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Asset Pricing with Heterogeneous Preferences, Beliefs, and Portfolio Constraints

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Abstract

Portfolio constraints are widespread and have significant effects on asset prices. This paper studies the effects of constraints in a dynamic economy populated by investors with different risk aversions and beliefs about the rate of economic growth. The paper provides a comparison of various constraints and conditions under which these constraints help match certain empirical facts about asset prices. Under these conditions, borrowing and short-sale constraints decrease stock return volatilities, whereas limited stock market participation constraints amplify them. Moreover, borrowing constraints generate spikes in interest rates and volatilities and have stronger effects on asset prices than short-sale constraints.

Journal of Economic Literature Classification Numbers: D52, G12.
Keywords: heterogeneous investors, borrowing constraints, short-sale constraints, limited participation, volatility.

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1. Introduction

Portfolio constraints have long been considered among the key factors that affect investment decisions and asset prices. Moreover, the tightening of borrowing and short-sale constraints during the recent financial crisis sparked further interest in the effects of constraints on financial markets. The main objective of this paper is to provide a comparison of the effects of borrowing, short-sale, and limited stock market participation constraints on asset prices in a dynamic economy in which investors have different risk aversions and beliefs about the rate of economic growth. The paper shows that constraints and preferences interplay in complex ways, giving rise to spikes in interest rates and stock return volatilities. Furthermore, by relaxing the assumption of logarithmic constrained investors, popular in the literature, this paper demonstrates that investors' elasticities of intertemporal substitution (EIS) play a key role in matching certain empirical facts about asset prices and determining whether constraints increase or decrease asset prices and the volatilities of their returns.

The paper considers a pure-exchange Lucas (1978) economy with one tree populated by two investors with heterogeneous constant relative risk aversion (CRRA) preferences and heterogeneous beliefs about the mean growth rate of aggregate output. The investors trade in a riskless bond and a stock, representing a claim to the stream of dividends produced by the tree. One investor is unconstrained, whereas the other may face borrowing, short-sale, or limited stock market participation portfolio constraints. The paper studies the market prices of risk, interest rates, stock return volatilities and stock price-dividend ratios and derives them as functions of the constrained investor's share of the aggregate consumption, which acts as a state variable in the model.

The main results on the effects of borrowing constraints are as follows. First, under these constraints, interest rates and stock return volatilities are complex non-monotone functions of the state variable with spikes. The spikes are due to a kink in the constrained investor's portfolio weight at a time when constraints start to bind. Second, in economies where stock return volatilities are countercyclical (i.e., negatively correlated with the aggregate output) and exceed dividend volatilities, as in the data, constraints decrease volatilities and destroy their countercyclicality. The intuition is that constraints homogenize investors' portfolios, and hence, the volatility decreases toward that in a homogeneous-investor economy, in which stock and aggregate output volatilities coincide. Third, constraints lead to higher market prices of risk, compensating the unconstrained investor for holding more stocks to clear the market, and to lower interest rates because the constrained investor borrows less.

The effects of constraints on investors' wealth-consumption (W/C) and price-dividend (P/D) ratios depend on investors' EIS, which determine the relative strength of income and substitution effects.[1] When the substitution effect dominates, borrowing constraints tend to decrease W/C ratios because of the low opportunity cost of consumption. This is due to low interest rates and the fact that investors cannot take full advantage of investing in stocks because in the model the unconstrained investor is pessimistic, whereas the other investor is constrained. The opposite happens for the income effect. The paper shows that the P/D ratio is a weighted average of investors' W/C ratios and, hence, responds to constraints in a similar way. The paper formalizes the intuition by deriving closed-form approximations for W/C and P/D ratios, which capture the interaction between constraints and EIS. The intuition for other constraints is analyzed similarly.

Next, the paper contributes to the debate on the economic effects of short-sale bans. It shows that in economies where stock return volatilities are countercyclical and exceed dividend volatilities, short-sale bans decrease volatilities, in line with the anticipations of policymakers during the recent financial crisis [e.g., Beber and Pagano (2013)]. Moreover, short-sale bans preserve the countercyclicality of volatilities, decrease market prices of risk and increase interest rates. However, the effect of these constraints on asset prices and volatilities is small, in line with the empirical studies on short-sale bans during the 2007-2009 crisis [e.g., Beber and Pagano (2013); Boehmer, Jones, and Zhang (2013)].

[1]When the investment opportunities worsen, the substitution effect induces investors to consume more and save less because of the lower opportunity cost of consumption. The income effect induces them to do the opposite in order to have higher consumption in the future. For CRRA preferences with risk aversion γ, EIS $= 1/\gamma$. The substitution effect dominates for EIS > 1, and the income effect dominates for EIS < 1.

Finally, to isolate the pure effects of constraints, the paper considers an economy where both investors are identical except that one of them faces a limited participation constraint, that is, invests only a small fraction of wealth in stocks. This constraint is typical for pension funds, retail investors, and some mutual fund families. The main finding is that when the substitution effect dominates, the model generates countercyclical market prices of risk and stock return volatilities, procyclical interest rates and price-dividend ratios, excess volatility, and a negative correlation between risk premia and price-dividend ratios, consistent with the empirical literature. The effects on market prices of risk and interest rates are similar to those for borrowing constraints.

The paper offers a new tractable characterization of equilibrium for comparing the effects of different constraints. This characterization does not rely on a restrictive assumption of logarithmic constrained investors, as commonly employed in the literature. The derivation of equilibrium proceeds in two steps. First, all processes are derived in terms of the shadow costs of constraints from the first order conditions for consumption. Then, the shadow costs are found from the Kuhn-Tucker conditions of optimality. Finding the equilibrium reduces to solving a system of ordinary differential equations (ODEs) for investors' W/C ratios. In the unconstrained benchmark, the paper provides a new closed-form solution of the model.

There is a large body of literature on economies with constrained logarithmic investors. The results in the current paper on whether constraints increase or decrease market prices of risk and interest rates do not strongly depend on preferences and are consistent with this literature. Nevertheless, the results on the spiky non-monotone dynamics of interest rates and volatilities and the analysis of the cyclicality of P/D ratios and volatilities are new even for economies with logarithmic investors. Moreover, as shown in this paper, models with general preferences and models with logarithmic preferences may have opposite predictions regarding the effects of constraints on P/D ratios and stock return volatilities.

Below, we review the most closely related works. Detemple and Murthy (1997) and Basak and Croitoru (2000, 2006) study economies with various constraints where all investors are logarithmic. Coen-Pirani (2005) studies margin requirements in an economy with Epstein-Zin investors who have EIS $= 1$. In those papers, constraints do not affect stock prices because income and substitution effects cancel each other when investors have EIS $= 1$. Pavlova and Rigobon (2008) study a three-country economy with constrained logarithmic investors, but they use home bias as the source of investor heterogeneity.

Kogan, Makarov, and Uppal (2007) consider a model with one unconstrained CRRA investor and one logarithmic investor who cannot borrow. Their processes are deterministic, and there is no excess volatility. Gallmeyer and Hollifield (2008) study short-sale bans in a model with heterogeneous preferences and beliefs and a logarithmic constrained investor. They find that constraints increase (decrease) volatilities when the unconstrained investor has EIS > 1 (EIS < 1). In contrast to the above papers, this paper handles looser constraints and non-logarithmic preferences, and the volatility can go either way for any fixed EIS, depending on the constrained investor's consumption share.

Basak and Cuoco (1998) study market prices of risk and interest rates in an economy with limited participation and a logarithmic constrained investor. Hugonnier (2012) studies endogenous bubbles in the model of Basak and Cuoco (1998). Guvenen (2006, 2009) resolves several asset pricing anomalies in a production economy with limited participation, labor income, and investors with heterogeneous low EIS. The model in this paper generates qualitatively similar results when investors have homogeneous preferences and high EIS. The economy in Guvenen (2006, 2009) is more realistic than the extension of Basak and Cuoco (1998) in this paper. However, Guvenen (2009) notes that the mechanism in Basak and Cuoco (1998) is also important, although less significant quantitatively.

The current paper is related to the literature on margin constraints. Gârleanu and Pedersen (2011) focus on the mispricing of securities under margin constraints but do not study volatilities. Chabakauri (2013) employs the methodology of the current paper to study the effects of margin constraints on volatilities and correlations in an economy with two stocks and two heterogeneous CRRA investors but does not study the effects of short-sale bans, limited participation, and differences in beliefs. Because of complicated boundary conditions, the model with limited participation in the current paper is solved using techniques that are not covered in Chabakauri (2013). Prieto (2013) and Rytchkov (2014) consider related models but with time-varying margins and find similar effects.[2]

The paper also contributes to the literature on economies with heterogeneous unconstrained investors. Longstaff and Wang (2012) provide closed-form solutions for equilibrium when one investor is twice as risk averse as the other. This paper extends their solutions in terms of hypergeometric functions to general risk aversions and beliefs. Bhamra and Uppal (2014) provide solutions in terms of series expansions in a model with habit and demonstrate that the heterogeneity in preferences gives rise to excess volatility.

