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Walrasian Foundations for Equilibria in Segmented Markets

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Abstract
We study an economy with segmented financial markets and strategic arbitrageurs who link these markets. We show that the equilibrium of the arbitraged economy is asymptotically Walrasian in the sense that it converges to the equilibrium of an appropriately defined competitive economy with no arbitrageurs. The equilibrium of this competitive economy, called Walrasian equilibrium with restricted consumption, is related to - though not identical to - the well-known Walrasian equilibrium with restricted participation. This characterization serves to clarify the role that arbitrageurs play in integrating markets.

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1 Introduction

The Arrow-Debreu model provides an elegant and parsimonious theoretical foundation for the study of financial markets. It has proved to be not only the bedrock of textbook financial economic theory, but also the benchmark relative to which the role of "frictions", such as taxes, asymmetric information or limits of arbitrage, can be studied. In this paper we focus on one such friction, namely asset market segmentation.

The same or similar assets are often traded in many different locations and at many different prices. For instance, Mifid II in Europe and RegNMS in the US have given rise to significant market fragmentation in equities. As a result, any one stock is traded on many competing exchanges, multilateral trading facilities, electronic communication networks, dark pools, systematic internalizers, and so on.[1] Arbitrageurs, often high frequency traders, exploit the surplus gains from trade arising from this segmentation. These activities lead to some price alignment, the extent of which depends on the degree of competition among arbitrageurs. In the present paper, we study the properties of equilibria of an economy with segmented markets linked by arbitrageurs, as the number of arbitrageurs grows without bound. The main question we seek to answer is the following: Can an equilibrium of a segmented markets economy, with a high degree of competition in the arbitraging sector, be approximated by a variant of Walrasian equilibrium, and if so which notion of Walrasian equilibrium would serve this purpose?

Our point of departure is the model of strategic arbitrage in Rahi and Zigrand (2009). This is a two-period model of financial markets with multiple market segments or "exchanges". Markets may be incomplete on any given exchange and the set of tradable payoffs may differ across exchanges. Each exchange is populated by competitive investors who can trade only on that exchange, and have preferences that yield a local CAPM. In addition, there are arbitrageurs who can trade across exchanges and engage in Cournot competition. There is a unique Cournot-Walras equilibrium (CWE) of this economy. Our goal is to provide a Walrasian benchmark for the CWE, that relates the CWE to the equilibrium of an appropriately defined competitive economy with no arbitrageurs. This characterization serves to elucidate the role that arbitrageurs play in integrating markets.

A natural candidate for such a benchmark is the well-studied concept of Walrasian equilibrium with restricted participation, wherein all agents face the same asset price vector but can only trade payoffs that lie in their local asset span. However,

[1] While market fragmentation has accelerated in recent years, it has of course always been an important feature of the economic landscape. Allais (1967) argued for a more realistic "economy of markets" in lieu of a "market economy". In his Nobel speech he says: "... I was led to discard the Walrasian general model of the market economy, characterized at any time, whether there be equilibrium or not, by a single price system, the same for all the operators, - a completely unrealistic hypothesis, - and to establish the theory of economic evolution and general equilibrium, of maximum efficiency, and of the foundations of economic calculus, on entirely new bases resting on ... a new model, the model of the economy of markets (in the plural)".

except for a narrow class of economies, this equilibrium notion fails to approximate a CWE. This is because a CWE is asymptotically arbitrage-free: as the number of arbitrageurs goes to infinity, they collectively exhaust all the available arbitrage opportunities. At a restricted-participation equilibrium, on the other hand, there may be arbitrage opportunities that agents are unable to exploit because of their participation constraints.

Instead, we propose a subtly different notion of equilibrium, which we call Walrasian equilibrium with restricted consumption, wherein all agents face the same asset price vector but can only *consume* payoffs that lie in their local asset span. Thus agents can trade all the assets in the economy, but may have to discard consumption in some states in order to stay within the imposed span. There is a unique restricted-consumption equilibrium in our setting. It is arbitrage-free, with asset valuations that coincide with the subjective valuations of arbitrageurs at the CWE of the corresponding arbitraged economy. Furthermore, as the number of arbitrageurs grows without bound, the CWE converges to the restricted-consumption equilibrium. Finally, if a restricted-participation equilibrium exists, it coincides with the restricted-consumption equilibrium if and only if the former is arbitrage-free.

What does this tell us about the role that arbitrageurs play at a CWE? A restricted-participation equilibrium does capture the fact that arbitrageurs allow investors to trade their local assets with the rest of the world insofar as these overlap with assets traded elsewhere. But the connection that we establish with the restricted-consumption economy shows that arbitrageurs in fact allow investors to trade *all* the assets in the economy. Investors gain as a result, even if future consumption must be curtailed to respect their local asset market constraints.

Market segmentation has been the subject of a recent and growing literature. In classical general equilibrium, segmentation is captured by restricted-participation constraints on agents (see Polemarchakis and Siconolfi (1997) and Cass et al. (2001)). Strategic arbitrage in a general equilibrium setting is the subject of Zigrand (2004, 2006). Rahi and Zigrand (2009) specialize this framework to a CAPM setting to study security design by arbitrageurs. For an alternative approach to arbitrage in a segmented economy, see Gromb and Vayanos (2002, 2009), who study the dynamics of arbitrage between identical assets traded in two separate markets. An extended discussion of the segmented markets literature in finance, including empirical work, can be found in Rahi and Zigrand (2009). A broader "limits of arbitrage" literature considers settings in which arbitrageurs fail to eliminate mispricings due to constraints that they face. This research is surveyed in Gromb and Vayanos (2010).

The paper is organized as follows. We introduce the framework and notation in Section 2. In Section 3, we present the equilibrium of the arbitraged economy, the CWE. In Section 4, we propose Walrasian equilibrium with restricted consumption as the appropriate benchmark for this CWE. We analyze Walrasian equilibrium with restricted participation in Section 5, leading to explicit characterizations of valuation in the restricted-consumption economy in Section 7. In Section 8, we bring together the various preceding results to provide an overall picture of the sense in which the arbitraged economy is asymptotically Walrasian. Section 9 concludes.

