Mathematical Structures of Simple Voting Games

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Abstract

We aim to systematize the quasi-algebraic operations involving simple voting games (SVGs), by constructing an appropriate category, consisting of a class of objects and mappings (morphisms) between these objects, in terms of which all the operations involving SVGs can be defined in a natural way. But what should we take as the objects of the desired category? After trying an obvious solution, which turns out to be a dead end, we present the right solution. All the operations on SVGs fall naturally into place. We discover the remarkable central role played by the operation of SVG composition.
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Motivation

- Systematize the theory of SVGs and clarify its structure.

- Bring it into line with other mathematical theories: category theory is the \textit{structural} foundation of mathematics.

- Find connections with other branches of mathematics and obtain new results about SVGs
Terminology, notation

By “game” I mean simple voting game. A game is an ordered pair \((V, G)\), where \(V\) is a finite set – the set of voters, aka the assembly – and \(G\) is the set of winning coalitions.

I say that \((V, G)\) is a game on \(V\).

I often use sloppy notation, omitting \(V\) and writing \(G\) instead of \((V, G)\).

I denote by \(L_V\) the set of all games on \(V\).
Operations involving games

- Application of a game as decision rule to a division of the voters into “yes” and “no” voters.

- Composition of games, including the special cases of forming the meet and join of SVGs.

- Formation of Boolean subgames, including the special cases of forming subgames and reduced games.

- Adding dummy voters to a game.

- Transforming an SVG by forming voter blocs, whereby coalitions of voters amalgamate to form new single voters.
An obvious attempt

Let \( \varphi : V \to W \) be an arbitrary map from \( V \) to the finite set \( W \). For any game \( G \) on \( V \), define \( L \varphi G \) as a game on \( W \) by putting

\[
L \varphi G := \{ Y \subseteq W : \varphi^{-1}[Y] \in G \}.
\]

This seems promising. We do get a category whose objects are the games, and with mappings of the form \( L \varphi \) as morphisms. (The notation ‘\( L \varphi \)’ anticipates an insight that will transpire later on.)
The mapping \( L\varphi \) is a sort of homomorphism.

\( L\varphi G \) is the game on \( W \) resulting from \( G \) by formation of the blocs corresponding to the partition \( \{ \varphi^{-1}\{\{w\}\} : w \in W \} \) of \( V \).

Moreover, if \( w \in W - \varphi[V] \) (ie, \( \varphi^{-1}\{\{w\}\} = \emptyset \)) then \( w \) is a dummy in \( L\varphi G \).

If \( \varphi \) is injective (one-to-one) but not surjective (onto) then \( L\varphi G \) is essentially \( G \) with added dummies.

So this takes care of bloc formation and adding dummies.

**But it doesn’t take care of any of the other operations: application of a game to a division of the voters, composition, Boolean subgames.**
An insight:

$L\varphi$ is defined “in the same way” not just for one particular game $G$, but for all games in $L_V$ and it maps $L_V$ into $L_W$. This is conveyed by the following diagram:

\[
\begin{array}{ccc}
V & \xrightarrow{\varphi} & W \\
\downarrow & & \downarrow \\
L_V & \xrightarrow{L\varphi} & L_W
\end{array}
\]

The significance of the downward arrows will become clear later. Moreover, $L_V$ and $L_W$ are lattices, in fact distributive lattices; and $L\varphi$ respects the lattice structure.
So the idea is to look at a category whose objects are not individual games, but lattices of the form $L_V$ for all finite sets $V$, and whose morphisms are not just mappings of the form $L\varphi$ but all mappings between these objects that respect their structure as lattices. We denote this category by $G$.

This is analogous to the insight of Peano who – following ideas of Grassmann – realized that to get a satisfactory vector algebra you must take as objects not individual vectors but vector spaces, and focus on the mappings between vector spaces that respect their structure, namely linear mappings.
Recall the definition of the lattice operations in $L_V$

$$(V, G) \lor (V, H) := (V, G \cup H), \quad (V, G) \land (V, H) := (V, G \cap H).$$
Liberalizing the definition of the $L_V$

For technical reasons that will become apparent later, we must liberalize the definition of the $L_V$, admitting games that are usually excluded because they are not useful as decision rules.

