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THE BASS AND TOPOLOGICAL STABLE RANKS OF THE BOHL ALGEBRA ARE INFINITE

RAYMOND MORTINI, RUDOLF RUPP, AND AMOL SASANE

Abstract. The Bohl algebra B is the ring of linear combinations of functions $t^k e^{\lambda t}$ on the real line, where k is any nonnegative integer, and λ is any complex number, with pointwise operations. We show that the Bass stable rank and the topological stable rank of B (where we use the topology of uniform convergence) are infinite.

1. Introduction

The aim of this article is to investigate two specific algebraic-analytic properties, named Bass stable rank and topological stable rank, for a particular algebra of functions, called the Bohl algebra. Stable ranks are concepts that originate from algebraic and topological K-theory. We give the pertinent definitions below. Bohl functions (although not under this name) appear in every introductory course to analysis and arise in finite dimensional control theory as impulse responses of systems described by linear constant coefficient ordinary differential equations.

Definition 1.1 (Bohl algebra B)**.** Let $\mathbb{N} := \{0, 1, 2, 3, \cdots\}$ denote the set of nonnegative integers, and $\mathbb{N}^* := \{1, 2, 3, \cdots\}$. The *Bohl algebra* B is the complex algebra of functions on the real line $\mathbb{R}$,

$$\mathrm{B} := \left\{ \sum_{n=1}^{N} c_n t^{m_n} e^{\lambda_n t} : N \in \mathbb{N}^*,\ c_n, \lambda_n \in \mathbb{C},\ m_n \in \mathbb{N} \right\},$$

with pointwise operations of addition, scalar multiplication and multiplication.

In view of the fact that for $\mathrm{Re}\, s > \mathrm{Re}\, \lambda$, the integral

$$F(s) := \int_0^\infty t^n e^{\lambda t}\, e^{-st}\, dt = \frac{n!}{(s-\lambda)^{n+1}}$$

converges absolutely, we see that the Bohl algebra is essentially the class of functions that have a strictly proper rational Laplace transform (that is, rational functions for which the degree of the denominator is strictly bigger

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than the degree of the numerator). They arise naturally in the study of linear differential equations with constant coefficients since

$$\left(\frac{d}{dt}-\lambda\right)^{k+1} t^k e^{\lambda t} = 0.$$

See for example [2, Definition 2.5 and Theorem 2.6].

In this article we investigate some algebraic analytic properties of the Bohl algebra. Let us also mention that each function in B is the trace on $\mathbb{R}$ of an entire function of finite exponential order. In this light, we therefore obtain also some information on an interesting subring of the ring $H(\mathbb{C})$ of entire functions.

Definition 1.2 (Bass stable rank)**.** Let R be a commutative unital ring with identity element 1. We assume that $1 \neq 0$, that is R is not the trivial ring $\{0\}$.

(1) An n-tuple $(f_1, \dots, f_n) \in R^n$ is said to be *invertible* (or *unimodular*), if there exists $(x_1, \dots, x_n) \in R^n$ such that the Bézout equation
$$\sum_{j=1}^{n} x_j f_j = 1$$
is satisfied. The set of all invertible n-tuples is denoted by $U_n(R)$. Note that $U_1(R) = R^{-1}$.

(2) An $(n+1)$-tuple $(f_1, \dots, f_n, g) \in U_{n+1}(R)$ is called *reducible* if there exists $(a_1, \dots, a_n) \in R^n$ such that
$$(f_1 + a_1 g, \dots, f_n + a_n g) \in U_n(R).$$

(3) The *Bass stable rank* of R, denoted by $\operatorname{bsr} R$, is the smallest integer n such that every element in $U_{n+1}(R)$ is reducible. If no such n exists, then $\operatorname{bsr} R = \infty$.

Note that if $\operatorname{bsr} R = n$, where $n < \infty$, and if $m \geq n$, then every invertible $(m+1)$-tuple $(f_1, \dots, f_n, g) \in R^{m+1}$ is reducible [8, Theorem 1]. Our first main result is the following.

Theorem 1.3. *The Bass stable rank of the Bohl algebra is infinite.*

An analogue of the Bass stable rank for topological rings was introduced by Rieffel in [7].