2 The Setup

We consider an economy with two dates, 0 and 1, and a single physical consumption good. Assets are traded at date 0, in several locations or "exchanges", and pay off at date 1. Uncertainty is parametrized by the state space $S := \{1, \ldots, S\}$.

Associated with each exchange is a group of competitive investors who can trade only on that exchange. Investor $i \in I^k := \{1, \ldots, I^k\}$ on exchange $k \in K := \{1, \ldots, K\}$ has endowments $(\omega_0^{k,i}, \omega^{k,i}) \in \mathbb{R} \times \mathbb{R}^S$, and preferences which allow a quasilinear quadratic representation,

$$U^{k,i}(x_0^{k,i}, x^{k,i}) = x_0^{k,i} + \sum_{s \in S} \pi_s \left[x_s^{k,i} - \frac{1}{2} \beta^{k,i} (x_s^{k,i})^2 \right],$$

where $x_0^{k,i} \in \mathbb{R}$ is consumption at date 0, $x^{k,i} \in \mathbb{R}^S$ is consumption at date 1, and π_s is the probability (common across agents) of state s. The coefficient $\beta^{k,i}$ is positive.

In addition, there is a set of arbitrageurs $N := \{1, \ldots, N\}$ who can trade both within and across exchanges. Arbitrageurs are imperfectly competitive. They have no endowments, and they care only about date 0 consumption.

Asset payoffs on exchange k are given by a full column rank payoff matrix R^k of dimension $S \times J^k$. The asset span on exchange k is the column space of R^k, which we denote by $\langle R^k \rangle$. Asset spans may differ across exchanges, and we do not assume that markets are complete on any exchange. Assets are in zero net supply.

We refer to this economy as the *arbitraged economy*, in order to distinguish it from variants of a competitive economy with no arbitrageurs that we will consider later in the paper.

The interaction between price-taking investors and strategic arbitrageurs involves a Nash equilibrium concept with a Walrasian fringe, pioneered by Gabszewicz and Vial (1972) (for a survey, see Mas-Colell (1982)). Let $y^{k,n}$ be the supply of assets on exchange k by arbitrageur n, and $y^k := \sum_{n \in N} y^{k,n}$ the aggregate arbitrageur supply on exchange k. For given y^k, $q^k(y^k)$ is the market-clearing asset price vector on exchange k, with the asset demand of investor i on exchange k denoted by $\theta^{k,i}(q^k)$.

Definition 2.1 *A Cournot-Walras equilibrium (CWE) of the arbitraged economy is an array of asset price functions, asset demand functions, and arbitrageur supplies,* $\{q^k : \mathbb{R}^{J^k} \to \mathbb{R}^{J^k},\ \theta^{k,i} : \mathbb{R}^{J^k} \to \mathbb{R}^{J^k},\ y^{k,n} \in \mathbb{R}^{J^k}\}_{k \in K,\, i \in I^k,\, n \in N}$*, such that*

1. *Investor optimization: For given* q^k, $\theta^{k,i}(q^k)$ *solves*

$$\max_{\theta^{k,i} \in \mathbb{R}^{J^k}} \; x_0^{k,i} + \sum_{s \in S} \pi_s \left[x_s^{k,i} - \frac{\beta^{k,i}}{2} (x_s^{k,i})^2 \right]$$

subject to the budget constraints:

$$x_0^{k,i} = \omega_0^{k,i} - q^k \cdot \theta^{k,i},$$
$$x^{k,i} = \omega^{k,i} + R^k \theta^{k,i}.$$

2. *Arbitrageur optimization: For given* $\{q^k(y^k), \{y^{k,n'}\}_{n' \neq n}\}_{k \in K}$, $y^{k,n}$ *solves*

$$\max_{y^{k,n} \in \mathbb{R}^{J^k}} \sum_{k \in K} {y^{k,n}}^\top q^k\Big(y^{k,n} + \sum_{n' \neq n} y^{k,n'}\Big)$$

$$s.t. \quad \sum_{k \in K} R^k y^{k,n} \leq 0.$$

3. *Market clearing:* $\{q^k(y^k)\}_{k \in K}$ *solves*

$$\sum_{i \in I^k} \theta^{k,i}(q^k(y^k)) = y^k, \qquad \forall k \in K.$$

Note that investors take asset prices as given, while arbitrageurs compete Cournot-style. Arbitrageurs maximize date 0 consumption, i.e. profits from their arbitrage trades, subject to a no-default constraint at date 1.

It is convenient to cast our analysis of equilibrium prices in terms of state-price deflators. To this end, we introduce some more notation. Let $\Pi := \text{diag}\,(\pi_1, \ldots, \pi_S)$. For $x \in \mathbb{R}^S$, the $L^2(\Pi)$-norm of x is $\|x\|_2 := (x^\top \Pi x)^{\frac{1}{2}}$. Let

$$P^k := R^k({R^k}^\top \Pi R^k)^{-1} {R^k}^\top \Pi.$$

Since P^k is idempotent, it is a projection. Indeed, it is an orthogonal projection in $L^2(\Pi)$ onto the asset span $\langle R^k \rangle$.

A vector $p \in \mathbb{R}^S$ is a state-price deflator[2] for (q^k, R^k) if $q^k = {R^k}^\top \Pi p$. If markets are incomplete on exchange k, there is a multiplicity of state-price deflators p, all of which satisfy $q^k = {R^k}^\top \Pi p$. Hence, it is often useful to identify the valuation functional for exchange k by the projected state-price deflator $P^k p$. Clearly, if p is a state-price deflator for (q^k, R^k), so is $P^k p$, since ${R^k}^\top \Pi P^k p = {R^k}^\top \Pi p$. Indeed, $P^k p$ is the unique state-price deflator that is also marketed, i.e. in the span $\langle R^k \rangle$.

