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SETS OF UNIT VECTORS WITH SMALL SUBSET SUMS

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ABSTRACT. We say that a family $\{\boldsymbol{x}_i \mid i \in [m]\}$ of vectors in a Banach space X satisfies the *k-collapsing condition* if $\left\|\sum_{i\in I} \boldsymbol{x}_i\right\| \leq 1$ for all k-element subsets $I \subseteq \{1, 2, \dots, m\}$. Let $\overline{\mathcal{C}}(k, d)$ denote the maximum cardinality of a k-collapsing family of unit vectors in a d-dimensional Banach space, where the maximum is taken over all spaces of dimension d. Similarly, let $\overline{\mathcal{CB}}(k, d)$ denote the maximum cardinality if we require in addition that $\sum_{i=1}^{m} \boldsymbol{x}_i = \boldsymbol{o}$. The case $k = 2$ was considered by Füredi, Lagarias and Morgan (1991). These conditions originate in a theorem of Lawlor and Morgan (1994) on geometric shortest networks in smooth finite-dimensional Banach spaces. We show that $\overline{\mathcal{CB}}(k, d) = \max\{k + 1, 2d\}$ for all $k, d \geq 2$. The behaviour of $\overline{\mathcal{C}}(k, d)$ is not as simple, and we derive various upper and lower bounds for various ranges of k and d. These include the exact values $\overline{\mathcal{C}}(k, d) = \max\{k + 1, 2d\}$ in certain cases.

We use a variety of tools from graph theory, convexity and linear algebra in the proofs: in particular the Hajnal–Szemerédi Theorem, the Brunn–Minkowski inequality, and lower bounds for the rank of a perturbation of the identity matrix.

0. NOTATION

Let $[n]$ denote the set $\{1, 2 \dots, n\}$, $|A|$ the cardinality of the set A, and $\binom{S}{k}$ the set $\{A \subseteq S \mid |A| = k\}$ of k-subsets of S. Let $d \geq 2$ and $m > k \geq 2$ be integers. Given expressions A and B that depend (in particular) on d, we use the notation $A = O(B)$ or $A \ll B$ to indicate that $A \leq CB$ for some absolute constant $C > 0$ and sufficiently large d, and $A = o(B)$ or $A \lll B$ to indicate that $A/B \to 0$ as $d \to \infty$. We use $A \sim B$ to mean $A/B \to 1$ as $d \to \infty$.

Let $X = X^d$ denote a d-dimensional real Banach space with norm $\|\cdot\|$. We denote the convex hull of a subset $A \subseteq X$ by $\operatorname{conv}(A)$. The boundary of A is the set

$$\partial A = \{\boldsymbol{x} \in X \mid \boldsymbol{x} \text{ is a limit point of } A \text{ and of } X \setminus A\}.$$

Throughout the paper we use the term *Minkowski space* for finite-dimensional real Banach space. Denote the closed ball with centre $\boldsymbol{c}$ and radius r by

$$B(\boldsymbol{c}, r) = \{\boldsymbol{x} \in X \mid \|\boldsymbol{x} - \boldsymbol{c}\| \leq r\}.$$

The *unit ball* of X is $B_X := B(\boldsymbol{o}, 1)$. Denote the dual of X by X^*. The elements of X^* are the *linear functionals* over X, that is, linear functions

$$\boldsymbol{x}^*\colon X \to \mathbb{R}, \quad \boldsymbol{x} \mapsto \langle \boldsymbol{x}^*, \boldsymbol{x}\rangle,$$

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with norm

$$\|\boldsymbol{x}^*\|^* := \sup\{\langle \boldsymbol{x}^*, \boldsymbol{x}\rangle \mid \boldsymbol{x} \in B_X\}.$$

Any $\boldsymbol{x} \in X$ has a *dual unit vector*: a functional $\boldsymbol{x}^* \in X^*$ such that $\|\boldsymbol{x}^*\|^* = 1$ and $\langle \boldsymbol{x}^*, \boldsymbol{x}\rangle = \|\boldsymbol{x}\|$. It is well-known that if the norm of a finite-dimensional X is *smooth*, that is, if $\|\cdot\|$ is differentiable on $X \setminus \{\boldsymbol{o}\}$, then X^* is strictly convex, that is, the boundary of B_{X^*} does not contain a line segment. Also, if X is strictly convex, then X^* is smooth. Recall that a space is smooth iff any $\boldsymbol{x} \in X \setminus \{\boldsymbol{o}\}$ has a unique dual unit vector.

Define the (multiplicative) *Banach-Mazur distance* between two Minkowski spaces X and Y of the same dimension as

$$d_{\mathrm{BM}}(X, Y) = \inf\{\lambda \geq 1 \mid B_Y \subseteq T(B_X) \subseteq \lambda B_Y \text{ for some linear } T\colon X \to Y\}.$$

Denote the coordinates of $\boldsymbol{x} \in \mathbb{R}^d$ by $\boldsymbol{x} = (\boldsymbol{x}(1), \dots, \boldsymbol{x}(d))$. Let $p \in [1, \infty)$. The space ℓ_p^d is $\mathbb{R}^d$ with the norm

$$\|\boldsymbol{x}\|_p = \|(\boldsymbol{x}(1), \boldsymbol{x}(2), \dots, \boldsymbol{x}(d))\|_p := \Big(\sum_{i=1}^{d} |\boldsymbol{x}(i)|^p\Big)^{1/p},$$

and the space ℓ_∞^d is $\mathbb{R}^d$ with the norm

$$\|\boldsymbol{x}\|_\infty = \|(\boldsymbol{x}(1), \boldsymbol{x}(2), \dots, \boldsymbol{x}(d))\|_\infty := \max\{|\boldsymbol{x}(i)| \mid i \in [d]\}.$$

1. Introduction

Definition 1. *A family* $\{\boldsymbol{x}_i \mid i \in [m]\}$ *of* m *(not necessarily distinct) vectors in some Minkowski space* X *satisfies the* k-collapsing condition *if*

$$\Big\|\sum_{i\in I} \boldsymbol{x}_i\Big\| \leq 1 \quad \textit{for all } I \in \binom{[m]}{k},$$

the full collapsing condition

$$\Big\|\sum_{i\in I} \boldsymbol{x}_i\Big\| \leq 1 \quad \textit{for all } I \subseteq [m],$$

the strong balancing condition *if*

$$\sum_{i=1}^{m} \boldsymbol{x}_i = \boldsymbol{o},$$

and the weak balancing condition *if*

$$\boldsymbol{o} \textit{ is in the relative interior of } \operatorname{conv}\{\boldsymbol{x}_i \mid i \in [m]\}.$$

In previous work by Füredi, Lagarias, Morgan, Lawlor and the present author [13, 25, 31, 32], the full collapsing condition and the 2-collapsing condition with or without the strong or the weak balancing condition were considered. Surprisingly, the 2-collapsing condition together with strong or weak balancing were often enough to give bounds on the size of the family of vectors that were still tight for the strong collapsing condition. The question then arises whether for instance similar results hold for instance for the 3-collapsing condition. In this paper we study the k-collapsing condition for any $k \geq 2$ with or without the strong balancing condition.