Definition 1.4 (Topological stable rank)**.** Let R be a commutative unital ring endowed with a topology $\mathcal{T}$. (We do not assume that the topology is compatible with the algebraic operations $+$ and $\cdot$ in R.) The *topological stable rank*, $\operatorname{tsr}_{\mathcal{T}} R$, of $(R, \mathcal{T})$ is the least integer n for which $U_n(R)$ is dense in R^n, or infinite if no such n exists.

For the algebra B of Bohl functions, we work with the topology of uniform convergence, that is, a basis of open sets is given by the family $(V_{f,\epsilon})_{f \in \mathrm{B}, \epsilon > 0}$, where
$$V_{f,\epsilon} := \left\{ g \in \mathrm{B} : \|f - g\|_\infty := \sup_{t \in \mathbb{R}} |f(t) - g(t)| < \epsilon \right\}.$$

Our second main result is the following.

Theorem 1.5. *The topological stable rank of* B *(with respect to the uniform topology) is infinite.*

The Bohl algebra is therefore a natural example of a subalgebra of $C(\mathbb{R},\mathbb{C})$ that shows that these notions of stable ranks are not order preserving: in fact

$$\operatorname{sr}\mathbb{C}[z]\,|_{\mathbb{R}} = 2,\ \operatorname{sr}\mathrm{B} = \infty \text{ and } \operatorname{sr}C(\mathbb{R},\mathbb{C}) = 1,$$

where sr denotes either the Bass or the topological stable rank.
The organization of the paper is as follows: in Section 2 we show that the Bass stable rank of B is infinite, and finally in Section 3, we show that the topological stable rank of B is infinite.

2. Bass stable rank of B is infinite

Definition 2.1 (The algebra AP)**.** Let $C_b(\mathbb{R},\mathbb{C})$ denote the set of bounded, continuous functions on $\mathbb{R}$ with values in $\mathbb{C}$, and let AP be the uniform closure in $C_b(\mathbb{R},\mathbb{C})$ of the set of all functions of the form

$$Q(t) := \sum_{j=1}^{N} a_j e^{i\lambda_j t},$$

where $a_j \in \mathbb{C}$, $\lambda_j \in \mathbb{R}$ and $N \in \mathbb{N}^*$. We call Q a *generalized trigonometric polynomial*. AP is the set of *almost periodic functions*.

Lemma 2.2. *Let the map* $\Psi : \mathrm{B} \to \mathrm{AP}$ *be defined as follows. If* $f \in \mathrm{B}$*, and*

$$f(t) = \sum_{j=1}^{J} c_j t^{k_j} e^{(\alpha_j + i\beta_j)t}, \quad t \in \mathbb{R}, \tag{2.1}$$

where the $c_j \in \mathbb{C}$, $\alpha_j, \beta_j \in \mathbb{R}$ *and* k_j *are nonnegative integers, then we define* $\Psi(f) \in$AP *by*

$$(\Psi(f))(t) = \sum_{j=1}^{J} c_j e^{i\beta_j t}, \quad t \in \mathbb{R}.$$

Then $\Psi : \mathrm{B} \to \mathrm{AP}$ *is well-defined and a ring homomorphism.*

Proof. By the linear independence of $\{t^k e^{\lambda t} : \lambda \in \mathbb{C},\ k \in \mathbb{N}\}$, it follows that f has a unique decomposition of the form given in (2.1), and so the map Ψ is well-defined.

Let $f \in \mathrm{B}$, and $K \subset \mathbb{N}$, $\Lambda \subset \mathbb{C}$ be finite sets such that

$$f(t) = \sum_{\substack{k \in K \\ \lambda \in \Lambda}} f_{k,\lambda} t^k e^{\lambda t},$$

where the coefficients $f_{k,\lambda}$ are complex numbers. Let $c, \lambda_* \in \mathbb{C}$, $k_* \in \mathbb{N}$. We have two cases:

1° $k_* \notin K$ or $\lambda_* \notin \Lambda$. Then

$$\begin{aligned}\Psi(f+ct^{k_*}e^{\lambda_* t}) &= \Psi\Bigg(\sum_{\substack{k\in K\\ \lambda\in\Lambda}} f_{k,\lambda}t^k e^{\lambda t}+ct^{k_*}e^{\lambda_* t}\Bigg)=\sum_{\substack{k\in K\\ \lambda\in\Lambda}} f_{k,\lambda}e^{i\cdot \mathrm{Im}(\lambda)\cdot t}+ce^{i\cdot\mathrm{Im}(\lambda_*)\cdot t}\\ &= \Psi\Bigg(\sum_{\substack{k\in K\\ \lambda\in\Lambda}} f_{k,\lambda}t^k e^{\lambda t}\Bigg)+\Psi(ct^{k_*}e^{\lambda_* t})=\Psi(f)+\Psi(ct^{k_*}e^{\lambda_* t}).\end{aligned}$$

2° $k_* \in K$ and $\lambda_* \in \Lambda$. Then

$$\begin{aligned}&\Psi(f+ct^{k_*}e^{\lambda_* t})\\ &\quad= \Psi\Bigg(\sum_{\substack{k\in K\\ \lambda\in\Lambda}} f_{k,\lambda}t^k e^{\lambda t}+ct^{k_*}e^{\lambda_* t}\Bigg)\\ &\quad= \Psi\Bigg(\sum_{\substack{k\in K\setminus\{k_*\}\\ \lambda\in\Lambda\setminus\{\lambda_*\}}} f_{k,\lambda}t^k e^{\lambda t}+\sum_{k\in K\setminus\{k_*\}} f_{k,\lambda_*}t^k e^{\lambda_* t}+\sum_{\lambda\in\Lambda\setminus\{\lambda_*\}} f_{k_*,\lambda}t^{k_*}e^{\lambda t}\\ &\qquad\qquad +(f_{k_*,\lambda_*}+c)t^{k_*}e^{\lambda_* t}\Bigg)\\ &\quad= \sum_{\substack{k\in K\setminus\{k_*\}\\ \lambda\in\Lambda\setminus\{\lambda_*\}}} f_{k,\lambda}e^{i\cdot\mathrm{Im}(\lambda)\cdot t}+\sum_{k\in K\setminus\{k_*\}} f_{k,\lambda_*}e^{i\cdot\mathrm{Im}(\lambda_*)\cdot t}+\sum_{\lambda\in\Lambda\setminus\{\lambda_*\}} f_{k_*,\lambda}e^{i\cdot\mathrm{Im}(\lambda)\cdot t}\\ &\qquad\qquad +(f_{k_*,\lambda_*}+c)e^{i\cdot\mathrm{Im}(\lambda_*)\cdot t}\\ &\quad= \sum_{\substack{k\in K\\ \lambda\in\Lambda}} f_{k,\lambda}e^{i\cdot\mathrm{Im}(\lambda)\cdot t}+ce^{i\cdot\mathrm{Im}(\lambda_*)\cdot t}\end{aligned}$$

$$= \Psi\Bigg(\sum_{\substack{k\in K\\ \lambda\in\Lambda}} f_{k,\lambda}t^k e^{\lambda t}\Bigg)+\Psi(ct^{k_*}e^{\lambda_* t})=\Psi(f)+\Psi(ct^{k_*}e^{\lambda_* t}).$$

By the cases 1° and 2°, we have for all $f,g\in \mathrm{B}$, $\Psi(f+g)=\Psi(f)+\Psi(g)$.

Now let $g\in\mathrm{B}$, and $L\subset\mathbb{N}$, $\Omega\subset\mathbb{C}$ be finite sets such that

$$g(t)=\sum_{\substack{\ell\in L\\ \omega\in\Omega}} g_{\ell,\omega}t^\ell e^{\omega t},$$

