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Asymptotic Glosten–Milgrom Equilibrium[*]

Cheng Li[†] and Hao Xing[†]

Abstract. This paper studies the Glosten–Milgrom model whose risky asset value admits an arbitrary discrete distribution. In contrast to existing results on insider models, the insider's optimal strategy in this model, if it exists, is not of feedback type. Therefore, a weak formulation of equilibrium is proposed. In this weak formulation, the inconspicuous trade theorem still holds, but the optimality for the insider's strategy is not enforced. However, the insider can employ some feedback strategy whose associated expected profit is close to the optimal value, when the order size is small. Moreover, this discrepancy converges to zero when the order size diminishes. The existence of such a weak equilibrium is established, in which the insider's strategy converges to the Kyle optimal strategy when the order size goes to zero.

Key words. Glosten–Milgrom model, Kyle model, nonexistence, occupation time, weak convergence

AMS subject classifications. 60G55, 60F05, 49N90

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1. Introduction. In the theory of market microstructure, two models, due to Kyle [16] and Glosten and Milgrom [13], are particularly influential. In the Kyle model, buy and sell orders are batched together by a market maker, who sets a unique price at each auction date. In the Glosten–Milgrom model, buy and sell orders are executed by the market maker individually, and hence bid and ask prices appear naturally. In both models,[1] an informed agent (insider) trades to maximize her expected profit utilizing her private information on the asset fundamental value, while another group of noise traders trade independently of the fundamental value. The cumulative demand of these noise traders is modeled by a Brownian motion in the Kyle model, cf. [2], and by the difference of two independent Poisson processes, whose jump size is scaled by the order size, in the Glosten–Milgrom model.

When the fundamental value, described by a random variable $\tilde{v}$, has an arbitrary continuous distribution,[2] Back [2] establishes a unique equilibrium between the insider and the market maker. Moreover, the cumulative demand process in the equilibrium connects elegantly to the theory of filtration enlargement; cf. [18]. However, much less is known about equilibrium in the Glosten–Milgrom model. Back and Baruch [3] consider a Bernoulli distributed $\tilde{v}$. In this case, the insider's optimal strategy is constructed in [9]. Equilibrium with general distribution of $\tilde{v}$, as Cho [11] puts it, "will be a great challenge to consider."

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[1]A profit maximizing informed agent is introduced in the Glosten–Milgrom model in [3].

[2]Models with discrete distributed $\tilde{v}$ can be studied similarly as in [2].

In this paper, we consider the Glosten–Milgrom model whose risky asset value $\tilde{v}$ has a discrete distribution,

$$(1.1) \qquad \mathbb{P}(\tilde{v} = v_n) = p_n, \quad n = 1, \ldots, N,$$

where $N \in \mathbb{N} \cup \{\infty\}$, $(v_n)_{n=1,\ldots,N}$ is an increasing sequence and $p_n \in (0,1)$ with $\sum_{n=1}^{N} p_n = 1$. This generalizes the setting in [3], where $N = 2$ is considered, i.e., $\tilde{v}$ has a Bernoulli distribution.

In models of insider trading, *inconspicuous trade theorem* is commonly observed; cf., e.g., [16], [2], [4], [3], [10], and [8] for equilibria of the Kyle type, and [9] for the Glosten–Milgrom equilibrium with Bernoulli distributed fundamental value. The inconspicuous trade theorem states that when the insider is trading optimally in equilibrium, the cumulative net orders from both insider and noise traders have the same distribution as the net orders from noise traders, i.e., the insider is able to hide her trades among noise trades. As a consequence, this allows the market maker to set the trading price only considering current cumulative noise trades. Moreover, in all aforementioned studies, the insider's optimal strategy is of *feedback form*, which only depends on the current cumulative total order. This functional form is associated to optimizers of the Hamilton–Jacobi–Bellman (HJB) equation for the insider's optimization problem. However, the situation is dramatically different in the Glosten–Milgrom model with N in (1.1) at least 3. Theorem 2.6 below shows that, given the aforementioned pricing mechanism, the insider's optimal strategy, if it exists, does *not* correspond to optimizers of the HJB equation. This result is a consequence of the difference between bid and ask prices in the Glosten–Milgrom model, which is in contrast to the unique price in the Kyle model.

Therefore, to establish equilibrium in these Glosten–Milgrom models, we propose a weak formulation of equilibrium in Definition 2.11, which is motivated by the convergence of Glosten–Milgrom equilibria to the Kyle equilibrium, as the order size diminishes and the trading intensities increase to infinity; cf. [3] and [9]. In this weak formulation, the insider still trades to enforce the inconspicuous trading theorem, but the insider's strategy may not be optimal. However, the insider can employ some feedback strategy so that the loss to her expected profit (compared to the optimal value) is small for a small order size. Moreover, this gap converges to zero when the order size vanishes. We call this weak formulation *asymptotic Glosten–Milgrom equilibrium* and establish its existence in Theorem 2.12.

In the asymptotic Glosten–Milgrom equilibrium, the insider's strategy is constructed explicitly in section 5, using a similar construction as in [9]. Using this strategy, the insider trades toward a middle level of an interval, driving the total demand process into this interval at the terminal date. This bridge behavior is widely observed in the aforementioned studies on insider trading. On the other hand, the insider's strategy is of feedback form. Hence, the insider can determine her trading intensity only using the current cumulative total demand. Moreover, as the order size diminishes, the family of suboptimal strategies converge to the optimal strategy in the Kyle model; cf. Theorem 2.13. In such an asymptotic Glosten–Milgrom equilibrium, the insider loses some expected profit. The expression of this profit loss is quite interesting mathematically: it is the difference of two stochastic integrals with respect to (scaled) Poisson occupation time. As the order size vanishes, both integrals converge to the same stochastic integral with respect to Brownian local time, and hence their difference vanishes.

The paper is organized as follows. The main results are presented in section 2. The mismatch between the insider's optimal strategy and optimizers for the HJB equation is proved in section 3. Then a family of suboptimal strategies are characterized and constructed in sections 4 and 5. Finally, the existence of asymptotic equilibrium is established in section 6, and a technical result is proved in the appendix.

2. Main results.

2.1. The model. We consider a continuous time market for a risky and a risk free asset. The risk free interest rate is normalized to 0, i.e., the risk free asset is regarded as the numéraire. We assume that the *fundamental value* of the risky asset $\tilde{v}$ has a discrete distribution of type (1.1). This fundamental value will be revealed to all market participants at a finite time horizon, say 1, at which point the market will terminate.

The microstructure of the market and the interaction of market participants are modeled similarly to [3], which we recall below. There are three types of agents: uninformed/noise traders, an informed trader/insider, and a market maker, all of whom are risk neutral. These agents share the same view toward future randomness of the market, but they possess different information. Therefore, the probability space $(\Omega, \mathbb{P})$ with different filtration accommodates the following processes:

- *Noise traders* trade for liquidity or hedging reasons which are independent of the fundamental value $\tilde{v}$. The cumulative demand Z is described by the difference of two independent jump processes Z^B and Z^S which are the cumulative buy and sell orders, respectively. Therefore, $Z = Z^B - Z^S$ and it is independent of $\tilde{v}$. Noise traders only submit orders of fixed sized δ every time they trade. As in [3], Z^B/δ and Z^S/δ are assumed to be independent Poisson processes with constant intensity β. Let $(\mathcal{F}_t^Z)_{t \in [0,1]}$ be the smallest filtration generated by Z and satisfying the usual conditions. Then $(\mathcal{F}_t^Z)_{t \in [0,1]}$ describes the information structure of noise traders.
- The *insider* knows the fundamental value $\tilde{v}$ at time 0 and observes the market price for the risky asset between time 0 and 1. The insider also submits orders of fixed size δ in every trade and tries to maximize her expected profit. The cumulative demand from the insider is denoted by $X := X^B - X^S$, where X^B and X^S are cumulative buy and sell orders, respectively. Since the insider observes the market price of the risky asset, she can back out the dynamics of noise orders; cf. discussions after Definition 2.1. Therefore, the information structure of the insider $\mathcal{F}_t^I$ includes $\mathcal{F}_t^Z$ and $\sigma(\tilde{v})$ for any $t \in [0,1]$.
- A competitive *market maker* only observes the aggregation of the informed and noise trades, so he cannot distinguish between informed and noise trades. Given $Y := X + Z$, the information of the market maker is $(\mathcal{F}_t^Y)_{t \in [0,1]}$ generated by Y and satisfies the usual conditions. As the market maker is risk neutral, the competition will force him to set the market price as $\mathbb{E}[\tilde{v}|\mathcal{F}_t^Y]$, $t \in [0,1]$.

In order to define equilibrium in the market, let us first describe admissible actions for the market maker and the insider. The market maker looks for a Markovian pricing mechanism, in which the price of the risky asset at time t is set using cumulative order Y_t and a pricing rule p.

Definition 2.1. *A function* $p : \delta\mathbb{Z} \times [0,1] \to \mathbb{R}$ *is a* pricing rule *if*

(i) $y \mapsto p(y,t)$ *is strictly increasing for each* $t \in [0,1)$*;*

(ii) $\lim_{y \to -\infty} p(y,t) = v_1$ *and* $\lim_{y \to \infty} p(y,t) = v_N$ *for each* $t \in [0,1]$*;*

(iii) $t \mapsto p(y,t)$ *is continuous for each* $y \in \delta\mathbb{Z}$*.*

The monotonicity of $y \mapsto p(y,t)$ in (i) is natural. It implies that the market price is higher whenever the demand is higher. Moreover, because of the monotonicity, the insider fully observes the uninformed orders Z by inverting the price process and subtracting her orders from the total orders. Item (ii) means that the range of the pricing rule is wide enough to price in every possibility of fundamental value.

The insider trades to maximize her expected profit. Her admissible strategy is defined as follows.

Definition 2.2. *The strategy* $(X^B, X^S; \mathcal{F}^I)$ *is* admissible *if*

(i) $\mathcal{F}^I$ *is a filtration satisfying the usual conditions and generated by* $\sigma(\tilde{v})$*,* $\mathcal{F}^Z$*, and* $\mathcal{H}$*, where* $(\mathcal{H}_t)_{t \in [0,1]}$ *is a filtration independent of* $\tilde{v}$ *and* $\mathcal{F}^Z$*;*

(ii) X^B *and* X^S *with* $X_0^B = X_0^S = 0$ *are* $\mathcal{F}^I$*-adapted and integrable*[3] *increasing point processes with jump size* δ*;*

(iii) *the* $(\mathcal{F}^I, \mathbb{P})$*-dual predictable projections of* X^B *and* X^S *are absolutely continuous with respect to time, and hence* X^B *and* X^S *admit* $\mathcal{F}^I$*-intensities* θ^B *and* θ^S*, respectively;*

(iv) $\mathbb{E}[\int_0^1 |p(Y_t,t)| \, |dX_t^i - \delta\theta_t^i dt|] < \infty$ *for* $i \in \{B, S\}$ *and the pricing rule* p *fixed by the market maker. Here,* $|X^i - \int_0^\cdot \delta\theta^i dt|$ *is the variation of the compensated point process.*

This set of admissible strategies is similar to [9, Definition 2.2]. Item (i) assumes that the insider is allowed to possess additional information $\mathcal{H}$, independent of $\tilde{v}$ and $\mathcal{F}^Z$, which she uses to generate her mixed strategy. Item (iv) implies $\delta\mathbb{E}[\int_0^1 |p(Y_t,t)| \, \theta_t^i dt] < \infty$, and hence the expected profit of the insider is finite. Item (ii) does not exclude the insider trading at the same time with noise traders. When the insider submits an order at the same time when an uniformed order arrives but in the opposite direction, assuming the market maker only observes the net demand implies that such pair of trades goes unnoticed by the market maker. This pair of opposite orders will be executed without a need for a market maker. Hence, the market maker only knows the transaction when there is a need for him. Henceforth, when the insider makes a trade at the same time with an uninformed trader but in an opposite direction, we say the insider *cancels* the noise trades. On the other hand, item (ii) also allows the insider to trade at the same time with noise traders in the same direction. We say that the insider *tops up* noise orders in this situation. However, the insider does not submit such orders in equilibrium, even when equilibrium is defined in a weak sense; cf. Remark 4.6 below. The assumption that the insider is allowed to trade at the same time as noise traders is different from assumptions for the Kyle model where the insider's strategy is predictable. This additional freedom for the insider is not the source for Theorem 2.6 below, which states that optimizers for the insider's HJB equation do not correspond to the optimal strategy; see Remark 2.8 below.

As described in the last paragraph, the insider's cumulative buy orders may consist of three components: $X^{B,B}$ arrives at different time than those of Z^B, $X^{B,T}$ arrives at the same time as some orders of Z^B, and $X^{B,S}$ cancels some orders of Z^S. Sell orders X^S are defined analogously. Therefore, $X^B = X^{B,B} + X^{B,T} + X^{B,S}$ and $X^S = X^{S,S} + X^{S,T} + X^{S,B}$.

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[3]That is, $\mathbb{E}[X_1^B]$ and $\mathbb{E}[X_1^S]$ are both finite.

As mentioned earlier, the insider aims to maximize her expected profit. Given an admissible trading strategy $X = X^B - X^S$, the associated profit at time 1 of the insider is given by

$$\int_0^1 X_{t-} dp(Y_t, t) + (\tilde{v} - p(Y_1, 1))X_1.$$

The last term appears due to a potential discrepancy between the market price and the liquidation value. Since X is of finite variation and $X_0 = 0$, applying integration by parts rewrites the profit as

$$\int_0^1 (\tilde{v} - p(Y_t, t)) \, dX_t^B - \int_0^1 (\tilde{v} - p(Y_t, t)) \, dX_t^S$$

$$= \int_0^1 (\tilde{v} - p(Y_{t-} + \delta, t)) \, dX_t^{B,B} + \int_0^1 (\tilde{v} - p(Y_{t-} + 2\delta, t)) \, dX^{B,T}$$

$$+ \int_0^1 (\tilde{v} - p(Y_{t-}, t)) \, dX_t^{B,S} - \int_0^1 (\tilde{v} - p(Y_{t-} - \delta, t)) \, dX_t^{S,S}$$

$$- \int_0^1 (\tilde{v} - p(Y_{t-} - 2\delta, t)) \, dX^{S,T} - \int_0^1 (\tilde{v} - p(Y_{t-}, t)) \, dX_t^{S,B},$$

where Y increases (resp., decreases) δ when $X^{B,B}$ (resp., $X^{S,S}$) jumps by δ, Y increases (resp., decreases) 2δ when $X^{B,T}$ (resp., $X^{S,T}$) jumps at the same time with Z^B (resp., Z^S), and Y is unchanged when $X^{S,B}$ (resp., $X^{B,S}$) jumps at the same time with Z^B (resp., Z^S). Define

$$a(y,t) := p(y + \delta, t) \qquad \text{and} \qquad b(y,t) := p(y - \delta, t),$$

which can be viewed as ask and bid prices, respectively. Then the expected profit of the insider conditional on her information can be expressed as

(2.1)

$$\mathbb{E}\left[\int_0^1 (\tilde{v} - a(Y_{t-}, t)) \, dX_t^{B,B} + \int_0^1 (\tilde{v} - p(Y_{t-}, t)) \, dX_t^{B,S} \right.$$

$$+ \int_0^1 (\tilde{v} - a(Y_{t-} + \delta, t)) \, dX_t^{B,T} - \int_0^1 (\tilde{v} - b(Y_{t-} - \delta, t)) \, dX_t^{S,T}$$

$$\left. - \int_0^1 (\tilde{v} - b(Y_{t-}, t)) \, dX_t^{S,S} - \int_0^1 (\tilde{v} - p(Y_{t-}, t)) \, dX_t^{S,B} \, \middle| \, \tilde{v} \right].$$

Having described the market structure, an equilibrium between the market maker and the insider is defined as in [3].

Definition 2.3. A Glosten–Milgrom equilibrium *is a quadruplet* $(p, X^B, X^S, \mathcal{F}^I)$ *such that*

(i) *given* $(X^B, X^S; \mathcal{F}^I)$, *p is a rational pricing rule, i.e.,* $p(Y_t, t) = \mathbb{E}[\tilde{v}|\mathcal{F}_t^Y]$ *for* $t \in [0, 1]$;

(ii) *given p,* $(X^B, X^S; \mathcal{F}^I)$ *is an admissible strategy maximizing* (2.1).

When $N = 2$, [9] establishes the existence of Glosten–Milgrom equilibria. In equilibrium the pricing rule is

(2.2)
$$p(y,t) = \mathbb{E}^{\mathbb{P}^y}[P(Z_{1-t})], \quad (y,t) \in \delta\mathbb{Z} \times [0,1].$$

Here, $\mathbb{P}^y$ is a probability measure under which Z is the difference of two independent Poisson processes and $\mathbb{P}^y(Z_0 = y) = 1$. P is a nondecreasing function such that $P(Z_1)$ has the same distribution as $\tilde{v}$. Moreover, the optimal strategy of the insider are given by jump processes $X^{i,j}$, $i \in \{B, S\}$ and $j \in \{B, T, S\}$, with intensities $\delta\,\theta^{i,j}(Y_{t-}, t)$, $t \in [0, 1]$. These intensities are deterministic functions of the state variable Y, and hence this control strategy is a *feedback control* and it corresponds to optimizers of the insider's HJB equation. However, when $N \geq 3$, Theorem 2.6 below shows that, given the pricing rule (2.2), the optimal strategy *does not* correspond to optimizers in the HJB equation for some values of $\tilde{v}$. This result is surprising because it is contrast to existing results in the Kyle and Glosten–Milgrom equilibria; cf. [16], [2], [4], [3], [10], [8], and [9]. This mismatch is a consequence of the discrete state space of the demand process in the Glosten–Milgrom model. The discrete state space yields different bid and ask prices, which is in contrast to the unique price in the Kyle model. See Remark 2.7 below for more discussion.

2.2. Nonexistence of a feedback optimal control. To state the aforementioned result, we introduce additional notation. For each $\delta > 0$, let $\Omega^\delta = \mathbb{D}([0, 1], \delta\mathbb{Z})$ be the space of $\delta\mathbb{Z}$-valued càdlàg functions on $[0, 1]$ with coordinate process Z^δ; $(\mathcal{F}_t^{Z,\delta})_{t \in [0,1]}$ is the minimal right continuous and complete filtration generated by Z^δ, and $\mathbb{P}^\delta$ is the probability measure under which Z^δ is the difference of two independent Poisson processes starting from 0 with the same jump size δ and intensity β^δ. We denote by $\mathbb{P}^{\delta,y}$ the probability measure under which $Z_0^\delta = y$ a.s. Henceforth, the superscript δ indicates the trading size in the Glosten–Milgrom model.

For the fundamental value $\tilde{v}^\delta$, let us first consider the following family of distributions.

Assumption 2.4. Given $\tilde{v}^\delta$ of type (1.1), there exists a $\delta\mathbb{Z} \cup \{-\infty, \infty\}$-valued strictly increasing sequence $(a_n^\delta)_{n=1,\ldots,N+1}$[4] with $a_1^\delta = -\infty$, $a_{N+1}^\delta = \infty$, and $\bigcup_{n=1}^{N}[a_n^\delta, a_{n+1}^\delta) = \delta\mathbb{Z} \cup \{-\infty\}$, such that

$$\mathbb{P}(\tilde{v}^\delta = v_n) = \mathbb{P}^\delta\left(Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta)\right), \qquad n = 1, \ldots, N. \tag{2.3}$$

For any $\tilde{v}$ with discrete distribution (1.1), Lemma 6.1 below shows that there exists a sequence $(\tilde{v}^\delta)_{\delta>0}$, each satisfying Assumption 2.4 and converging to $\tilde{v}$ in law as $\delta \downarrow 0$. Therefore, any $\tilde{v}$ of type (1.1) can be approximated by a $\tilde{v}^\delta$ satisfying Assumption 2.4. Given $\tilde{v}^\delta$ satisfying Assumption 2.4, define

$$h_n^\delta(y, t) := \mathbb{P}^{\delta,y}\left(Z_{1-t}^\delta \in [a_n^\delta, a_{n+1}^\delta)\right), \quad y \in \delta\mathbb{Z}, \quad t \in [0, 1], \quad n \in \{1, \ldots, N\}, \tag{2.4}$$

and

$$p^\delta(y, t) := \sum_{n=1}^{N} v_n h_n^\delta(y, t) = \mathbb{E}^{\delta,y}\left[P(Z_{1-t}^\delta)\right], \tag{2.5}$$

where the expectation is taken under $\mathbb{P}^{\delta,y}$ and

$$P(y) = v_n \quad \text{when } y \in [a_n^\delta, a_{n+1}^\delta). \tag{2.6}$$

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[4]When $N = \infty$, $N + 1 = \infty$.

Then (2.3) implies that $\tilde{v}^\delta$ and $P(Z_1^\delta)$ have the same distribution. If p^δ is chosen as the pricing rule, it has the same form as in (2.2). Finally, we impose a technical condition on p^δ. This assumption is clearly satisfied when N is finite.

