}^I)$-semimartingale with $\mathbb{P}^v(\lim_{t\to 1} Y_t = H^{-1}(1, V)) = 1$ for every $v \in f(\mathbb{R})$ and

$$
dY_t = \sigma\, dB_t^Y - \frac{\sigma^2\rho}{2N} Y_t H_y(t, Y_t)\, dt, \qquad t \in [0, 1]
$$

under $\mathcal{F}^M$.

PROOF. We will first show that there exists a unique weak solution to (5.1) on $[0, T]$ for any $T < 1$. Then Proposition IX.3.2 in [39] will imply the uniqueness of strong solutions since if Y^1 and Y^2 are two strong solutions, then $Y^1 - Y^2$ satisfies

$$
Y_t^1 - Y_t^2 = \int_0^t b\big(s, Y_s^1, Y_s^2, V\big)\, ds
$$

for some deterministic function b and, therefore, its local time process at level 0 is identically 0. The strong uniqueness combined with a weak solution will lead to the existence of a unique strong solution by a result due to Yamada and Watanabe (see Corollary 5.3.23 in [26]). To show the existence of a weak solution, fix $T < 1$ and let $N_t := p(t, \zeta_t; 1, H^{-1}(1, v))$ for $t \le T$ where $v \in \mathbb{R}$ and ζ is the unique strong solution of

$$
d\zeta_t = \sigma\, d\beta_t - \frac{\sigma^2\rho}{2N} \zeta_t H_y(t, \zeta_t)\, dt
$$

on $[0, 1]$ under a probability measure $\tilde{\mathbb{P}}$, where β is a $\tilde{\mathbb{P}}$-Brownian motion as established in Theorem 4.1. The same theorem also gives p as the transition density of ζ. Then $(N_t)_{t\in[0,T]}$ is a strictly positive and bounded martingale with respect to the natural filtration of ζ as a consequence of Itô formula and the estimates on p obtained in (4.14). Thus, $\frac{N_T}{N_0}$ has expectation 1 under $\tilde{\mathbb{P}}$ and defines an equivalent change of measure on the σ-algebra $\mathcal{F}_T^{\zeta}$. Since

$$
dN_t = \sigma N_t \frac{p_y}{p}\big(t, \zeta_t; 1, H^{-1}(1, v)\big)\, d\beta_t,
$$

then it follows from Girsanov's theorem that under the new measure, $\mathbb{Q}^T$,

$$
d\zeta_t = \sigma\, dW_t + \left\{-\frac{\sigma^2\rho}{2N} \zeta_t H_y(t, \zeta_t) + \sigma^2 \frac{p_y}{p}\big(t, \zeta_t; 1, H^{-1}(1, v)\big)\right\} dt
$$

for some $\mathbb{Q}^T$-Brownian motion. Thus, ζ, as a solution of the above under $\mathbb{Q}^T$ is a weak solution of (5.1) on $[0, T]$. Moreover, the weak uniqueness holds since the distribution of ζ under $\mathbb{Q}^T$ has a one-to-one correspondence with the distribution of ζ under the original measure via the change of measure martingale $p(t, \zeta_t; 1, H^{-1}(1, v))$. More precisely, for any bounded function F and points $0 = t_0 < \cdots < t_n = T$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^T}[F(\zeta_{t_1}, \dots, \zeta_{t_n})] \\
&\quad = \mathbb{E}^{\tilde{\mathbb{P}}}\left[F(\zeta_{t_1}, \dots, \zeta_{t_n}) \frac{p(T, \zeta_T; 1, H^{-1}(1, v))}{p(0, 0; 1, H^{-1}(1, v))}\right] \\
&\quad = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} F(y_1, \dots, y_n) \\
&\qquad \times \frac{p(0, 0; t_1, y_1) \cdots p(t_{n-1}, y_{n-1}; t_n, y_n) p(T, y_n; 1, H^{-1}(1, v))}{p(0, 0; 1, H^{-1}(1, v))} \, dy_1 \cdots dy_n.
\end{aligned} \tag{5.2}$$

Hence, we conclude the existence of a unique strong solution, Y^T, of (5.1) over the interval $[0, T]$ under $\mathbb{P}^v$. Define Y by $Y_t = Y_t^T \mathbf{1}_{t \leq T}$ and observe that due to the uniqueness of strong solutions Y is well defined and is the unique process that solves (5.1).