We shall also use the term state-price deflator to describe subjective, as opposed to equilibrium, valuations. Thus a state-price deflator p^A for an arbitrageur implies the subjective asset valuation ${R^k}^\top \Pi p^A$ on exchange k, which is in general not equal to the equilibrium asset price vector q^k.

We say that state-price deflators p and p' are *equivalent*, denoted by $p \equiv p'$, if $P^k p = P^k p'$, for all $k \in K$. Equivalent state-price deflators imply the same asset valuation on any given exchange.

[2]We do not restrict state prices to be nonnegative, since we will have occasion to consider economies with no arbitrageurs later in the paper. In such economies there may be unexploited arbitrage opportunities, and hence negative state prices, in equilibrium. See Example 5.1.

3 Cournot-Walras Equilibrium

Let $\beta^k := [\sum_{i\in I^K}(\beta^{k,i})^{-1}]^{-1}$, $\omega^k := \sum_{i\in I^k}\omega^{k,i}$, and $\mathbf{1} := (1,\ldots,1)^\top$, an $S\times 1$ vector of ones. It is shown in Rahi and Zigrand (2009) that $p^{k,i} := \mathbf{1}-\beta^{k,i}\omega^{k,i}$ is a no-trade state-price deflator for agent (k,i), i.e. $\theta^{k,i}=0$ at $q^k = {R^k}^\top \Pi p^{k,i}$, and $p^k := \mathbf{1}-\beta^k\omega^k$ is an autarky state-price deflator for exchange k, i.e. $\sum_{i\in I^k}\theta^{k,i}=0$ at $q^k = {R^k}^\top\Pi p^k$. Indeed, for given arbitrageur supply y^k,

$$q^k(y^k) = {R^k}^\top \Pi[p^k - \beta^k R^k y^k].$$

Thus $p^k - \beta^k R^k y^k$ is a state-price deflator for exchange k. The autarky state-price deflator p^k is obtained by setting $y^k = 0$. The parameter β^k measures the "depth" of exchange k: it is the price impact of a unit of arbitrageur trading. Notice that we can interpret equilibrium prices as risk-neutral prices ${R^k}^\top\Pi\mathbf{1}$ from which a risk-aversion discount $\beta^k {R^k}^\top \Pi(\omega^k + R^k y^k)$ is subtracted.

We begin with a preliminary result. We say that a vector $x\in\mathbb{R}^S$ satisfies condition **C** if

(C1) $x \geq 0$;

(C2) $\sum_{k\in K}\frac{1}{\beta^k}P^k(p^k - x)\leq 0$; and

(C3) $x\cdot\left[\sum_{k\in K}\frac{1}{\beta^k}P^k(p^k-x)\right] = 0.$[3]

Lemma 3.1 *Suppose $x, y\in\mathbb{R}^S$ both satisfy condition* **C**. *Then $x \equiv y$.*

Proof In order to save on notation, we use the following shorthand:

$$A := \sum_{k\in K}\frac{1}{\beta^k}\Pi P^k, \tag{1}$$

$$b := \sum_{k\in K}\frac{1}{\beta^k}\Pi P^k p^k. \tag{2}$$

The vectors x and y satisfy condition **C** if and only if

$$x\geq 0, \qquad Ax - b\geq 0, \qquad x^\top(Ax-b)=0, \tag{3}$$

$$y\geq 0, \qquad Ay - b\geq 0, \qquad y^\top(Ay-b)=0. \tag{4}$$

Since ΠP^k is positive semidefinite for all k, A is positive semidefinite as well. Hence,

$$(x-y)^\top A(x-y)\geq 0, \tag{5}$$

[3]Notice that each of the S terms that are summed up in the inner product must be less than or equal to zero, due to **C1** and **C2**. Hence all of these terms must in fact be zero.

or, equivalently,

$$y^\top Ax \le \frac{1}{2}(x^\top Ax + y^\top Ay). \tag{6}$$

Furthermore, since $y \ge 0$, from (3) and (4) we have $y^\top Ax \ge y^\top b = y^\top Ay$, and similarly $y^\top Ax \ge x^\top b = x^\top Ax$. Therefore, (6) must hold with equality, and hence so must (5), i.e. $\sum_k \frac{1}{\beta^k}(x-y)^\top \Pi P^k(x-y) = 0$. Again using the fact that ΠP^k is positive semidefinite for all k, this implies that $(x-y)^\top \Pi P^k(x-y) = 0$, or $\|P^k(x-y)\|_2^2 = 0$, for all k. Hence, $P^k(x-y) = 0$, for all k. □

In particular, the lemma tells us that all state-price deflators that satisfy condition **C** induce the same asset valuation on any given exchange.

We now present our CWE characterization. It turns out that there is a unique[4] CWE which is symmetric, with all arbitrageurs acting alike and sharing the same subjective asset valuations. These valuations can be described in terms of a (subjective) state-price deflator p^A, common across all arbitrageurs, where p^A_s is each arbitrageur's marginal shadow value of consumption in state s.[5]

Proposition 3.1 (CWE) *There is a unique CWE.*

1. *There exists* p^A *satisfying condition* **C** *such that equilibrium arbitrageur supplies of state-contingent consumption are given by*

$$R^k y^{k,n} = \frac{1}{(1+N)\beta^k} \cdot P^k\left(p^k - p^A\right), \qquad k \in K.$$

2. *Equilibrium asset prices on exchange* k *are given by* $q^k = R^{k\top}\Pi\hat{p}^k$, *where*

$$\hat{p}^k := \frac{1}{1+N}p^k + \frac{N}{1+N}p^A. \tag{7}$$

   Thus $\hat{p}^k$ *is an equilibrium state-price deflator for exchange* k.