Assumption 2.5. There exist positive constants C and n such that $|p^\delta(y,t)| \leq C(1+|y|^n)$ for any $(y,t) \in \delta\mathbb{Z} \times [0,1]$.

Given the pricing rule (2.5), let us first study the insider's optimization problem and derive the associated HJB equation via a heuristic argument. In this derivation, the superscript δ is omitted to simplify notation. Definition 2.2(iii) implies that $X^{i,j} - \delta \int_0^\cdot \theta_r^{i,j} dr$ defines an $\mathcal{F}^I$-martingale for $i \in \{B,S\}$ and $j \in \{B,T,S\}$. On the other hand, Definition 2.2(iv) and [7, Chapter I, T6] combined imply that $\int_0^\cdot (\tilde{v} - p(Y_{r-} + \delta, r))(dX_r^{B,B} - \delta\theta_r^{B,B} dr) = \int_0^\cdot (\tilde{v} - p(Y_r, r))(dX_r^{B,B} - \delta\theta_r^{B,B} dr)$ is an $\mathcal{F}^I$-martingale. A similar argument applied to other terms allows us to rewrite (2.1) as

$$\delta\mathbb{E}\bigg[\int_0^1 (\tilde{v} - p(Y_{r-} + \delta, r))\theta_r^{B,B}\, dr + \int_0^1 (\tilde{v} - p(Y_{r-}, r))\theta_r^{B,S}\, dr$$
$$+ \int_0^1 (\tilde{v} - p(Y_{r-} + 2\delta, r))\theta_r^{B,T}\, dr - \int_0^1 (\tilde{v} - p(Y_{r-} - \delta, r))\theta_r^{S,S}\, dr$$
$$- \int_0^1 (\tilde{v} - p(Y_{r-}, r))\theta_r^{S,B}\, dr - \int_0^1 (\tilde{v} - p(Y_{r-} - 2\delta, r))\theta_r^{S,T}\, dr\,\Big|\,\tilde{v}\bigg].$$

This motivates us to define the following value function for the insider:

(2.7)
$$V^\delta(\tilde{v}, y, t) := \sup_{\theta^{i,j}; i \in \{B,S\}, j \in \{B,T,S\}}$$
$$\delta\,\mathbb{E}\bigg[\int_t^1 (\tilde{v} - p(Y_{r-} + \delta, r))\theta_r^{B,B}\, dr + \int_t^1 (\tilde{v} - p(Y_{r-}, r))\theta_r^{B,S}\, dr$$
$$+ \int_t^1 (\tilde{v} - p(Y_{r-} + 2\delta, r))\theta_r^{B,T}\, dr - \int_t^1 (\tilde{v} - p(Y_{r-} - \delta, r))\theta_r^{S,S}\, dr$$
$$- \int_t^1 (\tilde{v} - p(Y_{r-}, r))\theta_r^{S,B}\, dr - \int_t^1 (\tilde{v} - p(Y_{r-} - 2\delta, r))\theta_r^{S,T}\, dr\,\Big|\, Y_t = y, \tilde{v}\bigg]$$

for $\tilde{v} = \{v_1, \ldots, v_N\}$, $y \in \delta\mathbb{Z}$, $t \in [0,1)$. The terminal value of V^δ is defined as $V^\delta(\tilde{v}, y, 1) = \lim_{t \to 1} V^\delta(\tilde{v}, y, t)$.[5] Lemma 3.2 and Proposition 4.4 below show that the optimization problem in (2.7) is well defined and nontrivial, i.e., $0 < V^\delta < \infty$ for each $\delta > 0$. Let us now derive the HJB equation which V^δ satisfies via a heuristic argument. Note that the positive (resp., negative) part of Y is $Y^B := X^{B,B} + X^{B,T} + Z^B - X^{S,B}$ (resp., $Y^S := X^{S,S} + X^{S,T} + Z^S - X^{B,S}$). Hence, $Y^B - \delta \int_0^\cdot (\beta - \theta_r^{S,B} - \theta_r^{B,T})\, dr - \delta \int_0^\cdot \theta_r^{B,B}\, dr - 2\delta \int_0^\cdot \theta_r^{B,T}\, dr$ (resp., $Y^S - \delta \int_0^\cdot (\beta - \theta_r^{B,S} - \theta_r^{S,T})\, dr - \delta \int_0^\cdot \theta_r^{S,S}\, dr - 2\delta \int_0^t \theta_r^{S,T}\, dr$) is an $\mathcal{F}^I$-martingale.[6] Then applying Itô's formula to

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[5]Since the set of admissible control is unbounded, the HJB equation associated to (2.7) usually admits a *boundary layer*, i.e., $\lim_{t \to 1} V^\delta(\tilde{v}, y, t)$ is not identically zero even if there is no terminal profit in (2.1). Such a phenomenon also shows up in the Kyle model; see [2].

[6]As discussed after Definition 2.2, the set of jumps of $X^{B,S}$ and $X^{S,T}$ (resp., $X^{S,B}$ and $X^{B,T}$) arrive at the same time as some jumps of Z^S (resp., Z^B), and then we necessarily have $\theta^{B,S} + \theta^{S,T} \leq \beta$ (resp., $\theta^{S,B} + \theta^{B,T} \leq \beta$).

$V^\delta(\tilde{v}, Y_r, r)$ and employing the standard dynamic programming arguments yield the following formal HJB equation for V^δ:

$$(2.8) \qquad -V_t(v_n, y, t) - H(v_n, y, t, V) = 0, \quad n \in \{1, \dots, N\}, \quad (y, t) \in \delta\mathbb{Z} \times [0, 1),$$

where the Hamilton H is defined as (the $\tilde{v}$ argument is omitted in H to simplify notation)

$$
\begin{aligned}
(2.9) \quad H(v_n, y, t, V) := {} & (V(y + \delta, t) - 2V(y, t) + V(y - \delta, t))\beta \\
& + \sup_{\theta^{B,B} \geq 0} \left[V(y + \delta, t) - V(y, t) + (v_n - p(y + \delta, t))\delta \right] \theta^{B,B} \\
& + \sup_{\theta^{B,T} \geq 0} \left[V(y + 2\delta, t) - V(y + \delta, t) + (v_n - p(y + 2\delta, t))\delta \right] \theta^{B,T} \\
& + \sup_{\theta^{B,S} \geq 0} \left[V(y, t) - V(y - \delta, t) + (v_n - p(y, t))\delta \right] \theta^{B,S} \\
& + \sup_{\theta^{S,S} \geq 0} \left[V(y - \delta, t) - V(y, t) - (v_n - p(y - \delta, t))\delta \right] \theta^{S,S} \\
& + \sup_{\theta^{S,T} \geq 0} \left[V(y - 2\delta, t) - V(y - \delta, t) - (v_n - p(y - 2\delta, t))\delta \right] \theta^{S,T} \\
& + \sup_{\theta^{S,B} \geq 0} \left[V(y, t) - V(y + \delta, t) - (v_n - p(y, t))\delta \right] \theta^{S,B}.
\end{aligned}
$$

Optimizers $\theta^{i,j}$, $i \in \{B, S\}$ and $j \in \{B, T, S\}$, in (2.9), are deterministic functions of v_n, y, and t; hence they are of feedback form. They are expected to be the optimal control intensities for (2.7). This is indeed the case in many existing results in the Kyle model and the Glosten–Milgrom model (with $N = 2$); compare [16], [2], [4], [3], and [9]. However, when $N \geq 3$ in the Glosten–Milgrom model, the following theorem shows any optimizers in (2.9) are *not* the optimal intensities when $\tilde{v}$ is neither v_1 nor v_N.

Theorem 2.6. *Let $N \geq 3$ and $\tilde{v}^\delta$ satisfy Assumption 2.4. Let p^δ in (2.5) be the pricing rule and satisfy Assumption 2.5. Then any optimizers $\theta^{i,j}(y, t)$, $i \in \{B, S\}, j \in \{B, T, S\}$, and $(y, t) \in \delta\mathbb{Z} \times [0, 1)$, for (2.9) are not the optimal strategy for (2.7) when $\tilde{v}^\delta = v_n$ for $1 < n < N$.*

Remark 2.7. When $\tilde{v}^\delta = v_1$ (resp., v_N), the insider knows the risky asset is always overpriced (resp., underpriced). Hence she always sells (resp., buys) in equilibrium. This situation is exactly the same as [9]. For when $\tilde{v}^\delta$ is neither minimal nor maximal, let us briefly describe the proof of Theorem 2.6 here. To ensure (2.8) is well posed, H must be finite for all $(y, t) \in \delta\mathbb{Z} \times [0, 1)$. Hence

$$(2.10) \quad (p(y, t) - v_n)\delta \leq V(y + \delta, t) - V(y, t) \leq (p(y + \delta, t) - v_n)\delta \quad \text{for all } (y, t) \in \delta\mathbb{Z} \times [0, 1),$$

where the second inequality comes from the first three maximizations in (2.9) and the first inequality comes from the last three. Since the optimal value V is positive, then $\theta^{i,j} \equiv 0$, $i \in \{B, S\}$, and $j \in \{B, T, S\}$ in (2.9) does not correspond to the optimal strategy, hence there must exist (y_0, t_0) such that one inequality in (2.10), say, the first one, is an equality. However, in this case, the discrete state space forces the first inequality to be an equality for *all* $(y, t) \in \delta\mathbb{Z} \times [0, 1)$, which implies the second inequality in (2.10) is strict for all (y, t), due to $p(y + \delta, t) > p(y, t)$. Therefore the optimizers in the first three maximizations in (2.9) must

be identically zero, which means the associated point process X does not have positive jumps. On the other hand, the dynamic programming principle and the boundary layer of (2.8) at $t = 1$ force $Y_1 = Z_1 + X_1 \in [a_n^\delta + \delta, a_{n+1}^\delta]$ a.s. This can never happen when X does not have positive jumps. Therefore, Theorem 2.6 is the joint effort of the discrete state space and the boundary layer of the HJB equation.

Remark 2.8. The statement of Theorem 2.6 remains valid when the insider is prohibited from trading with noise traders at the same time; i.e., $X^{B,T}, X^{B,S}, X^{S,T}, X^{S,B}$ are all zero. In this case, the second, third, fifth, and sixth maximizations do not present in (2.9). However, the first and fourth maximizations therein still lead to (2.10). Hence the same argument as in the previous remark still applies.

Remark 2.9. Examples of control problems without optimal feedback control exist in the literature of the optimal control theory; cf., e.g., [21, Chapter 3, p. 246] and [17, Example 1.1]. In these cases, the notion of *relaxed control* is employed to prove the existence of a relaxed optimal control; cf. [17] and references therein. For the insider's optimization problem, instead of $\{\theta : \delta\mathbb{Z} \times [0,1] \to \mathbb{R}_+\}$, the control set can be relaxed to $\{\theta : \delta\mathbb{Z} \times [0,1] \to \mathcal{M}^1(\mathbb{R}_+)\}$, where $\mathcal{M}^1(\mathbb{R}_+)$ is the set of all probability measures in $\mathbb{R}_+$. It is interesting to investigate whether (2.7) admits an optimal control in this relaxed set. We leave this topic to future studies.

2.3. Asymptotic Glosten–Milgrom equilibrium. To establish equilibrium of Glosten–Milgrom type when the risky asset $\tilde{v}$ has general discrete distribution (1.1) with $N \geq 3$, we introduce a weak form of equilibrium in what follows. To motivate this definition, we recall the convergence of Glosten–Milgrom equilibria as the order size decreasing to zero and intensity of noise trades increasing to infinity; cf. [3, Theorem 3] and [9, Theorem 5.3].

Proposition 2.10. *For any Bernoulli distributed $\tilde{v}$ (i.e., $N = 2$ in (1.1)), there exists a sequence of Bernoulli distributed random variables $\tilde{v}^\delta$ such that*

(i) *$\tilde{v}^\delta$ converges to $\tilde{v}$ in law as $\delta \downarrow 0$;*

(ii) *for each $\delta > 0$, model with $\tilde{v}^\delta$ as the fundamental value of the risky asset admits a Glosten–Milgrom equilibrium $(p^\delta, X^{B,\delta}, X^{S,\delta}, \mathcal{F}^{I,\delta})$;*

(iii) *when the intensity of Poisson process is given by $\beta^\delta := (2\delta^2)^{-1}$, $X^{B,\delta} - X^{S,\delta} \xrightarrow{\mathcal{L}} X^0$, as $\delta \downarrow 0$, where X^0 is the optimal strategy in the Kyle model and $\xrightarrow{\mathcal{L}}$ represents the weak convergence of stochastic processes.*[7]

This result motivates us to define the following weak form of Glosten–Milgrom equilibrium.

Definition 2.11. *For any $\tilde{v}$ with discrete distribution (1.1), an asymptotic Glosten–Milgrom equilibrium is a sequence $(\tilde{v}^\delta, p^\delta, X^{B,\delta}, X^{S,\delta}, \mathcal{F}^{I,\delta})_{\delta>0}$ such that*

(i) *$\tilde{v}^\delta$ converges to $\tilde{v}$ in law as $\delta \downarrow 0$;*

(ii) *for each $\delta > 0$, given $(\tilde{v}^\delta, X^{B,\delta}, X^{S,\delta}, \mathcal{F}^{I,\delta})$ and setting $Y^\delta := Z^\delta + X^{B,\delta} - X^{S,\delta}$, p^δ is a rational pricing rule, i.e., $p^\delta(Y_t^\delta, t) = \mathbb{E}[\tilde{v}^\delta \,|\, \mathcal{F}_t^{Y^\delta}]$ for $t \in [0,1]$;*

(iii) *given $(\tilde{v}^\delta, p^\delta)$ and $\beta^\delta = (2\delta^2)^{-1}$, let $\mathcal{J}^\delta(X^B, X^S)$ be an insider's expected profit associated to the admissible strategy (X^B, X^S). Then*

$$\sup_{(X^B, X^S) \text{ admissible}} \mathcal{J}^\delta(X^B, X^S) - \mathcal{J}^\delta(X^{B,\delta}, X^{S,\delta}) \to 0 \quad \text{as } \delta \downarrow 0.$$

---

[7]Refer to [5] or [14] for the definition of weak convergence of stochastic processes.

In the above definition, rationality of the pricing mechanism is not compromised. However, optimality of the insider's strategy is not enforced. Instead, item (iii) requires that when the order size is small, the loss of the insider's expected profit by employing the strategy $(X^{B,\delta}, X^{S,\delta}; \mathcal{F}^{\delta,I})$ is small, compared to the optimal value. Moreover, this discrepancy converges to zero when the order size vanishes. Therefore if the insider is willing to give up a small amount of expected profit, she can employ the strategy $(X^{B,\delta}, X^{S,\delta}; \mathcal{F}^{I,\delta})$ to establish a suboptimal equilibrium. The following result establishes the existence of equilibrium in the above weak sense.

Theorem 2.12. *Assume that $\tilde{v}$ satisfies* (1.1) *with $N < \infty$. Then the asymptotic Glosten–Milgrom equilibrium exists.*

In this asymptotic equilibrium, the pricing rule is given by (2.5). When the order size is δ, the insider employs the strategy $(X^{B,\delta}, X^{S,\delta}; \mathcal{F}^{I,\delta})$ with $\mathcal{F}^{I,\delta}$-intensities

$$
(2.11) \quad
\begin{aligned}
&\delta\beta^\delta \sum_{n=1}^{N} \mathbb{I}_{\{\tilde{v}^\delta = v_n\}} \left[\frac{h_n^\delta(Y_{t-}^\delta + \delta, t)}{h_n^\delta(Y_{t-}^\delta, t)} - 1 \right]_+ + \delta\beta^\delta \sum_{n=1}^{N} \mathbb{I}_{\{\tilde{v}^\delta = v_n\}} \left[\frac{h_n^\delta(Y_{t-}^\delta - \delta, t)}{h_n^\delta(Y_{t-}^\delta, t)} - 1 \right]_-, \\
&\delta\beta^\delta \sum_{n=1}^{N} \mathbb{I}_{\{\tilde{v}^\delta = v_n\}} \left[\frac{h_n^\delta(Y_{t-}^\delta - \delta, t)}{h_n^\delta(Y_{t-}^\delta, t)} - 1 \right]_+ + \delta\beta^\delta \sum_{n=1}^{N} \mathbb{I}_{\{\tilde{v}^\delta = v_n\}} \left[\frac{h_n^\delta(Y_{t-}^\delta + \delta, t)}{h_n^\delta(Y_{t-}^\delta, t)} - 1 \right]_-,
\end{aligned}
$$

respectively. In particular, when the fundamental value is v_n, the insider trades toward the middle level $m_n^\delta := (a_n^\delta + a_{n+1}^\delta - \delta)/2$ of the interval $[a_n^\delta, a_{n+1}^\delta)$: when the total demand is less than m_n^δ, the insider only places buy orders by either complementing noise buy orders or canceling some of noise sell orders; when the total demand is larger than m_n^δ, the insider does exactly the opposite. More specifically, Lemma 5.2 below shows that $y \mapsto h_n^\delta(y, t)$ is strictly increasing when $y < m_n^\delta$ and strictly decreasing when $y > m_n^\delta$. Therefore, when $Y_{t-}^\delta < m_n^\delta$, (2.11) implies that $X^{B,B,\delta}$ has intensity $\frac{1}{2\delta}(\frac{h_n^\delta(Y_{t-}^\delta + \delta, t)}{h_n^\delta(Y_{t-}^\delta, t)} - 1)$, $X^{B,S,\delta}$ has intensity $\frac{1}{2\delta}(1 - \frac{h_n^\delta(Y_{t-}^\delta - \delta, t)}{h_n^\delta(Y_{t-}^\delta, t)})$; meanwhile intensities of $X^{S,S,\delta}$ and $X^{S,B,\delta}$ are both zero. When $Y_{r-}^\delta > m_n^\delta$, intensities can be read out from (2.11) similarly. Even though Theorem 2.6 remains valid when the insider is prohibited from trading at the same time with noise traders, the strategy constructed above depends on the possibility of canceling orders. However, in this strategy, the insider never tops up noise orders, i.e., $X^{B,T} = X^{S,T} \equiv 0$. This allows the market maker to employ a rational pricing mechanism so that Definition 2.11(ii) is satisfied; cf. Remark 4.6 below.

The processes $(X^{B,\delta}, X^{S,\delta}; \mathcal{F}^{I,\delta})$ with intensities (2.11) will be constructed explicitly in section 5. The insider employs a sequence of independent random variables with uniform distribution on $[0, 1]$ to construct her mixed strategy. This sequence of random variables is also independent of Z^δ and $\tilde{v}^\delta$. This construction is a natural extension of [9]. In this construction, whenever a noise order arrives, the insider uses a uniform distributed random variable to decide whether to submit an opposite canceling order. Hence this strategy is adapted to insider's filtration, rather than predictable as in the Kyle model. Such a canceling strategy is called *input regulation* and has been studied extensively in the queueing theory literature; see, e.g., [7, Chapter VII, section 3].
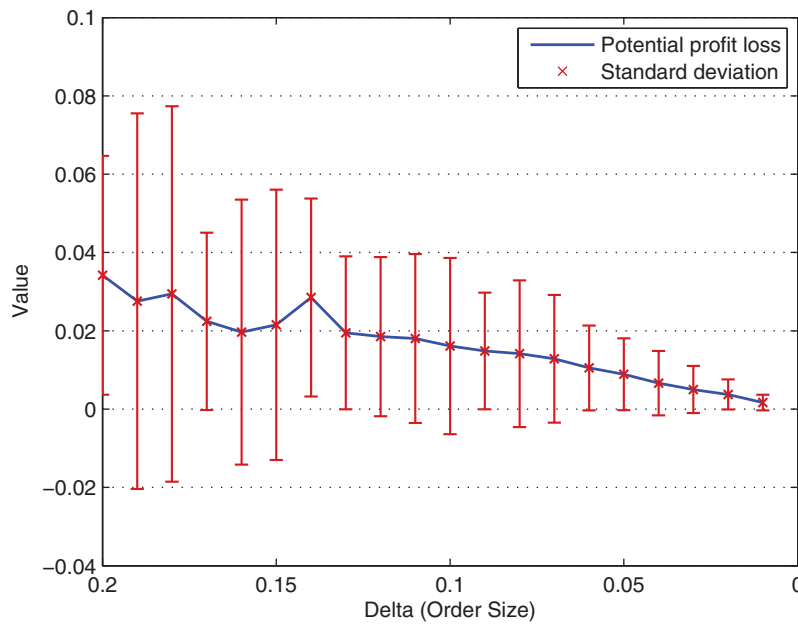
Figure 1. *The mean and standard deviation of the upper bound for the insider's expected profit loss. The figure is generated by Monte Carlo simulation with 10^5 paths.*

When the fundamental value is v_n and the insider follows the aforementioned strategy, the total demand at time 1 will end up in the interval $[a_n^\delta, a_{n+1}^\delta)$. Therefore the insider's private information is fully, albeit gradually, revealed to the public so that the trading price does not jump when the fundamental value is announced. On the other hand, the total demand, in its own filtration, has the same distribution of the demand from noise traders, i.e., the insider is able to hide her trades among the noise trades. This is another manifestation of the *inconspicuous trading theorem* commonly observed in the insider trading literature (cf., e.g., [16], [2], [4]).