This paper is organized as follows. Section 2 presents the economic setup and defines the equilibrium. Section 3 solves for equilibrium and discusses its properties. Section 4 provides the analysis of equilibrium, and Section 5 concludes. Online Appendices A, B, C and D provide the proofs, details of the numerical method, evaluation of the quality of approximations, and additional results for the case of logarithmic investors, respectively.

2. Economic setup

Consider a continuous-time infinite-horizon Lucas (1978) economy with one consumption good and two investors A and B with heterogeneous CRRA preferences. The uncertainty is represented by a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$ on which a Brownian motion w is defined. The stochastic processes are adapted to filtration $\{\mathcal{F}_t, t \in [0, \infty)\}$ generated by w, where sigma-field $\mathcal{F}_t$ represents the time-t information.

The investors trade continuously in a riskless bond in zero net supply and a stock in net supply of one unit. The stock is a claim to an exogenous stream of dividends D_t, which follows a geometric Brownian motion (GBM):

$$dD_t = D_t\big[\mu_D dt + \sigma_D dw_t\big], \tag{1}$$

where mean growth rate μ_D and volatility σ_D are constants. Dividend D_t is equal to the aggregate output in the economy at date t, and its growth rate μ_D can be interpreted as the rate of economic growth.

Investors observe D_t but disagree about mean growth rate μ_D. They have their own probability spaces $(\Omega, \{\mathcal{F}_t^i\}, \mathbb{P}^i)$ with *subjective probability measures* $\mathbb{P}^i$, which are equivalent to the true measure $\mathbb{P}$. Under investor i's beliefs dividends D_t evolve as follows:

$$dD_t = D_t\big[\mu_D^i dt + \sigma_D dw_t^i\big], \quad i = A, B, \tag{2}$$

where w^i is a Brownian motion under investor i's measure $\mathbb{P}^i$. Because investors agree on dividend growth dD_t/D_t, Equations (2) imply $\mu_D^A dt + \sigma_D dw_t^A = \mu_D^B dt + \sigma_D dw_t^B$, and hence,

$$dw_t^B = dw_t^A - \Delta_D dt, \tag{3}$$

where $\Delta_D = (\mu_D^B - \mu_D^A)/\sigma_D$. The heterogeneity of beliefs is needed to make short-sale constraints binding, whereas the other constraints in the paper bind even with identical beliefs. For simplicity, investors do not update their beliefs [e.g., Yan (2008)].

The paper looks for equilibria in which bond B_t and stock S_t prices follow the processes:

$$\begin{aligned} dB_t &= B_t r_t dt, && (4)\\ dS_t + D_t dt &= S_t\big[\mu_t dt + \sigma_t dw_t\big], &&\\ &= S_t\big[\mu_t^i dt + \sigma_t dw_t^i\big], \quad i = A, B, && (5) \end{aligned}$$

where interest rate r_t, stock mean return μ_t and volatility σ_t are determined in equilibrium and are adapted to $\mathcal{F}_t$, and the bond price at time 0 is normalized to $B_0 = 1$. The investors agree on asset prices

[2]Other related works in discrete time include Kiyotaki and Moore (1997), Brunnermeier and Pedersen (2009), Gromb and Vayanos (2002, 2010), Brumm et al (2013), who study margin and collateral constraints, and Buss et al (2013), who consider a model with habit and several regulatory measures.

but disagree on stock mean return. Similarly to Basak (2000, 2005), from Equations (3) and (5) it can be easily shown that Equation (3) for Brownian motions w_t^B and w_t^A imposes the following *consistency condition*:

$$\frac{\mu_t^B - \mu_t^A}{\sigma_t} = \frac{\mu_D^B - \mu_D^A}{\sigma_D} \equiv \Delta_D, \tag{6}$$

where Δ_D is called a *disagreement process.*

2.1. Portfolio constraints and investors' optimization

The investors choose consumption c_{it} and an investment policy $\{\alpha_{it}, \theta_{it}\}$, where α_{it} and θ_{it} denote the fractions of wealth invested in bonds and stocks, respectively, and hence, $\alpha_{it} + \theta_{it} = 1$. Processes c_{it}, α_{it}, θ_{it} are adapted to time-t information. The investors maximize expected discounted utilities over consumption with discount $\rho > 0$

$$\max_{c_{it},\, \theta_{it}} \mathbb{E}^i\Big[\int_0^\infty e^{-\rho t}\frac{c_{it}^{1-\gamma_i}}{1-\gamma_i}dt\Big], \tag{7}$$

where $\mathbb{E}^i[\cdot]$ is expectation under measure $\mathbb{P}^i$, subject to a self-financing budget constraint

$$dW_{it} = \Big[W_{it}\Big(r_t + \theta_{it}(\mu_t^i - r_t)\Big) - c_{it}\Big]dt + W_{it}\theta_{it}\sigma_t dw_t^i, \quad i = A, B, \tag{8}$$

and subject to portfolio $\theta_i \in \Theta_i$ and wealth $W_{it} > 0$ constraints. For $\gamma_i = 1$, the utility in (7) is replaced with $\ln(c_{it})$. At $t = 0$, investor B is endowed with s units of stock and $-b$ units of bond, and investor A is endowed with $1 - s$ units of stock and b units of bond.

Investor A is unconstrained, that is, $\Theta_A = \mathbb{R}$, whereas investor B faces constraint

$$\theta_B \in \Theta_B = \{\theta : \underline{\theta} \le \theta \le \overline{\theta}\}. \tag{9}$$

The paper focuses on three special cases of constraint (9): 1) borrowing constraint $\theta_B \le \tilde{\theta}$, with $\tilde{\theta} \ge 1$; 2) short-sale constraint $\theta_B \ge \tilde{\theta}$, with $\tilde{\theta} \le 0$; and 3) limited participation constraint $\theta_B \le \tilde{\theta}$, with $0 \le \tilde{\theta} < 1$.

2.2. Equilibrium

Definition. *An equilibrium is a set of processes* $\{r_t, \mu_t^i, \sigma_t\}_{i\in\{A,B\}}$ *and of consumption and investment policies* $\{c_{it}^*, \alpha_{it}^*, \theta_{it}^*\}_{i\in\{A,B\}}$ *such that consumption and investment policies solve dynamic optimization problem* (7) *for each investor, given processes* $\{r_t, \mu_t^i, \sigma_t\}_{i\in\{A,B\}}$*, and consumption and securities markets clear, that is,*

$$c_{At}^* + c_{Bt}^* = D_t, \quad \theta_{At}^* W_{At}^* + \theta_{Bt}^* W_{Bt}^* = S_t, \quad \alpha_{At}^* W_{At}^* + \alpha_{Bt}^* W_{Bt}^* = 0, \tag{10}$$

where W_{At}^* *and* W_{Bt}^* *denote wealths under optimal strategies.*

Instead of stock mean return μ, the paper reports the *market price of risk* $\kappa = (\mu - r)/\sigma$, from which μ can be easily recovered. The paper also studies stock P/D and investors' W/C ratios $\Psi = S/D$ and $\Phi_i = W_i^*/c_i^*$, respectively. Throughout the paper, $i = A, B$ is used as a superscript for processes on which the investors disagree (e.g., μ_D^i, μ_t^i, κ_t^i, w_t^i) and as a subscript for processes on which they agree (e.g., θ_{it}, W_{it}, Φ_{it}, etc).

All processes are derived as functions of investor B's *share of aggregate consumption*, $y = c_B^*/D$, similarly to the related literature [e.g., Detemple and Murthy (1997); Gârleanu and Pedersen (2011)]. The paper studies Markovian equilibria in which state variable y follows a Markovian Itô process (under true probability measure $\mathbb{P}$):

$$dy_t = -y_t[\mu_{yt}dt + \sigma_{yt}dw_t], \tag{11}$$

where the drift μ_y and volatility σ_y are determined in equilibrium as functions of y.

3. Characterization of equilibrium

This section derives the equilibrium in economies with constraints in two steps. The first step solves for investors' optimal consumptions in a partial equilibrium setting in which investment opportunities are taken as given. The second step derives the equilibrium processes from the market clearing conditions in terms of shadow costs of constraints, which are found from Kuhn-Tucker conditions of optimality.

Following Cvitanić and Karatzas (1992), the constrained investor's optimization is solved in a fictitious unconstrained economy, in which prices B_t and S_t follow the dynamics:

$$dB_t = B_t\Big(r_t + \delta(\nu_t^*)\Big)dt, \tag{12}$$

$$dS_t + D_t dt = S_t\Big[\Big(\mu_t^{B} + \nu_t^* + \delta(\nu_t^*)\Big)dt + \sigma_t dw_t^{B}\Big], \tag{13}$$

where *adjustment* ν^* can be interpreted as the shadow cost of a constraint, and $\delta(\cdot)$ is the *support function* for the set of admissible portfolio weights Θ_B, defined as

$$\delta(\nu) = \sup_{\theta \in \Theta_B} (-\nu\theta). \tag{14}$$

Adjustments ν^* belong to the support function's *effective domain*, given by $\Upsilon = \{\nu \in \mathbb{R} : \delta(\nu) < \infty\}$, and can be obtained from complementary slackness condition $\nu^*\theta_B^* + \delta(\nu^*) = 0$ [e.g., Cvitanić and Karatzas (1992); Karatzas and Shreve (1998)]. Table 1 provides effective domains Υ and support functions $\delta(\cdot)$ for different constraints.