3. *The equilibrium profit of each arbitrageur is given by*

$$\sum_{k\in K} q^k \cdot y^{k,n} = \frac{1}{(1+N)^2} \cdot \sum_{k\in K} \frac{1}{\beta^k}\,\|P^k(p^k - p^A)\|_2^2.$$

4. *The equilibrium demands of investors for state-contingent consumption are given by*

$$R^k\theta^{k,i} = \frac{1}{\beta^{k,i}}P^k(p^{k,i} - \hat{p}^k), \qquad k \in K,\ i \in I^k. \tag{8}$$

[4]By uniqueness we mean that the equilibrium allocation and pricing functional on each exchange are unique. There may, of course, be multiple state-price deflators that induce the same equilibrium pricing functional.

[5]Formally, p^A_s is the Lagrange multiplier associated with the arbitrageur's no-default constraint in state s.

Proof Items 1–4 are established in Rahi and Zigrand (2009). From Lemma 3.1, any choice of p^A that satisfies condition **C** gives us the same asset valuation, i.e. $P^k p^A$ is unique. It follows that asset supplies are unique as well, and so is the CWE. □

Arbitrageurs supply consumption in state s to exchange k if the price that agents on exchange k are willing to pay for a unit of state s consumption, p_s^k, exceeds the arbitrageurs' shadow willingness to pay, p_s^A, once the excess willingness to pay is projected onto the span of the permissible assets. This supply is higher the deeper is exchange k (i.e. the lower is β^k).

Note that the arbitrageur valuation p^A is nonnegative (since it satisfies **C**) and does not depend on the number of arbitrageurs N. As N grows without bound, the equilibrium valuation on every exchange converges to p^A, i.e. $\lim_{N\to\infty} q^k = R^{k\top}\Pi p^A$, for all k. At the same time, individual arbitrageur trades vanish, as do total arbitrageur profits.

Our goal in the rest of the paper is to provide a Walrasian benchmark for a CWE. We will show that the arbitraged economy is asymptotically Walrasian, in the sense that the CWE of this economy converges to the equilibrium of an appropriately defined competitive economy with no arbitrageurs. This characterization turns out to be very useful in clarifying the role of arbitrageurs in integrating markets.

4 Equilibrium with Restricted Consumption

In this section we analyze a competitive economy with no arbitrageurs which has the following convenient property: a Walrasian equilibrium state-price deflator of this economy is equivalent to p^A, the arbitrageurs' subjective state-price deflator in the arbitraged economy.

Definition 4.1 *A Walrasian equilibrium with restricted consumption (WERC) is a state-price deflator p^{RC}, and portfolios $\{\theta^{k,i}, \varphi^{k,i,\ell}\}_{k\in K,\, i\in I^k,\, \ell\in K}$, such that*

1. *Investor optimization: For given $q^k = R^{k\top}\Pi p^{RC}$, $k \in K$, $\{\theta^{k,i}, \{\varphi^{k,i,\ell}\}_{\ell\in K}\}$ solves*

$$\max_{\theta^{k,i}\in\mathbb{R}^{J^k},\, \varphi^{k,i,\ell}\in\mathbb{R}^{J^\ell}} \quad x_0^{k,i} + \sum_{s\in S}\pi_s\Big[x_s^{k,i} - \frac{\beta^{k,i}}{2}(x_s^{k,i})^2\Big]$$
$$s.t. \quad x_0^{k,i} = \omega_0^{k,i} - q^k\cdot\theta^{k,i} - \sum_{\ell\in K} q^\ell\cdot\varphi^{k,i,\ell},$$
$$x^{k,i} = \omega^{k,i} + R^k\theta^{k,i},$$
$$\sum_{\ell\in K} R^\ell\varphi^{k,i,\ell} \geq 0.$$

2. Market clearing:

$$\sum_{k\in K,\, i\in I^k} R^k\theta^{k,i} + \sum_{k\in K,\, i\in I^k,\, \ell\in K} R^\ell\varphi^{k,i,\ell} = 0.$$

At a WERC, agents can trade any asset in the economy, facing a common state-price deflator p^{RC}, but agents on exchange k can consume payoffs in $\langle R^k\rangle$ only. For agent (k,i), the portfolio that leads to future consumption is $\theta^{k,i}$. He can choose, in addition, an auxiliary portfolio $\{\varphi^{k,i,\ell}\}_{\ell\in K}$, provided the payoff of this portfolio is nonnegative. As we shall explain later (in particular, see Example 5.1 and the ensuing discussion), the auxiliary portfolio mimics the role played by arbitrageurs at a CWE, by allowing investors access to global markets but not to additional consumption outside their local asset span.

Given asset payoffs $\{R^k\}_{k\in K}$, we say that asset prices $\{q^k\}_{k\in K}$ are globally weakly arbitrage-free if an agent with access to all the asset markets in the economy is unable to construct a weak arbitrage, i.e. for any portfolio $\{z^k\}_{k\in K}$ satisfying $\sum_{k\in K} R^k z^k \geq 0$, we have $\sum_{k\in K} q^k\cdot z^k \geq 0$. By the fundamental theorem of asset pricing, this is the case if and only if there exists $\psi\geq 0$ such that $q^k = {R^k}^\top\Pi\psi$, for all k. Clearly, due to the auxiliary portfolio, there cannot be a global weak arbitrage at a WERC. Hence an equilibrium state-price deflator p^{RC} can always be chosen to be nonnegative. Moreover, as the following proposition shows, there is a unique WERC and the asset valuation at this WERC coincides with the asset valuation of arbitrageurs at the corresponding CWE.