The insider's strategy discussed above is of feedback form. The insider can determine her trades only using the current total cumulative demand (and some additional randomness coming from the sequence of independent and identically distributed (iid) uniform distributed random variables which are also independent of the fundamental value and the noise trades). Even though this strategy is not optimal, its associated expected profit is close to the optimal value when the order size is small. Moreover the discrepancy converges to zero as the order size diminishes.

The following numeric example illustrates the convergence of the upper bound for the insider's expected profit loss as the order size decreases to zero. In this example, $\tilde{v}$ takes values in $\{1, 2, 3\}$ with probability 0.55, 0.35, and 0.1, respectively. The expected profit in the Kyle–Back equilibrium is 0.512. Compared to this, Figure 1 shows that the loss to the insider's expected profit is small.

Finally, similar to Proposition 2.10(iii), the insider's net order in the asymptotic Glosten–Milgrom equilibrium converges to the optimal strategy in the Kyle model as the order size decreases to zero.

Theorem 2.13. *Let* $(X^{B,\delta}, X^{S,\delta}, \mathcal{F}^{I,\delta})_{\delta>0}$ *be the sequence of the insider's strategy in Theorem 2.12. Then*

$$X^{B,\delta} - X^{S,\delta} \xrightarrow{\mathcal{L}} X^0 \quad \text{as } \delta \downarrow 0,$$

where X^0 *is the optimal strategy in Kyle model.*

3. Optimizers in the HJB equation are not optimal control. Theorem 2.6 will be proved in this section. Let us first make the heuristic argument for the HJB equation rigorous by using the dynamic programming principle and standard arguments for viscosity solutions. To this end, recall the domain of Hamilton:

$$\mathrm{dom}(H) := \{(v_n, y, t, V) \in \{v_1, \dots, v_N\} \times \delta\mathbb{Z} \times [0,1] \times \mathbb{R}$$
$$- \text{ valued functions} \,|\, H(v_n, y, t, V) < \infty \}.$$

Observe that control variables for (2.9) are chosen in $[0, \infty)$. Hence $(v_n, y, t, V) \in \mathrm{dom}(H)$ if

$$(3.1) \qquad\qquad V(y+\delta, t) - V(y,t) + (v_n - p(y+\delta, t))\delta \leq 0,$$

$$(3.2) \qquad\qquad V(y-\delta, t) - V(y,t) - (v_n - p(y-\delta, t))\delta \leq 0.$$

Moreover, when $(v_n, y, t, V) \in \mathrm{dom}(H)$, the Hamilton is reduced to

$$(3.3) \qquad\qquad H(v_n, y, t, V) = (V(y+\delta, t) - 2V(y,t) + V(y-\delta, t))\beta.$$

Hence (2.8) reads

$$(3.4) \qquad -V_t - (V(y+\delta, t) - 2V(y,t) + V(y-\delta, t))\beta = 0 \quad \text{in } \mathrm{dom}(H).$$

Proposition 3.1. *The following statements hold for* V^δ, $\delta > 0$:
(i) V^δ *is a viscosity solution of* (2.8).
(ii) $(v_n, y, t, V^\delta) \in dom(H)$ *for any* $n \in \{1, \dots, N\}$ *and* $(y, t) \in \delta\mathbb{Z} \times [0,1)$. *Hence* V^δ *satisfies* (3.1), (3.2), *and is a viscosity solution of* (3.4).
(iii) $t \mapsto V^\delta(y, t)$ *is continuous on* $[0, 1]$.
(iv) $V^\delta(y, t) = \mathbb{E}^{\mathbb{P}^{\delta,y}}[V^\delta(Z_{s-t}, s)]$ *for any* $y \in \delta\mathbb{Z}$, *and* $0 \leq t \leq s \leq 1$.

The proof is postponed to Appendix A, where the dynamic programming principle together with the definition of viscosity solutions is recalled. The proof of Theorem 2.6 also requires the following result.

Lemma 3.2. *For any* $\delta > 0$, $n \in \{1, \dots, N\}$, *and* $(y, t) \in \delta\mathbb{Z} \times [0,1)$, $V^\delta(v_n, y, t) > 0$.

Proof. Without loss of generality, we fix $\delta = 1$, $\tilde{v} = v_n$ for some $n \in \{1, \dots, N\}$, and $(y, t) = (0, 0)$. The superscript δ is omitted throughout this proof. When $n > 1$, let us construct a strategy where the insider buys once the asset is underpriced. Consider

$$\tau := \inf\{r : p(Z_{r-} + 1, r) < v_n)\} \wedge 1 \quad \text{and} \quad \sigma := \inf\{r > \tau : \Delta Y_r \neq 0\} \wedge 1.$$

Here τ is the first time that the asset is underpriced and σ is the arrival time of the first order after τ. The insider employs a strategy with intensity $\theta_r^{B,B} = \mathbb{I}_{\{\tau \leq r \leq \sigma\}}$ and all other intensities zero. Then the associated expected profit is

$$\mathbb{E}\left[\int_0^1 (v_n - a(Y_{r-}, r))\, \mathbb{I}_{\{\tau \leq r \leq \sigma\}}\, dr\right] = \mathbb{E}\left[\int_\tau^\sigma (v_n - p(Z_{r-} + 1, r))\, dr\right] > 0,$$

where the inequality follows from the definition of τ and the fact that $\mathbb{P}(\tau < 1) > 0$ due to Definition 2.1(ii). When $n = 1$, set $\tau := \inf\{t : p(Z_{t-} - 1, t) > v_1\} \wedge 1$ and $\theta_t^{S,S} = \mathbb{I}_{\{\tau \leq t \leq \sigma\}}$. An argument similar to the above shows that this selling strategy also leads to positive expected profit. Therefore, in both cases, $V > 0$ is verified. ∎

Proof of Theorem 2.6. Without loss of generality, we set $\delta = 1$ and omit the superscript δ throughout the proof.

Step 1. For any $n \in \{1, \ldots, N\}$, either one of the following situations holds:

- (3.1) holds as an equality and (3.2) is a strict inequality at all $(y, t) \in \mathbb{Z} \times [0, 1)$;
- (3.2) holds as an equality and (3.1) is a strict inequality at all $(y, t) \in \mathbb{Z} \times [0, 1)$.

To prove the assertion, observe from (3.1) and (3.2) that

$$p(y, t) - v_n \leq V(y + 1, t) - V(y, t) \leq p(y + 1, t) - v_n, \quad (y, t) \in \mathbb{Z} \times [0, 1).$$

Since $y \mapsto p(y, t)$ is strictly increasing for any $t \in [0, 1)$, there exists $\eta(y, t) \in [0, 1]$ such that

$$V(y + 1, t) - V(y, t) = p(y, t) + \eta(y, t) \left(p(y + 1, t) - p(y, t)\right) - v_n, \quad (y, t) \in \mathbb{Z} \times [0, 1).$$

Assume that either (3.1) or (3.2) holds as an equality at some point. If such assumption fails, both inequalities in (3.1) and (3.2) are strict at all points in $\mathbb{Z} \times [0, 1)$. Then all optimizers in (2.9) are identically zero, with the associated expected profit of zero. Since $V > 0$ (cf. Lemma 3.2), these trivial optimizers are not optimal strategies for (2.7). Hence the statement of the theorem is already confirmed in this trivial situation. Let us now assume (3.2) holds as an equality at $(y_0 + 1, t_0)$; we will show (3.2) is an identity. On the other hand, combining the identity in (3.2) and the strict monotonicity of $y \mapsto p(y, t)$, we obtain

$$V(y + 1, t) - V(y, t) = p(y, t) - v_n < p(y + 1, t) - v_n, \quad (y, t) \in \mathbb{Z} \times [0, 1),$$

and hence the inequality (3.1) is always strict. The other situation where (3.1) is an identity and (3.2) is strict can be proved analogously.

Since (3.2) holds as an equality at $(y_0 + 1, t_0)$, then, for any $s \in (t_0, 1)$,

$$\mathbb{E}^{y_0}\left[p(Z_{s-t_0}, s)\right] - v_n = p(y_0, t_0) - v_n = V(y_0 + 1, t_0) - V(y_0, t_0)$$
$$= \mathbb{E}^{y_0}\left[V(Z_{s-t_0} + 1, s) - V(Z_{s-t_0}, s)\right],$$

where the first identity follows from (2.5) and the Markov property of Z, and the third identity is obtained after applying Proposition 3.1(iv) twice. On the other hand, the definition of $\eta(y, t)$ yields

$$\mathbb{E}^{y_0}\left[V(Z_{s-t_0} + 1, s) - V(Z_{s-t_0}, s)\right]$$
$$= \mathbb{E}^{y_0}\left[p(Z_{s-t_0}, s) + \eta(Z_{s-t_0}, s)\left(p(Z_{s-t_0} + 1, s) - p(Z_{s-t_0}, s)\right)\right] - v_n.$$

The last two identities combined imply

$$(3.5) \qquad \mathbb{E}^{y_0}\left[\eta(Z_{s-t_0}, s)\left(p(Z_{s-t_0} + 1, s) - p(Z_{s-t_0}, s)\right)\right] = 0.$$

Recall that $\eta \geq 0$, $p(\cdot + 1, s) - p(\cdot, s) > 0$ for any $s < 1$, and the distribution of Z_{s_0-t} has positive mass on each point in $\mathbb{Z}$. We then conclude from (3.5) that $\eta(y, s) = 0$ for any $y \in \mathbb{Z}$. Since s is arbitrarily chosen,

$$(3.6) \qquad \eta(y, s) = 0 \quad \text{for any } y \in \mathbb{Z}, t_0 < s < 1.$$

Now fix s; the previous identity yields, for any $t < s$ and $y \in \mathbb{Z}$,

$$V(y+1,t) - V(y,t) = \mathbb{E}^y \left[V(Z_{s-t}+1,s) - V(Z_{s-t},s) \right]$$
$$= \mathbb{E}^y \left[p(Z_{s-t},s) \right] - v_n = p(y,t) - v_n,$$

where Proposition 3.1(iv) is applied twice again to obtain the first identity. Therefore $\eta(y,t) = 0$ for any $y \in \mathbb{Z}$ and $t \leq s$, which combined with (3.6) implies (3.2) is an identity.

Step 2. Fix $1 < n < N$. When (3.2) is an identity, any optimizers in (2.9) are shown not to be the optimal strategy for (2.7). When (3.1) is an identity, a similar argument leads to the same conclusion. Combined with the result in Step 1, the statement of the theorem is confirmed.

When (3.2) is an identity, sending $t \to 1$, $V(y,1)$, defined as $\lim_{t \to 1} V(y,t)$, satisfies

$$V(y-1,1) - V(y,1) = v_n - P(y-1).$$

The previous identity and (2.6) combined imply that $V(y,1)$ is strictly decreasing when $y < a_n + 1$, constant when $y \in [a_n + 1, a_{n+1}+1)$, and strictly increasing when $y \geq a_{n+1}+1$. Thus $y \mapsto V(y,1)$ attains its minimum value when $y \in [a_n + 1, a_{n+1}]$. Let $(\hat{X}^B, \hat{X}^S)$ be the point processes whose $\mathcal{F}^I$-intensities are optimizers $\hat{\theta}^{i,j}$, $i \in \{B,S\}$ and $j \in \{B,T,S\}$, in (2.9), and set $\hat{Y} = Z + \hat{X}^B - \hat{X}^S$. Assuming that $(\hat{X}^B, \hat{X}^S)$ is the optimal strategy for (2.7), DPP (i) in Appendix A implies

$$V(y,t) \geq \mathbb{E}^{y,t}\Bigg[V(\hat{Y}_1,1) + \int_t^1 (v_n - p(\hat{Y}_{r-}+1,r))d\hat{X}_r^{B,B} + \int_t^1 (v_n - p(\hat{Y}_{r-}+2,r))d\hat{X}_r^{B,T}$$
$$+ \int_t^1 (v_n - p(\hat{Y}_{r-},r))d\hat{X}_r^{B,S} - \int_t^1 (v_n - p(\hat{Y}_{r-}-1,r))d\hat{X}_r^{S,S}$$
$$- \int_t^1 (v_n - p(\hat{Y}_{r-}-2,r))d\hat{X}_r^{S,T} - \int_t^1 (v_n - p(\hat{Y}_{r-},r))d\hat{X}_r^{S,B} \Bigg],$$

where the expectation is taken under $\mathbb{P}^{y,t}$ with $\mathbb{P}^{y,t}(\hat{Y}_t = y) = 1$. However, the value function $V(y,t)$ is exactly the expected profit when the insider employs the optimal strategy $(\hat{X}^B, \hat{X}^S)$. Therefore, the previous identity yields

$$\mathbb{E}^{y,t}[V(\hat{Y}_1,1)] = 0.$$

Recall that $V(\cdot,1)$, as a limit of positive functions, is nonnegative, and it achieves the minimum at $[a_n+1, a_{n+1}]$. The previous identity implies $V(y,1) = 0$ when $y \in [a_n+1, a_{n+1}]$ and

$$(3.7) \qquad\qquad\qquad \hat{Y}_1 \in [a_n+1, a_{n+1}], \qquad \mathbb{P}^{y,t}\text{-a.s.},$$

However, when (3.2) is an identity and (3.1) is a strict inequality, any optimizer of (2.9) satisfies $\hat{\theta}^{B,B} = \hat{\theta}^{B,S} \equiv 0$, i.e., $\hat{X}^B \equiv 0$. Therefore, $\hat{Y} = Z^B - Z^S - \hat{X}^S$ with only negative controlled jumps from $\hat{X}^S$ cannot compensate Z^S to satisfy (3.7), where $[a_n+1, a_{n+1}]$ is a finite interval in $\mathbb{Z}$ when $1 < n < N$. ∎

4. A suboptimal strategy. We start to prepare the proof of Theorem 2.12 from this section.

For the rest of the paper, $N < \infty$, assumed in Theorem 2.12, is enforced unless stated otherwise.

In this section we are going to characterize a suboptimal strategy of feedback form in the Glosten–Milgrom model with order size δ, such that the pricing rule (2.5) is rational. To simplify presentation, we will take $\delta = 1$, and hence omit all superscript δ, throughout this section. Scaling all processes by δ gives the desired processes when the order size is δ.

The following standing assumption on distribution of $\tilde{v}$ will be enforced throughout this section.

Assumption 4.1. There exists a strictly increasing sequence $(a_n)_{n=1,\ldots,N+1}$ such that
  (i) $a_n \in \mathbb{Z} \cup \{-\infty, \infty\}$, $a_1 = -\infty$, $a_{N+1} = \infty$, and $\cup_{n=1}^N [a_n, a_{n+1}) = \mathbb{Z} \cup \{-\infty\}$;
  (ii) $\mathbb{P}(Z_1 \in [a_n, a_{n+1})) = \mathbb{P}(\tilde{v} = v_n)$, $n = 1, \ldots, N$;
  (iii) the middle level $m_n = (a_n + a_{n+1} - 1)/2$ of the interval $[a_n, a_{n+1})$ is not an integer.

Items (i) and (ii) have already been assumed in Assumption 2.4. Item (iii) is a technical assumption which facilitates the construction of the suboptimal strategy. In the next section, when an arbitrary $\tilde{v}$ of distribution (1.1) is considered and the order size δ converges to zero, a sequence $(a_n^\delta)_{n=1,\ldots,N+1,\delta>0}$ together with a sequence of random variables $(\tilde{v}^\delta)_{\delta>0}$ will be constructed, such that Assumption 4.1 is satisfied for each δ and $\tilde{v}^\delta$ converges to $\tilde{v}$ in law. To simplify notation, we denote by $\underline{m}_n := \lfloor (a_n + a_{n+1} - 1)/2 \rfloor$ the largest integer smaller than m_n and by $\overline{m}_n := \lceil (a_n + a_{n+1} - 1)/2 \rceil$ the smallest integer larger than m_n. Assumption 4.1(iii) implies $a_n \le \underline{m}_n < m_n < \overline{m}_n < a_{n+1}$ and $\overline{m}_n - \underline{m}_n = 1$ when both a_n and a_{n+1} are finite.

Let us now define a function U, which relates to the expected profit of a suboptimal strategy and also dominates the value function V. First the Markov property Z implies that p is continuously differentiable in the time variable and satisfies[8]

$$
(4.1) \qquad \begin{aligned} p_t + (p(y+1,t) - 2p(y,t) + p(y-1,t))\, \beta &= 0, \quad (y,t) \in \mathbb{Z} \times [0,1), \\ p(y,1) &= P(y). \end{aligned}
$$

Define

$$
(4.2) \quad U(v_n, y, 1) := \sum_{j=y}^{a_n - 1} (v_n - A(j))\, \mathbb{I}_{\{y \le \underline{m}_n\}} + \sum_{j=a_{n+1}}^{y} (B(j) - v_n)\, \mathbb{I}_{\{y \ge \overline{m}_n\}}, \quad y \in \mathbb{Z}, 1 \le n \le N,
$$

where $A(y) := P(y+1)$ and $B(y) := P(y-1)$ can be considered as ask and bid pricing functions right before time 1. Since $(v_n)_{n=1,\ldots,N}$ is increasing, $U(\cdot, \cdot, 1)$ is nonnegative and

$$
(4.3) \qquad U(v_n, y, 1) = 0 \quad \Longleftrightarrow \quad y \in [a_n - 1, a_{n+1} + 1).
$$

Given $U(\cdot, \cdot, 1)$ as above, U is extended to $t \in [0,1)$ as follows:

$$
(4.4) \qquad U(v_n, y, t) := U(v_n, y, 1) + \beta \int_t^1 (p(y,r) - p(y-1,r))\, dr, \quad y \ge \overline{m}_n,
$$

$$
(4.5) \qquad U(v_n, y, t) := U(v_n, y, 1) + \beta \int_t^1 (p(y+1,r) - p(y,r))\, dr, \quad y \le \underline{m}_n,
$$

for $t \in [0,1)$ and $n = 1, \ldots, N$. Since N is finite, p is bounded, and hence U takes finite value.

---

[8]This follows from the same argument as in [9, footnote 4].

Proposition 4.2. *Let Assumption* 4.1 *hold. Suppose that the market maker chooses p in* (2.5) *as the pricing rule. Then for any insider's admissible strategy* $(X^B, X^S; \mathcal{F}^I)$, *with* $\mathcal{F}^I$-*intensities* $\theta^{i,j}, i \in \{B, S\}$ *and* $j \in \{B, T, S\}$, *the associated expected profit function* $\mathcal{J}(v_n, y, t; X^B, X^S)$ *satisfies*

$$(4.6) \qquad \mathcal{J}(v_n, y, t; X^B, X^S) \le U(v_n, y, t) - L(v_n, y, t), \quad n \in \{1, \dots, N\}, (y, t) \in \mathbb{Z} \times [0, 1],$$

where

$$
\begin{aligned}
L(v_n, y, t) := \mathbb{E}^y \Bigg[\int_t^1 (v_n - p(\underline{m}_n, r)) \left[\left(\beta - \theta_r^{B,S} + \theta_r^{S,S} \right) \mathbb{I}_{\{Y_{r-}=\overline{m}_n\}} \right. \\
\left. + \theta_r^{S,T} \mathbb{I}_{\{Y_{r-}=\overline{m}_n+1\}} \right] dr \, \Big| \tilde{v} = v_n \Bigg] \\
(4.7) \\
- \mathbb{E}^y \Bigg[\int_t^1 (v_n - p(\overline{m}_n, r)) \left[\left(\beta - \theta_r^{S,B} + \theta_r^{B,B} \right) \mathbb{I}_{\{Y_{r-}=\underline{m}_n\}} \right. \\
\left. + \theta_r^{B,T} \mathbb{I}_{\{Y_{r-}=\underline{m}_n-1\}} \right] dr \, \Big| \tilde{v} = v_n \Bigg].
\end{aligned}
$$

Moreover (4.6) *is an identity when the following conditions are satisfied:*
  (i) $Y_1 \in [a_n - 1, a_{n+1} + 1)$ *a.s. when* $\tilde{v} = v_n$;
  (ii) $X_t^{S,S} = X_t^{S,B} \equiv 0$ *when* $Y_{t-} \le \underline{m}_n$, $X_t^{B,B} = X_t^{B,S} \equiv 0$ *when* $Y_{t-} \ge \overline{m}_n$, $\theta^{B,T} \equiv 0$ *when* $y \ge \underline{m}_n$, *and* $\theta^{S,T} \equiv 0$ *when* $y \le \overline{m}_n$.