Next, we want to extend the process Y to time-1 by considering its limit. This limit exists in view of Theorem 2.2 in [16]. Note that Assumption 2.2 of [16] is satisfied since $(t, y) \mapsto p(t, y; u, z)$ is $C^{1,2}$ on $[0, u) \times \mathbb{R}$ and $p(t, y; u, z) = q(\sigma^2(u - t), z - y) r(t, y; u, z)$, where q is the transition density of standard Brownian motion and r is a strictly positive function with exponential bounds given by (4.14), which in particular implies p generates a time inhomogeneous Feller semigroup. Moreover, the same result also yields $\mathbb{P}^v(\lim_{t \to 1} Y_t = H^{-1}(1, V)) = 1$.

To show the semimartingale property of Y let $z = H^{-1}(1, v)$ and recall from (4.10) that

$$\begin{aligned}
&r(t, y; 1, z) \\
&\quad = \mathbb{E}^{\mathbb{Q}_{t \to 1}^{y \to z}}\left[\exp\left(-\frac{\rho}{2N} \int_t^1 X_s H_y(s, X_s) \, dX_s - \frac{\sigma^2 \rho^2}{8N^2} \int_t^1 X_s^2 H_y^2(s, X_s) \, ds\right)\right].
\end{aligned}$$

In view of the Markov property of Brownian bridges, we have

$$\begin{aligned}
&r(t, X_t; 1, z) \\
&\quad = \mathbb{E}^{\mathbb{Q}_{0 \to 1}^{0 \to z}}\left[\exp\left(-\frac{\rho}{2N} \int_t^1 X_s H_y(s, X_s) \, dX_s \right.\right. \\
&\qquad \left.\left. - \frac{\sigma^2 \rho^2}{8N^2} \int_t^1 X_s^2 H_y^2(s, X_s) \, ds\right) \Big| \mathcal{F}_t^X\right],
\end{aligned}$$

where $(\mathcal{F}^X)$ is the usual augmentation of the natural filtration of X since both $\int_t^1 X_s H_y(s, X_s) \, dX_s$ and $\int_t^1 X_s^2 H_y^2(s, X_s) \, ds$ are measurable with respect to $\sigma(X_u; u \in [t, 1])$.

Therefore,

$$L_t := r(t, X_t; 1, z) \exp\left(-\frac{\rho}{2N}\int_0^t X_s H_y(s, X_s)\, dX_s - \frac{\sigma^2\rho^2}{8N^2}\int_0^t X_s^2 H_y^2(s, X_s)\, ds\right)$$

is a $\mathbb{Q}_{0\to 1}^{0\to z}$-martingale. Moreover, it is strictly positive due to (4.12). Thus, we can define $\mathbb{P}^{0,z}$ on $\mathcal{F}_1^X$ via $\frac{d\mathbb{P}^{0,z}}{d\mathbb{Q}_{0\to 1}^{0\to z}} = L_1$.

Recall that under $\mathbb{Q}_{0\to 1}^{0\to z}$ X solves the following SDE:

$$X_t = \sigma W_t^z + \int_0^t \frac{z - X_s}{1-s}\, ds,$$

where W^z is a $\mathbb{Q}_{0\to 1}^{0\to z}$-Brownian motion. Thus, a straightforward application of Girsanov's theorem yields that X solves (5.1) once B is replaced with the $\mathbb{P}^{0,z}$-Brownian motion defined by the Girsanov transform since

$$\frac{dL_t}{L_t} = \left(\frac{r_x(t, X_t; 1, z)}{r(t, X_t; 1, z)} - \frac{\sigma\rho}{2N} X_t H_y(t, X_t)\right) dW_t^z.$$

Since semimartingale property is preserved under equivalent changes of measure and the strong uniqueness holds for the solutions of (5.1), we obtain the desired semimartingale property of its unique solution.

Having shown the semimartingale property it remains to demonstrate the claimed representation of Y under $\mathcal{F}^M$. Suppose that ξ is a solution of

$$\xi_t = \sigma\beta_t - \int_0^t \frac{\sigma^2\rho}{2N}\xi_s H_y(s, \xi_s)\, ds. \tag{5.3}$$

Then ξ has the transition density p. If one enlarges the filtration of ξ with ξ_1, then under the enlarged filtration ξ has the following decomposition:

$$d\xi_t = \sigma\, dW_t + \left\{-\frac{\sigma^2\rho}{2N}\xi_t H_y(t, \xi_t) + \sigma^2 \frac{p_y}{p}(t, \xi_t; 1, \xi_1)\right\} dt, \qquad t \in [0, 1), \tag{5.4}$$

where $(W_t)_{t\in[0,1)}$ is a Brownian motion in the enlarged filtration independent of ξ_1 (see Theorem 1.6 in [34]).