Proposition 4.1 (WERC) *There is a unique WERC with $p^{RC}\equiv p^A$, and*

$$R^k\theta^{k,i} = \frac{1}{\beta^{k,i}}P^k(p^{k,i}-p^{RC}), \qquad k\in K,\, i\in I^k. \tag{9}$$

Proof The Lagrangian for agent (k,i)'s optimization problem is

$$\begin{aligned}\mathcal{L} &= \omega_0^{k,i} - q^k\cdot\theta^{k,i} - \sum_{\ell\in K} q^\ell\cdot\varphi^{k,i,\ell} + \mathbf{1}^\top\Pi(\omega^{k,i}+R^k\theta^{k,i}) \\ &\quad - \frac{\beta^{k,i}}{2}(\omega^{k,i}+R^k\theta^{k,i})^\top\Pi(\omega^{k,i}+R^k\theta^{k,i}) + {\psi^{k,i}}^\top\Pi\sum_{\ell\in K}R^\ell\varphi^{k,i,\ell},\end{aligned}$$

where $\psi^{k,i} \in \mathbb{R}^S$ is a vector of Lagrange multipliers. Writing $q^k = {R^k}^\top \Pi p^{RC}$, the first-order conditions are equivalent to:[6]

$$\theta^{k,i} = \frac{1}{\beta^{k,i}} ({R^k}^\top \Pi R^k)^{-1} {R^k}^\top \Pi (p^{k,i} - p^{RC}), \tag{10}$$

$$ {R^\ell}^\top \Pi \psi^{k,i} = q^\ell = {R^\ell}^\top \Pi p^{RC}, \qquad \forall \ell \in K, \tag{11}$$

$$\psi^{k,i} \geq 0, \tag{12}$$

$$\sum_{\ell \in K} R^\ell \varphi^{k,i,\ell} \geq 0, \tag{13}$$

$$\psi^{k,i} \cdot \left(\sum_{\ell \in K} R^\ell \varphi^{k,i,\ell} \right) = 0. \tag{14}$$

In addition, we have the market-clearing condition:

$$\sum_{k,i} R^k \theta^{k,i} = - \sum_{k,i,\ell} R^\ell \varphi^{k,i,\ell}. \tag{15}$$

A WERC is completely characterized by equations (10)–(15). Equation (10) gives us the desired allocation (9), which in turn implies that

$$\sum_{k,i} R^k \theta^{k,i} = \sum_k \frac{1}{\beta^k} P^k (p^k - p^{RC}). \tag{16}$$

Equations (11) and (12) are the usual no-arbitrage conditions. In particular, ${R^\ell}^\top \Pi \psi^{k,i}$ is independent of (k, i), so we can choose $\psi^{k,i}$ to be the same for all (k, i), and p^{RC} equal to this common value. Thus

$$p^{RC} = \psi^{k,i} \geq 0. \tag{17}$$

Equations (13)–(17) together imply that p^{RC} satisfies condition **C**. Since p^A also satisfies condition **C** (Proposition 3.1(1)), we see from Lemma 3.1 that $p^{RC} \equiv p^A$, and moreover that the implied asset valuation is unique. The equilibrium allocation is then uniquely determined by (9). □

Proposition 4.1 shows that a WERC is the appropriate Walrasian foundation for a CWE. Before expanding on this theme, we consider another, more familiar, notion of restricted Walrasian equilibrium.

5 Equilibrium with Restricted Participation

Segmented asset markets have been widely studied in the general equilibrium literature in the context of a Walrasian economy with restricted participation. In

[6] In view of (12), equation (14) holds if and only if each of the S terms that are summed up in the inner product is zero.

such an economy, agents face a common state-price deflator p^{RP}, but agents on exchange k can trade claims in $\langle R^k \rangle$ only. In this section, we show that valuation in a restricted-participation economy differs in a subtle way from valuation in the restricted-consumption economy studied above. In general, Walrasian equilibrium with restricted participation is not a suitable benchmark for a CWE, as it captures only a subset of trades that are mediated by arbitrageurs in the arbitraged economy.

Definition 5.1 *A Walrasian equilibrium with restricted participation (WERP) is a state-price deflator p^{RP}, and portfolios $\{\theta^{k,i}\}_{k\in K,\, i\in I^k}$, such that*

1. *Investor optimization: For given $q^k = R^{k\top}\Pi p^{RP}$, $\theta^{k,i}$ solves*

$$\begin{aligned}\max_{\theta^{k,i}\in\mathbb{R}^{J^k}} \quad & x_0^{k,i} + \sum_{s\in S}\pi_s\left[x_s^{k,i} - \frac{\beta^{k,i}}{2}(x_s^{k,i})^2\right]\\ s.t. \quad & x_0^{k,i} = \omega_0^{k,i} - q^k\cdot\theta^{k,i},\\ & x^{k,i} = \omega^{k,i} + R^k\theta^{k,i}.\end{aligned}$$

2. *Market clearing:*

$$\sum_{k\in K,\, i\in I^k} R^k\theta^{k,i} = 0.$$

Defining

$$\lambda^k := \frac{\frac{1}{\beta^k}}{\sum_{j=1}^K \frac{1}{\beta^j}},$$

we have the following characterization of a WERP, analogous to Proposition 4.1 for a WERC:

Proposition 5.1 (WERP) *p^{RP} is a WERP state-price deflator if and only if it solves*

$$\sum_{k\in K}\lambda^k P^k(p^k - p^{RP}) = 0. \tag{18}$$

The corresponding net trades of state-contingent consumption are given by

$$R^k\theta^{k,i} = \frac{1}{\beta^{k,i}}P^k(p^{k,i} - p^{RP}), \qquad k\in K,\, i\in I^k. \tag{19}$$

If a WERP exists, it is unique.

Proof At a WERP, agent (k,i) solves

$$\max_{\theta^{k,i}\in\mathbb{R}^{J^k}} \omega_0^{k,i} - q^k\cdot\theta^{k,i} + \mathbf{1}^\top\Pi(\omega^{k,i} + R^k\theta^{k,i}) - \frac{\beta^{k,i}}{2}(\omega^{k,i} + R^k\theta^{k,i})^\top\Pi(\omega^{k,i} + R^k\theta^{k,i}).$$

Writing $q^k = R^{k\top}\Pi p^{RP}$, the first-order condition yields the optimal portfolio,

$$\theta^{k,i} = \frac{1}{\beta^{k,i}}(R^{k\top}\Pi R^k)^{-1}R^{k\top}\Pi(p^{k,i} - p^{RP}),$$

which gives us (19). The market-clearing condition is

$$\sum_{k,i} R^k\theta^{k,i} = \sum_k \frac{1}{\beta^k}P^k(p^k - p^{RP}) = 0,$$

which is equivalent to (18).