Before proving this result, let us derive equations that U satisfies. The following result shows that U satisfies (3.4) except when $y = \overline{m}_n$ and $y = \underline{m}_n$, and U satisfies the identity in either (3.1) or (3.2) depending on whether $y \le \underline{m}_n$ or $y \ge \overline{m}_n$.

Lemma 4.3. *The function U satisfies the following equations (here $\tilde{v} = v_n$ is fixed and the dependence on $\tilde{v}$ is omitted in U):*

$$(4.8) \qquad U_t + (U(y+1, t) - 2U(y, t) + U(y-1, t)) \beta = 0, \quad y > \overline{m}_n \text{ or } y < \underline{m}_n,$$

$$(4.9) \qquad U_t + (U(y+1, t) - 2U(y, t) + U(y-1, t)) \beta = (p(\underline{m}_n, t) - v_n)\beta, \quad y = \overline{m}_n,$$

$$(4.10) \qquad U_t + (U(y+1, t) - 2U(y, t) + U(y-1, t)) \beta = (v_n - p(\overline{m}_n, t))\beta, \quad y = \underline{m}_n,$$

$$(4.11) \qquad U(y, t) - U(y+1, t) - (v_n - p(y, t)) = 0, \quad y \ge \overline{m}_n,$$

$$(4.12) \qquad U(y, t) - U(y-1, t) + (v_n - p(y, t)) = 0, \quad y \le \underline{m}_n.$$

Proof. We will verify these equations only when $y \ge \overline{m}_n$. The remaining equations can be proved similarly. First (4.2) implies

$$U(y+1, 1) - U(y, 1) = B(y+1) - v_n = P(y) - v_n, \quad y \ge \overline{m}_n.$$

Combining the previous identity with (4.4),

$$
\begin{aligned}
U(y+1, t) - U(y, t) &= U(y+1, 1) - U(y, 1) \\
&\quad + \beta \int_t^1 (p(y+1, r) - 2p(y, r) + p(y-1, r)) \, dr \\
&= p(y, t) - v_n,
\end{aligned}
$$

where (4.1) is used to obtain the second identity. This verifies (4.11). When $y > \overline{m}_n$, summing up (4.11) at y and $y + 1$ and taking the time derivative in (4.4) yield

$$
\begin{aligned}
U_t &+ \left(U(y+1,t) - 2U(y,t) + U(y-1,t)\right)\beta \\
&= -\beta(p(y,t) - p(y-1,t)) + \beta(p(y,t) - p(y-1,t)) \\
&= 0,
\end{aligned}
$$

which confirms (4.8) when $y > \overline{m}_n$. When $y = \overline{m}_n$, observe from (4.2), (4.4), and (4.5) that $U(\overline{m}_n, \cdot) = U(\underline{m}_n, \cdot)$. Then

$$
\begin{aligned}
U_t &+ \left(U(y+1,t) - 2U(y,t) + U(y-1,t)\right)\beta \\
&= -\beta\left(p(\overline{m}_n,t) - p(\underline{m}_n,t)\right) + \beta\left(U(\overline{m}_n+1,t) - U(\overline{m}_n,t)\right) \\
&= -\beta\left(p(\overline{m}_n,t) - p(\underline{m}_n,t)\right) - \beta\left(v_n - p(\overline{m}_n,t)\right) \\
&= \beta\left(p(\underline{m}_n,t) - v_n\right),
\end{aligned}
$$

where the second identity follows from (4.11). ∎

Proof of Proposition 4.2. Throughout the proof the $\tilde{v} = v_n$ is fixed and the dependence on $\tilde{v}$ is omitted in U. Let $Y^B = Z^B + X^{B,B} + X^{B,T} - X^{S,B}$ and $Y^S = Z^S + X^{S,S} + X^{S,T} - X^{B,S}$ be positive and negative parts of Y, respectively. Then $Y^B - \int_0^\cdot (\beta - \theta_r^{S,B} - \theta_r^{B,T})dr - \int_0^\cdot \theta_r^{B,B} dr - 2\int_0^\cdot \theta_r^{B,T} dr$ and $Y^S - \int_0^\cdot (\beta - \theta_r^{B,S} - \theta_r^{S,T})dr - \int_0^\cdot \theta_r^{S,S} dr - 2\int_0^\cdot \theta_r^{S,T} dr$ are $\mathcal{F}^I$-martingales. Applying Itô's formula to $U(Y.,\cdot)$, we obtain

(4.13)

$$
\begin{aligned}
U(Y_1,1) &= U(y,t) + \int_t^1 U_t(Y_{r-},r)dr \\
&\quad + \int_t^1 \left[U(Y_r,r) - U(Y_{r-},r)\right]dY_r^B + \int_t^1 \left[U(Y_r,r) - U(Y_{r-},r)\right]dY_r^S \\
&= U(y,t) + \int_t^1 \left[U_t(Y_{r-},r) + \left(U(Y_{r-}+1,r) - 2U(Y_{r-},r) + U(Y_{r-}-1,r)\right)\beta\right]dr \\
&\quad + \int_t^1 \left[U(Y_{r-}+1,r) - U(Y_{r-},r)\right]\left(\theta_r^{B,B} - \theta_r^{S,B}\right)dr \\
&\quad + \int_t^1 \left[U(Y_{r-}+2,r) - U(Y_{r-}+1,r)\right]\theta_r^{B,T}dr \\
&\quad + \int_t^1 \left[U(Y_{r-}-1,r) - U(Y_{r-},r)\right]\left(\theta_r^{S,S} - \theta_r^{B,S}\right)dr \\
&\quad + \int_t^1 \left[U(Y_{r-}-2,r) - U(Y_{r-}-1,r)\right]\theta_r^{S,T}dr + M_1 - M_t,
\end{aligned}
$$

where

$$
\begin{aligned}
M &= \int_0^\cdot \left[U(Y_r,r) - U(Y_{r-},r)\right]d\left(Y_r^B - \int_0^r \left(\beta - \theta_u^{S,B} + \theta_u^{B,B} + \theta_u^{B,T}\right)du\right) \\
&\quad + \int_0^\cdot \left[U(Y_r,r) - U(Y_{r-},r)\right]d\left(Y_r^S - \int_0^r \left(\beta - \theta_u^{B,S} + \theta_u^{S,S} + \theta_u^{S,T}\right)du\right).
\end{aligned}
$$

Since (4.11) and (4.12) imply $U(y+1,t) - U(y,t)$ is either $p(y,t) - v_n$ or $p(y+1,t) - v_n$, which are both bounded from below by $v_1 - v_n$ and from above by $v_N - v_n$, M is an $\mathcal{F}^I$-martingale (cf. [7, Chapter I, T6]). On the right-hand side of (4.13), splitting the second integral on $\{Y_{r-} \geq \overline{m}_n\}$, $\{Y_{r-} = \underline{m}_n\}$, and $\{Y_{r-} < \underline{m}_n\}$, splitting the fourth integral on $\{Y_{r-} > \overline{m}_n\}$, $\{Y_{r-} = \overline{m}_n\}$, and $\{Y_{r-} \leq \underline{m}_n\}$, utilizing $U(\overline{m}_n, \cdot) = U(\underline{m}_n, \cdot)$, as well as different equations in Lemma 4.3 in different regions, we obtain

$$
U(Y_1, 1) = U(y,t) + \int_t^1 (p(\underline{m}_n, r) - v_n)\,\beta\mathbb{I}_{\{Y_{r-}=\overline{m}_n\}}dr + \int_t^1 (v_n - p(\overline{m}_n, r))\,\beta\mathbb{I}_{\{Y_{r-}=\underline{m}_n\}}dr
$$

$$
- \int_t^1 (v_n - p(Y_{r-}, r))\,\mathbb{I}_{\{Y_{r-}\geq\overline{m}_n\}}(\theta_r^{B,B} - \theta_r^{S,B})dr
$$

$$
- \int_t^1 (v_n - p(Y_{r-}+1, r))\,\mathbb{I}_{\{Y_{r-}<\underline{m}_n\}}(\theta_r^{B,B} - \theta_r^{S,B})dr
$$

$$
- \int_t^1 (v_n - p(Y_{r-}+1, r))\,\mathbb{I}_{\{Y_{r-}\geq\underline{m}_n\}}\theta_r^{B,T}dr
$$

$$
- \int_t^1 (v_n - p(Y_{r-}+2, r))\,\mathbb{I}_{\{Y_{r-}<\underline{m}_n-1\}}\theta_r^{B,T}dr
$$

$$
+ \int_t^1 (v_n - p(Y_{r-}-1, r))\,\mathbb{I}_{\{Y_{r-}>\overline{m}_n\}}(\theta_r^{S,S} - \theta_r^{B,S})dr
$$

$$
+ \int_t^1 (v_n - p(Y_{r-}, r))\,\mathbb{I}_{\{Y_{r-}\leq\underline{m}_n\}}(\theta_r^{S,S} - \theta_r^{B,S})dr
$$

$$
+ \int_t^1 (v_n - p(Y_{r-}-2, r))\,\mathbb{I}_{\{Y_{r-}>\overline{m}_n+1\}}\theta_r^{S,T}dr
$$

$$
+ \int_t^1 (v_n - p(Y_{r-}-1, r))\,\mathbb{I}_{\{Y_{r-}\leq\overline{m}_n\}}\theta_r^{S,T}dr + M_1 - M_t.
$$

Rearranging the previous identity by putting the profit of (X^B, X^S) to the left-hand side, we obtain

$$
U(y,t) - U(Y_1, 1) - K - L + M_1 - M_t
$$

(4.14)
$$
= \int_t^1 (v_n - p(Y_{r-}+1, r))\theta_r^{B,B}dr + \int_t^1 (v_n - p(Y_{r-}+2, r))\,\theta_r^{B,T}dr
$$

$$
+ \int_t^1 (v_n - p(Y_{r-}, r))\theta_r^{B,S}dr - \int_t^1 (v_n - p(Y_{r-}-1, r))\theta_r^{S,S}dr
$$

$$
- \int_t^1 (v_n - p(Y_{r-}-2, r))\,\theta_r^{S,T}dr - \int_t^1 (v_n - p(Y_{r-}, r))\theta_r^{S,B}dr,
$$

where

$$K = \int_t^1 \left(p(Y_{r-}+1,r) - p(Y_{r-},r) \right) \mathbb{I}_{\{Y_{r-} \geq \overline{m}_n\}} \theta_r^{B,B} dr$$

$$+ \int_t^1 \left(p(Y_{r-},r) - p(Y_{r-}-1,r) \right) \mathbb{I}_{\{Y_{r-} \geq \overline{m}_n\}} \theta_r^{B,S} dr$$

$$+ \int_t^1 \left(p(Y_{r-}+2,r) - p(Y_{r-}+1,r) \right) \mathbb{I}_{\{Y_{r-} \geq \underline{m}_n\}} \theta_r^{B,T} dr$$

$$+ \int_t^1 \left(p(Y_{r-},r) - p(Y_{r-}-1,r) \right) \mathbb{I}_{\{Y_{r-} \leq \underline{m}_n\}} \theta_r^{S,S} dr$$

$$+ \int_t^1 \left(p(Y_{r-}+1,u) - p(Y_{r-},r) \right) \mathbb{I}_{\{Y_{r-} \leq \underline{m}_n\}} \theta_r^{S,B} dr$$

$$+ \int_t^1 \left(p(Y_{r-}-1,r) - p(Y_{r-}-2,r) \right) \mathbb{I}_{\{Y_{r-} \leq \overline{m}_n\}} \theta_r^{S,T} dr,$$

$$L = \int_t^1 \left[v_n - p(\underline{m}_n,r) \right] \left[(\beta - \theta_r^{B,S} + \theta_r^{S,S})\mathbb{I}_{\{Y_{r-}=\overline{m}_n\}} + \theta_r^{S,T}\mathbb{I}_{\{Y_{r-}=\overline{m}_n+1\}} \right] dr$$

$$- \int_t^1 \left[v_n - p(\overline{m}_n,r) \right] \left[(\beta - \theta_r^{S,B} + \theta_r^{B,B})\mathbb{I}_{\{Y_{r-}=\underline{m}_n\}} + \theta_r^{B,T}\mathbb{I}_{\{Y_{r-}=\underline{m}_n-1\}} \right] dr.$$

Taking conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t^I, Y_t = y]$ on both sides of (4.14), the left-hand side is the expected profit $\mathcal{J}(X^B, X^S)$, while on the right-hand side, both $U(\cdot, 1)$ and K are nonnegative (cf. Definition 2.1(i)). Therefore (4.6) is verified. To attain the identity in (4.6), we need (i) $Y_1 \in [a_n - 1, a_{n+1} + 1)$ a.s. so that $U(Y_1, 1) = 0$ a.s. follows from (4.3); (ii) $\theta^{B,B} = \theta^{B,S} \equiv 0$ when $y \geq \overline{m}_n$, $\theta^{S,S} = \theta^{S,B} \equiv 0$ when $y \leq \underline{m}_n$, $\theta^{B,T} \equiv 0$ when $y \geq \underline{m}_n$, and $\theta^{S,T} \equiv 0$ when $y \leq \overline{m}_n$. ∎

Come back to the statement of Proposition 4.2. If the insider chooses a strategy such that both conditions in (i) and (ii) are satisfied, then the identity in (4.6) is attained, hence the expected profit of this strategy is $U - L$. On the other hand, define $U^S : \{v_1, \ldots, v_N\} \times \mathbb{Z} \times [0, 1] \to \mathbb{R}$ via

$$(4.15) \qquad U^S(v_n, y, t) = \begin{cases} U(v_n, y, t), & y \geq \overline{m}_n, \\ U(v_n, y-1, t), & y \leq \underline{m}_n. \end{cases}$$

The next result shows that U^S dominates the value function V, and therefore $U^S - U + L$ is the upper bound of the potential loss of the expected profit. In section 6, we will prove this potential loss converges to zero as $\delta \downarrow 0$. Therefore, when the order size is small, the insider loses little expected profit by employing a strategy satisfying Proposition 4.2(i) and (ii).

Proposition 4.4. *Let Assumption* 4.1 *hold. Then* $V \leq U^S$, *hence* $V < \infty$, *on* $\{v_1, \ldots, v_N\} \times \mathbb{Z} \times [0, 1]$.

Proof. Fix v_n and omit it as the first argument of U^S and U throughout the proof. We first verify

$$(4.16) \qquad U^S(y,t) - U^S(y+1,t) - (v_n - p(y,t)) = 0,$$

$$(4.17) \qquad U_t^S + \left(U^S(y+1,t) - 2U^S(y,t) + U^S(y-1,t)\right)\beta = 0,$$

for any $(y,t) \in \mathbb{Z} \times [0,1)$. Indeed, when $y \geq \overline{m}_n$, (4.16) is exactly (4.11). When $y = \underline{m}_n$,

$$U^S(\underline{m}_n,t) - U^S(\overline{m}_n,t) = U(\underline{m}_n - 1,t) - U(\overline{m}_n,t)$$
$$= U(\underline{m}_n - 1,t) - U(\underline{m}_n,t) = v_n - p(\underline{m}_n,t),$$

where the second identity follows from $U(\overline{m}_n,t) = U(\underline{m}_n,t)$ and the third identity holds due to (4.12). When $y < \underline{m}_n$,

$$U^S(y,t) - U^S(y+1,t) = U(y-1,t) - U(y,t) = v_n - p(y,t),$$

where (4.12) is utilized again to obtain the second identity. Therefore (4.16) is confirmed for all cases. As for (4.17), (4.16) yields

$$U^S(y+1,t) - 2U^S(y,t) + U^S(y-1,t) = p(y,t) - p(y-1,t).$$

On the other hand, we have from (4.4) and (4.5) that

$$U_t^S(y,t) = \begin{cases} U_t(y,t) = -\beta(p(y,t) - p(y-1,t)), & y \geq \overline{m}_n, \\ U_t(y-1,t) = -\beta(p(y,t) - p(y-1,t)), & y \leq \underline{m}_n. \end{cases}$$

Therefore (4.17) is confirmed after combining the previous two identities.

Now note that $U^S(\cdot,1) \geq 0$; moreover U^S satisfies (4.16) and (4.17). The assertion $V \leq U^S$ follows from the same argument as in the high type of [9, Proposition 3.2]. ∎

Having studied the insider's optimization problem, let us turn to the market maker. Given $(X^B, X^S; \mathcal{F}^I)$, Definition 2.11(ii) requires the pricing rule to be rational. This leads to another constraint on $(X^B, X^S; \mathcal{F}^I)$.

Proposition 4.5. *If there exists an admissible strategy $(X^B, X^S; \mathcal{F}^I)$ such that*
(i) $Y^B = Z^B + X^{B,B} + X^{B,T} - X^{S,B}$ *and* $Y^S = Z^S + X^{S,S} + X^{S,T} - X^{B,S}$ *are independent $\mathcal{F}^Y$-adapted Poisson processes with common intensity β;*
(ii) $[Y_1 \in [a_n, a_{n+1})] = [\tilde{v} = v_n]$, $n = 1, \ldots, N$.
Then the pricing rule (2.5) is rational.

Proof. For any $t \in [0,1]$,

$$p(Y_t, t) = \mathbb{E}^{Y_t}[P(Z_{1-t})] = \mathbb{E}\left[P(Z_1) \mid Z_t = Y_t\right] = \mathbb{E}\left[P(Y_1) \mid \mathcal{F}_t^Y\right] = \mathbb{E}[\tilde{v} \mid \mathcal{F}_t^Y],$$

where the third identity holds since Y and Z have the same distribution, and the fourth identity follows from (ii) and (2.6). ∎

Remark 4.6. If the insider places a buy (resp., sell) order when a noise buy (resp., sell) order arrives, Proposition 4.5(i) cannot be satisfied. Therefore in the asymptotic equilibrium the insider will not trade in the same direction as the noise traders, i.e., $X^{B,T} = X^{S,T} \equiv 0$, so that the market maker can employ a rational pricing rule.

Concluding this section, we need to construct point processes $(X^B, X^S; \mathcal{F}^I)$ which simultaneously satisfy conditions in Proposition 4.2(ii) and Proposition 4.5(i) and (ii).[9] This construction is a natural extension of [9, section 4], where $N = 2$ is considered, and will be presented in the next section.

5. Construction of a point process bridge. In this section, we will construct point processes X^B and X^S on a probability space $(\Omega, \mathcal{F}^I, (\mathcal{F}_t^I)_{t\in[0,1]}, \mathbb{P})$ such that $X^{B,T} = X^{S,T} \equiv 0$, due to Remark 4.6, and satisfy

(i) $Y^B = Z^B + X^{B,B} - X^{S,B}$ and $Y^S = Z^S + X^{S,S} - X^{B,S}$ are independent $\mathcal{F}^Y$-adapted Poisson processes with common intensity β;

(ii) $X_t^{B,B} = X_t^{B,S} \equiv 0$ when $Y_{t-} \geq \overline{m}_n$, $X_t^{S,S} = X_t^{S,B} \equiv 0$ when $Y_{t-} \leq \underline{m}_n$;

(iii) $[Y_1 \in [a_n, a_{n+1}]] = [\tilde{v} = v_n]$ $\mathbb{P}$-a.s. for $n = 1, \ldots, N$.

The construction is a natural extension of [9], where $N = 2$ is considered. As in [9], X^B and X^S are constructed using two independent sequences of iid random variables $(\eta_i)_{i\geq 1}$ and $(\zeta_i)_{i\geq 1}$ with uniform distribution on $[0,1]$; moreover they are independent of Z and $\tilde{v}$. The insider uses $(\eta_i)_{i\geq 1}$ to randomly contribute either buy or sell orders and uses $(\zeta_i)_{i\geq 1}$ to randomly cancel noise orders. Throughout this section Assumption 4.1 is enforced. Moreover, we set $\delta = 1$ and hence suppress the superscript δ. Otherwise X^B and X^S can be scaled by δ to obtain the desired processes.

In the following construction, we will define a probability space $(\Omega, \mathcal{F}^I, (\mathcal{F}_t^I)_{t\in[0,1]}, \mathbb{P})$ on which Y takes the form

$$(5.1) \qquad Y = Z + \sum_{n=1}^{N} \mathbb{I}_{A_n}(X^B - X^S).$$

Here Z is the difference of two independent $\mathcal{F}^I$-adapted Poisson processes with intensity β, $A_n \in \mathcal{F}_0^I$ such that $\mathbb{P}(A_n) = \mathbb{P}(Z_1 \in [a_n, a_{n+1}))$ for each $n = 1, \ldots, N$.