where $g_{\ell,\omega}$ are complex numbers. Then we have

$$\begin{aligned}\Psi(f\cdot g) &= \Psi\Bigg(\sum_{\substack{k\in K\\ \lambda\in\Lambda}} f_{k,\lambda}t^k e^{\lambda t}\cdot\sum_{\substack{\ell\in L\\ \omega\in\Omega}} g_{\ell,\omega}t^\ell e^{\omega t}\Bigg)=\Psi\Bigg(\sum_{\substack{k\in K,\ell\in L\\ \lambda\in\Lambda,\omega\in\Omega}} f_{k,\lambda}g_{\ell,\omega}t^{k+\ell}e^{(\lambda+\omega)t}\Bigg)\\
&= \sum_{\substack{k\in K,\ell\in L\\ \lambda\in\Lambda,\omega\in\Omega}} \Psi\Big(f_{k,\lambda}g_{\ell,\omega}t^{k+\ell}e^{(\lambda+\omega)t}\Big) \quad \text{(by additivity of } \Psi)\\
&= \sum_{\substack{k\in K,\ell\in L\\ \lambda\in\Lambda,\omega\in\Omega}} f_{k,\lambda}g_{\ell,\omega}e^{i\cdot\mathrm{Im}(\lambda+\omega)\cdot t}\\
&= \sum_{\substack{k\in K\\ \lambda\in\Lambda}} f_{k,\lambda}e^{i\cdot\mathrm{Im}(\lambda)\cdot t}\cdot\sum_{\substack{\ell\in L\\ \omega\in\Omega}} g_{\ell,\omega}e^{i\cdot\mathrm{Im}(\omega)\cdot t}=\Psi(f)\cdot\Psi(g)\end{aligned}$$

□

We will prove Theorem 1.3 using the following result (see [5, Theorem 3.5] for a proof).

Proposition 2.3. *For all $N\in\mathbb{N}^*$, there exists an $F=(f_1,\cdots,f_N,g)$, where $f_1,\cdots,f_N,g$ are generalized trigonometric polynomials, which is invertible in AP^n with inverse an $(N+1)$-tuple of generalized trigonometric polynomials, but not reducible in* AP*.*

We recall here the explicit tuple given in the proof of [5, Theorem 3.5]. Given $N\in\mathbb{N}^*$, let $\{\lambda_1,\cdots,\lambda_{4N}\}$ be any set of positive real numbers that is linearly independent over $\mathbb{Q}$. For $j=1,\cdots,2N$ and $s\in\mathbb{N}^*$, let

$$f_j(t)=(e^{i\lambda_{2j-1}t})^s+(e^{i\lambda_{2j}t})^s-1, \quad \text{and} \quad g(t)=\frac{1}{4}-\sum_{j=1}^{N} f_j(t)f_{N+j}(t).$$

Then it is clear that $F:=(f_1,\cdots,f_N,g)$ has an inverse which is an $(N+1)$-tuple of generalized trigonometric polynomials. It is shown in [5, Theorem 3.5] that F is not reducible in AP.

Proof of Theorem 1.3. We will prove the claim by contradiction. Suppose that the Bass stable rank of B is at most N for some $N\in\mathbb{N}$. By Proposition 2.3, there exists an $F=(f_1,\cdots,f_N,g)\in\mathrm{AP}^n$ which is invertible as an element of AP with inverse an $(N+1)$-tuple of generalized trigonometric polynomials, but not reducible in AP. Since every generalized trigonometric polynomial is a Bohl function, it follows that $F\in\mathrm{B}^{N+1}$ and F is invertible with an inverse $G\in\mathrm{B}^{N+1}$. As the Bass stable rank of B is assumed to be at most N, there exist elements $h_1,\cdots,h_N\in\mathrm{B}$ and $x_1,\cdots,x_N\in\mathrm{B}$ such that

$$(f_1+h_1g)x_1+\cdots+(f_N+h_Ng)x_N=1.$$

Applying the ring homomorphism Ψ from Lemma 2.2, we obtain

$$(f_1 + \Psi(h_1) \cdot g) \cdot \Psi(x_1) + \cdots + (f_N + \Psi(h_N) \cdot g) \cdot \Psi(x_N) = 1.$$

But this means that $(f_1, \cdots, f_N, g)$ is reducible in AP, a contradiction. $\square$

Remark 2.4. The *Krull dimension of* R is defined to be the supremum of the lengths of all increasing chains $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ of prime ideals in R. An inequality due to Heitmann [3] says that the Bass stable rank of a ring is at most the Krull dimension+2, and so it follows that the Krull dimension of B is infinite. In particular, it is not Noetherian.