The advantage of the fictitious-economy approach is that investor B's optimization can be solved using the results from complete-market portfolio choice literature [e.g., Liu (2007)]. To understand the intuition, suppose that B faces constraint $\theta_{Bt} \leq \tilde{\theta}$, where $\tilde{\theta} > 1$, and hence holds less wealth in stocks than in the unconstrained economy. Investor B's portfolio choice can be replicated in an unconstrained economy with lower risk premia and higher interest rates by properly choosing adjustment ν^* so that $\delta(\nu^*) \geq 0$ and $\nu^* \leq 0$.

[Table 1 about here]

Suppose adjustment ν^* is given. Then, investor B's problem can be solved in the fictitious complete-market economy. The state price densities of investors in their respective unconstrained real and fictitious economies follow the dynamics below [e.g., Duffie (2001)]:

$$d\xi_{At} = -\xi_{At}[r_t dt + \kappa_t^{A} dw_t^{A}], \quad d\xi_{Bt} = -\xi_{Bt}[(r_t + \delta(\nu_t^*))dt + (\kappa_t^{B} + \nu_t^*/\sigma_t)dw_t^{B}], \tag{15}$$

where $\kappa^i = (\mu^i - r)/\sigma$. The investors' first-order conditions (FOC) equate their marginal utilities and state price densities and are given by:

$$e^{-\rho t}(c_{it}^*)^{-\gamma_i} = \psi_i \xi_{it}, \quad i = A, B. \tag{16}$$

Substituting optimal consumptions $c_{it}^* = (\psi_i e^{\rho t}\xi_{it})^{-1/\gamma_i}$ into consumption clearing condition $c_{At}^* + c_{Bt}^* = D_t$, applying Itô's lemma to both sides and then matching the dt and dw_t terms gives the equilibrium processes for κ and r in terms of investor B's consumption share y and adjustment ν^*. Lemma 1 reports the results.

Lemma 1 (Equilibrium processes in terms of shadow costs of constraints).
1) *If there exists a Markovian equilibrium described by the FOC in Equations* (16), *then market price of risk* $\kappa = (\mu - r)/\sigma$, *interest rate* r, *volatility* σ_y *and drift* μ_y *of consumption share* $y = c_B^*/D$, *and volatility* σ *of stock returns are functions of* y *and* ν^*, *given by:*

$$\kappa_t = \Gamma_t \sigma_D + \frac{\mu_D - \bar{\mu}_{Dt}}{\sigma_D} - \Gamma_t \frac{y_t}{\gamma_B}\frac{\nu_t^*}{\sigma_t}, \tag{17}$$

$$r_t = \rho + \Gamma_t \bar{\mu}_{Dt} - \frac{\Gamma_t \Pi_t}{2}\sigma_D^2 - \Gamma_t \frac{y_t}{\gamma_B}\delta(\nu_t^*) \tag{18}$$
$$+\Gamma_t^3 \frac{y_t(1-y_t)}{\gamma_B^2\gamma_A^2}\Big[\Big((\gamma_B-\gamma_A)\sigma_D - \Delta_D\Big)\Big(\Delta_D + \frac{\nu_t^*}{\sigma_t}\Big) + \frac{1}{2}\Big(1+\frac{\gamma_B\gamma_A}{\Gamma_t}\Big)\Big(\Delta_D^2 - \Big(\frac{\nu_t^*}{\sigma_t}\Big)^2\Big)\Big],$$
$$\sigma_{yt} = \Gamma_t \frac{1-y_t}{\gamma_B\gamma_A}\Big((\gamma_B-\gamma_A)\sigma_D - \Delta_D - \frac{\nu_t^*}{\sigma_t}\Big), \tag{19}$$
$$\mu_{yt} = \mu_D - \sigma_{yt}\sigma_D - \frac{r_t + \delta(\nu_t^*) - \rho}{\gamma_B} - \frac{\mu_D - \mu_D^B}{\sigma_D}(\sigma_D - \sigma_{yt}) - \frac{1+\gamma_B}{2}(\sigma_D - \sigma_{yt})^2, \tag{20}$$
$$\sigma_t = \sigma_D - y_t\sigma_{yt}\frac{\Psi'(y_t)}{\Psi(y)}, \tag{21}$$

where $\bar{\mu}_D = (\Gamma_t y_t/\gamma_B)\mu_D^B + (\Gamma_t(1-y_t)/\gamma_A)\mu_D^A$ is the average subjective growth rate, Γ_t and Π_t are the risk aversion and prudence parameters of a representative investor, given by:

$$\Gamma_t = \frac{\gamma_B\gamma_A}{\gamma_B(1-y_t)+\gamma_A y_t}, \qquad \Pi_t = \Gamma_t^2\Big(\frac{1+\gamma_B}{\gamma_B^2}y_t + \frac{1+\gamma_A}{\gamma_A^2}(1-y_t)\Big). \tag{22}$$

2) *The equilibrium processes for κ^i and μ_y^i under investors' subjective beliefs are given by:*

$$\kappa_t^B = \Gamma_t\Big(\sigma_D - \frac{y_t}{\gamma_B}\frac{\nu_t^*}{\sigma_t} + \frac{1-y_t}{\gamma_A}\Delta_D\Big), \quad \kappa_t^A = \kappa_t^B - \Delta_D, \tag{23}$$
$$\mu_{yt}^B = \mu_D^B - \sigma_{yt}\sigma_D - \frac{r_t + \delta(\nu_t^*) - \rho}{\gamma_B} - \frac{1+\gamma_B}{2}(\sigma_D - \sigma_{yt})^2, \ \mu_{yt}^A = \mu_{yt}^B - \Delta_D\sigma_{yt}, \tag{24}$$

and processes r_t, σ_y, and σ do not depend on subjective beliefs.

Table 1 gives the signs of ν^*. For example, for the borrowing constraint [case (c) in Table 1] $\nu^* \le 0$. Therefore, Equation (17) implies that κ increases under the constraint, provided that $\sigma > 0$, which can be verified after computing the equilibrium. Intuitively, constrained investor B holds less stocks than without constraints, and hence investor A should hold more stocks to clear the market. Consequently, κ increases to compensate investor A for excessive risk taking. The impact of constraints on r is ambiguous because it is a quadratic function of adjustment ν^*. On the one hand, under the borrowing constraint r should go down because investor B borrows less. On the other hand, it should go up because investor A now holds more stocks and, hence, is less willing to lend. Interestingly, in the unconstrained case [case (a) in Table 1] $\nu_t^* = 0$, and hence, Lemma 1 provides closed-form equilibrium processes, similar to those in Basak (2000, 2005).

Proposition 1 below completes the characterization of equilibrium. In particular, it provides equations for W/C and P/D ratios and for adjustment ν^*, which is derived from Kuhn-Tucker conditions of optimality. For simplicity of exposition, the proposition reports the results only for borrowing and short-sale constraints, and the case of limited stock market participation constraints is discussed in Section 4.3.

Proposition 1 (Equilibrium with borrowing and short-sale constraints).
1) *Suppose, there exists a Markovian equilibrium with wealth-consumption ratios $\Phi_i(y) \in C^1[0,1] \cap C^2(0,1)$. Then, the stock price-dividend ratio is given by $\Psi(y) = (1-y)\Phi_A(y) + y\Phi_B(y)$, and investor i's value function J_i and portfolio weight θ_i^* are given by:*

$$J_i(y_t, t) = e^{-\rho t}\frac{W_{it}^{1-\gamma_i}\Phi_i(y_t)^{\gamma_i}}{1-\gamma_i}, \tag{25}$$
$$\theta_i^*(y_t;\nu_t^*) = \frac{1}{\gamma_i\sigma_t}\Big(\kappa^i + 1_{\{i=B\}}\frac{\nu_t^*}{\sigma_t} - \gamma_i y_t\sigma_{yt}\frac{\Phi_i'(y_t)}{\Phi_i(y_t)}\Big), \tag{26}$$

where $i = A, B$, $1_{\{i=B\}}$ is an indicator function, and wealth-consumption ratios $\Phi_A(y)$ and $\Phi_B(y)$ satisfy *ordinary differential equations (ODEs)*

$$\begin{aligned}
\frac{y^2\sigma_y^2}{2}\Phi_A'' &- y\Big(\mu_y^A + \frac{1-\gamma_A}{\gamma_A}\kappa^A\sigma_y\Big)\Phi_A' + \Big(\frac{1-\gamma_A}{2\gamma_A}(\kappa^A)^2 + (1-\gamma_A)r - \rho\Big)\frac{\Phi_A}{\gamma_A} + 1 = 0, && (27)\\
\frac{y^2\sigma_y^2}{2}\Phi_B'' &- y\Big(\mu_y^B + \frac{1-\gamma_B}{\gamma_B}\Big(\kappa^B + \frac{\nu^*}{\sigma}\Big)\sigma_y\Big)\Phi_B' \\
&+ \Big(\frac{1-\gamma_B}{2\gamma_B}\Big(\kappa^B + \frac{\nu^*}{\sigma}\Big)^2 + (1-\gamma_B)(r + \delta(\nu^*)) - \rho\Big)\frac{\Phi_B}{\gamma_B} + 1 = 0, && (28)
\end{aligned}$$