On the other hand, since $\mathbb{E}[\xi_t | \mathcal{F}_t^\xi] = \xi_t$ for $t < 1$, we must have

$$\sigma\beta_t - \int_0^t \frac{\sigma^2\rho}{2N}\xi_s H_y(s, \xi_s)\, ds = \mathbb{E}[\sigma W_t | \mathcal{F}_t^\xi] + \mathbb{E}\left[\int_0^t \sigma^2 \frac{p_y}{p}(s, \xi_s; 1, \xi_1)\, ds \middle| \mathcal{F}_t^\xi\right] - \int_0^t \frac{\sigma^2\rho}{2N}\xi_s H_y(s, \xi_s)\, ds.$$

Since the projection of a martingale onto a smaller filtration is still a martingale, from the above equation we conclude that $\mathbb{E}[\int_0^t \sigma^2 \frac{p_y}{p}(s, \xi_s; 1, \xi_1)\, ds | \mathcal{F}_t^\xi]$ is an $\mathcal{F}^\xi$-martingale which is equivalent to

$$\mathbb{E}\left[\int_t^u \sigma^2 \frac{p_y}{p}(s, \xi_s; 1, \xi_1)\, ds \Big| \mathcal{F}_t^\xi\right] = 0 \qquad \forall u \in [t, 1). \tag{5.5}$$

Observe that by Theorem 4.1 the distribution of ξ_1 and that of $H^{-1}(1, V)$ coincide. Since we have established the uniqueness in law of solutions to (5.1) and V is independent of B, we can conclude that the processes Y and ξ have the same distribution. Thus, from (5.5) it follows that for $u < 1$

$$\begin{aligned}\mathbb{E}\left[\int_t^u \sigma^2 \frac{p_y}{p}\big(s, Y_s; 1, H^{-1}(1, V)\big)\, ds \Big| \mathcal{F}_t^Y\right] &= \mathbb{E}\left[\int_t^u \sigma^2 \frac{p_y}{p}(s, Y_s; 1, Y_1)\, ds \Big| \mathcal{F}_t^Y\right] \\ &= 0.\end{aligned}$$

The above implies that $\mathbb{E}[\int_0^t \sigma^2 \frac{p_y}{p}(s, Y_s; 1, H^{-1}(1, V))\, ds | \mathcal{F}_t^Y]$ is an $\mathcal{F}^Y$-martingale. Therefore, Y has the following decomposition with respect to $\mathcal{F}^Y$:

$$Y_t = M_t - \int_0^t \frac{\sigma^2 \rho}{2N} Y_s H_y(s, Y_s)\, ds, \qquad t \in [0, 1),$$

where M is an $\mathcal{F}^Y$-martingale. On the other hand, $[M, M]_t = [Y, Y]_t = \sigma^2 t$. Thus, by Lévy's characterisation, $M_t = \sigma B_t^Y$. Note that $(B_t^Y)_{t \in [0,1)}$ is a uniformly integrable martingale; thus, we can define $B_1^Y = \lim_{t \to 1} B_t^Y$ so that $(B_t^Y)_{t \in [0,1]}$ is a Brownian motion. This establishes the desired decomposition on $[0, 1]$ as Y converges to a finite limit as $t \to 1$. □

Theorem 5.1 in conjunction with Proposition 3.1 establish the existence of an equilibrium in our model.

THEOREM 5.2. *Let H^* and p be the functions defined in Theorem 4.1, and*

$$X_t^* = \int_0^t \left\{ -\frac{\sigma^2 \rho}{2N} Y_s^* H_y^*(s, Y_s^*) + \sigma^2 \frac{p_y}{p}\big(s, Y_s^*; 1, H^{*-1}(1, V)\big) \right\} ds.$$

Then (H^*, X^*) is an equilibrium.