Before constructing X^B and X^S satisfying desired properties, let us draw some intuition from the theory of filtration enlargement. Let us define $(\mathbb{D}([0,1], \mathbb{Z}), \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t\in[0,1]}, \overline{\mathbb{P}})$ as the canonical space where $\mathbb{D}([0,1], \mathbb{Z})$ is $\mathbb{Z}$-valued càdlàg functions, $\overline{\mathbb{P}}$ is a probability measure under which Z^B and Z^S are independent Poisson processes with intensities β, $(\overline{\mathcal{F}}_t)_{t\in[0,1]}$ is the minimal filtration generated by Z^B and Z^S satisfying the usual conditions, and $\overline{\mathcal{F}} = \vee_{t\in[0,1]} \overline{\mathcal{F}}_t$. Let us denote by $(\mathcal{G}_t)_{t\in[0,1]}$ the filtration $(\overline{\mathcal{F}}_t)_{t\in[0,1]}$ enlarged with a sequence of random variables $(\mathbb{I}_{\{Z_1 \in [a_n, a_{n+1})\}})_{n=1,\ldots,N}$.

In order to find the $\mathcal{G}$-intensities of Z^B and Z^S, we use a standard enlargement of filtration argument which can be found, e.g., in [18]. To this end, recall $h_n(y, t) = \overline{\mathbb{P}}[Z_1 \in [a_n, a_{n+1}) \mid Z_t = y]$. Note that h_n is strictly positive on $\mathbb{Z} \times [0, 1)$. Moreover the Markov property of Z implies h_n is continuously differentiable in the time variable and satisfies

$$(5.2) \qquad \begin{aligned} \partial_t h_n + (h_n(y+1, t) - 2h_n(y, t) + h_n(y-1, t))\, \beta &= 0, \quad (y, t) \in \mathbb{Z} \times [0, 1), \\ h_n(y, 1) &= \mathbb{I}_{\{y \in [a_n, a_{n+1})\}}. \end{aligned}$$

---

[9]Note that Proposition 4.5(ii) implies Proposition 4.2(i).

Lemma 5.1. *The $\mathcal{G}$-intensities of Z^B and Z^S at $t \in [0,1)$ are given by*

$$\sum_{n=1}^{N} \mathbb{I}_{\{Z_1 \in [a_n, a_{n+1})\}} \frac{h_n(Z_{t-}+1, t)}{h_n(Z_{t-}, t)} \beta \quad and \quad \sum_{n=1}^{N} \mathbb{I}_{\{Z_1 \in [a_n, a_{n+1})\}} \frac{h_n(Z_{t-}-1, t)}{h_n(Z_{t-}, t)} \beta,$$

respectively.

Proof. We will calculate only the intensity for Z^B. The intensity of Z^S can be obtained similarly. All expectations are taken under $\overline{\mathbb{P}}$ throughout this proof. For $s \le t < 1$, take an arbitrary $E \in \overline{\mathcal{F}}_s$ and denote $M_t^B := Z_t^B - \beta t$. The definition of h_n and the $\overline{\mathcal{F}}$-martingale property of M^B imply

$$\mathbb{E}\left[(M_t^B - M_s^B)\mathbb{I}_E \mathbb{I}_{\{Z_1 \in [a_n, a_{n+1})\}}\right]$$
$$= \mathbb{E}\left[(M_t^B - M_s^B)\mathbb{I}_E h_n(Z_t, t)\right]$$
$$= \mathbb{E}\left[\mathbb{I}_E(\langle M^B, h_n(Z., \cdot)\rangle_t - \langle M^B, h_n(Z., \cdot)\rangle_s)\right]$$
$$= \mathbb{E}\left[\mathbb{I}_E \int_s^t \beta \left(h_n(Z_{r-}+1, r) - h_n(Z_{r-}, r)\right) dr\right]$$
$$= \mathbb{E}\left[\mathbb{I}_E \int_s^t \beta \, \mathbb{I}_{\{Z_1 \in [a_n, a_{n+1})\}} \frac{h_n(Z_{r-}+1, r) - h_n(Z_{r-}, r)}{h_n(Z_{r-}, r)} dr\right].$$

These computations for each $n = 1, \ldots, N$ imply that

$$M^B - \int_s^{\cdot} \beta \sum_{n=1}^{N} \mathbb{I}_{\{Z_1 \in [a_n, a_{n+1})\}} \frac{h_n(Z_{r-}+1, r) - h_n(Z_{r-}, r)}{h_n(Z_{r-}, r)} dr$$

defines a $\mathcal{G}$-martingale. Therefore the $\mathcal{G}$-intensity of Z^B follows from $Z_t^B = M_t^B + \beta t$. ∎

To better understand intensities in the previous lemma, let us collect several properties for h_n.

Lemma 5.2. *Let Assumption* 4.1 *hold. The following properties hold for each h_n, $n = 1, \ldots, N$:*

(i) $h_n(\cdot, \cdot) = h_n(2m_n - \cdot, \cdot)$*; in particular, $h_n(\overline{m}_n, \cdot) = h_n(\underline{m}_n, \cdot)$.*

(ii) $y \mapsto h_n(y, t)$ *is strictly increasing when $y \le \underline{m}_n$ and strictly decreasing when $y \ge \overline{m}_n$.* *Here, when $n = 1$ (resp., $n = N$), $\overline{m}_n = \underline{m}_n = -\infty$ (resp., $\overline{m}_n = \underline{m}_n = \infty$).*

Proof. Recall that $a_n + a_{n+1} - 1 = 2m_n$. Then

$$h_n(y, t) = \overline{\mathbb{P}}[Z_1 \in [a_n, a_{n+1}) \mid Z_t = y] = \overline{\mathbb{P}}[y + Z_{1-t} \in [a_n, a_{n+1})]$$
$$= \overline{\mathbb{P}}[2m_n - y - Z_{1-t} \in (2m_n - a_{n+1}, 2m_n - a_n]]$$
$$= \overline{\mathbb{P}}[2m_n - y - Z_{1-t} \in [a_n, a_{n+1})] = h_n(2m_n - y, t),$$

where the last identity holds since Z and $-Z$ have the same distribution. This verifies (i). To prove (ii), rewrite $h_n(y, t) = \overline{\mathbb{P}}[Z_{1-t} \in [a_n - y, a_{n+1} - y)]$. Then the statement (ii) follows from the fact that $y \mapsto \mathbb{P}(Z_{1-t} = y)$ is strictly increasing when $y \le 0$ and strictly decreasing when $y \ge 0$. ∎

In what follows, given $A_n \in \mathcal{F}_0^I$ such that $\mathbb{P}(A_n) = \overline{\mathbb{P}}(Z_1 \in [a_n, a_{n+1}))$, $(X^B, X^S; \mathcal{F}^I)$ on A_n will be constructed so that $\mathcal{F}^I$-intensity of Y^B (resp., Y^S) on A_n match $\mathcal{G}$-intensities of

Z^B (resp., Z^S) on $[Z_1 \in [a_n, a_{n+1})]$. Matching these intensities ensures that $(X^B, X^S; \mathcal{F}^I)$ satisfies desired properties; cf. Proposition 5.5 below. Recall $Y^B = Z^B + X^{B,B} - X^{S,B}$ and $Y^S = Z^S + X^{S,S} - X^{B,S}$. Subtracting β from $\mathcal{G}$-intensities of Z^B (resp., Z^S) in Lemma 5.1, we can read out intensities of $X^{B,B} - X^{S,B}$ (resp., $X^{S,S} - X^{B,S}$). Since property (ii) at the beginning of this section implies that θ^B and θ^S are never positive at the same time, when the intensity of $X^{B,B} - X^{S,B}$ is positive, the insider contributes buy orders $X^{B,B}$ with such intensity; otherwise the insider submits sell orders $X^{S,B}$ with the same intensity to cancel some noise buy orders from Z^B. Applying the same strategy to $X^{S,S} - X^{B,S}$ and utilizing Lemma 5.2, we read out $\mathcal{F}^I$-intensities for $X^{i,j}$, $i, j \in \{B, S\}$:

Corollary 5.3. *Suppose that $\mathcal{F}^I$-intensities of Y^B and Y^S match $\mathcal{G}$-intensities of Z^B and Z^S, respectively; moreover $X_t^{B,B} = X_t^{B,S} \equiv 0$ when $Y_{t-} \geq \overline{m}_n$ and $X_t^{S,S} = X_t^{S,B} \equiv 0$ when $Y_{t-} \leq \underline{m}_n$. Then $\mathcal{F}^I$-intensities of $X^{i,j}$, $i, j \in \{B, S\}$, have the following form on A_n when $Y_{t-} = y$:*

$$\theta^{B,B}(y,t) = \left(\frac{h_n(y+1,t)}{h_n(y,t)} - 1\right)_+ \beta, \qquad \theta^{B,S}(y,t) = \left(\frac{h_n(y-1,t)}{h_n(y,t)} - 1\right)_- \beta,$$

$$\theta^{S,S}(y,t) = \left(\frac{h_n(y-1,t)}{h_n(y,t)} - 1\right)_+ \beta, \qquad \theta^{S,B}(y,t) = \left(\frac{h_n(y+1,t)}{h_n(y,t)} - 1\right)_- \beta.$$

In particular, $\theta^{i,j}$, $i, j \in \{B, S\}$, satisfies the following properties:

(i) $\theta^{B,B}(y,\cdot) = \theta^{B,S}(y,\cdot) \equiv 0$, $\theta^{S,S}(y,\cdot) > 0$, and $\theta^{S,B}(y,\cdot) > 0$, when $y \geq \overline{m}_n$; $\theta^{S,S}(y,\cdot) = \theta^{S,B}(y,\cdot) \equiv 0$, $\theta^{B,B}(y,\cdot) > 0$, and $\theta^{B,S}(y,\cdot) > 0$, when $y \leq \underline{m}_n$;

(ii) $\theta^{B,B}(\cdot,\cdot) = \theta^{S,S}(2m_n - \cdot, \cdot)$, $\theta^{B,S}(\cdot,\cdot) = \theta^{S,B}(2m_n - \cdot, \cdot)$;

(iii) $\theta^{B,B}(\underline{m}_n, \cdot) = \theta^{S,S}(\overline{m}_n, \cdot) \equiv 0$.

As described in Corollary 5.3, when $A_n \in \overline{\mathcal{F}}_0$ is fixed, the state space is divided into two domains, $\mathcal{S} := \{y \in \mathbb{Z} : y \geq \overline{m}_n\}$ and $\mathcal{B} := \{y \in \mathbb{Z} : y \leq \underline{m}_n\}$. As Y makes excursions into these two domains, either X^S or X^B is active. In the following construction, we will focus on the domain $\mathcal{B}$ and construct inductively jumps of X^B until Y leaves $\mathcal{B}$. When Y makes excursions into $\mathcal{S}$, X^S can be constructed similarly.

When Y is in $\mathcal{B}$, one of the goals of X^B is to make sure that Y_1 ends up in the interval $[a_n, a_{n+1})$. In order to achieve this goal, X^B will add some jumps in addition to the jumps coming from Z^B. However, this by itself will not be enough since Y also jumps downward due to Z^S. Thus, X^B also needs to cancel some of the downward jumps from Z^S. Therefore X^B consists of two components $X^{B,B}$ and $X^{B,S}$, where $X^{B,B}$ complements jumps of Z^B and $X^{B,S}$ cancels some jumps of Z^S. Let us denote by $(\tau_i)_{i \geq 1}$ the sequence of jump times for Y. These stopping times will be constructed inductively as follows. Given $\tau_{i-1} < 1$ and $Y_{\tau_{i-1}} \leq \underline{m}_n$, the next jump time τ_i happens at the minimum of the following three random times:

- the next jump of Z^B,
- the next jump of $X^{B,B}$,
- the next jump of Z^S which is not canceled by a jump of $X^{B,S}$.

Here $X^{B,B}$ and $X^{B,S}$ need to be constructed so that their intensities $\theta^{B,B}(Y_{t-}, t)$ and $\theta^{B,S}(Y_{t-}, t)$ match the forms in Corollary 5.3. This goal is achieved by employing two independent sequences of iid random variables $(\eta_i)_{i \geq 1}$ and $(\zeta_i)_{i \geq 1}$ with uniform distribution on $[0, 1]$. They are also independent of $\overline{\mathcal{F}}$ and $(A_n)_{n=1,\ldots,N}$. These two sequences will be used to

generate a random variable ν_i and another sequence of Bernoulli random variables $(\xi_{j,i})_{j\geq 1}$ taking values in $\{0,1\}$. Let $(\sigma_i^+)_{i\geq 1}$ and $(\sigma_i^-)_{i\geq 1}$ be jump time of Z^B and Z^S, respectively. Then, after τ_{i-1}, the next jump of Z^B is at $\sigma_{Z^B_{\tau_{i-1}}+1}^+$, the next jump of $X^{B,B}$ is at ν_i, and the next jump of Z^S not canceled by jumps of $X^{B,S}$ is at $\tau_i^- = \min\{\sigma_j^- > \tau_{i-1} : \xi_{j,i} = 1\}$. Then the next jump of Y is at

$$\tau_i = \sigma_{Z^B_{\tau_{i-1}}+1}^+ \wedge \nu_i \wedge \tau_i^-.$$

The construction of ν_i and $(\xi_{j,i})_{j\geq 1}$ using $(\eta_i)_{i\geq 1}$ and $(\zeta_i)_{i\geq 1}$ is exactly the same as in [9, section 4], only replacing h therein by h_n.

All aforementioned construction is performed in a filtrated probability space $(\Omega, \mathcal{F}^I, (\mathcal{F}_t^I)_{t\in[0,1]}, \mathbb{P})$ such that there exist $(A_n)_{n=1,\ldots,N} \in \mathcal{F}_0^I$ with $\mathbb{P}(A_n) = h_n(0,0)$ and two independent sequences of iid $\mathcal{F}^I$-measurable random variables $(\eta_i)_{i\geq 1}$ and $(\zeta_i)_{i\geq 1}$ with uniform distribution on $[0,1]$; moreover these two sequences are independent of both Z and $(A_n)_{n=1,\ldots,N}$. These requirements can be satisfied by extending $\overline{\mathcal{F}}_0$ (resp., $\overline{\mathcal{F}}$) to $\mathcal{F}_0^I$ (resp., $\mathcal{F}^I$). As for the filtration $(\mathcal{F}_t^I)_{t\in[0,1]}$, we require that it is right continuous and complete under $\mathbb{P}$, and moreover Z, as the difference of two independent Poisson processes with intensity β, is adapted to $(\mathcal{F}_t^I)_{t\in[0,1]}$. Therefore Z is independent of $(A_n)_{n=1,\ldots,N}$, since Z has independent increments. Finally, we also assume that $(\mathcal{F}_t^I)_{t\in[0,1]}$ is rich enough so that $(\nu_i)_{i\geq 1}$ and $(\tau_i^-)_{i\geq 1}$ discussed above are $\mathcal{F}^I$-stopping times.

An argument similar to [9, Lemma 4.3] yields the following.

Lemma 5.4. *Given point processes $(X^B, X^S; \mathcal{F}^I)$ constructed above, the $\mathcal{F}^I$-intensities of Y^B and Y^S at $t \in [0,1)$ are given by*

$$\sum_{n=1}^N \mathbb{I}_{A_n} \frac{h_n(Y_{t-} + 1, t)}{h_n(Y_{t-}, t)} \beta \quad and \quad \sum_{n=1}^N \mathbb{I}_{A_n} \frac{h_n(Y_{t-} - 1, t)}{h_n(Y_{t-}, t)} \beta,$$

respectively.

Now we are ready to verify that our construction is as desired.

Proposition 5.5. *The process Y as constructed above satisfies the following properties:*

(i) $[Y_1 \in [a_n, a_{n+1}]] = A_n$ *a.s. for $n = 1, \ldots, N$;*

(ii) Y^B *and Y^S are independent Poisson processes with intensity β with respect to the natural filtration $(\mathcal{F}_t^Y)_{t\in[0,1]}$ of Y;*

(iii) $(X^B, X^S; \mathcal{F}^I)$ *is admissible in the sense of Definition 2.2.*

Proof. To verify that Y satisfies the desired properties, let us introduce an auxiliary process $(\ell_t)_{t\in[0,1)}$:

$$\ell_t := \sum_{n=1}^N \mathbb{I}_{A_n} \frac{h_n(0,0)}{h_n(Y_t, t)}, \quad t \in [0,1).$$

When $n = 2, \ldots, N-1$, there is only almost surely a finite number of positive (resp., negative) jumps of Y on A_n when $Y. \geq \overline{m}_n$ (resp., $Y. \leq \underline{m}_n$). Therefore Y_t is finite on these A_n when $t < 1$ is fixed. When $n = 1$ (resp., $n = N$), there is a finite number of positive (resp., negative) jumps of Y on A_1 (resp., A_N) before t. Hence $Y_t < \infty$ on A_1 (resp., $Y_t > -\infty$ on A_N). This analysis implies $h_n(Y_t, t) > 0$ on A_n for each $n = 1, \ldots, N$ and $t < 1$. Therefore $(\ell_t)_{t\in[0,1)}$ is a well-defined positive process with $\ell_0 = 1$.

To prove (i), we first show that ℓ is a positive $\mathcal{F}^I$-local martingale on $[0, 1)$. To this end, the Itô formula yields that

$$d\ell_t = \sum_{n=1}^{N} \mathbb{I}_{A_n} \ell_{t-} \left[\frac{h_n(Y_{t-}, t) - h_n(Y_{t-} + 1, t)}{h_n(Y_{t-} + 1, t)} dM_t^B + \frac{h_n(Y_{t-}, t) - h_n(Y_{t-} - 1, t)}{h_n(Y_{t-} - 1, t)} dM_t^S \right],$$

where $t \in [0, 1)$. Here

$$M^B = Y^B - \beta \int_0^{\cdot} \sum_{n=1}^{N} \mathbb{I}_{A_n} \frac{h_n(Y_{r-} + 1, r)}{h_n(Y_{r-}, r)} dr,$$

$$M^S = Y^S - \beta \int_0^{\cdot} \sum_{n=1}^{N} \mathbb{I}_{A_n} \frac{h_n(Y_{r-} - 1, r)}{h_n(Y_{r-}, r)} dr$$

are all $\mathcal{F}^I$-local martingales. Define $\zeta_m^+ = \inf\{t \in [0, 1] : Y_t = m\}$ and $\zeta_m^- = \inf\{t \in [0, 1] : Y_t = -m\}$. Consider the sequence of stopping time $(\eta_m)_{m \geq 1}$:

$$\eta_m := \left(\mathbb{I}_{\cup_{n=2}^{N-1} A_n} \zeta_m^+ \wedge \zeta_m^- + \mathbb{I}_{A_1} \zeta_m^+ + \mathbb{I}_{A_N} \zeta_m^- \right) \wedge (1 - 1/m).$$

It follows from the definition of h_n that each $h_n(Y_t, t)$ on A_n is bounded away from zero uniformly in $t \in [0, \eta_m]$. This implies that ℓ^{η_m} is bounded, hence ℓ^{η_m} is an $\mathcal{F}^I$-martingale. The construction of Y yields $\lim_{m \to \infty} \eta_m = 1$. Therefore, ℓ is a positive $\mathcal{F}^I$-local martingale, hence also a supermartingale, on $[0, 1)$.

Define $\ell_1 := \lim_{t \to 1} \ell_t$, which exists and is finite due to Doob's supermartingale convergence theorem. This implies $h_n(Y_{1-}, 1) > 0$ on A_n. On the other hand, the construction of Y yields Y^S (resp., Y^B) does not jump at time 1 $\mathbb{P}$-a.s. when $Y_{1-} \leq \underline{m}_n$ (resp., $Y_{1-} \geq \overline{m}_n$). Therefore $h_n(Y_1, 1) > 0$ on A_n. However, $h_n(\cdot, 1)$ by definition can only be either 0 or 1. Hence $Y_1 \in [a_n, a_{n+1})$ on A_n for each $n = 1, \ldots, N$, and the statement (i) is confirmed.

As for the statement (ii), we will prove that Y^B is an $\mathcal{F}^Y$-adapted Poisson process. The similar argument can be applied to Y^S as well. In view of the $\mathcal{F}^I$-intensity of Y^B calculated in Lemma 5.4, one has that, for each $i \geq 1$,

$$Y_{\cdot \wedge \tau_i \wedge 1}^B - \beta \left(\int_0^{\cdot \wedge \tau_i \wedge 1} \sum_{n=1}^{N} \mathbb{I}_{A_n} \frac{h_n(Y_{u-} + 1, u)}{h_n(Y_{u-}, u)} du \right)$$

is an $\mathcal{F}^I$-martingale, where τ_i is the ith jump time of Y. We will show in the next paragraph that, when stopped at $\tau_i \wedge 1$, Y^B is a Poisson process in $\mathcal{F}^Y$ by showing that $(Y_{\tau_i \wedge t}^B - \beta(\tau_i \wedge t))_{t \in [0,1]}$ is an $\mathcal{F}^Y$-martingale. (Here note that τ_i is an $\mathcal{F}^Y$-stopping time.) This in turn will imply that Y^B is a Poisson process with intensity β on $[0, \tau \wedge 1)$ where $\tau = \lim_{i \to \infty} \tau_i$ is the explosion time. Since Poisson process does not explode, this will further imply $Y_{\tau \wedge 1}^B < \infty$ and, therefore, $\tau \geq 1$, $\mathbb{P}$-a.s.