where r, κ^i, σ_y, μ_y^i and σ are given in Lemma 1 and $\delta(\cdot)$ is given in Table 1. Boundary values $\Phi_i(0)$ and $\Phi_i(1)$ are positive and bounded, and are given by

$$\begin{aligned}
\Phi_A(0) &= \frac{\gamma_A}{\rho - (1-\gamma_A)\Big(\frac{\kappa^A(0)^2}{2\gamma_A} + r(0)\Big)}, \quad \Phi_B(0) = \frac{\gamma_B}{\rho - (1-\gamma_B)\Big(\frac{(\kappa^B(0)+\frac{\bar{\nu}^*}{\sigma_D})^2}{2\gamma_B} + r(0) - \tilde{\theta}\bar{\nu}^*\Big)}, \\
\Phi_A(1) &= \frac{\gamma_A}{\rho - (1-\gamma_A)\Big(\frac{\kappa^A(1)^2}{2\gamma_A} + r(1)\Big)}, \quad \Phi_B(1) = \frac{\gamma_B}{\rho - (1-\gamma_B)\Big(\frac{\kappa^B(1)^2}{2\gamma_B} + r(1)\Big)},
\end{aligned} \qquad (29)$$

where $\bar{\nu}^* = \min\Big(0, \tilde{\theta}\sigma_D^2(\gamma_B\tilde{\theta} - \gamma_A - \Delta_D/\sigma_D)\Big)/\tilde{\theta}$, $\kappa^i(0) = \gamma_A\sigma_D + 1_{\{i=B\}}\Delta_D$, $\kappa^i(1) = \gamma_B\sigma_D - 1_{\{i=A\}}\Delta_D$, $r(0) = \rho + \gamma_A\mu_D^A - 0.5\gamma_A(1+\gamma_A)\sigma_D^2$, $r(1) = \rho + \gamma_B\mu_D^B - 0.5\gamma_B(1+\gamma_B)\sigma_D^2$.

2) *In the above equilibrium, for borrowing constraint $\theta_B \le \tilde{\theta}$ with $\tilde{\theta} \ge 1$ and short-sale constraint $\theta_B \ge \tilde{\theta}$ with $\tilde{\theta} < 0$, adjustment ν^* satisfies Kuhn-Tucker optimality conditions*

$$\nu_t^*(\theta^*(y_t;\nu_t^*)/\tilde{\theta} - 1) = 0, \quad \nu_t^*\tilde{\theta} \le 0, \quad \theta_B^*(y_t;\nu_t^*)/\tilde{\theta} - 1 \le 0, \qquad (30)$$

and is given by the following expression:

$$\nu_t^* = \begin{cases} 0, & \text{if } \theta_B^*(y_t;\nu_t^*)/\tilde{\theta} < 1, \\ \dfrac{(\gamma_B - \gamma_A)\sigma_D^2 - \Delta_D\sigma_D}{\tilde{\theta} + g_1(y_t)(1-\tilde{\theta})} - \dfrac{(1-\tilde{\theta})\sigma_D^2 g_1(y_t)/g_2(y_t)}{(\tilde{\theta} + g_1(y_t)(1-\tilde{\theta}))^2}, & \text{if } \theta_B^*(y_t;\nu_t^*)/\tilde{\theta} = 1, \nu_t^*\tilde{\theta} < 0, \end{cases} \qquad (31)$$

where θ_B^* is given by Equation (26), $g_1(y) = \Big(1 + y\Phi_B'(y)/\Phi_B(y)\Big)/\Big(1 + y\Phi_B'(y)/\Phi_B(y) - y\Psi'(y)/\Psi(y)\Big)$, and $g_2(y) = \Big(\Gamma(1-y)/(\gamma_A\gamma_B)\Big)\Big(1 + y\Phi_B'(y)/\Phi_B(y)\Big)$.

3) *Suppose there exist bounded and positive functions $\Phi_i(y) \in C^1[0,1] \cap C^2(0,1)$ that satisfy ODEs (27)–(28) and boundary conditions (29), processes κ^i, ν^*/σ, σ, and θ_i^* are bounded and $|\sigma| > 0$. Then, there exists a Markovian equilibrium in which value functions and portfolio weights are given by equations (25) and (26), respectively, equilibrium processes κ^i, r, and σ are as in Lemma 1, and consumptions c_i^* satisfy first-order conditions (16).*

The expressions for value functions J_i, optimal portfolio weights θ_i^*, and ODEs for wealth-consumption ratios Φ_i are derived using dynamic programming in the same way as in the complete-market partial equilibrium economy in Liu (2007). The main contribution of Proposition 1 is Equation (31) for adjustment ν^*. This equation reveals that ν^* is a function of Φ_i, Ψ and their derivatives, and hence, Equations (27)–(28) comprise a system of quasilinear ODEs for the W/C ratios.[3] Equations (27)–(28) are solved by finite-difference methods, which are discussed in Appendix B, for parameters γ_i, ρ, μ_D^i and σ_D such that boundary values $\Phi_i(0)$ and $\Phi_i(1)$ are positive and finite.

[3]Remark A.1 in Appendix A provides conditions under which there can be multiple ν^*. However, these conditions do not hold in the calibrations in Section 4. The multiplicity of equilibria remains an open question. Furthermore, Lemma 1 and Proposition 1 remain valid for time-varying bounds $\tilde{\theta}_t = \tilde{\theta}(y_t, \sigma_t)$.

To provide the intuition for the effect of constraints on equilibrium, from Equation (26) for portfolio weights θ_A^* and θ_B^* and Equations (19) and (23) for σ_y and κ^i, after simple algebra, we obtain the following expression for consumption share volatility σ_y:

$$\sigma_{yt} = \frac{(\theta_{At}^* - \theta_{Bt}^*)\sigma_t}{\dfrac{1}{1-y_t} - y_t\dfrac{\Phi_A'(y_t)}{\Phi_A(y_t)} + y_t\dfrac{\Phi_B'(y_t)}{\Phi_B(y_t)}}. \tag{32}$$

Equation (32) shows that volatility σ_y quantifies portfolio heterogeneity and, therefore, it decreases with tighter constraints because the constraints homogenize portfolio weights θ_A^* and θ_B^*. Equation (21) for volatility σ demonstrates that smaller σ_y decreases the distance $|\sigma - \sigma_D|$. As a result, the constraints decrease (increase) volatility σ when volatility σ in the unconstrained model is higher (lower) than the dividend volatility σ_D.

In the case of no-borrowing constraint with $\tilde{\theta} = 1$, investors cannot lever up and invest all their wealth in stock. Therefore, their portfolios become homogeneous so that $\theta_A^* = \theta_B^* = 1$, and hence, Equation (32) implies that $\sigma_y = 0$. Then, Equation (21) for σ implies that $\sigma = \sigma_D$, as in Kogan, Makarov, and Uppal (2007). Moreover, Equation (31) for ν^* gives $\nu_t^* = (\gamma_B - \gamma_A)\sigma_D^2 - \Delta_D\sigma_D$, and hence, the equilibrium processes in Lemma 1 and Proposition 1 can be obtained in closed form, although they are not reported for brevity.

The unconstrained economy is a convenient benchmark for a model with constraints. Proposition 2 below provides closed-form expressions for equilibrium processes in the unconstrained economy for arbitrary risk aversions γ_A and γ_B. This proposition generalizes the results of Longstaff and Wang (2012) for an economy with $\gamma_A = 2\gamma_B$. Proposition 2 also provides the probability density function (PDF) of consumption share y in closed form.

Proposition 2 (Equilibrium in unconstrained economy).
1) *In the unconstrained economy, the equilibrium processes κ, r, σ_y, μ_y and σ are given by Equations* (17)–(21) *with $\nu^* = 0$, and the price-dividend ratio is given by:*

$$\begin{aligned}
\Psi_t \;&= \frac{1}{|a_2|\sqrt{2b}}\Big[-\frac{1}{\gamma_B + \varphi_-}\,{}_2F_1\Big(\Big(1-\frac{\gamma_A}{\gamma_B}\Big)\varphi_- - \gamma_A, 1, 1-\gamma_A - \frac{\gamma_A}{\gamma_B}\varphi_-; 1-y_t\Big) \\
&+ \Big(1-\frac{\gamma_A}{\gamma_B}\Big)\frac{1-y_t}{1-\gamma_A-\frac{\gamma_A}{\gamma_B}\varphi_-}\,{}_2F_1\Big(\Big(1-\frac{\gamma_A}{\gamma_B}\Big)\varphi_- + 1-\gamma_A, 1, 2-\gamma_A-\frac{\gamma_A}{\gamma_B}\varphi_-; 1-y_t\Big) \\
&+ \frac{\gamma_A}{\gamma_B}\frac{1}{\varphi_+}\,{}_2F_1\Big(\Big(1-\frac{\gamma_A}{\gamma_B}\Big)\varphi_+ - \gamma_A, 1, 1+\varphi_+; y_t\Big) \\
&+ \Big(1-\frac{\gamma_A}{\gamma_B}\Big)\frac{1-y_t}{\varphi_+}\,{}_2F_1\Big(\Big(1-\frac{\gamma_A}{\gamma_B}\Big)\varphi_+ + 1-\gamma_A, 1, 1+\varphi_+; y_t\Big)\Big],
\end{aligned} \tag{33}$$

where ${}_2F_1(x_1, x_2, x_3; y)$ is a hypergeometric function given by Equation (A.35) *in Appendix A, $\varphi_\pm = (a_1 \pm |a_2|\sqrt{2b})/a_2^2$, $a_1 = \Big[(\gamma_A - \gamma_B)\Big(\mu_D^A - 0.5\sigma_D^2\Big) - 0.5\Delta_D^2 - \Big((\gamma_B - \gamma_A)\sigma_D - \Delta_D\Big)(1-\gamma_A)\sigma_D\Big]/\gamma_B$, $a_2 = -\Big((\gamma_B - \gamma_A)\sigma_D - \Delta_D\Big)/\gamma_B$, $b = \rho - (1-\gamma_A)\mu_D^A + 0.5\gamma_A(1-\gamma_A)\sigma_D^2 + 0.5a_1^2/a_2^2$.*