First, like Taylor and Zwicker in *Simple Games*, we admit into each $L_V$ a *bottom* and a *top* game which are, respectively, a game in which no coalition is winning, and a game in which every coalition (including the empty one!) is winning:

$$
\bot_V := (V, \emptyset), \quad \top_V := (V, \varnothing V).
$$

And we insist that morphisms of our category $G$ respect these trivial games; so if $f : L_V \to L_W$ is a morphism of $G$, it must not only respect the lattice operations $\lor$ and $\land$,

$$
f(G \lor H) = fG \lor fH, \quad f(G \land H) = fG \land fH,
$$

but also obey

$$
f \bot_V = \bot_W, \quad f \top_V = \top_W.
$$
In addition, unlike anyone else, we admit the degenerate object $L_\emptyset$, the lattice of games without any voters. There are exactly two such ‘rubberstamp’ games, $\bot_\emptyset$ and $\top_\emptyset$. They play the role of truth values, false and true.
For $A \subseteq V$ we denote by $[A]$ the game that has $A$ as its sole minimal winning coalition (MWC). In this game a bill is passed iff all members of $A$ vote for it. The voters in $V - A$ are dummies. In lattice-algebraic terms, $[A]$ is a principal member of $L_V$.

In particular, if $a \in V$, $\{a\}$ is the dictatorial game with $a$ as dictator.

Here is what the 3 simplest objects of $G$ look like:
\[ L_{\{a,b\}} \]

\[ T_{\{a,b\}} \]

\[ \{a\} \lor \{b\} \]

\[ \{a\} \land \{b\} \]

\[ \perp_{\{a,b\}} \]
Characterization of the $L_V$

**Theorem** Any game $G$ on $V$ can be presented as a join of a set of pairwise incomparable principal games:

$$G = \bigvee_{i=1}^{k} [A_i],$$

where $k \geq 0$ and $i \neq j \Rightarrow A_j \not\subset A_i$.

Moreover, this presentation is unique (up to the order of the $A_i$).

But a principal game $[A]$ can be presented as a meet of dictatorial games:

$$[A] = \bigwedge_{x \in A} [\{x\}].$$

Hence we have:
Characterization of the $L_V$ (continued)

Join normal form theorem: Any game $G$ on $V$ can be presented as

$$G = \bigvee_{i=1}^{k} \bigwedge R_i,$$

where $k \geq 0$ and each $R_i$ is a set of dictatorial games

such that $i \neq j \Rightarrow R_j \not\subseteq R_i$.

Moreover, this presentation is unique (up to the order of the $R_i$ and the order of the dictatorial games in each $R_i$).

This provides a characterization of the $L_V$: Let $L$ be a bounded lattice. Suppose there are $n$ elements in $L$ – call them ‘atoms’ – such that any element $g$ of $L$ has a unique JNF presentation as a join of meets of atoms similar to the above, then $L$ is isomorphic (in the category of all bounded lattices) to $L_V$ with $|V| = n$. 
The category $G$; Main Lemma

Recall that $G$ is the category whose objects are the $L_V$ for all finite sets $V$ and whose morphisms are the mappings between these objects that respect their structure as bounded lattices.

Main Lemma A morphism $f : L_V \to L_W$ is uniquely determined by the images under $f$ of the dictatorial games $\{\{v\}\} : v \in V$. Moreover, these images, namely $\{f(\{v\}) : v \in V\}$, can be chosen freely as arbitrary games in the codomain $L_W$.

So in $G$ the dictatorial games play a role of free generators, analogous to a basis of a vector space in the category of vector spaces: to determine a linear transformation, you can choose freely the images of the basis vectors, and this determines the transformation uniquely. But a vector space has infinitely many bases, whereas in $L_V$ the dictatorial games are the only ‘basis’.

We have an explicit formula for $fG$, where $G \in L_V$, in terms of the $f(\{v\})$:

$$fG = \{Y \subseteq W : \{v \in V : Y \in f(\{v\})\} \in G\}.$$

Another form of this is

$$\forall Y \subseteq W : Y \in fG \Leftrightarrow \{v \in V : Y \in f(\{v\})\} \in G.$$
The category $G$; Another way of writing $fG$

Without loss of generality, we take $V = \hat{n} := \{1, 2, \ldots, n\}$. (This is the canonical assembly of cardinality $n$).

Let $W$ be any finite set and let $f : L_{\hat{n}} \to L_W$ be a morphism in our category.