Moreover, under $\mathcal{F}^M$ the equilibrium demand evolves as

$$Y_t^* = \sigma B_t^Y - \frac{\sigma^2 \rho}{2N} \int_0^t Y_s^* H_y^*(s, Y_s^*)\, ds.$$

PROOF. Note that H^* is a bounded function being a solution of heat equation with a bounded terminal condition. Thus, conditions (2.1) and (2.2) are automatically satisfied. Moreover, Theorem 5.1 yields that X^* is a $(P^v, \mathcal{F}^I)$-semimartingale and $\mathbb{P}^v(H^*(1, Y_1^*) = v) = 1$. Thus, Proposition 3.1 yields that X is admissible and optimal strategy for the insider given H^*.

Thus, it remains to verify the zero-utility gain condition of the market makers, that is, to prove that $U(G)$ is an $\mathcal{F}^M$-martingale. Recall from Theorem 5.1 that with this choice of X^*, Y^* solves

$$dY_t = \sigma\, dB_t^Y - \frac{\sigma^2\rho}{2N} Y_t H_y^*(t, Y_t)\, dt. \tag{5.6}$$

Thus, Itô's formula together with the conditions on H^* yields

$$U(G_t) = -\exp\biggl(\frac{\sigma\rho}{N}\int_0^t Y_s^* H_y^*(s, Y_s^*)\, dB_s^Y - \frac{\sigma^2\rho^2}{2N^2}\int_0^t \bigl(Y_s^* H_y^*(s, Y_s^*)\bigr)^2\, ds\biggr).$$

Clearly, $-U(G)$ is an exponential local martingale.

Next, observe in view of the absolute continuity relationship between the laws of Y and σW as established in Lemma 4.2 $(\frac{1}{M_t^*})_{t\in[0,1]}$ is a strictly positive $\mathbb{P}$-martingale, where

$$M_t^* = \exp\biggl(-\frac{\rho}{2N}\int_0^t Y_s^* H_y\bigl(s, Y_s^*\bigr)\, dY_s - \frac{\sigma^2\rho^2}{8N^2}\int_0^t \bigl(Y_s^* H_y\bigl(s, Y_s^*\bigr)\bigr)^2\, ds\biggr).$$

Therefore, if we define an equivalent measure, $\mathbb{Q}$, on $\mathcal{F}_1^{Y^*}$ by $\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{1}{M_1^*}$, then $W^Y := \frac{Y^*}{\sigma}$ is a $\mathbb{Q}$-Brownian motion. Consequently, $U(G)$ is a $\mathbb{P}$-martingale if and only if $U(G)M^*$ is $\mathbb{Q}$-martingale.

On the other hand, a straightforward application of integration by parts formula yields

$$d\bigl(-U(G_t)M_t^*\bigr) = -U(G_t)M_t^* \frac{\sigma\rho}{2N} Y_t H_y(t, Y_t)\, dW_t^Y,$$

that is, $-U(G)M^*$ is the stochastic exponential of $\int_0^\cdot \frac{\sigma\rho}{2N} Y_t H_y(t, Y_t)\, dW_t^Y$. Moreover,

$$\left|\int_0^1 Y_t H_y(t, Y_t)\, dW_t^Y\right| \le K\bigl(1 + \bigl|W_1^Y\bigr|\bigr)$$

by (4.5). Since $|W_1^Y|$ has all exponential moments, we conclude that $-U(G)M^*$ is a $\mathbb{Q}$-martingale using Kazamaki's criterion (see, e.g., Theorem III.44 in [38]). $\square$

The above theorem shows that the equilibrium demand process has a drift in its own filtration. This is in contrast with the other possible generalisations found in the literature (for the change in the pattern of private information arrival see [5], for a risk averse insider see [8] and for competition among insiders see [4]) of the original models of [29] and [2] lead to equilibria with total demand being a martingale in its own filtration.

Moreover, as $H_y > 0$ the equilibrium total demand process is mean reverting. This suggests a theoretical explanation for the emergence of mean reversion in the specialists' inventories, which has strong empirical support (see, e.g., [12, 23, 24] and [33]): The mean reversion appears as a result of the insider's reaction to the market maker's demand for risk sharing. The speed of mean reversion is not constant and depends on the market makers' level of effective risk aversion, $\frac{\rho}{N}$, as well as the level of informational asymmetry, σ, in a nontrivial way due to the definition of H_y. This theoretical implication is in line with the empirical findings of [23] who observe that the speed of mean reversion depends on the inventory levels of the market makers in the London Stock Exchange.