2) *If $(\gamma_B - \gamma_A)\sigma_D - \Delta_D \neq 0$ the probability density function of consumption share y_τ at time $\tau > t$ conditional on consumption share y_t at time t is given by:*

$$p(y, \tau; y_t, t) = \frac{1}{\sqrt{2\chi_2^2(\tau - t)\pi}}\Big(\frac{\gamma_B}{y} + \frac{\gamma_A}{1-y}\Big)\exp\Big\{-\frac{\Big(\gamma_B \ln\frac{y}{y_t} - \gamma_A\ln\frac{1-y}{1-y_t} - \chi_1(\tau - t)\Big)^2}{2\chi_2^2(\tau - t)}\Big\}, \tag{34}$$

and $\chi_1 = (\gamma_A - \gamma_B)\Big(\mu_D - 0.5\sigma_D^2\Big) + 0.5\Big[\Big((\mu_D^A - \mu_D)/\sigma_D\Big)^2 - \Big((\mu_D^B - \mu_D)/\sigma_D\Big)^2\Big]$, $\chi_2 = (\gamma_B - \gamma_A)\sigma_D - \Delta_D$.

Moreover, the equilibrium is Markovian in D_t, and y_t is given by

$$y_t = 1 - f\left(\frac{y_0}{(1-y_0)^{\gamma_A/\gamma_B}}\left(\frac{D_t}{D_0}\right)^{-\chi_2/(\gamma_B\sigma_D)}\exp\left\{-\left(\frac{\Delta_D^2}{2\gamma_B}+\frac{\Delta_D}{\gamma_B}\frac{\mu_D^A-0.5\sigma_D^2}{\sigma_D}\right)t\right\}\right), \tag{35}$$

where function $f(z)$ satisfies equation $zf(z)^{\gamma_B/\gamma_A} + f(z) = 1$.

The distribution of consumption share (34) is *non-stationary.* Parameter χ_1 is a survival index introduced in Yan (2008), who derives the distribution of y by Monte Carlo simulations in a model with $\gamma_A = \gamma_B$. As $\tau \to \infty$, only investor B survives if $\chi_1 > 0$, only investor A survives if $\chi_1 < 0$, and each investor survives with probability 0.5 when $\chi_1 = 0$.

The non-stationarity of y gives rise to policy implications of constraints. Suppose $\gamma_B = \gamma_A$, investor A has correct beliefs and investor B is irrationally pessimistic. Consider a policymaker who knows the true value of μ_D and believes that investor B will perfectly learn μ_D from an unanticipated event at a random date T. Proposition 2 implies that $\chi_1 < 0$, and hence, with high probability, investor B arrives at date T with low consumption share y. Intuitively, pessimist B shorts stocks. However, shorting turns out to be sub-optimal ex post at date T. Imposing short-sale bans restricts trading on incorrect beliefs and slows down the extinction of B. Therefore, imposing constraints can be an important tool for improving welfare and for bringing asset prices closer to their fare values, provided that policymakers know economic conditions better than market participants.

4. Analysis of equilibrium

This section studies the equilibrium processes and shows that the effects of constraints depend on the elasticities of intertemporal substitutions of investors A and B, given by $1/\gamma_A$ and $1/\gamma_B$, respectively. The section also discusses which constraints help explain 1) the countercyclicality of market prices of risk, risk premia and volatilities, 2) the procyclicality of price-dividend ratios, and 3) excess stock return volatilities, observed in the data [e.g., Ferson and Harvey (1991); Shiller (1981); Campbell and Shiller (1988); Schwert (1989); Campbell and Cochrane (1999)]. For brevity, the economic intuition is provided only for borrowing constraints because for the other constraints it can be analyzed in a similar way. In the main analysis, process X is called *countercyclical* (*procyclical*) if $\mathrm{cov}_t(dX_t, dD_t) < 0$ ($\mathrm{cov}_t(dX_t, dD_t) > 0$) [e.g., Longstaff and Wang (2012)]. Throughout this Section, $\rho = 2\%$, $\mu_D = 1.8\%$ and $\sigma_D = 3.2\%$ [e.g., Basak and Cuoco (1998); Cambell (2003)].

4.1. Equilibrium with borrowing constraints

Consider an economy with borrowing constraint $\theta_B \le \tilde{\theta}$, where $\tilde{\theta} \ge 1$, in which investor A is pessimistic and believes that $\mu_D^A = 0.65\mu_D$, and investor B has correct belief $\mu_D^B = \mu_D$. Figure 1 shows equilibrium processes as functions of y for different upper bounds $\tilde{\theta}$. To disentangle the effects of EIS and heterogeneity in preferences, Panels (a.i)-(d.i), (a.ii)-(d.ii), (a.iii)-(d.iii) show the results for three cases: i) $\gamma_A = \gamma_B = 0.8$; ii) $\gamma_A = \gamma_B = 3$; and iii) $\gamma_A = 3$, $\gamma_B = 1.5$, respectively. The heterogeneity of beliefs is crucial for studying the role of EIS because without it the constraints do not bind in cases i and ii.

Investor B's consumption share y is procyclical because investor B is (weakly) less risk averse and more optimistic than investor A and, hence, holds more stocks in equilibrium. Therefore, positive (negative) dividend shocks shift consumption to investor B (investor A), and hence, $\mathrm{cov}_t(dy_t, dD_t) > 0$. Consequently, the periods with high (low) y_t correspond to *good times* (*bad times*). In equilibrium, investor B borrows from the pessimistic and risk averse investor A. Therefore, the constraints bind when investor B's share y is below a certain threshold $\bar{y}$, in which case share $1 - y$ of credit provider A is sufficiently high, so that investor B can easily lever up until the constraints become binding.[4]

Panels (a.i)-(a.iii) show the market price of risk κ in cases i-iii. In all cases, κ increases when the constraints bind, consistent with the intuition in Section 3. Moreover, κ is a decreasing function of procyclical variable y and, hence, is countercyclical. Equity premium $\mu - r$ is also countercyclical, although it is not reported for brevity. Intuitively, in bad times κ is high because investor A dominates in the economy (i.e., y is low) and requires higher compensation for risk to clear the market because A is pessimistic and weakly more risk averse than B. Similarly, in good times, κ is low because optimist B dominates.

Panels (b.i)-(b.iii) show that interest rate r decreases with tighter constraints because of lower demand for borrowing. Panel (b.iii) demonstrates that the heterogeneity in risk aversions gives rise to a non-monotone dependence of r on state variable y even without constraints. This is because increasing consumption share y of investor B has two opposite effects. On the one hand, r increases because investor B has a high expectation of economic growth and low precautionary savings. On the other hand, r decreases because B has higher EIS than investor A and, hence, is more willing to save for consumption smoothing.

Imposing borrowing constraints gives rise to a complex non-monotone pattern in r with a spike in case iii, which is due to a kink in investor B's portfolio strategy when the constraint starts to bind. Intuitively, as discussed above, r decreases in times with low y when the constraint binds. Then, r reverts upward to the unconstrained rate as y increases because the constraint stops binding; it then coincides with the downward sloping unconstrained rate. As a result, r increases before and decreases after the constraint stops binding, giving rise to a spike.

Another economic implication of the non-monotonicity of r is that it cannot be forecast by P/D ratios Ψ because Ψ is a monotone function of y. In contrast to r, there is a negative correlation between κ and Ψ in cases i and iii.[5] These findings are consistent with empirical results from a vector autoregressive model in Campbell and Ammer (1993). In particular, their results imply that P/D ratios do not forecast interest rates but forecast stock excess returns, and stock excess returns are negatively correlated with P/D ratios.

[**Figure 1 about here**]

Panels (c.i)-(c.iii) reveal the effect of EIS on P/D ratio Ψ. In particular, Ψ decreases with tighter constraints when EIS > 1 [Panel (c.i)], and vice versa, when EIS < 1 [Panels (c.ii) and (c.iii)]. The intuition is as follows. Tightening the access to credit has two opposite effects. On the one hand, the investment opportunities improve because κ increases. On the other hand, they become worse because r decreases. The second effect is stronger for both investors because A is pessimistic and weakly more risk averse and, as a result, invests small fraction of wealth in stocks, whereas B is constrained and, hence, cannot take full advantage of the high κ. Therefore, when the substitution effect dominates (i.e., $\gamma_i < 1$), the wealth-consumption ratios Φ_i tend to decrease because of low opportunity costs of consumption. The opposite happens when the income effect dominates (i.e., $\gamma_i > 1$). From Proposition 1, $\Psi = (1-y)\Phi_A + y\Phi_B$, and hence, Ψ behaves similarly to Φ_i.