In Section 1.1 we survey previous results in order to sketch a context for the work presented here. The new results of this paper are summarised in Section 1.2. Section 1.3 contains an overview of the remaining sections.

1.1. **Previous work.** The full collapsing and strong balancing conditions of Definition 1 originate in a theorem of Lawlor and Morgan [25] on geometric shortest networks in smooth Minkowski spaces. We next describe their work.

Given a family $N = \{\boldsymbol{p}_i \mid i \in [n]\}$ of points in a Minkowski space X, a *Steiner tree* is a (finite) tree $T = (V, E)$ such that $N \subseteq V \subset X$. The points in $V \setminus N$ (if any) are called the *Steiner points* of T. The *length* $\ell(T)$ of a tree is the sum $\sum_{\boldsymbol{xy}\in E} \|\boldsymbol{x} - \boldsymbol{y}\|$ of the edge lengths. A *Steiner minimal tree* of N is a Steiner tree of N that minimises $\ell(T)$. By a compactness argument [7] any finite family of points in a Minkowski space has at least one Steiner minimal tree. The following theorem characterises the edges that are incident to a Steiner point of a Steiner minimal tree when the underlying Minkowski space is smooth.

Theorem 2 (Lawlor and Morgan [25])**.** *Let $N = \{\boldsymbol{p}_i \mid i \in [n]\}$ be a family of points, all different from the origin $\boldsymbol{o}$, in a smooth Minkowski space X. Let $\boldsymbol{p}_i^*$ be the dual unit vector of $\boldsymbol{p}_i$, $i \in [n]$. Then the Steiner tree that joins $\boldsymbol{o}$ to each $\boldsymbol{p}_i$ by straight-line segments is a Steiner minimal tree of N if and only if the family $\{\boldsymbol{p}_i^* \mid i \in [n]\}$ satisfies the full collapsing condition and the strong balancing condition in the dual space X^*.*

Since the dual of a smooth Minkowski space is strictly convex, a natural problem suggested by Theorem 2 is to find an upper bound on the cardinality of a family of unit vectors satisfying the full collapsing and strong balancing conditions in a strictly convex Minkowski space.

Theorem 3 (Lawlor and Morgan [25])**.** *Let $N = \{\boldsymbol{x}_i \mid i \in [n]\}$ be a family of unit vectors satisfying the full collapsing condition and the strong balancing condition in a d-dimensional strictly convex Minkowski space. Then $n \leq d + 1$. This bound is tight.*

Combined with Theorem 2 this implies that the degree of a Steiner point in any Steiner minimal tree in a d-dimensional smooth Minkowski space is bounded from above by $d + 1$.

The following theorem characterises the edges incident to an arbitrary point of a Steiner minimal tree in a smooth Minkowski space. Observe that if $\boldsymbol{p}$ is a Steiner point of a Steiner minimal tree $T = (V, E)$ of the point family N, then T is still a Steiner minimal tree of $N \cup \{\boldsymbol{p}\}$ (but with $\boldsymbol{p}$ not a Steiner point anymore). Therefore, the condition in this characterisation should be logically weaker than the characterisation appearing in Theorem 2, and it turns out that the full balancing condition has to be dropped.

Theorem 4 ([32])**.** *Let $N = \{\boldsymbol{p}_i \mid i \in [n]\}$ be a family of points, all different from the origin $\boldsymbol{o}$, in a smooth Minkowski space X. Let $\boldsymbol{p}_i^*$ be the dual unit vector of $\boldsymbol{p}_i$, $i \in [n]$. Then the Steiner tree that joins $\boldsymbol{o}$ to each $\boldsymbol{p}_i$ by straight-line segments is a Steiner minimal tree of $N \cup \{\boldsymbol{o}\}$ if and only if the family $\{\boldsymbol{p}_i^* \mid i \in [n]\}$ satisfies the full collapsing condition in the dual space X^*.*

The following is a strengthening of Theorem 3:

Theorem 5 ([32])**.** *Let $N = \{\boldsymbol{x}_i \mid i \in [n]\}$ be a family of unit vectors in a d-dimensional strictly convex Minkowski space satisfying the strong collapsing condition. Then $n \leq d + 1$.*

Therefore, all points in a Steiner minimal tree in a smooth d-dimensional Minkowski space have degree at most $d+1$. Generalising Theorems 2 and 4 to non-smooth Minkowski spaces is much more involved. There the degrees of Steiner points can be as large as 2^d; see [35] for a further discussion. We now leave the original motivation of Steiner minimal trees behind and continue to survey previous work on the various collapsing and balancing conditions.

After the paper of Lawlor and Morgan [25], Füredi, Lagarias and Morgan [13] introduced the 2-collapsing and weak balancing conditions, and used classical combinatorial convexity to study these conditions. They showed the following.

Theorem 6 (Füredi, Lagarias and Morgan [13]). *Let $N = \{\boldsymbol{x}_i \mid i \in [n]\}$ be a family of unit vectors in a d-dimensional Minkowski space X satisfying the 2-collapsing and weak balancing conditions. Then $n \leq 2d$, with equality only if N consists of a basis of X and its negative.*

They also mention without proof that if N is a family of $2d$ unit vectors in a d-dimensional Minkowski space satisfying the full collapsing and the strong balancing condition, then the space is isometric to ℓ_∞^d. We extend the above theorem to the k-collapsing condition, requiring however the strong balancing condition instead of the weak one (Theorem 20). The proof is completely different.

For strictly convex norms Füredi, Lagarias and Morgan [13] obtained the following stronger conclusion (thus weakening the hypotheses of Theorem 3 in a different way from Theorem 5).

Theorem 7 (Füredi, Lagarias and Morgan [13]). *Let $N = \{\boldsymbol{x}_i \mid i \in [n]\}$ be a family of unit vectors in a d-dimensional strictly convex Minkowski space satisfying the 2-collapsing condition and the weak balancing condition. Then $n \leq d+1$.*

Without any balancing condition or condition on the norm, they showed the following:

Theorem 8 (Füredi, Lagarias and Morgan [13]). *Let $N = \{\boldsymbol{x}_i \mid i \in [n]\}$ be a family of unit vectors in a d-dimensional Minkowski space X satisfying the 2-collapsing condition. Then $n \leq 3^d - 1$.*

This exponential behaviour for the 2-collapsing condition without any balancing condition is necessary:

Theorem 9 (Füredi, Lagarias and Morgan [13]). *For each sufficiently large $d \in \mathbb{N}$ there exists a strictly convex and smooth d-dimensional Minkowski space with a family N of at least 1.02^d unit vectors that satisfies the following strengthened 2-collapsing condition: $\|\boldsymbol{x}+\boldsymbol{y}\| < 1$ for all $\{\boldsymbol{x},\boldsymbol{y}\} \in \binom{N}{2}$.*

We construct similar exponential lower bounds for the k-collapsing condition (Theorem 32).