3. Topological stable rank of B is infinite

Lemma 3.1. *Let* $f_{AP}(t) := c_1 e^{i\xi_1 t} + \cdots + c_n e^{i\xi_n t}$, $t \in \mathbb{R}$, *where* $\xi_1, \cdots, \xi_n$ *are distinct real numbers, and* $c_1, \cdots, c_n$ *are nonzero complex numbers. Then there exists an* $\epsilon > 0$ *and a sequence* $(t_k)_{k\in\mathbb{N}}$ *such that*

$$\lim_{k\to\infty} t_k = +\infty,$$

and $|f(t_k)| \geq \epsilon$ *for all* $k \in \mathbb{N}$.

Proof. By the linear independence of $e^{i\xi_1 t}, \cdots, e^{i\xi_n t}$, it follows that there is a $t_* \in \mathbb{R}$ such that $f_{AP}(t_*) \neq 0$. Let $2\epsilon := |f_{AP}(t_*)|$. Then by using the fact that the set of ϵ-translation numbers of almost periodic functions are relatively dense, it follows that there exists a $T > 0$ such that every interval of length T contains an ϵ-translation number of f_{AP} [1, Theorem 1.9, 1.10]. Thus, for every $k \in \mathbb{N}$, the interval $[kT, (k+1)T]$ contains a number t'_k such that $\|f_{AP}(\cdot) - f_{AP}(\cdot + t'_k)\|_\infty < \epsilon$. But this implies $|f_{AP}(t_*) - f_{AP}(t_* + t'_k)| \leq \epsilon$, and so with $t_k = t'_k + t_*$, we have $|f_{AP}(t_k)| \geq \epsilon$. $\square$

Lemma 3.2. *If*

$p_1, \cdots, p_n$ *are polynomials, and*
$\xi_1, \cdots, \xi_n$ *distinct real numbers,*

such that $f_I(t) := p_1(t)e^{i\xi_1 t} + \cdots + p_n(t)e^{i\xi_n t}$, $t \in [0, \infty)$, *satisfies, for some* $M > 0$,

$$\sup_{t\in[0,\infty)} |f_I(t)| \leq M,$$

then the p_j *are all constant.*

Proof. We argue by contradiction. Let us suppose that the polynomials p_j are arranged in decreasing order of degree, and that the degree of p_1 is strictly positive. Let i_* be the index such that

$$d := \deg(p_1) = \cdots = \deg(p_{i_*}) > \deg(p_{i*+1}) \geq \cdots \geq \deg(p_n).$$

If the degrees all coincide, we put $i^* = n$. If the leading coefficients of $p_1, \cdots, p_{i_*}$ are $c_1, \cdots, c_{i_*}$, then

$$\Big|\underbrace{c_1 e^{i\xi_1 t} + \cdots + c_{i_*} e^{i\xi_{i*} t}}_{=:f_{AP}} + t^{-d} \sum_{\substack{k,\xi \\ k<d}} c_{k,\xi} t^k e^{i\xi t}\Big| \leq \frac{M}{|t|^d}. \tag{3.1}$$

Now if we take t to be the terms of a sequence $(t_k)_{k\in\mathbb{N}^*}$ corresponding to f_{AP} as in the previous Lemma 3.1, and by passing to the limit as $k \to \infty$, we get the contradiction to the fact that

$$\inf_{k\in\mathbb{N}^*} |f_{AP}(t_k)| =: \epsilon > 0.$$

So the degree of each p_j must be zero, that is they are constants. □

Lemma 3.3. *Let $F := F_I + F_R$, where*

$F_I(t) := p_1(t)e^{i\xi_1 t} + \cdots + p_n(t)e^{i\xi_n t}$,

$F_R(t) := q_1(t)e^{z_1 t} + \cdots + q_m(t)e^{z_m t}$,

$\xi_1, \cdots, \xi_n$ *are distinct real numbers,*

$z_1, \cdots, z_m$ *are distinct complex numbers, each with a nonzero real part,*

$p_1, \cdots, p_n, q_1, \cdots, q_m$ *are polynomials.*

Suppose, moreover, that $\|F\|_\infty < M$ for some $M > 0$, where the supremum is taken over $\mathbb{R}$. Then $q_1, \cdots, q_m = 0$, that is $F_R \equiv 0$, and $p_1, \cdots, p_n$ are constants. Hence F is in AP.