The above intuition can be formalized by noting that because the coefficients in front of derivatives $\Phi_i'(y)$ and $\Phi_i''(y)$ in ODEs (27)–(28) are small, Φ_i are approximately given by [Appendix C evaluates the quality of these approximations]:

$$\Phi_A \approx \frac{\gamma_A}{\rho - (1-\gamma_A)\Big(\frac{(\kappa_t^A)^2}{2\gamma_A} + r_t\Big)}, \quad \Phi_B \approx \frac{\gamma_B}{\rho - (1-\gamma_B)\Big(\frac{(\kappa_t^B + \bar{\nu}_t^*/\sigma_t)^2}{2\gamma_B} + r_t - \tilde{\theta}\bar{\nu}_t^*\Big)}. \tag{36}$$

[4]Thresholds $\bar{y}$ in Figure 1 appear relatively high. For example, $\bar{y} \approx 0.6$ for a constraint with $\tilde{\theta} = 1.8$. However, investor B's share y tends to increase over time because investor A is irrationally pessimistic and loses wealth, as argued in Section 3. Therefore, a high $\bar{y}$ does not indicate that a constraint is tight at all times, and periods of loose and binding constraints frequently alternate when share y approaches $\bar{y}$.

[5]Because the model is non-stationary, predictability regressions for κ and Ψ are not considered here.

Equations (36) demonstrate that the effect of constraints on investment opportunities is quantified by $(\kappa_t^A)^2/(2\gamma_A) + r_t$ for investor A and by $(\kappa_t^B + \bar{\nu}_t^*/\sigma_D)^2/(2\gamma_B) + r_t - \tilde{\theta}\bar{\nu}_t^*$ for investor B. It can be verified that these quantities decrease with tighter constraints for wide ranges of consumption share y. Coefficients $1-\gamma_A$ and $1-\gamma_B$ in approximations (36) explain why the effects of constraints flip depending on whether $\gamma_i < 1$ or $\gamma_i > 1$.

The P/D ratio Ψ appears to be procyclical (i.e., an increasing function of y) over large intervals in the case of high EIS when $\gamma_A = \gamma_B = 0.8$ [Panel (c.i)] and is countercyclical in the case of low EIS when $\gamma_A = \gamma_B = 3$ [Panel (c.ii)]. Interestingly, Ψ is procyclical in the case of heterogeneous preferences, even for low EIS, when $\gamma_A = 3, \gamma_B = 1.5$ [Panel (c.iii)]. The intuition can be analyzed similarly as above by invoking Equations (36). The procyclicality (countercyclicality) of Ψ makes σ_t higher [Panels (d.i) and (d.iii)] (lower [Panel (d.ii)]) than the dividend volatility σ_D. This is because the stock price is given by $S_t = \Psi_t D_t$, and hence, the volatility goes up (down) when Ψ and D move in the same direction (opposite directions).

Panels (d.i)-(d.iii) show that $|\sigma_t - \sigma_D|$ decreases with tighter constraints and that σ_t reaches the maximum or minimum when the constraints start to bind. That is, σ_t decreases (increases) if in the unconstrained benchmark $\sigma_t > \sigma_D$ ($\sigma_t < \sigma_D$), consistent with the discussion in Section 3. Therefore, the unconstrained benchmark helps predict the direction of the effect of constraints without computing the equilibrium with constraints.

Our analysis generates three additional surprising results. First, the model with low risk aversions $\gamma_A = \gamma_B = 0.8$ (i.e., high homogeneous EIS) generates the dynamics of asset prices consistent with the data, as discussed above. Second, the latter model also generates high κ and low r despite low risk aversions. Third, the effects on Panels (a.i)-(d.i) for $\gamma_A = \gamma_B = 0.8$ and on Panels (a.iii)-(d.iii) for $\gamma_A = 3, \gamma_B = 1.5$ are qualitatively similar, and hence, cases i and iii are competing models for explaining the dynamics of asset prices. However, cases i and iii make opposite predictions on whether constraints increase or decrease Ψ, which can be used to disentangle them empirically.

Cases i and iii generate the following testable predictions about the tightening of constraints: 1) σ_t becomes less countercyclical; 2) σ_t and r_t have spikes around the time when the constraints start to bind; and 3) σ_t decreases. Hardouvelis and Perestiani (1992) and Hardouvelis and Theodossiou (2002) provide empirical evidence showing that tighter access to credit decreases σ_t, consistent with the last prediction.

Making investor B logarithmic has significant qualitative effects on equilibrium. For example, increasing γ_B in case i by a mere 0.2 to $\gamma_B = 1$ makes Ψ countercyclical and decreases σ_t below σ_D [see Panels (c.i)-(d.i), Figure 5 in Appendix D], in contrast to empirical findings. Similarly, making $\gamma_A = 3$ and $\gamma_B = 1$ reverses the results for Ψ and σ in case ii in Panels (c.ii)-(d.ii) of Figure 1 [see Panels (c.ii)-(d.ii), Figure 5 in Appendix D]. Furthermore, when both investors are logarithmic, constraints have no effects on Ψ and σ [see Panels (a.iii)-(d.iii), Figure 5 in Appendix D]. Finally, in case i of Figure 1, the effect of constraints on σ_t depends on consumption share y [Panel (d.i)]. In particular, holding the EIS fixed, constraints decrease σ_t when y is large and increase σ_t when y is small. This new effect disappears when B is logarithmic because then σ_t becomes uniformly higher (lower) than σ_D when $\gamma_A < 1$ ($\gamma_A > 1$).

4.2. Equilibrium with short-sale constraints

This section studies the short-sale constraint $\theta_B \geq \tilde{\theta}$, where $\tilde{\theta} \leq 0$. For this constraint to bind, investor B is now pessimistic and investor A has correct beliefs, so that $\mu_D^B = 0.65\mu_D$ and $\mu_D^A = \mu_D$. Figure 2 shows the equilibrium for three cases: i) $\gamma_A = \gamma_B = 0.8$ [Panels (a.i)-(d.i)]; ii) $\gamma_A = \gamma_B = 3$ [Panels (a.ii)-(d.ii)]; iii) $\gamma_A = 1.5$, $\gamma_B = 3$ [Panels (a.iii)-(d.iii)]. Variable y is countercyclical because B is pessimistic and weakly more risk averse than A.

[Figure 2 about here]

Panels (a.i)-(a.iii) show that κ decreases with tighter constraints because unconstrained investor A holds a smaller fraction of wealth in stocks and, as a result, requires smaller compensation for risk.

Panels (b.i)-(b.iii) show that r increases because investor A increases investment in bonds. Moreover, κ is countercyclical, whereas r has a rich non-monotone pattern and spikes downward.

Similarly to the borrowing constraints, the short-sale bans generate plausible dynamics of asset prices in cases i and iii. Furthermore, volatilities σ decrease for all y in case iii and for y from a large interval in case i. However, in contrast to the borrowing constraints, the model predicts that short-sale bans do not destroy the countercyclicality of σ.

The model further predicts that the effects of short-sale bans on volatilities [Panels (d.i)-(d.iii)] and P/D ratios [Panels (c.i)-(c.iii)] are small because these bans cannot suppress trading on incorrect beliefs completely, and hence, investors' portfolios remain substantially heterogeneous. In contrast to short-sale bans, borrowing constraints have stronger asset pricing effects because they significantly curb the bilateral trades. For example, borrowing constraints with $\tilde{\theta} = 1$ make portfolios perfectly homogeneous, as shown in Section 3. Empirical studies by Beber and Pagano (2013) and Boehmer, Jones, and Zhang (2013) demonstrate that short-sale bans during the 2007-2009 financial crisis did not have any significant effects on asset prices. Although there can be multiple explanations for their findings, the above effects might have been contributing factors.

Making investor B logarithmic confounds the effects of EIS and reverses some of the results for the general case. For example, in a model with $\gamma_B = 1$, the short-sale ban increases (decreases) σ_t when $\gamma_A > 1$ ($\gamma_A < 1$) [e.g., Gallmeyer and Hollifield (2008)]. However, in the general case, the effects critically depend on consumption share y. For example, Panel (d.i) shows that the constraint can either increase or decrease σ_t depending on the value of y when $\gamma_A = \gamma_B = 0.8$.[6] The non-monotone pattern in interest rate r is preserved when investor B is logarithmic and $\gamma_A < 1$.