In an earlier paper [31] we applied the Brunn–Minkowski inequality to improve the upper bound of Theorem 8 as follows.

Theorem 10 ([31]). *Let $N = \{\boldsymbol{x}_i \mid i \in [n]\}$ be a family of unit vectors in a d-dimensional Minkowski space X satisfying the 2-collapsing condition. Then $n \leq 2^{d+1}+1$.*

In this paper we combine the Brunn–Minkowski inequality with the Hajnal-Szemerédi Theorem from graph theory to extend the above theorem to the k-collapsing condition (Theorem 30). In [13] it was asked whether there is an upper bound polynomial in d for the size of a collection of unit vectors in a d-dimensional Minkowski space satisfying the strong collapsing condition but not necessarily any balancing condition. This was subsequently answered as follows:

Theorem 11 ([31])**.** *Let $N = \{\boldsymbol{x}_i \mid i \in [n]\}$ be a family of unit vectors in a d-dimensional Minkowski space X satisfying the strong collapsing condition. Then $n \leq 2d$, with equality if and only if X is isometric to ℓ_∞^d, with N corresponding to $\{\pm\boldsymbol{e}_i \mid i \in [d]\}$ under any isometry.*

The analogous theorem for the strictly convex case is as follows:

Theorem 12 ([32])**.** *Let $N = \{\boldsymbol{x}_i \mid i \in [n]\}$ be a family of unit vectors in a d-dimensional strictly convex Minkowski space X satisfying the full collapsing condition. Then $n \leq d+1$. If, in addition, the balancing condition is not satisfied then $n \leq d$.*

The full collapsing condition is closely connected to certain notions from the local theory of Banach spaces. The *absolutely summing constant* or the 1-*summing constant* $\pi_1(X)$ of a Minkowski space X is defined to be the infimum of all $c > 0$ satisfying

$$\sum_{i=1}^{m} \|\boldsymbol{x}_i\| \leq c \max_{\epsilon_i = \pm 1} \Big\| \sum_{i=1}^{m} \epsilon_i \boldsymbol{x}_i \Big\|$$

where $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in X$. It is clear that $2\pi_1(X)$ is an upper bound to the number of unit vectors that satisfy the full collapsing condition. Deschaseaux [9] showed that $\pi_1(X) \leq d$ with equality iff X is isometric to ℓ_∞^d. This gives another proof of Theorem 11, apart from the characterisation of the family of unit vectors in the case of equality. Franchetti and Votruba [12] showed that if X is 2-dimensional then $2\pi_1(X)$ equals the perimeter of the unit circle. By a result of Gołąb [15] (see also [26]), the perimeter of the unit circle is less than 4 unless X is isometric to ℓ_∞^2. This implies the 2-dimensional case of Deschaseaux's theorem.

For $q \geq 2$, the *cotype q constant* $\kappa_q(X)$ of a Minkowski space X is defined to be the infimum of all $c > 0$ such that

$$\sum_{i=1}^{m} \|\boldsymbol{x}_i\|^q \leq c^q \Big(\frac{1}{2^m} \sum_{\epsilon_i = \pm 1} \Big\| \sum_{i=1}^{m} \epsilon_i \boldsymbol{x}_i \Big\|^q \Big)$$

where $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in X$. Thus, $(2\kappa_q(X))^q$ is an upper bound for the number of vectors satisfying the full collapsing condition. For instance, bounds on the cotype 2 constants for ℓ_p^d (essentially consequences of the Khinchin inequalities) give upper bounds independent of the dimension for fixed $p \in [1, \infty)$. Details may be found in [32].

A more general question was asked by Sidorenko and Stechkin [29, 30] and Katona and others [19, 20, 21, 22, 23], where the '≤ 1' in the collapsing conditions is replaced by '$\leq \delta$' or '$< \delta$'. In this direction work was also done in [33]. We do not pursue this generalisation here, instead leaving it for a later investigation, as it will be seen that the arguments in this paper are already quite involved.

1.2. **Overview of new results.** In this paper we only consider the k-collapsing condition and strong balancing condition. It will be convenient to define the following discrete quantities.

Definition 13. *For any $k \geq 2$, define $\mathcal{C}_k(X)$ to be the largest m such that a family of m vectors in X of norm at least 1 exists that satisfies the k-collapsing condition.*

Also, define $\mathcal{CB}_k(X)$ to be the largest m such that a family of m vectors in X of norm at least 1 exists that satisfies the k-collapsing condition and the strong balancing condition.

Next define the numbers

$$\begin{aligned}
\overline{\mathcal{C}}(k,d) &:= \max\left\{\mathcal{C}_k(X^d) \,\middle|\, X^d \text{ is a } d\text{-dimensional Minkowski space}\right\},\\
\underline{\mathcal{C}}(k,d) &:= \min\left\{\mathcal{C}_k(X^d) \,\middle|\, X^d \text{ is a } d\text{-dimensional Minkowski space}\right\},\\
\overline{\mathcal{CB}}(k,d) &:= \max\left\{\mathcal{CB}_k(X^d) \,\middle|\, X^d \text{ is a } d\text{-dimensional Minkowski space}\right\},\\
\underline{\mathcal{CB}}(k,d) &:= \min\left\{\mathcal{CB}_k(X^d) \,\middle|\, X^d \text{ is a } d\text{-dimensional Minkowski space}\right\}.
\end{aligned}$$

A compactness argument shows that $\overline{\mathcal{C}}(k,d)$ and $\overline{\mathcal{CB}}(k,d)$ are always finite. Although the vectors occurring in Theorems 2 to 12 above are unit vectors, we weaken this to vectors of norm at least 1 in the above definition. Indeed, it turns out that the quantities $\overline{\mathcal{C}}(k,d)$ and $\overline{\mathcal{CB}}(k,d)$ stay exactly the same whether we require the vectors to be of norm ≥ 1 or $= 1$. See Corollary 40 in Section 5 for this non-trivial fact.

1.2.1. *Some general observations.* We first show that $\underline{\mathcal{C}}(k,d)$ and $\underline{\mathcal{CB}}(k,d)$ are easily determined. Since we have assumed $d \geq 2$, it follows that for any value of $k \geq 2$ there exist $k+1$ unit vectors that satisfy the strong balancing condition, hence also the k-collapsing condition.

Proposition 14. *Let $k,d \geq 2$. Then $\mathcal{C}_k(X^d) \geq \mathcal{CB}_k(X^d) \geq k+1$ for any d-dimensional X^d.*

In Section 2 we show that these inequalities cannot be improved in general:

Proposition 15. *$\mathcal{C}_k(\ell_2^d) = \mathcal{CB}_k(\ell_2^d) = k+1$ for any $k \geq 2$ and $d \geq 2$.*

Consequently, the lower numbers $\underline{\mathcal{C}}(k,d)$ and $\underline{\mathcal{CB}}(k,d)$ are known.