In order to show uniqueness, we adapt the argument in the proof of Lemma 3.1. Let x and y be two values of p^{RP} that satisfy equation (18). Using the shorthand notation defined in (1) and (2), we have $Ax - b = 0$ and $Ay - b = 0$. Hence $(x-y)^\top A(x-y) = 0$, i.e. $\sum_k \frac{1}{\beta^k}(x-y)^\top \Pi P^k(x-y) = 0$. Now since ΠP^k is positive semidefinite for each k, this implies that $(x-y)^\top\Pi P^k(x-y)= 0$, or $\|P^k(x-y)\|_2^2 = 0$, for all k. Hence, $P^k(x-y) = 0$, for all k. Thus x and y are equivalent state-price deflators which give us the same asset valuation on each exchange. Portfolios are then uniquely pinned down by (19). □

While there is always a nonnegative p^{RC}, by Proposition 4.1, this is not the case for p^{RP}. The following example illustrates:

Example 5.1 (WERP vs WERC) Consider an economy with two states of the world, two exchanges, and a single agent on each exchange. We refer to the agent on exchange k as agent k, $k = 1, 2$. The payoff matrices are

$$R^1 = \begin{bmatrix}1\\0\end{bmatrix}, \qquad R^2 = \begin{bmatrix}1\\1\end{bmatrix}.$$

The two exchanges are equally deep, with β^1 and β^2 both equal to $\bar\beta$, which satisfies

$$0 < \bar\beta < \frac{\pi_1}{1+\pi_1}. \tag{20}$$

Date 1 endowments are as follows: $\omega^1 = \mathbf{1}$ and $\omega^2 = (1/\bar\beta - 1)\mathbf{1}$. Autarky state-price deflators are, therefore, $p^1 = (1-\bar\beta)\mathbf{1}$ and $p^2 = \bar\beta\mathbf{1}$, respectively. Agent 1 values date 1 consumption more than agent 2. In autarky, $q^1 = (1-\bar\beta)\pi_1$, and $q^2 = \bar\beta$. The restriction (20) implies that $q^1 > q^2$. Thus there exist profit opportunities for arbitrageurs, buying on exchange 2 and delivering to exchange 1.

Now consider a WERP of this economy: agents face a common state-price deflator p^{RP}, but can only trade claims that lie in their local asset span. Since $\langle R^1\rangle \cap \langle R^2\rangle = \{0\}$, however, the two agents cannot trade with each other. Equilibrium asset prices are the same as in autarky. Since these prices allow for an arbitrage, albeit for a hypothetical agent with access to all markets, at least one of the state prices must be negative. The state-price deflator p^{RP} (which is unique since markets are complete in the integrated economy) solves $q^k = R^{k\top}\Pi p^{RP}$, $k = 1, 2$:

$$p^{RP} = \begin{bmatrix} 1 - \bar{\beta} \\ (1/\pi_2)[(1+\pi_1)\bar{\beta} - \pi_1] \end{bmatrix}.$$

It follows from (20) that $p_2^{RP} < 0$. Notice that this is not due to the non-monotonicity of quadratic utility. Equilibrium consumption at date 1 (which is just the initial endowment for both agents) is below the bliss point $1/\bar{\beta}$.

At the WERP, agents are unable to exploit the arbitrage opportunity, because doing so would take them outside their local asset span. In particular, if agent 1 were to buy the riskfree asset (which is underpriced from his perspective) from agent 2, he would end up with excess consumption in state 2. At the WERC, on the other hand, agents can arbitrage away the mispricing. Agent 1 simply disposes of the state 2 consumption good that he acquires from agent 2. Consequently $p_2^{RC} = 0$ (implying that $q^1 = q^2$). The equilibrium net trade of state-contingent consumption is given by $R^k\theta^k = \frac{1}{\beta^k}P^k(p^k - p^{RC})$, $k = 1, 2$. The projections P^1 and P^2 are:

$$P^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad P^2 = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix}.$$

Therefore, noting that $p_2^{RC} = 0$,

$$R^1\theta^1 = \frac{1 - \bar{\beta} - p_1^{RC}}{\bar{\beta}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad R^2\theta^2 = \frac{\bar{\beta} - \pi_1 p_1^{RC}}{\bar{\beta}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Market clearing for state 1 (in which there is no excess consumption) gives us $p_1^{RC} = \frac{1}{1+\pi_1}$, so that

$$R^1\theta^1 = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \qquad R^2\theta^2 = \begin{bmatrix} -\alpha \\ -\alpha \end{bmatrix},$$

where

$$\alpha := \frac{\pi_1}{\bar{\beta}(1+\pi_1)} - 1.$$

From (20) it follows that $\alpha > 0$. Agent 2 effectively sells α units of the riskfree asset to agent 1. Through this trade, agent 2 reduces his date 1 consumption by α in both states (and increases his date 0 consumption). Agent 1, on the other hand, is constrained by his local asset span to augment his date 1 consumption only in state 1. He increases his consumption in state 1 by α, while disposing of α units of state 2 consumption. It is easy to check that date 1 consumption at the WERC is below the bliss point $1/\bar{\beta}$ for both agents. Indeed, this would be the case even if agent 1 were allowed to consume the "excess" consumption in state 2; in other words, this consumption is in excess because it lies outside the permissible span, not because it takes the agent past his bliss point. ‖

The two notions of restricted Walrasian equilibrium differ in two key respects (both of which are captured by the auxiliary portfolios $\{\varphi^{k,i,\ell}\}$). First, at a WERP agents cannot trade claims outside their local asset span, while they can at a WERC. Second, the market-clearing condition at a WERC is weaker: at a WERC, we have $\sum_k R^k\theta^k \leq 0$, while at a WERP, $\sum_k R^k\theta^k = 0$. There may be arbitrage opportunities at a WERP that investors are unable to exploit due to their restricted-participation constraints. This is not the case at a WERC.