Closely related to the observation that the total order has a drift, is the fact that the equilibrium price is no longer a martingale under the physical measure. Moreover, Y^* and, therefore, $H^*(t, Y_t^*)$ are mean-reverting processes. This mean-reversion property of Y^* also entails that Kyle's conclusion of constant market depth, which is the order size necessary to move the prices by one unit, does not hold in this model. Indeed,

$$dH_y^*(t, Y_t^*) = H_{yy}^*(t, Y_t^*)\sigma\, dB_t^Y - \frac{\sigma^2\rho}{2N} Y_t^* H_y^*(t, Y_t^*) H_{yy}^*(t, Y_t^*)\, dt$$

implies that $H_y^*(t, Y_t^*)$ is not a martingale since H^* is not linear. In particular, if H^* is such that $H_{yy}^*(1, y) = -H_{yy}^*(1, -y)$ with $H_{yy}^*(1, y) \le 0$ for $y \ge 0$, then $yH_{yy}^*(t, y) \le 0$, and thus $H_y^*(t, Y_t^*)$ is a submartingale. Consequently, $\mathbb{E}[H_y^*(t, Y_t^*)]$ has an upward slope, that is, the executions costs increase in time in our model. This is consistent with the empirical findings of U-shaped patterns of execution costs on NYSE (see [32]).

6. Conclusion and further remarks. We have solved a long-standing open problem first posed by Subrahmanyam in [40] of existence of an equilibrium in a financial market with asymmetrically informed traders and risk averse market makers in a continuous-time version of a model first introduced by Kyle [29]. The equilibrium turns out to be the solution of a nonstandard FBSDE. We have solved this FBSDE by transforming it into a forward–backward system of stochastic and partial differential equations and employing a novel application of Schauder's fixed-point theorem.

Consistent with the empirical studies on the inventories of market makers we find that the risk aversion of market makers causes mean reversion in the equilibrium total demand (i.e., collective inventory of the market makers). This implies that the informed trader's strategy ceases to be *inconspicuous* and, therefore, provides the first example of an equilibrium in a Kyle-type model which does not satisfy inconspicuousness condition. The driving force behind this result is that the risk aversion of market makers makes them unwilling to bear risk. Instead of paying the extra compensation for the inventory risk, the informed trader chooses to absorb a part of large fluctuations in the market makers' inventories, that is, participates in a risk sharing.

We also show that the sensitivity of prices to the total order, which is the reciprocal of the market depth, can be a sub-martingale for certain model parameters.

This implies that the execution costs are, on average, increasing toward the end of a trading period, which is consistent with the empirical results obtained in [32].

Whereas, for general set of the parameters, the reciprocal of the market depth is not a sub-martingale, it is not a martingale either. This theoretical conclusion is in discord with the results obtained in [29], as well as in [8], who studies the effect of risk averse insider on the equilibrium, and in [5], who extend Kyle's model to the case when the informed trader receives a fluctuating signal over time. In fact, Kyle in [29] made a conjecture that:

> $[\cdots]$ neither increasing nor decreasing depth is consistent with behavior by the informed trader which is "stable" enough to sustain an equilibrium. If depth ever increases, the insider wants to destabilize prices (before the increase in depth) to generate unbounded profits. If depth ever decreases, the insider wants to incorporate all of his private information into the price immediately.

Thus, the results obtained from our model demonstrate that the necessity of risk sharing between the informed trader and the market makers makes exploitation of systematic movements in market depth unprofitable for the informed trader. Indeed, if the trader attempts to follow the strategy outlined by Kyle, that is, acquiring a large position when depth is lower in order to liquidate at more favourable price when depth is higher, she would be moving the total order away from its mean, leaving the market makers exposed to the risk of large orders. Violation of risk sharing would cause the market makers to adjust the prices eliminating favourable liquidation opportunities for the informed trader. Thus, contrary to Kyle, such a strategy does not lead to unbounded profits.

Moreover, the appearance of systematic changes in market depth as a result of market makers' risk aversion demonstrates that competition of the informed traders, as in [4], is not the only possible mechanism that can make the strategy proposed by Kyle unprofitable, thus leading to a drift in the reciprocal of the market depth.

These observations show that a mere introduction of risk averse market makers to the setting of [29] changes the equilibrium outcome fundamentally.

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