Corollary 16. *$\underline{\mathcal{C}}(k,d) = \underline{\mathcal{CB}}(k,d) = k+1$ for all $k,d \geq 2$.*

The rest of the paper is concerned with the upper numbers $\overline{\mathcal{C}}(k,d)$ and $\overline{\mathcal{CB}}(k,d)$ The family of d unit vectors and their negatives $\{\pm \boldsymbol{e}_1, \dots, \pm \boldsymbol{e}_d\}$ shows the following:

Proposition 17. *Let $k,d \geq 2$. Then*

$$\mathcal{C}_k(\ell_\infty^d) \geq \mathcal{CB}_k(\ell_\infty^d) \geq 2d.$$

Keeping in mind Proposition 14, we obtain a simple lower bound.

Corollary 18. *$\overline{\mathcal{C}}(k,d) \geq \overline{\mathcal{CB}}(k,d) \geq \max\{k+1, 2d\}$ for all $k,d \geq 2$.*

In Section 2 we show that equality is possible in Corollary 18:

Proposition 19. *For any $k \geq 2$ and $d \geq 2$,*

$$\mathcal{C}_k(\ell_\infty^d) = \mathcal{CB}_k(\ell_\infty^d) = \max\{k+1, 2d\}.$$

1.2.2. *Large k-collapsing families with the balancing condition.* It turns out that $X^d = \ell_\infty^d$ is an extremal case for the quantity $\mathcal{CB}_k(X^d)$.

Theorem 20. *For any $k \geq 2$ and $d \geq 2$,*

$$\overline{\mathcal{CB}}(k,d) = \max\{k+1, 2d\}.$$

If $d \geq 2$, $2 \leq k \leq 2d-2$ and $\mathcal{CB}_k(X^d) = 2d$, then any family of $2d$ vectors of norm at least 1 satisfying the k-collapsing and strong balancing conditions are *necessarily unit vectors consisting of a basis of X^d and its negative. If furthermore $d \leq k \leq 2d-2$, then the only space X^d for which $\mathcal{CB}_k(X^d) = 2d$ is ℓ_∞^d up to isometry. (If $2 \leq k \leq d-1$ then there are infinitely many non-isometric spaces X^d such that $\mathcal{CB}_k(X^d) = 2d$.)*

Cf. Theorem 6 above. The proof uses a reduction to $m \times m$ matrices that are perturbations of the identity matrix in a certain weak sense, together with results on lower bounds of the ranks of such matrices (Lemma 41). In order to apply these lower bounds we have to solve certain convex optimization problems (Lemmas 42 and 43). Analogous to Theorem 7 above we make the following conjecture.

Conjecture 21. *If X^d is a strictly convex d-dimensional Minkowski space then*

$$\overline{\mathcal{CB}}_k(X^d) \leq \max\{k+1, d+1\}.$$

This conjecture holds for $k = 2$ [13]. Also, for each $d \geq 2$ there exists a strictly convex d-dimensional space with $d+1$ unit vectors satisfying the strong collapsing condition, so this conjecture would give the best possible estimate if true. Analogous to Theorem 6 we may hope for a positive answer to the following question.

Question 22. *Can the strong balancing condition in Theorem 20 be replaced by the weak balancing condition? That is, if the family $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_m\}$ of unit vectors in a d-dimensional Minkowski space X^d satisfies the k-collapsing condition and weak balancing condition, is $m \leq \max\{k+1, 2d\}$?*

Our methods do not seem to offer any way of using the weak balancing condition. Again, it is known that the answer is yes when $k = 2$ [13].

1.2.3. *Large k-collapsing families without the balancing condition.* Estimating $\overline{\mathcal{C}}(k,d)$ is much harder. The same proof techniques as for $\overline{\mathcal{CB}}(k,d)$ work only up to a certain extent and the details become much trickier. Also, for k fixed, $\overline{\mathcal{C}}(k,d)$ grows exponentially in d as $d \to \infty$ (Theorems 23, 30, 32, summarised in Table 1). For $k = c\sqrt{d}$ with $c < 1$ we have that $\overline{\mathcal{C}}(k,d)$ has polynomial growth as $d \to \infty$ (Theorems 24 and 33), and for the range $\sqrt{d} < k < 2d - \sqrt{d/2}$, $\overline{\mathcal{C}}(k,d)$ is linear in d, in particular $\overline{\mathcal{C}}(k,d) = 2d$ when $0.45d < k < 2d - \sqrt{d/2}$ (Theorem 25). If $d \leq 7$, we obtain $\overline{\mathcal{C}}(k,d) = \max\{k+1, 2d\}$ for all but finitely many k (Theorem 26). For larger d and with k very large ($\gg d^{d+2}$) we obtain $\overline{\mathcal{C}}(k,d) = k+1$ (Theorem 27).

Theorem 23. *For $k \geq 2$ let γ_k be the unique (positive) solution to*

$$(1+x)^{1/x}\left(1+\frac{1}{x}\right) = k^2.$$

Then $\mathrm{e}/k^2 < \gamma_k < \mathrm{e}/(k^2 - \mathrm{e})$ and

$$\overline{\mathcal{C}}(k,d) < 1.33k^{2\gamma_k d+2}. \tag{1}$$

If $k < \sqrt{d}$ then

$$\overline{\mathcal{C}}(k,d) < \frac{k}{\sqrt{d}} k^{2\gamma_k d+2}.$$

In particular, if $k = c\sqrt{d}$ with $c < 1$, then $\overline{\mathcal{C}}(k,d) = O(d^{1+\mathrm{e}/c^2})$ as $d \to \infty$.

See Table 1 for the first few values of γ_k. Compare (1) with the lower bound (2) below in Theorem 32. The next theorem gives a slightly sharper result for k a small multiple of $\sqrt{d}$. See also the lower bound of Theorem 33 below.

k	γ_k	Upper bound from Theorem 23 $k^{2\gamma_k d}$	Upper bound from Theorem 30 $(1+\frac{2}{k})^d$	Lower bound from Theorem 32 $\left(1+\frac{1}{2(2k+1)^2}\right)^d$
2	1	4^d	$\mathbf{2^d}$	1.02^d
3	0.3541686	2.178^d	$\mathbf{1.667^d}$	1.0102^d
4	0.1854203	1.673^d	$\mathbf{1.5^d}$	1.0061^d
5	0.1149225	1.448^d	$\mathbf{1.4^d}$	1.0041^d
6	0.0784510	$\mathbf{1.325^d}$	1.334^d	1.0029^d
7	0.0570503	$\mathbf{1.249^d}$	1.286^d	1.0022^d
8	0.0433914	$\mathbf{1.198^d}$	1.25^d	1.0017^d
9	0.0341301	$\mathbf{1.162^d}$	1.223^d	1.0013^d

TABLE 1. Values of γ_k (defined in Theorem 23) together with the upper bounds to $\overline{\mathcal{C}}(k,d)$ given by Theorems 23 and 30 and the lower bound to $\overline{\mathcal{C}}(k,d)$ given by Theorem 32. The values of γ_k are rounded to the nearest decimal, of $k^{2\gamma_k}$ and $1+2/k$ are rounded up and of $1+1/(2(2k+1)^2)$ are rounded down. The numbers in bold denote the better of the two upper bounds for each value of k.