We proceed by projecting the above martingale into $\mathcal{F}^Y$ to see that

$$Y^B - \beta \int_0^{\cdot} \sum_{n=1}^{N} \mathbb{P}(A_n | \mathcal{F}_r^Y) \frac{h_n(Y_{r-} + 1, r)}{h_n(Y_{r-}, r)} dr$$

is an $\mathcal{F}^Y$-martingale when stopped at $\tau_i \wedge 1$. Therefore, it remains to show that, for almost all $t \in [0,1)$, on $[t \leq \tau_i]$,

$$
(5.3) \qquad \sum_{n=1}^{N} \mathbb{P}(A_n | \mathcal{F}_t^Y) \frac{h_n(Y_{t-} + 1, t)}{h_n(Y_{t-}, t)} = 1, \quad \mathbb{P}\text{-a.s.}.
$$

To this end, we will show, on $[t \leq \tau_i]$,

$$
(5.4) \qquad \mathbb{P}(A_n | \mathcal{F}_t^Y) = h_n(Y_t, t) \quad \text{for } t \in [0,1).
$$

Then (5.3) follows since $Y_t \neq Y_{t-}$ only for countably many times.

We have seen that $(\ell_{u \wedge \tau_i})_{u \in [0,t]}$ is a strictly positive $\mathcal{F}^I$-martingale for each i. Define a probability measure $\mathbb{Q}^i \sim \mathbb{P}$ on $\mathcal{F}_t^I$ via $d\mathbb{Q}^i/d\mathbb{P}|_{\mathcal{F}_t^I} = \ell_{\tau_i \wedge t}$. It follows from Girsanov's theorem that Y^B is a Poisson process when stopped at $\tau_i \wedge t$ and with intensity β under $\mathbb{Q}^i$. Therefore, they are independent from A_n under $\mathbb{Q}^i$. Then, for $t < 1$, we obtain from the Bayes's formula that

$$
(5.5) \qquad
\begin{aligned}
\mathbb{I}_{\{r \leq \tau_i \wedge t\}} \mathbb{P}(A_n | \mathcal{F}_r^Y) &= \mathbb{I}_{\{r \leq \tau_i \wedge t\}} \frac{\mathbb{E}^{\mathbb{Q}^i}[\mathbb{I}_{A_n} \ell_r^{-1} | \mathcal{F}_r^Y]}{\mathbb{E}^{\mathbb{Q}^i}[\ell_r^{-1} | \mathcal{F}_r^Y]} \\
&= \mathbb{I}_{\{r \leq \tau_i \wedge t\}} \frac{\mathbb{E}^{\mathbb{Q}^i}[\mathbb{I}_{A_n} \frac{h_n(Y_r, r)}{h_n(0,0)} | \mathcal{F}_r^Y]}{\mathbb{E}^{\mathbb{Q}^i}[\sum_{n=1}^{N} \mathbb{I}_{A_n} \frac{h_n(Y_r, r)}{h_n(0,0)} | \mathcal{F}_r^Y]} \\
&= \mathbb{I}_{\{r \leq \tau_i \wedge t\}} h_n(Y_r, r),
\end{aligned}
$$

where the third identity follows from the aforementioned independence of Y and A_n under $\mathbb{Q}^i$ along with the fact that $\mathbb{Q}^i$ does not change the probability of $\mathcal{F}_0^I$ measurable events so that $\mathbb{Q}^i(A_n) = \mathbb{P}(A_n) = h_n(0,0)$. As result, (5.4) follows from (5.5) after sending $i \to \infty$.

Since Y^B and Y^S are $\mathcal{F}^Y$-Poisson processes and they do not jump simultaneously by their construction, they are then independent. To show the strategy $(X^B, X^S; \mathcal{F}^I)$ constructed is admissible, it remains to show both $\mathbb{E}[X_1^B \mathbb{I}_{A_n}]$ and $\mathbb{E}[X_1^S \mathbb{I}_{A_n}]$ are finite for each $n = 1, \ldots, N$. To this end, for each n, $\mathbb{E}[X_1^B \mathbb{I}_{A_n}] = \mathbb{E}[X_1^{B,B} \mathbb{I}_{A_n}] + \mathbb{E}[X_1^{B,S} \mathbb{I}_{A_n}]$, where $\mathbb{E}[X_1^{B,S} \mathbb{I}_{A_n}] \leq \mathbb{E}[Z^S] < \infty$ and $\mathbb{E}[X_1^{B,B} \mathbb{I}_{A_n}] \leq \mathbb{E}[Y_1^B \mathbb{I}_{A_n}] + \mathbb{E}[X_1^{S,B} \mathbb{I}_{A_n}] \leq \mathbb{E}[Z_1^B \mid Z \in [a_n, a_{n+1})] + \mathbb{E}[Z_1^S] < \infty$. A similar argument also implies $\mathbb{E}[X_1^S \mathbb{I}_{A_n}] < \infty$. Finally, since $N < \infty$, p is bounded, Definition 2.2(iv) is verified using $\mathbb{E}[X_1^B \mathbb{I}_{A_n}], \mathbb{E}[X_1^S \mathbb{I}_{A_n}] < \infty$ for each $n \in \{1, \ldots, N\}$. ∎

6. Convergence. Collecting results from previous sections, we will prove Theorems 2.12 and 2.13 in this section. Let us first construct a sequence of random variables $(\tilde{v}^\delta)_{\delta > 0}$, each of which will be the fundamental value in the Glosten–Milgrom model with order size δ.

Adding to the sequence of canonical spaces $(\Omega^\delta, \mathcal{F}^{Z,\delta}, (\mathcal{F}_t^{Z,\delta})_{t \in [0,1]}, \mathbb{P}^\delta)$, defined at the beginning of section 2.2, we introduce $(\Omega^0, \mathcal{F}^0, (\mathcal{F}_t^0)_{t \in [0,1]}, \mathbb{P}^0)$, where $\Omega^0 = \mathbb{D}([0,1], \mathbb{R})$ is the space of $\mathbb{R}$-valued càdlàg functions on $[0,1]$ with coordinate process Z^0, and $\mathbb{P}^0$ is the Wiener measure. Denote by $\mathbb{P}^{0,y}$ the Wiener measure under which $Z_0^0 = y$ a.s. Let us now define a $\mathbb{R} \cup \{-\infty, \infty\}$-valued sequence $(a_n^0)_{n=1,\ldots,N+1}$ via

$$
a_1^0 = -\infty, \quad a_n^0 = \Phi^{-1}(p_1 + \cdots + p_{n-1}), \quad n = 2, \ldots, N+1,
$$

where $\Phi(\cdot) = \int_{-\infty}^{\cdot} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$. Using this sequence, one can define a pricing rule following the same recipe in (2.5):

$$(6.1) \qquad p^0(y, t) := \sum_{n=1}^{N} v_n h_n^0(y, t), \quad y \in \mathbb{R}, \quad t \in [0, 1], \quad n \in \{1, \ldots, N\},$$

where $h_n^0(y, t) := \mathbb{P}^{0,y}(Z_{1-t}^0 \in [a_n^0, a_{n+1}^0)) = \Phi(a_{n+1}^0 - y) - \Phi(a_n^0 - y)$.

As we will see later, this is exactly the pricing rule in the Kyle–Back equilibrium. Moreover, the sequence $(a_n^\delta)_{n=1,\ldots,N+1}$, associated to $(\tilde{v}^\delta)_{\delta>0}$ constructed below, converges to $(a_n^0)_{n=1,\ldots,N+1}$ as $\delta \downarrow 0$, helping to verify Definition 2.11(i).

Lemma 6.1. *For any $\tilde{v}$ with distribution* (1.1) *where N may not be finite, there exists a sequence of random variables $(\tilde{v}^\delta)_{\delta>0}$, each of which takes value in $\{v_1, \ldots, v_N\}$, such that*
   (i) *Assumption* 4.1 *is satisfied when $\tilde{v}$ therein is replaced by each $\tilde{v}^\delta$;*[10]
   (ii) *$Law(\tilde{v}^\delta) \Longrightarrow Law(\tilde{v})$, as $\delta \downarrow 0$. Here $\Longrightarrow$ represents the weak convergence of probability measures.*

Proof. For each $\delta > 0$, $\tilde{v}^\delta$ will be constructed by adjusting p_n in (1.1) to some p_n^δ, $n = 1, \ldots, N$. Starting from $[\tilde{v} = v_1]$, choose $a_1^\delta = -\infty$, $a_2^\delta = \inf\{y \in \delta\mathbb{Z} : \mathbb{P}^\delta(Z_1^\delta \le y) \ge p_1\}$, and set $\mathbb{P}^\delta(\tilde{v}^\delta = v_1) = \mathbb{P}^\delta(Z_1^\delta \in [a_1^\delta, a_2^\delta))$. Moving on to $[\tilde{v}^\delta = v_2]$, choose $a_3^\delta = \inf\{y \in \delta\mathbb{Z} : \mathbb{P}^\delta(Z_1^\delta \le y) \ge p_1 + p_2$ and $(a_2^\delta + y - \delta)/2 \notin \delta\mathbb{Z}\}$ and set $\mathbb{P}^\delta(\tilde{v}^\delta = v_2) = \mathbb{P}^\delta(Z_1^\delta \in [a_2^\delta, a_3^\delta))$. Following this step, we can define a_n^δ inductively. When $N < \infty$, we set $a_{N+1}^\delta = \infty$. This construction gives a sequence of random variables $(\tilde{v}^\delta)_{\delta>0}$ taking values in $\{v_1, \ldots, v_N\}$ such that $\mathbb{P}^\delta(\tilde{v}^\delta = v_n) = p_n^\delta := \mathbb{P}^\delta(Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta))$ with $\sum_{n=1}^{N} p_n^\delta = 1$; moreover each sequence $(a_n^\delta)_{n=1,\ldots,N+1}$ satisfies Assumption 4.1.

It remains to show $Law(\tilde{v}^\delta) \Longrightarrow Law(\tilde{v})$ as $\delta \downarrow 0$. To this end, note that a_n^δ is either the $(\sum_{i=1}^{n-1} p_i)$th quantile of the distribution of Z_1^δ or δ above this quantile. When β^δ is chosen as $1/(2\delta^2)$, it follows from [12, Chapter 6, Theorem 5.4] that $\mathbb{P}^\delta \Longrightarrow \mathbb{P}^0$, in particular, $Law(Z_1^\delta) \Longrightarrow Law(Z_1^0)$. Therefore,

$$(6.2) \qquad \lim_{\delta\downarrow 0} a_n^\delta = a_n^0, \quad n = 1, \ldots, N+1.$$

For any $\epsilon > 0$ and $n \in \{1, \ldots, N\}$, the previous convergence yields the existence of a sufficiently small $\delta_{\epsilon,n}$ such that $[a_n^0 + \epsilon, a_{n+1}^0 - \epsilon) \subseteq [a_n^\delta, a_{n+1}^\delta) \subseteq [a_n^0 - \epsilon, a_{n+1}^0 + \epsilon)$ for any $\delta \le \delta_{\epsilon,n}$. Hence

$$\mathbb{P}^\delta\left(Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta)\right) \le \mathbb{P}^\delta\left(Z_1^\delta \in [a_n^0 - \epsilon, a_{n+1}^0 + \epsilon)\right) \to \mathbb{P}^0\left(Z_1^0 \in [a_n^0 - \epsilon, a_{n+1}^0 - \epsilon)\right),$$

$$\mathbb{P}^\delta\left(Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta)\right) \ge \mathbb{P}^\delta\left(Z_1^\delta \in [a_n^0 + \epsilon, a_{n+1}^0 - \epsilon)\right) \to \mathbb{P}^0\left(Z_1^0 \in [a_n^0 + \epsilon, a_{n+1}^0 - \epsilon)\right),$$

as $\delta \downarrow 0$, where both convergences follow from $Law(Z_1^\delta) \Longrightarrow Law(Z_1^0)$ and the fact that the distribution of Z_1^0 is continuous. Since ϵ is arbitrarily chosen, utilizing the continuity of the distribution for Z_1^0 again, we obtain from the previous two inequalities

$$\lim_{\delta\downarrow 0} \mathbb{P}^\delta\left(Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta)\right) = \mathbb{P}^0\left(Z_1^0 \in [a_n^0, a_{n+1}^0)\right).$$

Hence $\lim_{\delta\downarrow 0} p_n^\delta = p_n^0$ for each $n \in \{1, \ldots N\}$ and $Law(\tilde{v}^\delta) \Rightarrow Law(\tilde{v})$. ∎

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[10]When the order size is δ, Assumption 4.1(iii) reads $(a_n^\delta + a_{n+1}^\delta - \delta)/2 \notin \delta\mathbb{Z}$.

After $(\tilde{v}^\delta)_{\delta>0}$ is constructed, it follows from sections 4 and 5 that a sequence of strategies $(X^{B,\delta}, X^{S,\delta}; \mathcal{F}^{I,\delta})_{\delta>0}$ exists, each of which satisfies conditions in Proposition 4.5. Hence p^δ in (2.5) is rational for each $\delta > 0$. It then remains to verify Definition 2.11(iii) to establish an asymptotic Glosten–Milgrom equilibrium.

Before doing this, we prove Theorem 2.13 first. Let us recall the Kyle–Back equilibrium. Following arguments in [16] and [2], the equilibrium pricing rule is given by (6.1) and the equilibrium demand satisfies the SDE

$$Y^0 = Z^0 + \sum_{n=1}^N \mathbb{I}_{\{\tilde{v}=v_n\}} \int_0^\cdot \frac{\partial_y h_n^0(Y_r^0, r)}{h_n^0(Y_r^0, r)} \, dr,$$

where Z^0 is a $\mathbb{P}^0$-Brownian motion modeling the demand from noise traders. Hence the insider's strategy in the Kyle–Back equilibrium is given by

$$X^0 = \sum_{n=1}^N \mathbb{I}_{\{\tilde{v}=v_n\}} \int_0^\cdot \frac{\partial_y h_n^0(Y_r^0, r)}{h_n^0(Y_r^0, r)} \, dr.$$

Proof of Theorem 2.13. As we have seen in Lemma 6.1, Assumption 4.1 is satisfied by each $\tilde{v}^\delta$. It then follows from Proposition 5.5(i) and (ii) that the distribution of Y^δ on $[\tilde{v}^\delta = v_n]$ is the same as the distribution of Z^δ conditioned on $Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta)$. Denote $Y^{0,n} = Y^0 \mathbb{I}_{\{\tilde{v}=v_n\}}$ as the cumulative demand in the Kyle–Back equilibrium when the fundamental value is v_n. The same argument as in [9, Lemma 5.4] yields

$$Law(Z^\delta \mid Z_1^\delta \in [a_n^\delta, a_{n+1}^\delta)) \Longrightarrow Law(Y^{0,n}), \quad \text{as } \delta \downarrow 0,$$

for each $n \in \{1, \dots, N\}$. It then follows

(6.3) $$Law(Y^\delta; \mathcal{F}^{I,\delta}) \Longrightarrow Law(Y^0; \mathcal{F}^{I,0}), \quad \text{as } \delta \downarrow 0,$$

where the filtration $\mathcal{F}^{I,0}$ is $\mathcal{F}^0$ initially enlarged by $\tilde{v}$. Recall from (5.1) that $Y^\delta = Z^\delta + X^{B,\delta} - X^{S,\delta}$, and moreover $Y^0 = Z^0 + X^0$. Combining (6.3) with $Law(Z^\delta) \Longrightarrow Law(Z^0)$, we conclude from [14, Proposition VI.1.23] that $Law(X^{B,\delta} - X^{S,\delta}) \Longrightarrow Law(X^0)$ as $\delta \downarrow 0$. ∎

In the rest of the section, Definition 2.11(iii) is verified for strategies $(X^{B,\delta}, X^{S,\delta}; \mathcal{F}^{I,\delta})_{\delta>0}$, which concludes the proof of Theorem 2.12. We have seen in Proposition 4.2 that the expected profit of the strategy $(X^{B,\delta}, X^{S,\delta}; \mathcal{F}^{I,\delta})$, constructed in section 5, satisfies

$$\mathcal{J}^\delta(v_n, 0, 0; X^{B,\delta}, X^{S,\delta}) = U^\delta(v_n, 0, 0) - L^\delta(v_n, 0, 0), \quad n \in \{1, \dots, N\},$$

where

(6.4)
$$L^\delta(v_n, 0, 0) = \delta\beta^\delta \, \mathbb{E}^{\delta,0}\left[\int_0^1 (v_n - p^\delta(\underline{m}_n^\delta, r)) \, \mathbb{I}_{\{Y_{r-}^\delta = \overline{m}_n^\delta\}} dr \, \middle| \, \tilde{v}^\delta = v_n\right]$$
$$- \delta\beta^\delta \, \mathbb{E}^{\delta,0}\left[\int_0^1 (v_n - p^\delta(\overline{m}_n^\delta, r)) \, \mathbb{I}_{\{Y_{r-}^\delta = \underline{m}_n^\delta\}} dr \, \middle| \, \tilde{v}^\delta = v_n\right].$$

This expression for L^δ follows from changing the order size in (4.7) from 1 to δ and utilizing $\theta^{B,S,\delta}(\overline{m}_n^\delta, \cdot) = \theta^{S,S,\delta}(\overline{m}_n^\delta, \cdot) = \theta^{S,B,\delta}(\underline{m}_n^\delta, \cdot) = \theta^{B,B,\delta}(\underline{m}_n^\delta, \cdot) = 0$ from Corollary 5.3(i) and (iii),

$\theta^{B,T,\delta} = \theta^{S,T,\delta} \equiv 0$ from Remark 4.6, and the expectations are taken under $\mathbb{P}^{\delta,0}$. Here $\underline{m}_n^{\delta} := \delta \lfloor (a_n + a_{n+1} - \delta)/2\delta \rfloor$ is the largest integer multiple of δ smaller than m_n^{δ} and $\overline{m}_n^{\delta} := \delta \lceil (a_n + a_{n+1} - \delta)/2\delta \rceil$ is the smallest integer multiple of δ larger than m_n^{δ}. To prove Theorem 2.13, let us first show

$$(6.5) \qquad \lim_{\delta \downarrow 0} L^{\delta}(v_n, 0, 0) = 0, \quad n \in \{1, \dots, N\}.$$

In the following development, we fix v_n and denote $L^{\delta} = L^{\delta}(v_n, 0, 0)$.

Before presenting technical proofs for (6.5), let us first introduce a heuristic argument. First, since $\beta^{\delta} = 1/(2\delta^2)$, (6.4) can be rewritten as

$$(6.6) \qquad L^{\delta} = \mathbb{E}^{\delta,0} \left[\overline{I}_1^{\delta,n} \,\middle|\, \tilde{v}^{\delta} = v_n \right] - \mathbb{E}^{\delta,0} \left[\underline{I}_1^{\delta,n} \,\middle|\, \tilde{v}^{\delta} = v_n \right],$$

where

$$\overline{I}_.^{\delta,n} = \int_0^{\cdot} (v_n - p^{\delta}(Y_{r-}^{\delta} - \delta, r)) \, d\mathcal{L}_r^{\delta, \overline{m}_n^{\delta}}, \quad \underline{I}_.^{\delta,n} = \int_0^{\cdot} (v_n - p^{\delta}(Y_{r-}^{\delta} + \delta, r)) \, d\mathcal{L}_r^{\delta, \underline{m}_n^{\delta}},$$

and $\mathcal{L}_.^{\delta,y} = \frac{1}{2\delta} \int_0^{\cdot} \mathbb{I}_{\{Y_{r-}^{\delta} = y\}} \, dr$ is the *scaled occupation time* of Y^{δ} at level y. Here Y^{δ} is, in its natural filtration, the difference of two independent Poisson $Y^{B,\delta}$ and $Y^{S,\delta}$ with jump size δ and intensity β^{δ}; cf. Proposition 5.5(ii). For the integrands in $\overline{I}_.^{\delta,n}$ and $\underline{I}_.^{\delta,n}$, we expect that $v_n - p^{\delta}(Y_.^{\delta} \pm \delta, \cdot) \xrightarrow{\mathcal{L}} v_n - p^0(Y_.^0, \cdot)$, where Y^0 is a $\mathbb{P}^0$-Brownian motion. As for the integrators, we will show both $\mathcal{L}_.^{\delta, \overline{m}_n^{\delta}}$ and $\mathcal{L}_.^{\delta, \underline{m}_n^{\delta}}$ converge weakly to $\mathcal{L}_.^{m_n}$, which is the Brownian local time at level $m_n := (a_n^0 + a_{n+1}^0)/2$. Then the weak convergence of both integrands and integrators yields

$$\overline{I}_.^{\delta,n} \text{ and } \underline{I}_.^{\delta,n} \xrightarrow{\mathcal{L}} I_.^{0,n} := \int_0^{\cdot} (v_n - p^0(Y_r^0, r)) \, d\mathcal{L}_r^{m_n}, \quad \text{as } \delta \downarrow 0.$$

Finally passing the previous convergence to conditional expectation, the two terms on the right-hand side of (6.6) cancel each other in the limit.