4.3. Equilibrium with limited stock market participation

Finally, consider an economy with limited stock market participation constraint $\theta_B \leq \tilde{\theta}$, where $\tilde{\theta} \leq 1$. The latter constraint binds even when the investors have identical preferences and beliefs. Therefore, this constraint helps evaluate the pure effects of constraints, which are not confounded by investor heterogeneity. To focus on the effects of EIS, both investors are assumed to have identical beliefs. Figure 3 shows the results for the following cases: i) $\gamma_A = \gamma_B = 0.8$ [Panels (a.i)-(d.i)]; ii) $\gamma_A = \gamma_B = 3$ [Panels (a.ii)-(d.ii)]; and iii) $\gamma_A = 3$, $\gamma_B = 1.5$ [Panels (a.iii)-(d.iii)]. Consumption share y_t is now countercyclical because investor B holds less stocks than A. Therefore, negative (positive) dividend shocks shift relative consumption to investor B (investor A), and hence, $\text{cov}_t(dy_t, dD_t) < 0$.

In contrast to borrowing constraints, the limit $y \to 1$ does not correspond to a one-investor economy because B cannot clear the market. The market is cleared by investor A with very small consumption share $1 - y \approx 0$, and hence, κ and r have a singularity at $y = 1$. As a result, the boundary conditions for ODEs (27)–(28) are now different from those in Proposition 1 and are derived in Section B.2 in Appendix B.[7]

[Figure 3 about here]

Limited participation constraints increase κ [Panels (a.i)-(a.iii)] and decrease r [Panels (b.i)-(b.iii)], decrease Ψ and make it procyclical when EIS > 1 [Panel (c.i)] and vice versa when EIS < 1 [Panels (c.ii)-(c.iii)]. A new and surprising result is that volatility σ_t increases, is countercyclical, and $\sigma_t > \sigma_D$ when EIS > 1 [Panel (c.i)] even without investor heterogeneity. The opposite happens when EIS < 1 [Panels (d.ii)-(d.iii)]. The conclusion from Figure 3 is that the economy with $\gamma_A = \gamma_B = 0.8$ better matches the dynamics of asset prices. The intuition for the results is similar to the case of borrowing constraints.

[6] The range of y for which the constraint increases the volatility becomes significantly wider when $\gamma_A = 0.8$ and $\gamma_B = 0.7$. However, these results are not reported for brevity.

[7] The verification of optimality of θ_i^* and c_i^* in Proposition 1 requires all processes to be bounded, which is violated for κ and r. Therefore, the verification of optimality remains an open question.

Guvenen (2009, 2011) notes similarities between models with limited participation and the external habit model of Campbell and Cochrane (1999) (CC'99). In particular, using the consumption clearing $c^*_{At} + c^*_{Bt} = D_t$, the FOC of investor A in Equation (16) can be rewritten as $e^{-\rho t}(D_t - c^*_{Bt})^{-\gamma_A} = \psi_A \xi_t$. Therefore, c^*_{Bt} is analogous to an external habit, and D_t is analogous to the consumption of a representative agent. Moreover, surplus-consumption ratio $s = (D_t - c^*_{Bt})/D_t$ of CC'99 is given by $s = 1 - y$, and, as a result, is procyclical, as in CC'99. Similarly to CC'99, our model gives rise to a large countercyclical κ, although the procyclicality of Ψ requires high EIS in our model. An important difference from CC'99 is that their model is stationary and r is constant, in contrast to this paper.

5. Conclusion

The main conclusion of the paper is that constraints have significant economic effects. In particular, the paper presents new conditions under which constraints increase or decrease market prices of risk, interest rates, stock return volatilities and price-dividend ratios and make them countercyclical or procyclical. The paper finds that borrowing and short-sale constraints decrease the stock return volatility and generate rich non-monotone patterns in equilibrium processes. Finally, the paper demonstrates that the limited participation constraint generates excess volatilities and dynamics of equilibrium processes consistent with the data even when investors have identical preferences and beliefs.

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Case	Constraint	Υ	$\delta(\nu \mid \nu \in \Upsilon)$
(a)	$\theta \in \mathbb{R}$	0	0
(b)	$\theta = 0$	$\mathbb{R}$	0
(c)	$\theta \leq \tilde{\theta},\ \tilde{\theta} > 0$	$\nu \leq 0$	$-\nu\tilde{\theta}$
(d)	$\theta \geq \tilde{\theta},\ \tilde{\theta} < 0$	$\nu \geq 0$	$-\nu\tilde{\theta}$
(e)	$\underline{\theta} \leq \theta \leq \overline{\theta},\ \underline{\theta} \leq 0$	$\mathbb{R}$	$\max(-\nu, 0)\overline{\theta} - \max(\nu, 0)\underline{\theta}$

Table 1
Effective domains and support functions
This table shows effective domains Υ and support functions $\delta(\nu)$ for various portfolio constraints $\theta \in \Theta$, which are defined as $\Upsilon = \{\nu \in \mathbb{R} : \delta(\nu) < \infty\}$ and $\delta(\nu) = \sup_{\theta \in \Theta}(-\nu\theta)$, respectively.

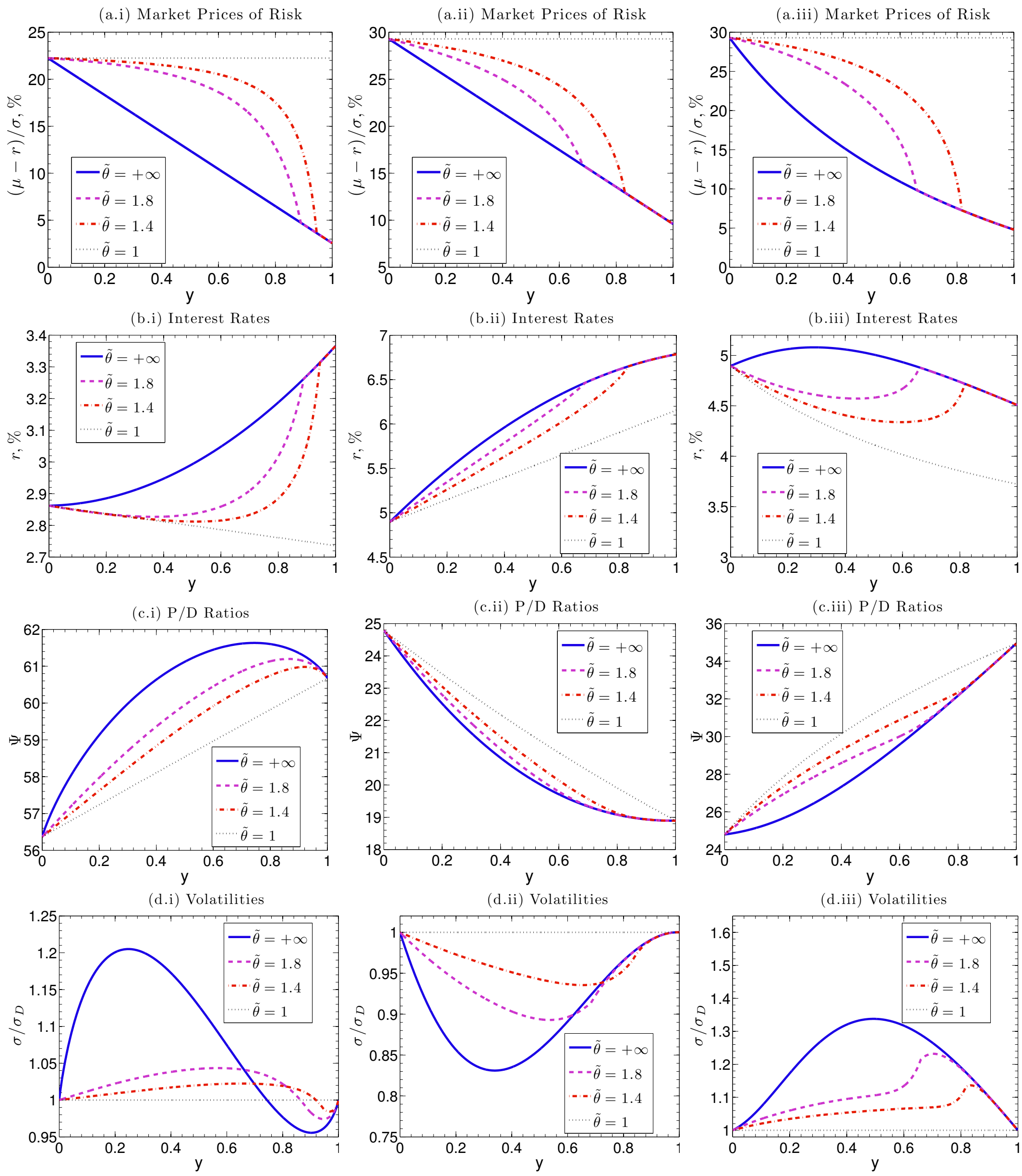

Figure 1
Equilibrium with borrowing constraint $\theta_{Bt} \leq \tilde{\theta}$, $\tilde{\theta} \geq 1$
Panels (a.i)–(d.i): $\gamma_A = \gamma_B = 0.8$ (high EIS); Panels (a.ii)–(d.ii): $\gamma_A = \gamma_B = 3$ (low EIS); Panels (a.iii)–(d.iii): $\gamma_A = 3, \gamma_B = 1.5$ (low EIS). Other parameters: $\mu_D^A = 0.65\mu_D$, $\mu_D^B = \mu_D$, $\mu_D = 1.8\%$, $\sigma_D = 3.2\%$, $\rho = 0.02$. Processes are functions of $y = c_B^*/D$, and $\text{cov}_t(dy_t, dD_t) > 0$.