Theorem 24. *For any $\varepsilon>0$ and $p\in\mathbb{N}$, $p\geq 2$, there exist d_0 and $c>0$ such that for all $d>d_0$, if*

$$\left((p!)^{-1/(2p)}+\varepsilon\right)\sqrt{d}<k\leq\sqrt{d}$$

then $\overline{\mathcal{C}}(k,d)<cd^p$.

For larger k we obtain almost optimal results. In particular, we obtain the exact result $\overline{\mathcal{C}}(k,d)=2d$ for $(\sqrt{6}-2)d+O(1)<k<2d-\sqrt{d/2}$.

Theorem 25. *Let $k\geq 3$ and $d\geq 2$.*

1. *If $\sqrt{d}<k\leq\frac{d+1}{2}$ then $\overline{\mathcal{C}}(k,d)\leq\frac{2d(k-1)^2}{k^2-d}=2d\left(1+\frac{d-2k+1}{k^2-d}\right)$.*
2. *If $-2d+\sqrt{6d^2+3d+1}\leq k\leq 2d-\sqrt{d/2}$ then $\overline{\mathcal{C}}(k,d)=2d$.*
3. *If $d\geq 3$ and $k>2d-\sqrt{d/2}$ then $\overline{\mathcal{C}}(k,d)\leq k+\frac{1+\sqrt{2d-3}}{2}$.*

For values of d up to 7 as $k\to\infty$ the same methods as used in proving Theorems 20, 23, 24 and 25 give the following exact values.

Theorem 26. *$\overline{\mathcal{C}}(k,d)=\max\{k+1,2d\}$ in the following cases:*

1. *$d=2$ and $k\geq 2$,*
2. *$d\in\{3,4,5\}$ and $k\geq 3$,*
3. *$d=6$ and $k\in\{3,4,5,\dots,10\}\cup\{17,18,19\dots\}$,*
4. *$d=7$ and $k\in\{3,4,5,\dots,12\}\cup\{41,42,43,\dots\}$.*

The proof method gives no information for $d\geq 8$ and k large. (The estimate $\overline{\mathcal{C}}(2,3)\leq 9$ is also obtained in the proof.) For arbitrary d, as long as k is large, we obtain the following using a completely different technique.

Theorem 27. *If $k\gg d^{d+2}$ then $\overline{\mathcal{C}}(k,d)=k+1$.*

The proof uses geometric tools from convexity, in particular the Brunn–Minkowski inequality and the theorem of Carathéodory. The hypothesis $k\gg d^{d+2}$ is most likely not best possible, but we need at least $k \geq 2d-1$ for the conclusion of this theorem to hold, as shown by the example of $k \leq 2d-2$ and the family $\{\pm \boldsymbol{e}_i \mid i \in [d]\}$ in ℓ_∞^d.

Conjecture 28. $\overline{\mathcal{C}}(k,d) = k+1$ *whenever* $k \geq 2d-1$.

By Theorem 26 this conjecture holds for $d \leq 5$. The next conjecture has non-empty content only for $d \geq 8$.

Conjecture 29. $\overline{\mathcal{C}}(k,d) = 2d$ *if* $2d - \sqrt{d/2} \leq k \leq 2d-2$.

Since Theorem 25 gives $\overline{\mathcal{C}}(k,d) = 2d$ for $(\sqrt{6}-2)d + O(1) < k < 2d - \sqrt{d/2}$, it is likely that the bound in Conjecture 29 already holds for values of k smaller than $(\sqrt{6}-2)d$. On the other hand, as implied by Theorem 33 below, we need at least $k > (\frac{1}{2} + o(1))\sqrt{d}$.

We show the following upper bound using a method closely related to the proof of Theorem 27. We again use the Brunn–Minkowski inequality, but combine it with the Hajnal–Szemerédi Theorem from graph theory:

Theorem 30. *For any* $k, d \geq 2$, $\overline{\mathcal{C}}(k,d) \leq k(1+\frac{2}{k})^d + k - 1$.

Asymptotically for fixed k as $d \to \infty$, this bound is better when $k \leq 5$ while for $k \geq 6$ Theorem 23 is better. See Table 1 for a comparison between the upper bounds given by Theorem 23 and Theorem 30 for $k = 2, \dots, 8$.

Related to Proposition 15 is the following result on spaces close to Euclidean space.

Proposition 31. *Let* $D = d_{\mathrm{BM}}(X^d, \ell_2^d)$ *be the Banach-Mazur distance between* X^d *and* ℓ_2^d. *Then for any* $k > D^2$,

$$\mathcal{C}_k(X^d) \leq \frac{k^2 - D^2}{k - D^2} = k + D^2 + \frac{D^4 - D^2}{k - D^2}.$$

In particular, if $D^2 \leq (2k-1)/(k+1)$ *then* $\mathcal{C}_k(X^d) = k+1$.

Its simple proof is at the end of Section 2. By John's theorem [17], $d_{\mathrm{BM}}(X^d, \ell_2^d) \leq \sqrt{d}$, from which follows $\mathcal{C}_k(X^d) \leq k + d + \frac{d^2-d}{k-d}$ if $k > d$. This estimate is worse, however, than the estimates of Theorems 25 and 26 whenever $k > d$. On the other hand, if $D = d_{\mathrm{BM}}(X, \ell_2^d)$ is sufficiently small, then Proposition 31 may give bounds better than Theorems 25. In particular, Proposition 31 is better than Theorem 25 in the range $d < k \leq 2d - \sqrt{d/2}$ if $d_{\mathrm{BM}}(X, \ell_2^d) \leq \sqrt{\frac{(2d-k)k}{2d-1}}$, and in the range $k > 2d - \sqrt{d/2}$ if $d_{\mathrm{BM}}(X, \ell_2^d) \leq (d/2)^{1/4}$.

1.2.4. *Bounding* $\overline{\mathcal{C}}(k,d)$ *from below.* We now turn to lower bounds. The first, generalising Theorem 9, uses a simple greedy construction of sets of almost orthogonal Euclidean unit vectors (Lemma 49 in Section 9).