Both notions of equilibrium capture the idea that arbitrageurs allow investors to trade their own claims abroad. The weaker restrictions implicit in a WERC mimic the allocational role that arbitrageurs play over and above their mediation of this obvious category of trades. Indeed, arbitrageurs allow investors to trade *any* claim available in the economy. Investors can thereby exploit good deals in the global markets, which relaxes their date 0 budget constraint. They are better off as a result, even if they have to discard consumption in some states at date 1 (via the arbitrageurs) to remain within their local asset span.

Notice that the state-price deflator p^{RC} need not be strictly positive. This is due to the fact that investors by construction behave as if they are satiated in those directions of the consumption space that lie outside the imposed span. The states in which investors dispose of consumption at a WERC are precisely those in which arbitrageurs dispose of consumption at the CWE of the corresponding arbitraged economy, due to their inability to bring it back to date 0 without disturbing their arbitrage portfolio. In these states, $p_s^{RC} = p_s^A$.

6 Equilibrium with Complete Markets

A third notion of Walrasian equilibrium, indeed the most natural one, that we will have occasion to consider is Walrasian equilibrium with complete markets (or WECM). Formally, a WECM is just a special case of a WERP, with $R^k = P^k = I$, for all k. We denote a WECM state-price deflator by p^{CM}. Due to market completeness, the state-price deflator associated with a given WECM is unique. In fact, there is a unique WECM:

Proposition 6.1 (WECM) *There is a unique WECM, with*

$$p^{CM} = \sum_{k\in K} \lambda^k p^k,$$

and

$$\theta^{k,i} = \frac{1}{\beta^{k,i}}(p^{k,i} - p^{CM}), \qquad k \in K,\ i \in I^k.$$

This result is immediate from Proposition 5.1. The state-price deflator p^{CM} can be interpreted as the investors' economy-wide average willingness to pay, with the willingness to pay on each exchange weighted by its relative depth.

7 Explicit Characterizations

We have argued above that while a WERC serves as a suitable Walrasian benchmark for a CWE, in general a WERP does not. This is because at a WERP there may be unexploited arbitrage opportunities, a situation that clearly cannot arise in an economy with an unbounded number of arbitrageurs. However, what if a WERP happens to be arbitrage-free?

Proposition 7.1 *Suppose a WERP exists. There is no global weak arbitrage at the WERP if and only if* $p^{RP} \equiv p^{RC}$.

Proof By the FTAP, there is no global weak arbitrage at the WERP if and only if there exists a nonnegative p^{RP}. If $p^{RP} \geq 0$, the equations characterizing the WERC, (10)–(15), are satisfied at $\psi^{k,i} = p^{RC} = p^{RP}$, and $\varphi^{k,i,\ell} = 0$, for all k, i, ℓ. Hence $p^{RP} \equiv p^{RC}$. Conversely, if $p^{RP} \equiv p^{RC}$, then p^{RP} can be chosen to be nonnegative. □

Thus, if there is an arbitrage-free WERP, it does serve as a suitable Walrasian benchmark for the corresponding CWE. While this is applicable in an admittedly narrow class of economies, it is nevertheless of interest. As we shall see shortly, the assumptions commonly made in the literature limit us to this set of economies. Moreover, under these assumptions, there is a simple closed-form solution for p^{RP}.

Arbitrage opportunities can arise at a WERP because of the participation constraints that investors face, as in Example 5.1, or because investors who could potentially exploit these opportunities are satiated. A sufficient condition for a WERP to be arbitrage-free is that there is an investor who has access to all asset markets in the economy, and that this investor is nonsatiated at the equilibrium. The market access condition in our setting is simply the following:

(S1) $\langle R^1 \rangle$ contains $\langle R^k \rangle$, for all $k \in K$.

It says that there is an exchange (which we take to be exchange 1 without loss of generality) that has maximal asset span, in that this span contains the spans of all other exchanges. Thus investors on exchange 1 can trade all the assets in the economy.

In order to state the nonsatiation condition in terms of the primitives of the economy, it is convenient to introduce some additional notation. Let

$$\beta := \left[\sum_k (\beta^k)^{-1} \right]^{-1}$$

$$Q^1 := \left[\lambda^1 I + \sum_{k \neq 1} \lambda^k P^k \right]^{-1},$$

$$Q^k := \left[\lambda^1 I + \sum_{k \neq 1} \lambda^k P^k \right]^{-1} P^k, \qquad k \neq 1.$$

The inverse in the definition of Q^k exists since the matrix

$$\lambda^1\Pi^{-1} + \sum_{k\neq 1} \lambda^k R^k ({R^k}^\top \Pi R^k)^{-1} {R^k}^\top$$

is positive definite, hence invertible. We will employ the following nonsatiation condition:

(N1) $\mathbf{1} - \beta \cdot \sum_{k\in K} Q^k \omega^k \geq 0$.

It says that the representative agent with aggregate preference parameter β is nonsatiated at the weighted aggregate endowment, $\sum_{k\in K} Q^k \omega^k$.

Proposition 7.2 *Under* **S1**, $p^{RP} \equiv \bar{p}^{RP}$, *where*

$$\bar{p}^{RP} := \sum_{k\in K} \lambda^k Q^k p^k. \tag{21}$$

Furthermore, $\bar{p}^{RP} \geq 0$ *if and only if* **N1** *holds so that, under* **S1** *and* **N1**, $p^{RC} \equiv p^{RP} \equiv \bar{p}^{RP}$.