Proof. In view of Lemma 3.2, it is enough to show that $q_1, \cdots, q_m = 0$. To achieve a contradiction, we suppose that none of the polynomials q_j appearing in F_R is the zero polynomial. There is no loss of generality in assuming that one of the z_j has a positive real part. (Since otherwise, we may just consider the function $\widetilde{F}$ defined by $\widetilde{F}(t) = F(-t)$, and then having got the desired conclusion for $\widetilde{F}$, it is also obtained for F.)

Also, we may assume the real numbers $\mathrm{Re}(z_j)$ are arranged in decreasing order. Let i_* be the index such that

$$\mathrm{Re}(z_1) = \cdots = \mathrm{Re}(z_{i_*}) > \mathrm{Re}(z_{i_*+1}) \geq \cdots \geq \mathrm{Re}(z_m),$$

where $\mathrm{Re}(z_1) > 0$. If the real parts are all the same, we let $i^* = m$. Let

$$f(t) := q_1(t)e^{i\mathrm{Im}(z_1)\cdot t} + \cdots + q_{i_*}(t)e^{i\mathrm{Im}(z_{i_*})\cdot t}.$$

Then we have

$$e^{\mathrm{Re}(z_1)\cdot t}\Big|f(t) + \frac{q_{i_*+1}(t)e^{z_{i_*+1}t} + \cdots + q_m(t)e^{z_m t}}{e^{\mathrm{Re}(z_1)\cdot t}} + \frac{f_I(t)}{e^{\mathrm{Re}(z_1)\cdot t}}\Big| \leq M,$$

where the central sum does not appear if $i^* = m$. Hence

$$\Big|f(t) + \frac{q_{i_*+1}(t)e^{z_{i_*+1}t} + \cdots + q_m(t)e^{z_m t}}{e^{\mathrm{Re}(z_1)\cdot t}} + \frac{f_I(t)}{e^{\mathrm{Re}(z_1)\cdot t}}\Big| \leq \frac{M}{e^{\mathrm{Re}(z_1)\cdot t}}.$$

Since

$$\lim_{t\to\infty} q(t)e^{-st} = 0$$

for $s > 0$, it follows that f is bounded on $[0, \infty)$, and so by the previous Lemma 3.2, $q_1, \cdots, q_{i_*}$ are constants. So in fact f is of the form described in Lemma 3.1. Hence there exists an $\epsilon > 0$ and a sequence $(t_k)_{k\in\mathbb{N}^*}$ such that $t_k \to \infty$ and $|f(t_k)| \geq \epsilon$. But the above inequality yields that $\lim_{t\to\infty} f(t) = 0$; a contradiction. □

Proposition 3.4. *The invertible elements in* B *are the exponentials* $ce^{\lambda t}$, *where* $c \in \mathbb{C} \setminus \{0\}$ *and* $\lambda \in \mathbb{C}$.

Proof. Let $f(t) = \sum_{j=1}^{n} p_j(t) e^{\lambda_j t} \in \mathrm{B}$. Consider the entire function

$$F(z) = \sum_{j=1}^{n} p_j(z) e^{\lambda_j z}.$$

Then, for all z with large modulus,

$$|F(z)| \leq C|z|^m \sum_{j=1}^{n} e^{\mathrm{Re}\,(\lambda_j z)} \leq C|z|^m \sum_{j=1}^{n} e^{|\lambda_j|\,|z|} \leq e^{b|z|+c}$$

for some constants $b, c \in \mathbb{R}^+ :=]0, \infty[$. Now, if f is invertible in B, there is $g \in \mathrm{B}$ such that $fg = 1$ (on $\mathbb{R}$). Being the traces of holomorphic functions, the equality $F(z)G(z) = 1$ holds in $\mathbb{C}$. In particular, F has no zeros in $\mathbb{C}$. Hence, by Hadamard's classical theorem on entire functions of exponential type (see [4, p. 84]), $F(z) = e^{\beta z + \alpha}$ for some $\alpha, \beta \in \mathbb{C}$. □