Proposition 6.2. *On the family of filtration* $(\mathcal{F}_t^{Y,\delta})_{t \in [0,1], \delta \geq 0}$, *generated by* $(Y^{\delta})_{\delta \geq 0}$,

$$p^{\delta}(Y_.^{\delta} \pm \delta, \cdot) \xrightarrow{\mathcal{L}} p^0(Y_.^0, \cdot) \quad \text{on } \mathbb{D}[0,1) \text{ as } \delta \downarrow 0.$$

Proof. To simplify presentation, we will prove

$$(6.7) \qquad p^{\delta}(Y_.^{\delta}, \cdot) \xrightarrow{\mathcal{L}} p^0(Y_.^0, \cdot) \quad \text{as } \delta \downarrow 0.$$

The assertions with $\pm \delta$ can be proved by replacing Y^{δ} by $Y^{\delta} \pm \delta$. First, applying Itô's formula and utilizing (4.1) yield

$$(6.8) \qquad \begin{aligned} p^{\delta}(Y_.^{\delta}, \cdot) = p^{\delta}(0,0) &+ \int_0^{\cdot} \frac{1}{\delta} \left(p^{\delta}(Y_{r-}^{\delta} + \delta, r) - p^{\delta}(Y_{r-}^{\delta}, r) \right) \, d\overline{Y}_r^{B,\delta} \\ &+ \int_0^{\cdot} \frac{1}{\delta} \left(p^{\delta}(Y_{r-}^{\delta} - \delta, r) - p^{\delta}(Y_{r-}^{\delta}, r) \right) \, d\overline{Y}_r^{S,\delta}, \end{aligned}$$

where $\overline{Y}_{\cdot}^{B,\delta} = Y_{\cdot}^{B,\delta} - \delta\beta^\delta\cdot$ and $\overline{Y}_{\cdot}^{S,\delta} = Y_{\cdot}^{S,\delta} - \delta\beta^\delta\cdot$ are compensated jump processes. For $p^\delta(0,0)$ on the right-hand side, the same argument in Lemma 6.1 yields $\lim_{\delta\downarrow 0} p^\delta(0,0) = p^0(0,0)$. As for the other two stochastic integrals, we will show that they converge weakly to

$$\frac{1}{\sqrt{2}} \int_0^\cdot \partial_y p^0(Y_r^0, r) dW_r^B \quad \text{and} \quad -\frac{1}{\sqrt{2}} \int_0^\cdot \partial_y p^0(Y_r^0, r) dW_r^S, \quad \text{respectively,}$$

where W^B and W^S are two independent Brownian motions. These estimates then imply the right-hand side of (6.8) converges weakly to

$$p^0(0,0) + \int_0^\cdot \partial_y p^0(Y_r^0, r) \, dW_r,$$

where $W = W^B/\sqrt{2} - W^S/\sqrt{2}$ is another Brownian motion. Since p^0 satisfies $\partial_t p^0 + \frac{1}{2}\partial_{yy}^2 p^0 = 0$, the previous process has the same law as $p^0(Y_\cdot^0, \cdot)$. Therefore (6.7) is confirmed.

To prove the aforementioned convergence of stochastic integrals, let us first derive the convergence of $(p^\delta(\cdot + \delta, \cdot) - p^\delta(\cdot, \cdot))/\delta$ on $\mathbb{R} \times [0, 1)$. To this end, it follows from (2.5) that

$$\frac{1}{\delta}(p^\delta(y + \delta, t) - p^\delta(y, t))$$

$$= \frac{1}{\delta} \sum_{n=1}^N v_n \left[\mathbb{P}^{\delta, y+\delta}(Z_{1-t}^\delta \in [a_n^\delta, a_{n+1}^\delta)) - \mathbb{P}^{\delta, y}(Z_{1-t}^\delta \in [a_n^\delta, a_{n+1}^\delta)) \right]$$

$$= \frac{1}{\delta} \sum_{n=1}^N v_n \left[\mathbb{P}^{\delta, y}(Z_{1-t}^\delta = a_n^\delta - \delta) - \mathbb{P}^{\delta, y}(Z_{1-t}^\delta = a_{n+1}^\delta - \delta) \right]$$

$$= \frac{1}{\delta} \sum_{n=1}^N v_n \left[\mathbb{P}^{1,0}\left(Z_{1-t}^1 = \frac{a_n^\delta - \delta - y}{\delta}\right) - \mathbb{P}^{1,0}\left(Z_{1-t}^1 = \frac{a_{n+1}^\delta - \delta - y}{\delta}\right) \right]$$

$$= \sum_{n=1}^N v_n \left[\frac{1}{\delta} e^{-\frac{1-t}{\delta^2}} I_{\left|\frac{a_n^\delta - \delta - y}{\delta}\right|}\left(\frac{1-t}{\delta^2}\right) - \frac{1}{\delta} e^{-\frac{1-t}{\delta^2}} I_{\left|\frac{a_{n+1}^\delta - \delta - y}{\delta}\right|}\left(\frac{1-t}{\delta^2}\right) \right]$$

$$\to \sum_{n=1}^N v_n \left[\frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(a_n^0 - y)^2}{2(1-t)}\right) - \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(a_{n+1}^0 - y)^2}{2(1-t)}\right) \right]$$

$$= \partial_y p^0(y, t), \quad \text{as } \delta \downarrow 0.$$

Here Z_{1-t}^1 is the difference of two independent Poisson random variables with common parameter $(1 - t)\beta^\delta = (1 - t)(2\delta^2)^{-1}$ under $\mathbb{P}^{1,0}$. Hence the fourth identity above follows from the probability distribution function of the *Skellam distribution*: $\mathbb{P}^{1,0}(Z_{1-t}^1 = k) = e^{-2\mu}I_{|k|}(2\mu)$, where $I_{|k|}(\cdot)$ is the *modified Bessel function of the second kind* and $\mu = (1 - t)(2\delta^2)^{-1}$; cf. [20]. The convergence above is locally uniformly in $\mathbb{R} \times [0, 1)$ according to [1, Theorem 2]. The last identity above follows from taking y derivative to $p^0(y, t) = \sum_{n=1}^N (\Phi(\frac{a_{n+1}^0 - y}{\sqrt{1-t}}) - \Phi(\frac{a_n^0 - y}{\sqrt{1-t}}))$; cf. (6.1). Combining the previous locally uniform convergence of $(p^\delta(\cdot + \delta, \cdot) - p^\delta(\cdot, \cdot))/\delta$ with the weak convergence $Y^\delta \xrightarrow{\mathcal{L}} Y^0$ in their natural filtration, we have from [5, Chapter 1,

Theorem 5.5]

$$\frac{1}{\delta}\left(p^\delta(Y^\delta_\cdot + \delta, \cdot) - p^\delta(Y^\delta_\cdot, \cdot)\right) \xrightarrow{\mathcal{L}} \partial_y p^0(Y^0_\cdot, \cdot) \quad \text{on } \mathbb{D}[0,1) \text{ as } \delta \downarrow 0.$$

As for the integrators in (6.8), $\overline{Y}^{B,\delta} \xrightarrow{\mathcal{L}} W^B/\sqrt{2}$ and $\overline{Y}^{S,\delta} \xrightarrow{\mathcal{L}} W^S/\sqrt{2}$. Moreover, both $(\overline{Y}^{B,\delta})_{\delta>0}$ and $(\overline{Y}^{S,\delta})_{\delta>0}$ are *predictable uniform tight* (P-UT), since $\langle \overline{Y}^{B,\delta} \rangle_t = \langle \overline{Y}^{S,\delta} \rangle_t = t/2$, for any $\delta > 0$; cf. [14, Chapter VI, Theorem 6.13(iii)]. Then combining weak convergence of both integrands and integrators, we obtain from [14, Chapter VI, Theorem 6.22] that

$$\int_0^\cdot \frac{1}{\delta}(p^\delta(Y^\delta_{r-} + \delta, r) - p^\delta(Y^\delta_{r-}, r))\, d\overline{Y}^{B,\delta}_r \xrightarrow{\mathcal{L}} \frac{1}{\sqrt{2}} \int_0^\cdot \partial_y p^0(Y^0_r, r)\, dW^B_r \text{ on } \mathbb{D}[0,1) \text{ as } \delta \downarrow 0.$$

A similar weak convergence holds for the other stochastic integral in (6.8) as well. Therefore the claimed weak convergence of stochastic integrals on the right-hand side of (6.8) is confirmed. ∎

Having studied the weak convergence of integrands in $\overline{I}^{\delta,n}$ and $\underline{I}^{\delta,n}$, let us switch our attention to the integrators $\mathcal{L}^{\delta,\overline{m}^\delta_n}$ and $\mathcal{L}^{\delta,\underline{m}^\delta_n}$.

Proposition 6.3. *On the family of filtration* $(\mathcal{F}^{Y,\delta}_t)_{t\in[0,1],\delta\geq 0}$, *for any* $n \in \{1,\ldots,N\}$,

$$\mathcal{L}^{\delta,\overline{m}^\delta_n} \xrightarrow{\mathcal{L}} \mathcal{L}^{m_n} \quad and \quad \mathcal{L}^{\delta,\underline{m}^\delta_n} \xrightarrow{\mathcal{L}} \mathcal{L}^{m_n} \quad on \ \mathbb{D}[0,1] \ as \ \delta \downarrow 0.$$

Proof. For simplicity of presentation, we will prove

(6.9) $$\mathcal{L}^{\delta,0} \xrightarrow{\mathcal{L}} \mathcal{L}^0 \quad \text{as } \delta \downarrow 0.$$

Since $\lim_{\delta\downarrow 0} \overline{m}^\delta_n = \lim_{\delta\downarrow 0} \underline{m}^\delta_n = m_n$ follows from (6.2), the statement of the proposition follows from replacing Y^δ by $Y^\delta - \overline{m}^\delta_n$ (or by $Y^\delta - \underline{m}^\delta_n$) and Y^0 by $Y^0 - m_n$ in the rest of the proof. To prove (6.9), applying Itô's formula to $|Y^\delta_\cdot|$ yields

(6.10)
$$\begin{aligned}
|Y^\delta_\cdot| &= \sum_{r\leq\cdot}\left(|Y^\delta_r| - |Y^\delta_{r-}|\right) \\
&= \int_0^\cdot \left(|Y^\delta_{r-} + \delta| - |Y^\delta_{r-}|\right) d(Y^{B,\delta}_r/\delta - \beta^\delta r) \\
&\quad + \int_0^\cdot \left(|Y^\delta_{r-} - \delta| - |Y^\delta_{r-}|\right) d(Y^{S,\delta}_r/\delta - \beta^\delta r) \\
&\quad + \int_0^\cdot \left(|Y^\delta_{r-} + \delta| + |Y^\delta_{r-} - \delta| - 2|Y^\delta_{r-}|\right) \beta^\delta dr \\
&= \int_0^\cdot \left(|Y^\delta_{r-} + \delta| - |Y^\delta_{r-}|\right) d\overline{Y}^{B,\delta}_r/\delta + \int_0^\cdot \left(|Y^\delta_{r-} - \delta| - |Y^\delta_{r-}|\right) d\overline{Y}^{S,\delta}_r/\delta \\
&\quad + \int_0^\cdot \frac{1}{\delta}\mathbb{I}_{\{Y^\delta_{r-}=0\}} dr,
\end{aligned}$$

where the third identity follows from $|y+\delta| + |y-\delta| - 2|y| = 2\delta\,\mathbb{I}_{\{y=0\}}$ for any $y \in \mathbb{R}$. On the other hand, the Tanaka formula for Brownian motion is

(6.11) $$|Y^0_\cdot| = \int_0^\cdot sgn(Y^0_r)\, dY^0_r + 2\mathcal{L}^0_\cdot,$$

where $sgn(x) = 1$ when $x > 0$ or -1 when $x \leq 0$.

The convergence (6.9) is then confirmed by comparing both sides of (6.10) and (6.11). To this end, since $Y^\delta \xrightarrow{\mathcal{L}} Y^0$ and the absolute value is a continuous function, then $|Y^\delta| \xrightarrow{\mathcal{L}} |Y^0|$ follows from [5, Chapter 1, Theorem 5.1]. Then (6.9) is confirmed as soon as we prove the martingale term on the right-hand side of (6.10) converges weakly to the martingale in (6.11), which we prove in the next result. ∎

Lemma 6.4. *Let* $M^\delta := \int_0^\cdot \left(|Y_{r-}^\delta + \delta| - |Y_{r-}^\delta| \right) d\overline{Y}_r^{B,\delta}/\delta + \int_0^\cdot \left(|Y_{r-}^\delta - \delta| - |Y_{r-}^\delta| \right) d\overline{Y}_r^{S,\delta}/\delta$ *and* $M^0 := \int_0^\cdot sgn(Y_r^0) \, dY_r^0$. *Then* $M^\delta \xrightarrow{\mathcal{L}} M^0$ *on* $\mathbb{D}[0,1]$ *as* $\delta \downarrow 0$.

Proof. Define $f^\delta(y) := \frac{1}{\delta}(|y + \delta| - |y|)$ for $y \in \mathbb{R}$ and observe

$$f^\delta(y) = \begin{cases} 1, & y \geq 0, \\ 2y/\delta + 1, & -\delta < y < 0, \\ -1, & y \leq -\delta. \end{cases}$$

It is clear that f^δ converges to $sgn(\cdot)$ locally uniformly on $\mathbb{R} \setminus \{0\}$. On the other hand, $Y^\delta \xrightarrow{\mathcal{L}} Y^0$ and the law of Y^0 is continuous. It then follows from [5, Chapter 1, Theorem 5.5] that $f^\delta(Y^\delta) \xrightarrow{\mathcal{L}} sgn(Y^0)$. As for the integrators $(\overline{Y}^{B,\delta})_{\delta > 0}$, as we have seen in the proof of Proposition 6.2, they converge weakly to $W^B/\sqrt{2}$ and are P-UT. Then [14, Chapter VI, Theorem 6.22] implies

$$\int_0^\cdot \left(|Y_{r-}^\delta + \delta| - |Y_{r-}^\delta| \right) d\overline{Y}_r^{B,\delta}/\delta \xrightarrow{\mathcal{L}} \frac{1}{\sqrt{2}} \int_0^\cdot sgn(Y_r^0) \, dW_r^B.$$

A similar argument yields

$$\int_0^\cdot \left(|Y_{r-}^\delta - \delta| - |Y_{r-}^\delta| \right) d\overline{Y}_r^{S,\delta}/\delta \xrightarrow{\mathcal{L}} -\frac{1}{\sqrt{2}} \int_0^\cdot sgn(Y_r^0) \, dW_r^S.$$

Here W^B and W^S are independent Brownian motions. Defining $W = W^B/\sqrt{2} - W^S/\sqrt{2}$, we obtain from the previous two convergences that

$$M^\delta \xrightarrow{\mathcal{L}} \int_0^\cdot sgn(Y_r^0) \, dW_r, \quad \text{which has the same law as } M^0. \quad ∎$$

Propositions 6.2 and 6.3 combined yield the weak convergence of $(\overline{I}^{\delta,n})_{\delta > 0}$ and $(\underline{I}^{\delta,n})_{\delta > 0}$. Moreover the sequence of local time in Proposition 6.3 also converges in expectation.

Corollary 6.5. *On the family of filtration* $(\mathcal{F}_t^{Y,\delta})_{t \in [0,1], \delta \geq 0}$, *for any* $n \in \{1, \ldots, N\}$,

$$\overline{I}^{\delta,n} \text{ and } \underline{I}^{\delta,n} \xrightarrow{\mathcal{L}} I^{0,n} \quad \text{on } \mathbb{D}[0,1) \text{ as } \delta \downarrow 0.$$

Proof. The statement follows from combining Propositions 6.2 and 6.3 and appealing to [14, Chapter VI, Theorem 6.22]. In order to apply the previous result, we need to show that both $(\mathcal{L}^{\delta,\overline{m}_n^\delta})_{\delta > 0}$ and $(\mathcal{L}^{\delta,\underline{m}_n^\delta})_{\delta > 0}$ are P-UT. This property will be verified for $(\mathcal{L}^{\delta,\overline{m}_n^\delta})_{\delta > 0}$. The same argument works for $(\mathcal{L}^{\delta,\underline{m}_n^\delta})_{\delta > 0}$ as well. To this end, since $\mathcal{L}^{\delta,\overline{m}_n^\delta}$ is a nondecreasing

process, $(\mathcal{L}^{\delta,\overline{m}_n^\delta})_{\delta>0}$ is P-UT as soon as $(Var(\mathcal{L}^{\delta,\overline{m}_n^\delta})_1)_{\delta>0}$ is tight, where $Var(X)$ is the variation of the process X, cf. [14, Chapter VI, 6.6]. Note $Var(\mathcal{L}^{\delta,\overline{m}_n^\delta})_1 = \mathcal{L}_1^{\delta,\overline{m}_n^\delta}$, since $\mathcal{L}^{\delta,\overline{m}_n^\delta}$ is nondecreasing. Then the tightness of $(Var(\mathcal{L}^{\delta,\overline{m}_n^\delta})_1)_{\delta>0}$ is implied by Proposition 6.3. ∎

Corollary 6.6. *For any $n \in \{1,\dots,N\}$ and $t \in [0,1]$,*

$$\lim_{\delta\downarrow 0} \mathbb{E}^{\delta,0}\left[\mathcal{L}_t^{\delta,\overline{m}_n^\delta}\right] = \lim_{\delta\downarrow 0} \mathbb{E}^{\delta,0}\left[\mathcal{L}_t^{\delta,\underline{m}_n^\delta}\right] = \mathbb{E}^{0,0}\left[\mathcal{L}_t^{m_n}\right].$$

Proof. For simplicity of presentation, we will prove $\lim_{\delta\downarrow 0} \mathbb{E}^{\delta,0}[\mathcal{L}_t^{\delta,0}] = \mathbb{E}^{0,0}[\mathcal{L}_t^0]$. Then the statement of the corollary follows from replacing Y_t^δ by $Y_t^\delta - \overline{m}_n^\delta$ or $Y_t^\delta - \underline{m}_n^\delta$ in the rest of the proof. Since the stochastic integrals in (6.10) are $\mathbb{P}^{\delta,0}$-martingales,

$$2\mathbb{E}^{\delta,0}[\mathcal{L}_t^{\delta,0}] = \mathbb{E}^{\delta,0}[|Y_t^\delta|].$$

Since $\mathbb{E}[(Y_t^\delta)^2] = t$ for any $\delta > 0$, $(|Y_t^\delta|; \mathbb{P}^{\delta,0})_{\delta>0}$ is uniformly integrable. It then follows from [12, Appendix, Proposition 2.3] and $Law(|Y_t^\delta|) \Longrightarrow Law(|Y_t^0|)$ that $\lim_{\delta\downarrow 0} \mathbb{E}^{\delta,0}[|Y_t^\delta|] = \mathbb{E}^{0,0}[|Y_t^0|]$. Therefore the claim follows since $\mathbb{E}^{0,0}[|Y_t^0|] = 2\mathbb{E}^{0,0}[\mathcal{L}_t^0]$; cf. (6.11). ∎

Collecting the previous results, the following result confirms (6.5).