Proof From (21),

$$\left[\lambda^1 I + \sum_{k\neq 1} \lambda^k P^k\right] \bar{p}^{RP} = \lambda^1 p^1 + \sum_{j\neq 1} \lambda^j P^j p^j.$$

Premultiplying both sides by P^1, and noting that **S1** implies that $P^1 P^k = P^k$:

$$\left[\lambda^1 P^1 + \sum_{k\neq 1} \lambda^k P^k\right] \bar{p}^{RP} = \lambda^1 P^1 p^1 + \sum_{j\neq 1} \lambda^j P^j p^j,$$

i.e.

$$\sum_k \lambda^k P^k (p^k - \bar{p}^{RP}) = 0.$$

Therefore, $p^{RP} = \bar{p}^{RP}$ solves (18). Recalling that $p^k = \mathbf{1} - \beta^k \omega^k$, it is easy to verify that $\bar{p}^{RP} = \mathbf{1} - \beta \cdot \sum_{k\in K} Q^k \omega^k$, so that $\bar{p}^{RP} \geq 0$ if and only if **N1** holds.[7] □

A sharper characterization of p^{RP} can be obtained under the following alternative set of conditions (we define $\omega := \sum_k \omega^k$):

(S2) Either (a) $\langle R^k \rangle = \langle R \rangle$, $k \in K$, or (b) $p^k - p^{CM} \in \langle R^k \rangle$, $k \in K$.

(N2) $\mathbf{1} - \beta\omega \geq 0$.

[7] Exactly the same proof goes through if we assume from the start that $P^1 = I$. Thus $\bar{p}^{RP}$ is the (unique) WERP state-price deflator of the economy in which asset payoffs are the same as in the original economy except that markets are complete on exchange 1.

Condition **S2**(a) specializes **S1** to the case in which all exchanges have the same asset span. **S2**(b) is the condition that characterizes equilibrium security design in a setting in which arbitrageurs choose the asset structure $\{R^k\}_{k\in K}$ prior to the Cournot game in which they carry out their arbitrage trades (see Rahi and Zigrand (2009)). Notice that **S1** and **S2** are not nested. Condition **N2** says that the representative investor for the whole economy with aggregate preference parameter β is weakly nonsatiated at the aggregate endowment ω.

Proposition 7.3 *Under* **S2**, $p^{RP} \equiv p^{CM}$. *Furthermore,* $p^{CM} \geq 0$ *if and only if* **N2** *holds so that, under* **S2** *and* **N2**, *we have* $p^{RC} \equiv p^{RP} \equiv p^{CM}$.

Proof If **S2**(a) holds, $P^k = P$, for all k. Then it is easy to see that $p^{RP} = p^{CM}$ solves (18).[8] Under **S2**(b), $P^k(p^k - p^{CM}) = p^k - p^{CM}$, so p^{CM} solves (18) in this case as well. Finally, note that $p^{CM} = \mathbf{1} - \beta\omega$, so that $p^{CM} \geq 0$ if and only if **N2** holds. □

While condition **S2** is quite restrictive, it is nevertheless more general than the assumption that the same assets are traded on every exchange, an assumption that is commonly made in the literature on arbitrage in asset markets.

8 Convergence to Walrasian Equilibrium

Recall that, in the arbitraged economy, as the number of arbitrageurs goes to infinity, asset prices converge to the arbitrageur valuation of assets at the CWE (see equation (7)). We have shown that the latter is just the asset valuation at the WERC of the corresponding Walrasian economy (Proposition 4.1). Comparing (8) and (9), we also see that the equilibrium allocation (for investors) in the arbitraged economy converges to the WERC allocation. Closed-form solutions for the WERC valuation can be derived under restrictions on preferences, endowments and the asset structure that ensure that the WERC and WERP coincide (Propositions 7.2 and 7.3). We summarize these observations in the following proposition, which makes precise the sense in which the arbitraged economy is asymptotically Walrasian:

[8] If markets are complete, p^{CM} is the unique solution to (18). If markets are incomplete, so that the common span $\langle R\rangle$ is a strict subset of $\mathbb{R}^S$, p^{CM} is still a solution, but it is not the only one. All solutions are of course equivalent to p^{CM}.

Proposition 8.1 (Convergence to Walrasian equilibrium)

1. *As the number of arbitrageurs N goes to infinity, the equilibrium valuation on exchange k in the arbitraged economy converges to the WERC valuation, i.e.* $\lim_{N\to\infty} q^k = {R^k}^\top \Pi p^{RC}$. *Under* **S1** *and* **N1**, *this is also the WERP valuation,* ${R^k}^\top \Pi \bar{p}^{RP}$. *Under* **S2** *and* **N2**, *it coincides with the WECM valuation,* ${R^k}^\top \Pi p^{CM}$.

2. *As the number of arbitrageurs N goes to infinity, the equilibrium allocation in the arbitraged economy converges to the WERC allocation. Under either* **S1** *and* **N1**, *or under* **S2** *and* **N2**, *this is also the WERP allocation.*

As stated in the proposition, under **S2** and **N2**, we get convergence to the WECM valuation. However, we do not get convergence to the WECM allocation unless it coincides with the WERP allocation. A sufficient condition for the latter is complete markets on each exchange ($\langle R^k \rangle = \mathbb{R}^S$, for all k), in addition to **N2**. For an economy in which investors on any given exchange have the same no-trade valuations, i.e. $p^{k,i} = p^k$, for all $i \in I^k$, **S2**(b) and **N2** suffice as well.

9 Conclusion

Given an economy with an arbitrary asset structure, if we view Walrasian equilibria as approximations to more complex equilibria with segmented markets and strategic arbitrageurs, the concept of Walrasian equilibrium with restricted consumption introduced in this paper is the appropriate benchmark, rather than the well-studied and intuitive notion of Walrasian equilibrium with restricted participation. The subtle difference between these two kinds of Walrasian equilibrium clarifies the sense in which arbitrageurs serve to integrate markets in the arbitraged economy.

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