Remark 3.5. A characterization of the invertible m-tuples $(f_1, \dots, f_m)$ in B seems to be "out of range". For example, if $(p_1, \dots, p_m)$ are polynomials in $\mathbb{C}[z_1, \dots, z_n]$ with

$$\bigcap_{j=1}^{m} Z(p_j) = \emptyset,$$

(with $Z(p_j)$ being the set of zeros of p_j in $\mathbb{C}^n$), then the Hilbert Nullstellensatz gives the existence of polynomials q_j such that

$$\sum_{j=1}^{m} p_j q_j = 1.$$

Now every such polynomial gives rise to uncountably many different Bohl functions: for example, put

$$f_j(t) := p_j\big(t, e^{\lambda_1^{(j)} t}, \dots, e^{\lambda_{n-1}^{(j)} t}\big),$$

where $\lambda_k^{(j)} \in \mathbb{C}$ and

$$g_j(t) := q_j\big(t, e^{\lambda_1^{(j)} t}, \dots, e^{\lambda_{n-1}^{(j)} t}\big).$$

Then

$$\sum_{j=1}^{m} f_j(t) g_j(t) = 1,$$

and so we have a solution to a Bézout equation in B.

Proof of Theorem 1.5 that $\operatorname{tsr}_{\tau_\infty} \mathbf{B} = \infty$**.** Let $n \in \mathbb{N}^*$ and, for $j = 1, \dots, n$, let

$$f_j(t) = e^{i\lambda_{2j-1}t} + e^{i\lambda_{2j}t} - 1,$$

where $\{\lambda_1, \dots, \lambda_{2n}\}$ is a set of reals linearily independent over $\mathbb{Q}$. Clearly each f_j is in the Bohl algebra, and hence the n-tuple $F := (f_1, \dots, f_n)$ belongs to $\mathrm{B}^n \cap \mathrm{AP}^n$. By [5, Theorem 5.1], F cannot be uniformly approximated by invertible tuples in AP. We claim that this F can't be approximated by invertible tuples in B, either. Suppose the contrary. Then there exists a sequence $(G^{(m)})_{m\in\mathbb{N}^*}$ of B-invertible n-tuples with $G^{(m)} = (g_1^{(m)}, \cdots, g_n^{(m)}) \in \mathrm{B}^n$ for each $m \in \mathbb{N}^*$, such that

$$\|f_j - g_j^{(m)}\|_\infty < \frac{1}{m}, \quad j = 1, \cdots, n, \; m \in \mathbb{N}^*.$$

Then, as each f_j is bounded, so is each $g_j^{(m)}$. So by Lemma 3.3, each $g_j^{(m)}$ is in AP. Hence $G^{(m)} \in \mathrm{AP}^n$ for each m. We know this $G^{(m)}$ is invertible as an element of B^n, and so there exists a $H = (h_1, \cdots, h_n) \in \mathrm{B}^n$ such that

$$G^{(m)} \cdot H = g_1^{(m)} h_1 + \cdots + g_n^{(m)} h_n = 1. \tag{3.2}$$

Applying the ring homomorphism Ψ from Lemma 2.2 in (3.2), we obtain

$$g_1^{(m)} \cdot \Psi(h_1) + \cdots + g_n^{(m)} \cdot \Psi(h_n) = 1.$$

Hence $G^{(m)}$ is invertible as an element of AP^n, a contradiction to the fact that F cannot be approximated by invertible tuples in AP. □

Recall that the Bass stable rank of the ring $H(\mathbb{C})$ of entire functions is one ([6]). Here is now, to the best of our knowledge, the first explicit subring of $H(\mathbb{C})$ having infinite stable rank.

Corollary 3.6. *The ring of all functions*

$$\sum_{j=1}^{n} c_j z^{n_j} e^{\lambda_j z}, \quad c_j, \lambda_j \in \mathbb{C}, \; n_j \in \mathbb{N},$$

has Bass stable rank infinity.

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