Theorem 32. *For all* $k \geq 2$ *and sufficiently large* d *depending on* k, *there exists a strictly convex and smooth* d*-dimensional Minkowski space* X^d *such that*

$$\mathcal{C}_k(X^d) \geq \left(1 + \frac{1}{2(2k+1)^2}\right)^d. \tag{2}$$

The proof in fact gives a norm that is a C^∞ function on $\mathbb{R}^d \setminus \{\boldsymbol{o}\}$. The lower bound (2) almost matches the upper bound (1) from Theorem 23 asymptotically in the sense that as $k \to \infty$ and $d \ggg \log k$, (2) implies that $\overline{\mathcal{C}}(k,d)^{1/d} - 1 \gg 1/k^2$, while (1) implies that $\overline{\mathcal{C}}(k,d)^{1/d} - 1 \ll (\log k)/k^2$. See the last column in Table 1. (Note that since $\overline{\mathcal{C}}(k,d) \geq k+1$, we need d to grow with k in order to have $\lim_{k\to\infty} \overline{\mathcal{C}}(k,d)^{1/d} = 1$, and in fact $\lim_{k\to\infty}(k+1)^{1/d} = 1$ iff $d \ggg \log k$.)

The second lower bound uses a well-known algebraic construction of almost orthogonal Euclidean vectors [18, 10] (Lemma 50 in Section 9).

Theorem 33. *For any $d \in \mathbb{N}$ let $q = q_d$ be the largest prime power such that $d \geq q^2 - q + 1$. (By the Prime Number Theorem, $q_d \sim \sqrt{d}$ as $d \to \infty$.) Then for each $c \in \mathbb{N}$ and $k \geq 2$ satisfying $c \leq q-2$ and*

$$k \leq \frac{q-1}{2c} - \frac{1}{2} \quad \left(\sim \frac{\sqrt{d}}{2c} \text{ as } d \to \infty\right)$$

there exists a d-dimensional Minkowski space X^d such that

$$\mathcal{C}_k(X^d) \geq q^{c+2} \quad (\sim d^{1+c/2} \text{ as } d \to \infty).$$

In particular, when $k \leq (\frac{1}{2} + o(1))\sqrt{d}$ as $d \to \infty$ we have $\overline{\mathcal{C}}_k(d) \gg d^{3/2}$. The lower bound of Theorem 33 is better than that of Theorem 32 when $k \gg \sqrt{d}/\log d$. For k a small multiple of $\sqrt{d}$, Theorems 23 and 24 give an upper bound polynomial in d while Theorem 33 gives a lower bound polynomial in d, but with a gap between the degrees of the polynomials. Nevertheless, Theorem 33 matches the bound (1) of Theorem 23 in a similar sense as in the discussion after Theorem 32, in that it implies that $\overline{\mathcal{C}}(k,d)^{1/d} - 1 \gg (\log k)/k^2$ as $k \to \infty$ and $k \sim \sqrt{d}/(2c)$, $c \in \mathbb{N}$.

1.3. **Organisation of the paper.** In Section 2 we use elementary averaging arguments involving coordinates and inner products to prove Proposition 19 on ℓ_∞^d, Proposition 15 on ℓ_2^d and Proposition 31 on spaces close to ℓ_2^d. In Section 3 we use the Brunn–Minkowski inequality and the Hajnal–Szemerédi Theorem to prove Theorem 30. This is followed in Section 4 by a proof of Theorem 27 which is along similar lines. In addition to the Brunn–Minkowski inequality it uses a metric consequence of Carathéodory's Theorem that may be of independent interest (Lemma 37). Then in Section 5 we reformulate the notion of a k-collapsing collection of vectors in terms of matrices. There we also prove a general version of a well-known result that bounds the rank of a matrix from below (Lemma 41). These results are applied in Section 6, where Theorem 20 is proved, and Section 7 where Theorems 25 and 26 are proved. These proofs are all somewhat technical and involve an application of Lemma 41 combined with convex optimisation. In Section 8 Theorems 23 and 24 are proved. The arguments are similar as in Sections 6 and 7 and use in addition a well-known bound on the rank of an integer Hadamard power of a matrix (Lemma 44). In Section 9 we derive the lower bounds of Theorems 32 and 33.

2. Two elementary averaging arguments

In This section we collect two simple arguments, one for ℓ_∞^d (Proposition 34), the other for ℓ_2^d (Lemma 35). These are the essential ingredients of the proofs of Propositions 19, 15 and 31.

Proposition 34. *Let $k,d\geq 2$. If $S=\{\boldsymbol{x}_i \mid i\in[m]\}\subset \ell_\infty^d$ is a k-collapsing family of $m>k+1$ vectors of norm at least 1, then $m\leq 2d$. If furthermore $m=2d$, then $S=\{\pm\boldsymbol{e}_1,\dots,\pm\boldsymbol{e}_d\}$.*

Proof. Suppose that there exist a coordinate $j\in[d]$ and two distinct indices $i\in[m]$ such that $\boldsymbol{x}_i(j)\geq 1$. Without loss of generality, $\boldsymbol{x}_{m-1}(1),\boldsymbol{x}_m(1)\geq 1$. By the k-collapsing condition, for any $I\in\binom{[m-2]}{k-2}$,

$$\sum_{i\in I}\boldsymbol{x}_i(1)\leq -2+\sum_{i\in I\cup\{m-1,m\}}\boldsymbol{x}_i(1)\leq -2+\Big\|\sum_{i\in I\cup\{m-1,m\}}\boldsymbol{x}_i\Big\|_\infty\leq -1.$$

Fix a $J\in\binom{[m-2]}{k}$ (note that $k\leq m-2$). It follows that

$$\binom{k-1}{k-3}\sum_{i\in J}\boldsymbol{x}_i(1)=\sum_{I\in\binom{J}{k-2}}\sum_{i\in I}\boldsymbol{x}_i(1)\leq -\binom{k}{k-2},$$

which gives

$$\sum_{i\in J}\boldsymbol{x}_i(1)\leq -\binom{k}{k-2}\Big/\binom{k-1}{k-3}=-k/(k-2)<-1,$$

hence $\left\|\sum_{i\in J}\boldsymbol{x}_i\right\|_\infty>1$, contradicting the k-collapsing condition.

Therefore, for each coordinate $j\in[d]$ there is at most one index $i\in[m]$ such that $\boldsymbol{x}_i(j)\geq 1$. Similarly, there is at most one $i\in[m]$ such that $\boldsymbol{x}_i(j)\leq -1$. Therefore, there are at most $2d$ pairs $(i,j)\in[m]\times[d]$ such that $|\boldsymbol{x}_i(j)|\geq 1$. On the other hand, since $\|\boldsymbol{x}_i\|_\infty\geq 1$ for each $i\in[m]$, there are at least m such pairs, which gives $m\leq 2d$.