Proposition 6.7. *For the strategies $(X^{B,\delta}, X^{S,\delta}; \mathcal{F}^{I,\delta})_{\delta>0}$ constructed in section 5,*

$$\lim_{\delta\downarrow 0} L^\delta(v_n, 0, 0) = 0, \quad n \in \{1,\dots,N\}.$$

Proof. Fix any $\epsilon \in (0,1)$. Corollary 6.5 implies that $Law(\overline{I}_{1-\epsilon}^{\delta,n}; \mathcal{F}^{Y,\delta}) \Longrightarrow Law(I_{1-\epsilon}^{0,n}; \mathcal{F}^0)$. Recall $Law(\tilde{v}^\delta) \Longrightarrow Law(\tilde{v})$ from Lemma 6.1. It then follows that

$$Law\left(\overline{I}_{1-\epsilon}^{\delta,n} \mathbb{I}_{\{\tilde{v}^\delta = v_n\}}; \mathcal{F}^{Y,\delta}\right) \Longrightarrow Law\left(I_{1-\epsilon}^{0,n} \mathbb{I}_{\{\tilde{v}=v_n\}}; \mathcal{F}^0\right).$$

On the other hand, since N is finite, p^δ is bounded uniformly in δ. Then there exists constant C such that $|\overline{I}_{1-\epsilon}^{\delta,n}| \mathbb{I}_{\{\tilde{v}^\delta = v_n\}} \leq C\mathcal{L}_{1-\epsilon}^{\delta,\overline{m}_n^\delta}$, where the expectation of the upper bound converges; cf. Corollary 6.6. Therefore appealing to [12, Appendix, Theorem 1.2] and utilizing $\lim_{\delta\downarrow 0} \mathbb{P}^\delta(\tilde{v}^\delta = v_n) = \mathbb{P}^0(\tilde{v} = v_n)$ from Lemma 6.1, we obtain

(6.12)
$$\mathbb{E}^{\delta,0}\left[\overline{I}_{1-\epsilon}^{\delta,n} \mid \tilde{v}^\delta = v_n\right] = \frac{\mathbb{E}^{\delta,0}\left[\overline{I}_{1-\epsilon}^{\delta,n} \mathbb{I}_{\{\tilde{v}^\delta = v_n\}}\right]}{\mathbb{P}^\delta(\tilde{v}^\delta = v_n)}$$
$$\to \frac{\mathbb{E}^{0,0}\left[I_{1-\epsilon}^{0,n} \mathbb{I}_{\{\tilde{v}=v_n\}}\right]}{\mathbb{P}^0(\tilde{v} = v_n)} = \mathbb{E}^{0,0}\left[I_{1-\epsilon}^{0,n} \mid \tilde{v} = v_n\right],$$

as $\delta \downarrow 0$. On the other hand, since $\lim_{\delta\downarrow 0} \mathbb{P}^\delta(\tilde{v}^\delta) = \mathbb{P}^0(\tilde{v} = v_n) > 0$, there exists a constant C such that

$$\mathbb{E}^{\delta,0}\left[\left|\overline{I}_1^{\delta,n} - \overline{I}_{1-\epsilon}^{\delta,n}\right| \mid \tilde{v}^\delta = v_n\right] \leq C\,\mathbb{E}^{\delta,0}\left[\mathcal{L}_1^{\delta,\overline{m}_n^\delta} - \mathcal{L}_{1-\epsilon}^{\delta,\overline{m}_n^\delta}\right] \to C\,\mathbb{E}^{0,0}\left[\mathcal{L}_1^{m_n} - \mathcal{L}_{1-\epsilon}^{m_n}\right],$$

as $\delta \downarrow 0$, where the convergence follows from applying Corollary 6.6 twice. For the difference of Brownian local time, Lévy's result (cf. [15, Chapter 3, Theorem 6.17]) yields

$$\mathbb{E}^{0,0}\left[\mathcal{L}_1^{m_n} - \mathcal{L}_{1-\epsilon}^{m_n}\right] = \mathbb{E}^{0,-m_n}\left[\mathcal{L}_1^0 - \mathcal{L}_{1-\epsilon}^0\right]$$
$$= \frac{1}{2}\mathbb{E}^{0,-m_n}\left[\sup_{r\leq 1} Y_r^0 - \sup_{r\leq 1-\epsilon} Y_r^0\right] = \sqrt{\frac{2}{\pi}}(1 - \sqrt{1-\epsilon}),$$

where Y^0 is a $\mathbb{P}^0$-Brownian motion and $\mathbb{E}^{0,y}[\sup_{r\leq t} Y_r^0] = \sqrt{2t/\pi} + y$ is utilized to obtain the third identity. Now the previous two estimates combined yield

(6.13) $\quad \limsup_{\delta\downarrow 0} \mathbb{E}^{\delta,0}\left[\left|\overline{I}_1^{\delta,n} - \overline{I}_{1-\epsilon}^{\delta,n}\right|\,\Big|\,\tilde{v}^\delta = v_n\right] \leq C(1 - \sqrt{1-\epsilon}) \quad$ for another constant C.

Estimates in (6.12) and (6.13) also hold when $\overline{I}^{\delta,n}$ is replaced by $\underline{I}^{\delta,n}$. These estimates then yield

$$\mathbb{E}^{\delta,0}\left[\overline{I}_1^{\delta,n} - \underline{I}_1^{\delta,n}\,|\,\tilde{v}^\delta = v_n\right]$$
$$\leq \mathbb{E}^{\delta,0}\left[\overline{I}_{1-\epsilon}^{\delta,n} - \underline{I}_{1-\epsilon}^{\delta,n}\,|\,\tilde{v}^\delta = v_n\right] + \mathbb{E}^{\delta,0}\left[\left|\overline{I}_1^{\delta,n} - \overline{I}_{1-\epsilon}^{\delta,n}\right|\,\Big|\,\tilde{v}^\delta = v_n\right]$$
$$+ \mathbb{E}^{\delta,0}\left[\left|\underline{I}_1^{\delta,n} - \underline{I}_{1-\epsilon}^{\delta,n}\right|\,\Big|\,\tilde{v}^\delta = v_n\right].$$

Sending $\delta \downarrow 0$ in the previous inequality, the first term on the right side vanishes in the limit, and because both conditional expectations converge to the same limit, the limit superior of both second and third terms are less than $C(1 - \sqrt{1-\epsilon})$. Now since ϵ is arbitrarily choose, sending $\epsilon \to 1$ yields $\limsup_{\delta\downarrow 0} \mathbb{E}^{\delta,0}[\overline{I}_1^{\delta,n} - \underline{I}_1^{\delta,n}\,|\,\tilde{v}^\delta = v_n] \leq 0$. A similar argument leads to $\liminf_{\delta\downarrow 0} \mathbb{E}^{\delta,0}[\overline{I}_1^{\delta,n} - \underline{I}_1^{\delta,n}\,|\,\tilde{v}^\delta = v_n] \geq 0$, which concludes the proof. ∎

Finally the proof of Theorem 2.12 is concluded.

Proof of Theorem 2.12. It remains to verify Definition 2.11(iii). Fix v_n and $(y,t) = (0,0)$ throughout the proof. We have seen from Proposition 4.4 that $V^\delta \leq U^{S,\delta}$. On the other hand, Proposition 4.2 yields $\mathcal{J}(X^{B,\delta}, X^{S,\delta}) = U^\delta - L^\delta$. Therefore

$$\sup_{(X^B,X^S)\text{ admissible}} \mathcal{J}^\delta(X^B, X^S) - \mathcal{J}^\delta(X^{B,\delta}, X^{S,\delta}) \leq U^{S,\delta} - U^\delta + L^\delta.$$

Since $\lim_{\delta\downarrow 0} L^\delta = 0$ is proved in Proposition 6.7, it suffices to show $\lim_{\delta\downarrow 0} U^{S,\delta} - U^\delta = 0$. To this end, from the definition of $U^{S,\delta}$,

(6.14) $\quad U^{S,\delta}(0,0) - U^\delta(0,0) = (U^\delta(-\delta,0) - U^\delta(0,0))\mathbb{I}_{\{0\leq \underline{m}_n^\delta\}} = \delta(v_n - p^\delta(0,0))\mathbb{I}_{\{0\leq \underline{m}_n^\delta\}}.$

The second identity above follows from (4.12), which reads $U^\delta(y,t) - U^\delta(y-1,t) + \delta(v_n - p^\delta(y,t)) = 0$ for $y \leq \underline{m}_n^\delta$ when the order size is δ. Therefore $\lim_{\delta\downarrow 0} U^{S,\delta} - U^\delta = 0$ is confirmed after sending $\delta \downarrow 0$ in (6.14). ∎

Appendix A. Viscosity solutions. Proposition 3.1 will be proved in this section. To simplify notation, $\delta = 1$ and $\tilde{v} = v_n$ are fixed throughout this section. First let us recall the

definition of the (discontinuous) viscosity solution to (2.8). Given a locally bounded function[11] $v : \mathbb{Z} \times [0,1] \rightarrow \mathbb{R}$, its *upper-semicontinuous envelope* v^* and *lower-semicontinuous envelope* v_* are defined as

$$(\text{A.1}) \qquad v^*(y,t) := \limsup_{t' \rightarrow t} v(y,t'), \quad v_*(y,t) := \liminf_{t' \rightarrow t} v(y,t'), \quad (y,t) \in \mathbb{Z} \times [0,1].$$

Definition A.1. *Let* $v : \mathbb{Z} \times [0,1] \rightarrow \mathbb{R}$ *be locally bounded.*
 (i) v *is a* (discontinuous) *viscosity subsolution of* (2.8) *if*

$$-\varphi_t(y,t) - H(y,t,v^*) \leq 0$$

for all $y \in \mathbb{Z}$ $t \in [0,1)$ *and any function* $\varphi : \mathbb{Z} \times [0,1] \rightarrow \mathbb{R}$ *continuously differentiable in the second variable such that* (y,t) *is a maximum point of* $v^* - \varphi$.
 (ii) v *is a* (discontinuous) *viscosity supersolution of* (2.8) *if*

$$-\varphi_t(y,t) - H(y,t,v_*) \geq 0$$

for all $y \in \mathbb{Z}$, $t \in [0,1)$ *and any function* $\varphi : \mathbb{Z} \times [0,1] \rightarrow \mathbb{R}$ *continuously differentiable in the second variable such that* (y,t) *is a minimum point of* $v_* - \varphi$.
 (iii) *We say that* v *is a* (discontinuous) *viscosity solution of* (2.8) *if it is both subsolution and supersolution.*

For the insider's optimization problem, let us recall the *dynamic programming principle* (cf., e.g., [19, Remark 3.3.3]). Given an admissible strategy (X^B, X^S), any $[t,1]$-valued stopping time τ, and the fundamental value v_n, denote the associated profit by

$$\begin{aligned}
\mathcal{I}_{t,\tau}^n &:= \int_t^\tau (v_n - p(Y_{r-} + 1, r)) dX_r^{B,B} + \int_t^\tau (v_n - p(Y_{r-} + 2, r)) dX_r^{B,T} \\
&\quad + \int_t^\tau (v_n - p(Y_{r-}, r)) dX_r^{B,S} - \int_t^\tau (v_n - p(Y_{r-} - 1, r)) dX_r^{S,S} \\
&\quad - \int_t^\tau (v_n - p(Y_{r-} - 2, r)) dX_r^{S,T} - \int_t^\tau (v_n - p(Y_{r-}, r)) dX_r^{S,B},
\end{aligned}$$

where $Y = Z + X^B - X^S$. Then the dynamic programming principle reads as follows:
 DPP (i) For any admissible strategy (X^B, X^S) and any $[t,1]$-valued stopping time τ,

$$V(y,t) \geq \mathbb{E}^{y,t}[V(\tau, Y_\tau) + \mathcal{I}_{t,\tau}^n].$$

DPP (ii) For any $\epsilon > 0$, there exists an admissible strategy (X^B, X^S) such that for all $[t,1]$-valued stopping time τ,

$$V(y,t) - \epsilon \leq \mathbb{E}^{y,t}[V(\tau, Y_\tau) + \mathcal{I}_{t,\tau}^n].$$

---

[11]Since the state space $\mathbb{Z}$ is discrete, v is locally bounded if $v(y,\cdot)$ is bounded in any bounded neighborhood of t and any fixed $y \in \mathbb{Z}$.

The viscosity solution property of the value function V follows from the dynamic programming principle and standard arguments in viscosity solutions (see, e.g., [19, Propositions 4.3.1 and 4.3.2][12]). Therefore Proposition 3.1(i) is verified.

Remark A.2. The proof of DPP (ii) utilizes the measurable selection theorem. To avoid this technical result, one could employ the *weak* dynamic programming principle in [6]. For the insider's optimization problem, the weak dynamic programming principle reads as follows:

WDPP (i) For any $[t, 1]$-valued stopping time τ,

$$V(y,t) \leq \sup_{(X^B, X^S)} \mathbb{E}^{y,t} \left[V^*(\tau, Y_\tau) + \mathcal{I}^n_{t,\tau} \right].$$

WDPP (ii) For any $[t, 1]$-valued stopping time τ and any upper-semicontinuous function φ on $\mathbb{Z} \times [0, 1]$ such that $V \geq \varphi$, then

$$V(y,t) \geq \sup_{(X^B, X^S)} \mathbb{E}^{y,t} \left[\varphi(\tau, Y_\tau) + \mathcal{I}^n_{t,\tau} \right].$$

Conditions A1, A2, and A3 from Assumption A in [6] are clearly satisfied in the current context. Condition A4 from Assumption A can be verified following the same argument in [6, Proposition 5.4]. Therefore the aforementioned weak dynamic programming principle holds. Hence the value function is a viscosity solution to (2.8) following from arguments similar to [6, section 5.2].

Now the proof of Proposition 3.1(ii) is presented. To prove $(v_n, y, t, V) \in \text{dom}(H)$, observe from the viscosity supersolution property of V that $H(v_n, y, t, V_*) < \infty$, hence $(v_n, y, t, V_*) \in \text{dom}(H)$. On the other hand, for any integrable intensities $\theta^{i,j}$, $i \in \{B, S\}$ and $j \in \{B, T, S\}$, due to Definition 2.2(iv), one can show $\mathbb{E}^{y,t}[\mathcal{I}^n_{t,1}]$ is a continuous function in t. As a supremum of a family of continuous function (cf. (2.7)), V is then lower-semicontinuous in t. Therefore $V_* \equiv V$, which implies $(v_n, y, t, V) \in \text{dom}(H)$ for any v_n, $(y, t) \in \mathbb{Z} \times [0, 1)$. It then follows from (3.1) and (3.2) that

(A.2) $V(y-1, t) + p(y-1, t) - v_n \leq V(y, t) \leq V(y-1, t) + p(y, t) - v_n$ for any $(y, t) \in \mathbb{Z} \times [0, 1)$.

Taking limit supremum in t in the previous inequalities and utilizing the continuity of $t \mapsto p(y, t)$, it follows that the previous inequalities still hold when V is replaced by V^*, which means $(v_n, y, t, V^*) \in \text{dom}(H)$ for any v_n, $(y, t) \in \mathbb{Z} \times [0, 1)$. As a result, $H(v_n, y, t, V_*)$ and $H(v_n, y, t, V^*)$ have the reduced form (3.3), where V is replaced by V_* and V^*, respectively. Hence Definition A.1 implies that V is a viscosity solution of (3.4).

To prove Proposition 3.1(iii) and (iv), let us first derive a comparison result for (3.4). The function $v : \mathbb{Z} \times [0, 1] \to \mathbb{R}$ has at most polynomial growth in its first variable if there exist C and n such that $|v(y, t)| \leq C(1 + |y|^n)$ for any $(y, t) \in \mathbb{Z} \times [0, 1]$.

Lemma A.3. *Assume that u (resp., v) has at most polynomial growth and that it is upper-semicontinuous viscosity subsolution (resp., lower-semicontinuous supersolution) to (3.4). If $u(\cdot, 1) \leq v(\cdot, 1)$, then $u \leq v$ in $\mathbb{Z} \times [0, 1)$.*

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[12]Therein the stopping time τ_m can be chosen as the first jump time of Y, where $Y_{t_m} = y$ for a sequence $(t_m)_m \to \bar{t}$.

Assume this comparison result for a moment. Inequality (A.2) and Assumption 2.5 combined imply that V is of at most polynomial growth. Then Lemma A.3 and (A.1) combined yield $V_* \leq V^* \leq V_*$, which implies the continuity of $t \mapsto V(y, t)$, and hence Proposition 3.1(iii) is verified. On the other hand, one can prove $\tilde{V}(y, t) := \mathbb{E}^{y,t}[V(Z_1, 1)]$ is of at most polynomial growth and is another viscosity solution to (3.4).[13] Then Lemma A.3 yields

$$V(y, t) = \tilde{V}(y, t) = \mathbb{E}^{y,t}[V(Z_1, 1)],$$

which confirms Proposition 3.1(iv) via the Markov property of Z.

Proof of Lemma A.3. For $\lambda > 0$, define $\tilde{u} = e^{\lambda t} u$ and $\tilde{v} = e^{\lambda t} v$. One can check $\tilde{u}$ (resp., $\tilde{v}$) is a viscosity subsolution (resp., supersolution) to

$$(A.3) \qquad -w_t + \lambda w - (w(y + 1, t) - 2w(y, t) + w(y - 1, t))\beta = 0.$$

Since the comparison result for (A.3) implies the comparison result for (2.8), it suffices to consider u (resp., v) as the viscosity subsolution (resp., supersolution) of (A.3).

Let C and n be constants such that $|u|, |v| \leq C(1 + |y|^n)$ on $\mathbb{Z} \times [0, 1]$. Consider $\psi(y, t) = e^{-\alpha t}(y^{2n} + \tilde{C})$ for some constants α and $\tilde{C}$. It follows that

$$-\psi_t + \lambda\psi + (\psi(y + 1, t) - 2\psi(y, t) + \psi(y - 1, t))\beta$$
$$> e^{-\alpha t}\left((\alpha + \lambda)(y^{2n} + \tilde{C}) - 2\beta y^{2n}\right) > 0$$

when $\alpha + \lambda > 2\beta$. Choosing α satisfying the previous inequality, then $v + \xi\psi$, for any $\xi > 0$, is a viscosity supersolution to (A.3). Once we show $u \leq v + \xi\psi$, the statement of the lemma then follows after sending $\xi \downarrow 0$.

Since both u and v have at most linear growth

$$(A.4) \qquad \lim_{|y| \to \infty} (u - v - \xi\psi)(y, t) = -\infty.$$

Replacing v by $v + \xi\psi$, we can assume that u (resp., v) is a viscosity subsolution (resp., supersolution) to (A.3) and

$$\sup_{\mathbb{Z} \times [0,1]} (u - v) = \sup_{\mathcal{O} \times [0,1]} (u - v) \quad \text{for some compact set } \mathcal{O} \subset \mathbb{Z}.$$

Then $u \leq v$ follows from the standard argument in viscosity solutions (cf., e.g., [19, Theorem 4.4.4]), which we briefly recall below.

Assume $M := \sup_{\mathbb{Z} \times [0,1]}(u - v) = \sup_{\mathcal{O} \times [0,1]}(u - v) > 0$ and the maximum is attained at $(\overline{x}, \overline{t}) \in \mathcal{O} \times [0, 1]$. For any $\epsilon > 0$, define

$$\Phi_\epsilon(x, y, t, s) := u(x, t) - v(y, s) - \phi_\epsilon(x, y, t, s),$$

where $\phi_\epsilon(x, y, t, s) := \frac{1}{\epsilon}[|x - y|^2 + |t - s|^2]$. The upper-semicontinuous function Φ_ϵ attains its maximum, denoted by M_ϵ, at $(x_\epsilon, y_\epsilon, t_\epsilon, s_\epsilon)$. One can show, using the same argument as in [19, Theorem 4.4.4],

$$M_\epsilon \to M \quad \text{and} \quad (x_\epsilon, y_\epsilon, t_\epsilon, s_\epsilon) \to (\overline{x}, \overline{x}, \overline{t}, \overline{t}) \in \mathcal{O}^2 \times [0, 1]^2 \quad \text{as } \epsilon \downarrow 0.$$

---

[13]Write $\tilde{V}(y, t) = \mathbb{E}^0[V(Z_{1-t} + y, 1)]$. One can utilize the Markov property of Z to show that $\tilde{V}$ is continuous differentiable and $\tilde{V}$ is a classical solution to (3.4).

Here $(x_\epsilon, y_\epsilon, t_\epsilon, s_\epsilon) \in \mathcal{O}^2 \times [0,1]^2$ for sufficiently small ϵ. Now observe that

- (x_ϵ, t_ϵ) is a local maximum of $(x,t) \mapsto u(x,t) - \phi_\epsilon(x, y_\epsilon, t, s_\epsilon)$;
- (y_ϵ, s_ϵ) is a local minimum of $(y,t) \mapsto v(y,s) + \phi_\epsilon(x_\epsilon, y, t_\epsilon, s)$.

Then the viscosity subsolution property of u and the supersolution property of v imply, respectively,

$$-\frac{2}{\epsilon}(t_\epsilon - s_\epsilon) + \lambda u(x_\epsilon, t_\epsilon) - (u(x_\epsilon + 1, t_\epsilon) - 2u(x_\epsilon, t_\epsilon) + u(x_\epsilon, t_\epsilon))\,\beta \le 0,$$

$$-\frac{2}{\epsilon}(t_\epsilon - s_\epsilon) + \lambda v(y_\epsilon, s_\epsilon) - (u(y_\epsilon + 1, s_\epsilon) - 2v(y_\epsilon, s_\epsilon) + v(y_\epsilon, s_\epsilon))\,\beta \ge 0.$$

Taking difference of the previous inequalities yields

$$(\lambda + 2\beta)(u(x_\epsilon, t_\epsilon) - v(y_\epsilon, s_\epsilon))$$
$$\le \beta\,(u(x_\epsilon + 1, t_\epsilon) + u(x_\epsilon - 1, t_\epsilon)) - \beta\,(v(y_\epsilon + 1, s_\epsilon) + v(y_\epsilon - 1, s_\epsilon)).$$

Sending $\epsilon \downarrow 0$ on both sides, we obtain

$$(\lambda + 2\beta)M = (\lambda + 2\beta)u(\overline{x}, \overline{t})$$
$$\le \beta\,\left(u(\overline{x} + 1, \overline{t}) - v(\overline{x} + 1, \overline{t})\right) + \beta\,\left(u(\overline{x} - 1, \overline{t}) - v(\overline{x} - 1, \overline{t})\right) \le 2\beta M,$$

which contradicts with $\lambda M > 0$. ∎

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