Let us put $H_i := f[\{i\}]$ for all $i \in \hat{n}$. Then using our formula for $fG$ we get, for all $G \in L_{\hat{n}}$:

\[ fG = G[H_1, H_2, \ldots, H_n]. \]

Here we use the notation for game composition defined (for a special case) by Shapley (1962) and in complete generality by Felsenthal and Machover (1998).
What this means is that \textit{the most general morphism} in our category $G$ produces as image of any game $G$ in its domain the composition of $G$ with the images (in its codomain) of the dictatorial games in its domain.

This result surprised us. We knew that composition is important; but we had not realized \textit{how} important. It is the most general operation on games!

I shall now show how the other operations listed in the beginning are obtained as special cases, by special choice of the $f|\{v\}$.
Bloc formation revisited

To define a morphism \( f : L_V \to L_W \), we may choose the images \( f[\{v\}] \) of the dictatorial games in \( L_V \) to be *completely arbitrary* games in \( L_W \). Let us now see what happens when we choose the latter to be *arbitrary dictatorial* games (in \( L_W \)).

So – as in our first obvious attempt (which led nowhere) – let us take any map \( \varphi : V \to W \), and consider the morphism \( f \) such that

\[
\forall v \in V : f[\{v\}] = [\{\varphi v\}] \text{ in } L_W.
\]

Putting this in our formula for \( fG \), we obtain

\[
fG = \{ Y \subseteq W : \varphi^{-1}[Y] \in G \},
\]

which is exactly the same as what we had for our old \( L\varphi G \). So this \( f \) is our old \( L\varphi \). As we know, it yields the operation of bloc formation, with optional added dummies.

The reason our first attempt failed is that game composition cannot be obtained as a special case of bloc formation, because the exact opposite is true.
The old diagram revisited

We draw the old diagram with some added decoration:

\[
\begin{array}{cccc}
V & \varphi \text{ (in FinSet)} & W \\
\downarrow L & & \downarrow L \\
L_V & \xrightarrow{L\varphi \text{ (in } G)} & L_W
\end{array}
\]

**FinSet** is the category of finite sets, with set mappings (such as \( \varphi \)) as morphisms. Those familiar with category theory will see at once that \( L \) is a **functor** from **FinSet** to **G**.

In fact, \( L \) is the left part of an adjointness relation; the corresponding right adjoint is the forgetful functor

\[ F : G \to \text{FinSet}. \]
Boolean subgames

Let $A$ and $N$ be disjoint subsets of $V$ and let $W = V - (A \cup N)$. In their book, Taylor and Zwicker define, for any game $G$ on $V$, the Boolean subgame of $G$ determined by $N$ and $A$, which we (but not they) denote by $\square^A_N G$ as the game on $W$ given by

$$\square^A_N G := \{Y \subseteq W : Y \cup A \in G\}.$$

**Explanation** Consider $G$ is a decision rule with $V$ as its set of voters. Suppose that voters belonging to subsets $A$ and $N$ of $V$ are committed in advance to voting “aye” and “nay” respectively, come what may. When a bill is put to the vote, the outcome will then depend only on the votes of the remaining voters, members of $W = V - (A \cup N)$. We are left with a decision rule with $W$ as the *de facto* set of voters. This rule is precisely $\square^A_N G$.

Special cases are:

- $A = \emptyset$. Then $\square^0_N G$ is the subgame of $G$ determined by $W$.
- $N = \emptyset$. Then $\square^A_\emptyset G$ is the reduced game of $G$ determined by $W$. 
Boolean subgames (continued)

It turns out that $\sqsubseteq_N^A$ is a morphism of $G$. We obtain the morphism

$$\sqsubseteq_N^A : L_V \to L_W$$

by choosing:

$$\sqsubseteq_N^A \{\{v\}\} := \begin{cases} \top_W & \text{if } v \in A, \\ \bot_W & \text{if } v \in N, \\ \{\{v\}\} \text{ on } W & \text{if } v \in W. \end{cases}$$
A very special case

With $V$, $A$ and $N$ as above, suppose $W = \emptyset$, so $V = A \cup N$. Then

$$\Box^A_N : L_V \to L_{\emptyset}.$$ 

In fact we obtain,

$$\Box^A_N G = \begin{cases} \top & \text{if } A \in G, \\ \bot & \text{if } A \notin G. \end{cases}$$

So $\Box^A_N$ is the operator that, when applied to the game $G$, yields the output (truth value) under $G$ of the division of $V$ in which $A$ is the coalition of “aye” voters and $N$ is the coalition of “nay” voters.