If we assume $m=2d$, then for each $j\in[d]$ there is exactly one $i\in[m]$ such that $\boldsymbol{x}_i(j)\geq 1$, and exactly one $i\in[m]$ such that $\boldsymbol{x}_i(j)\leq -1$. We may then renumber the $\boldsymbol{x}_i$ such that $\boldsymbol{x}_{2i-1}(i)\geq 1$ and $\boldsymbol{x}_{2i}(i)\leq -1$ for each $i\in[d]$. By the k-collapsing condition, for any $J\in\binom{[m-2]}{k-1}$,

$$\sum_{i\in J}\boldsymbol{x}_i(d)+1\leq\sum_{i\in J\cup\{2d-1\}}\boldsymbol{x}_i(d)\leq\Big\|\sum_{i\in J\cup\{2d-1\}}\boldsymbol{x}_i\Big\|_\infty\leq 1,$$

hence $\sum_{i\in J}\boldsymbol{x}_i(d)\leq 0$. Similarly, $\sum_{i\in J}\boldsymbol{x}_i(d)\geq 0$. Therefore, $\sum_{i\in J}\boldsymbol{x}_i(d)=0$ for each $J\in\binom{[m-2]}{k-1}$. Since $k-1<m-2$, it follows that $\boldsymbol{x}_i(d)=0$ for all $i\in[m-2]$ and $\boldsymbol{x}_{2d-1}(d)=1$, $\boldsymbol{x}_{2d}(d)=-1$. Similarly, $\boldsymbol{x}_i(j)=0$ for all i,j such that $i\notin\{2j-1,2j\}$, and $\boldsymbol{x}_{2j-1}(j)=1$, $\boldsymbol{x}_{2j}(j)=-1$. We conclude that $\boldsymbol{x}_{2i-1}=\boldsymbol{e}_i$ and $\boldsymbol{x}_{2i}=-\boldsymbol{e}_i$ for all $i\in[d]$. □

Proof of Proposition 19. By Propositions 14 and 17,

$$\mathcal{C}_k(\ell_\infty^d)\geq\mathcal{CB}_k(\ell_\infty^d)\geq\max\{k+1,2d\}.$$

Proposition 34 implies that $\mathcal{C}_k(\ell_\infty^d)\leq\max\{k+1,2d\}$. □

The next lemma occurs in an equivalent form in [19, Lemma 5].

Lemma 35. *Let $k\geq 2$ and $\lambda\in(0,\sqrt{k})$. Let $\boldsymbol{x}_1,\dots,\boldsymbol{x}_m$ be vectors in an inner product space such that $\|x_i\|_2\geq 1$ for all $i\in[m]$ and*

$$\Big\|\sum_{i\in I}\boldsymbol{x}_i\Big\|_2\leq\lambda\quad\textit{for all}\quad I\in\binom{[m]}{k}. \tag{3}$$

Then

$$m \leq \frac{k^2-\lambda^2}{k-\lambda^2}.$$

Proof. Square (3) and sum over all $I \in \binom{[m]}{k}$ to obtain

$$\begin{aligned}\binom{m}{k}\lambda^2 &\geq \binom{m-1}{k-1}\sum_{i=1}^{m}\|\boldsymbol{x}_i\|_2^2 + \binom{m-2}{k-2}\sum_{\{i,j\}\in\binom{[m]}{2}} 2\langle \boldsymbol{x}_i, \boldsymbol{x}_j\rangle \\ &= \left(\binom{m-1}{k-1}-\binom{m-2}{k-2}\right)\sum_{i=1}^{m}\|\boldsymbol{x}_i\|_2^2 + \binom{m-2}{k-2}\left\|\sum_{i=1}^{m}\boldsymbol{x}_i\right\|_2^2 \\ &\geq \left(\binom{m-1}{k-1}-\binom{m-2}{k-2}\right)m + 0,\end{aligned}$$

which simplifies to the conclusion of the theorem. □

Proof of Proposition 15. For the upper bound, set $\lambda = 1$ in Lemma 35. The lower bound follows from Proposition 14. □

Proof of Proposition 31. By the definition of the Banach-Mazur distance there exists an inner product on X^d such that $\|\boldsymbol{x}\| \leq \|\boldsymbol{x}\|_2 \leq D\|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in X^d$. Then apply Lemma 35 with $\lambda = D$. □

3. The Brunn–Minkowski inequality and graph colourings

In this section we prove Theorem 30 which gives an upper bound for $\overline{\mathcal{C}}(k,d)$ for fixed k that is exponential in d. The proofs of Theorems 27 and 30 are similar, but that of Theorem 30 is somewhat more straightforward, and that is why we consider it first. We first discuss the three main tools used in its proof. The first is the dimension-independent version of the Brunn–Minkowski inequality (see Ball [5]). The idea of using the Brunn–Minkowski inequality is from [31], where it is used in the case $k = 2$. Denote the volume (or d-dimensional Lebesgue measure) of a measurable set $A \subseteq \mathbb{R}^d$ by $\operatorname{vol}(A)$.

Brunn–Minkowski inequality. *If $A, B \subset \mathbb{R}^d$ are compact sets and $0 < \lambda < 1$, then*

$$\operatorname{vol}(\lambda A + (1-\lambda)B) \geq \operatorname{vol}(A)^{\lambda}\operatorname{vol}(B)^{1-\lambda}.$$

Induction immediately gives the following version for k sets:

k-fold Brunn–Minkowski inequality. *Let $A_1, A_2, \dots, A_k \subset \mathbb{R}^d$ be compact and $\lambda_1, \lambda_2, \dots, \lambda_k > 0$ such that $\sum_{i=1}^k \lambda_i = 1$. Then*

$$\operatorname{vol}(\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_k A_k) \geq \prod_{i=1}^{k}\operatorname{vol}(A_i)^{\lambda_i}.$$

The second tool is the Hajnal–Szemerédi Theorem. A *k-colouring* of a graph $G = (V, E)$ is a function $f\colon V \to [k]$ such that $f(x) \neq f(y)$ whenever $xy \in E$. Any k-colouring partitions the vertex set V into *colour classes* $f^{-1}(i)$, $i \in [k]$. A k-colouring of a graph on m vertices is called *equitable* if each colour class has cardinality $\lfloor m/k \rfloor$ or $\lceil m/k \rceil$. The following result was originally a conjecture of Erdős [11]. Although the original proof [16] was quite complicated and long, there is now a relatively simple, compact proof, due to Kierstead and Kostochka [24].

Hajnal–Szemerédi Theorem. *Let G be a graph with maximum degree Δ. Then for any $k > \Delta$, G has an equitable k-colouring.*

The third tool is the following simple consequence of the triangle inequality.

Lemma 36. *Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ be vectors of norm at least 1 in a normed space such that*

$$\Big\|\sum_{i=1}^{k} \boldsymbol{x}_i\Big\| \leq 1.$$

Then for each $i \in [k]$ there exists $j \in [k]$ such that $\|\boldsymbol{x}_i - \boldsymbol{x}_j\| \geq 1$.