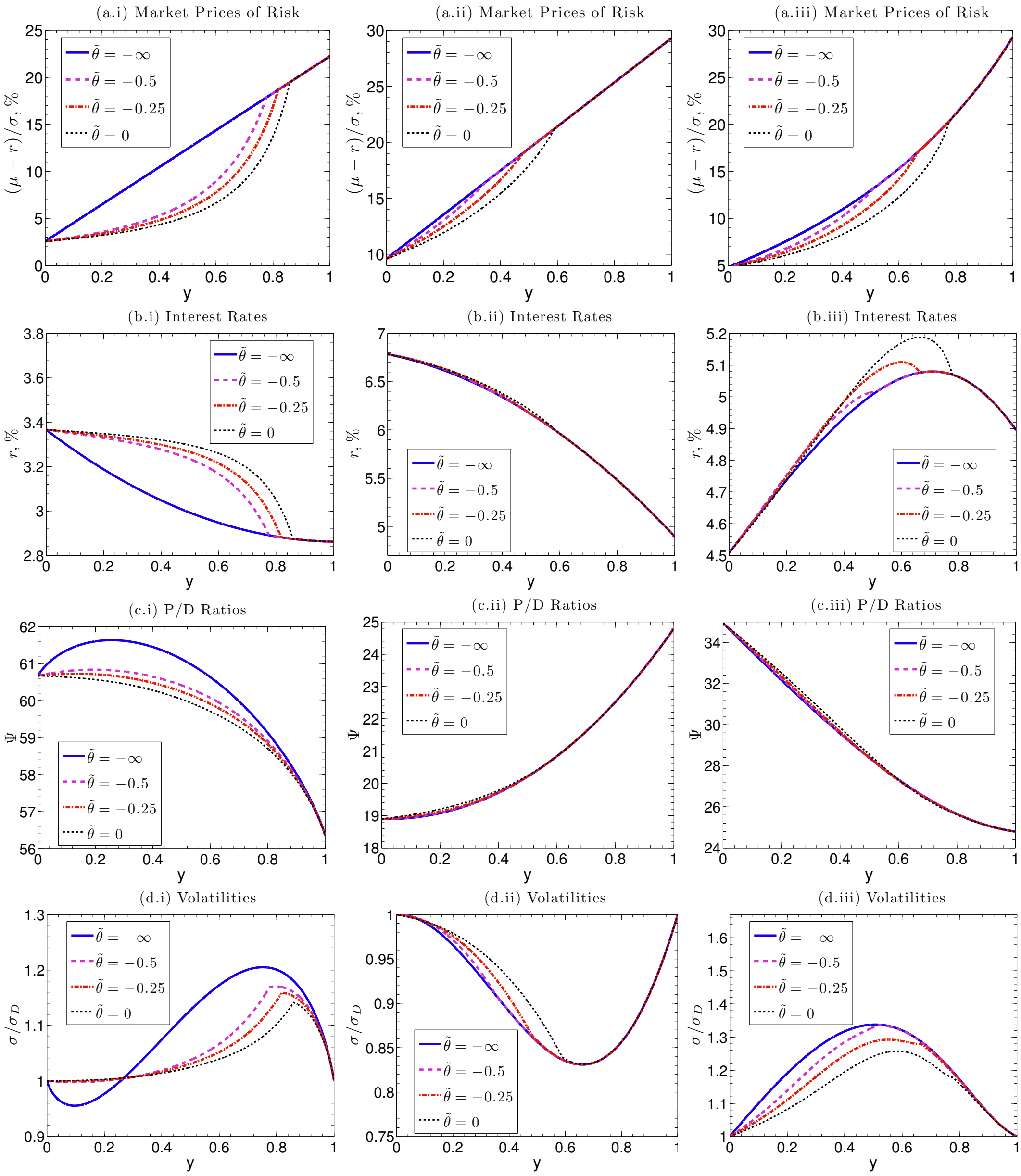

Figure 2
Equilibrium with short-sale constraint $\theta_{Bt} \geq \tilde{\theta}$, $\tilde{\theta} \leq 0$
Panels (a.i)–(d.i): $\gamma_A = \gamma_B = 0.8$ (high EIS); Panels (a.ii)–(d.ii): $\gamma_A = \gamma_B = 3$ (low EIS); Panels (a.iii)–(d.iii): $\gamma_A = 1.5, \gamma_B = 3$ (low EIS). Other parameters: $\mu_D^A = \mu_D$, $\mu_D^B = 0.65\mu_D$, $\mu_D = 1.8\%$, $\sigma_D = 3.2\%$, $\rho = 0.02$. Processes are functions of $y = c_B^*/D$, and $\text{cov}_t(dy_t, dD_t) < 0$.

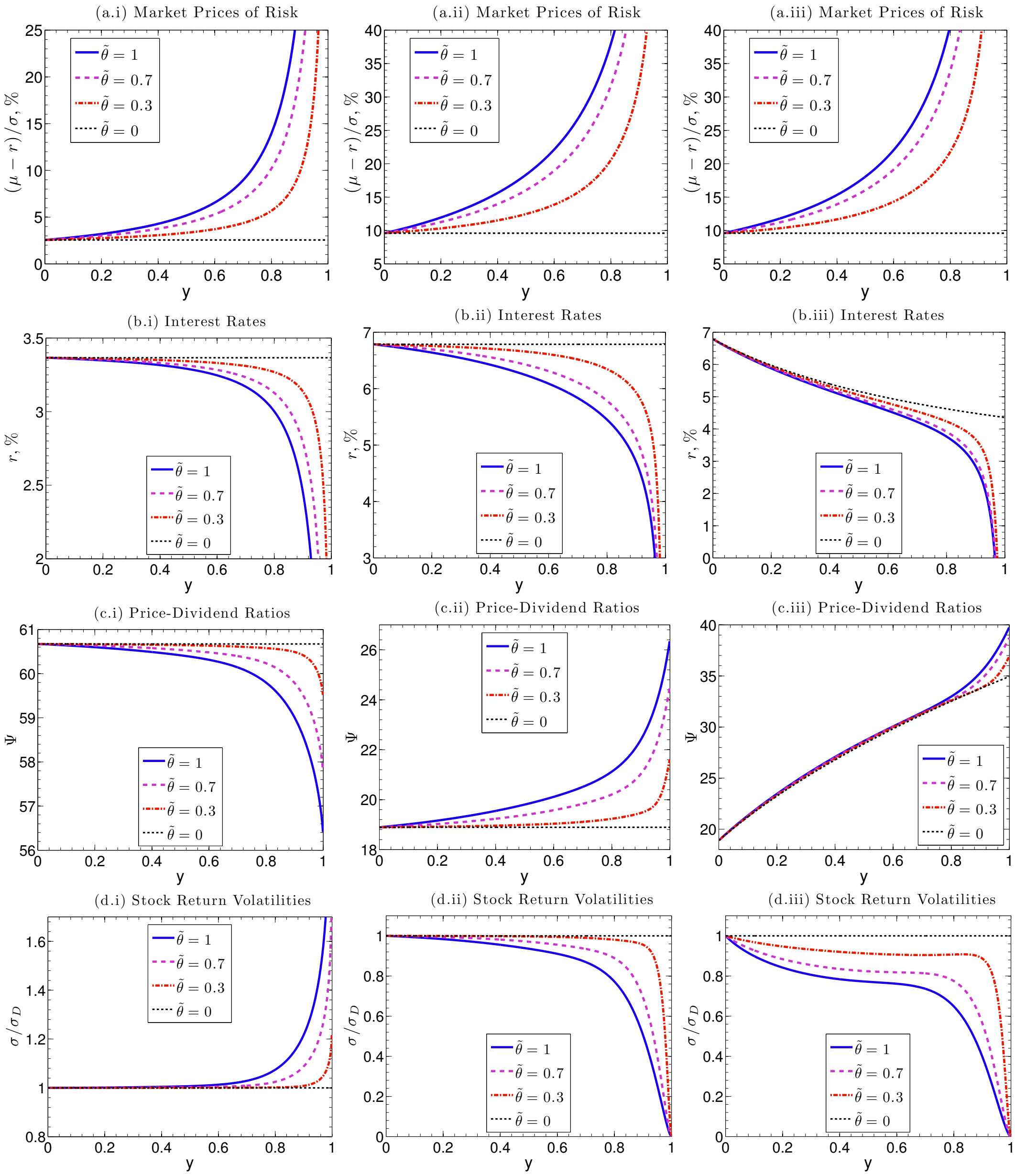

Figure 3
Equilibrium with limited stock market participation $\theta_{Bt} \leq \tilde{\theta}$, $\tilde{\theta} \leq 1$
Panels (a.i)–(d.i): $\gamma_A = \gamma_B = 0.8$ (high EIS); Panels (a.ii)–(d.ii): $\gamma_A = \gamma_B = 3$ (low EIS); Panels (a.iii)–(d.iii): $\gamma_A = 3, \gamma_B = 1.5$ (low EIS). Other parameters: $\mu_D^A = \mu_D^B = \mu_D = 1.8\%$, $\sigma_D = 3.2\%$, $\rho = 0.02$. Processes are functions of $y = c_B^*/D$, and $\text{cov}_t(dy_t, dD_t) < 0$.