Proof. By the triangle inequality and the hypotheses,

$$\begin{aligned} k \leq \|k\boldsymbol{x}_i\| = \Big\|\sum_{j=1}^{k} \boldsymbol{x}_j + \sum_{j=1}^{k}(\boldsymbol{x}_i - \boldsymbol{x}_j)\Big\| &\leq \Big\|\sum_{j=1}^{k} \boldsymbol{x}_j\Big\| + \sum_{j=1}^{k} \|\boldsymbol{x}_i - \boldsymbol{x}_j\| \\ &\leq 1 + \sum_{j=1}^{k} \|\boldsymbol{x}_i - \boldsymbol{x}_j\| = 1 + \sum_{\substack{j=1 \\ j\neq i}}^{k} \|\boldsymbol{x}_i - \boldsymbol{x}_j\|. \end{aligned}$$

The average distance between $\boldsymbol{x}_i$ and the other points is then bounded below:

$$\frac{1}{k-1}\sum_{\substack{j=1 \\ j\neq i}}^{k} \|\boldsymbol{x}_i - \boldsymbol{x}_j\| \geq 1,$$

which implies that $\|\boldsymbol{x}_i - \boldsymbol{x}_j\| \geq 1$ for some $j \neq i$. □

Proof of Theorem 30. Let $V = \{\boldsymbol{x}_i \mid i \in [m]\} \subset X^d$ be a k-collapsing family with each $\|\boldsymbol{x}_i\| \geq 1$. Define a graph G on V by joining $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ if $\|\boldsymbol{x}_i - \boldsymbol{x}_j\| < 1$. By Lemma 36, the maximum degree Δ of G is at most $k-2$. By the Hajnal–Szemerédi Theorem, G has an equitable k-colouring. This gives a partition $I_1, \dots, I_k$ of $[m]$ such that each $|I_t| \in \{q, q+1\}$, where $q := \lfloor m/k \rfloor$, and such that $\|\boldsymbol{x}_i - \boldsymbol{x}_j\| \geq 1$ whenever i, j are distinct elements from the same I_t. For each $t \in [k]$ let

$$S_t := \bigcup_{j\in I_t} B\Big(\boldsymbol{x}_j, 1/2\Big).$$

Then

$$\operatorname{vol}(S_t) = (1/2)^d\, |I_t| \operatorname{vol}(B_X). \tag{4}$$

By the k-collapsing property,

$$\frac{1}{k}(S_1 + \cdots + S_k) \subseteq B\left(\boldsymbol{o}, \frac{1}{2} + \frac{1}{k}\right). \tag{5}$$

Substitute (4) and (5) into the k-fold Brunn–Minkowski inequality

$$\prod_{t=1}^{k} \operatorname{vol}(S_t)^{1/k} \leq \operatorname{vol}\Big(\frac{1}{k}(S_1 + \cdots + S_k)\Big),$$

to obtain

$$\left(\prod_{t=1}^{k} |I_t|\right)^{1/k} \leq \left(1 + \frac{2}{k}\right)^d.$$

Set $r := m - kq$. There are r sets I_t of cardinality $q+1$ and $k-r$ of cardinality q. Therefore,

$$\left(\left(\frac{m-r}{k}+1\right)^r\left(\frac{m-r}{k}\right)^{k-r}\right)^{1/k} \leq \left(1+\frac{2}{k}\right)^d. \tag{6}$$

Instead of minimising the left-hand side over all $r \in \{0,1,\dots,k-1\}$, we simply weaken it to

$$\frac{m-r}{k} \leq \left(1+\frac{2}{k}\right)^d,$$

to obtain

$$m \leq k\left(1+\frac{2}{k}\right)^d + r \leq k\left(1+\frac{2}{k}\right)^d + k - 1. \qquad \square$$

By taking more care in minimising the left-hand side of (6) it is possible to find a slightly better upper bound. However, it is not possible to show that $\overline{\mathcal{C}}(k,d) \leq k\left(1+\frac{2}{k}\right)^d$ from only (6). For instance, the values $d=4$, $m=19$, $k=6$ satisfy (6), but not $m \leq k\left(1+\frac{2}{k}\right)^d$. (Of course $\overline{\mathcal{C}}(6,4) = 8$ by Theorem 26.)

4. The Brunn–Minkowski inequality and Carathéodory's theorem

In this section we prove Theorem 27, which considers k-collapsing sets when $k \ggg d$ as $d \to \infty$. We use the Brunn–Minkowski inequality in much the same way as in the proof of Theorem 30, but now coupled with Carathéodory's theorem from combinatorial convexity.

Carathéodory's Theorem. *Suppose that $\boldsymbol{p}$ is in the convex hull of a family $\{\boldsymbol{x}_i \mid i \in I\}$ of points in $\mathbb{R}^d$. Then $\boldsymbol{p} \in \operatorname{conv}\{\boldsymbol{x}_i \mid i \in J\}$ for some $J \subseteq I$ with $|J| \leq d+1$.*

Carathéodory's theorem is used to prove the following auxiliary result. The technique is very similar to an argument in [36] that bounds the number of vertices of edge-antipodal polytopes.

Lemma 37. *Let $d \geq 2$, $n \geq 1$ and $\{\boldsymbol{x}_i \mid i \in [n]\} \subset X^d$ be such that $\|\boldsymbol{x}_i\| \geq 1$ for each $i \in [n]$ and*

$$\operatorname{diam}\{\boldsymbol{x}_i \mid i \in [n]\} < 1 + 1/d. \tag{7}$$

Then

$$\left\|\frac{1}{n}\sum_{i=1}^n \boldsymbol{x}_i\right\| > 1/d^2. \tag{8}$$

Proof. Let $P := \operatorname{conv}\{\boldsymbol{x}_i \mid i \in [n]\}$. By convexity, the centroid $\frac{1}{n}\sum_{i=1}^n \boldsymbol{x}_i$ is in P. Choose $\boldsymbol{p} \in P$ of minimum norm. It is sufficient to prove that $\|\boldsymbol{p}\| > 1/d^2$. Suppose first that $\boldsymbol{p} = \boldsymbol{o}$. Then by Carathéodory's Theorem, $\boldsymbol{o} = \sum_{i\in J}\lambda_i\boldsymbol{x}_i$ where $J \subseteq [n]$, $|J| \leq d+1$, $\lambda_i \geq 0$ for each $i \in J$, and $\sum_{i\in J}\lambda_i = 1$. Note that $|J| \geq 2$. For any $j \in J$,

$$-\boldsymbol{x}_j = \sum_{i\in J\setminus\{j\}} \lambda_i(\boldsymbol{x}_i - \boldsymbol{x}_j),$$

hence, by the triangle inequality,

$$1 \leq \sum_{i\in J\setminus\{j\}} \lambda_i\|\boldsymbol{x}_i - \boldsymbol{x}_j\| \leq \sum_{i\in J\setminus\{j\}} \lambda_i \operatorname{diam} P = (1-\lambda_j)\operatorname{diam} P.$$

Summing over all $j \in J$, we obtain $|J| \le (|J|-1)\operatorname{diam} P$ and

$$\operatorname{diam} P \ge \frac{|J|}{|J|-1} \ge \frac{d+1}{d}.$$

However,

$$\operatorname{diam} P = \operatorname{diam}\{\boldsymbol{x}_i \mid i \in [n]\} < 1 + 1/d$$

by assumption, a contradiction. It follows that $\boldsymbol{p} \neq \boldsymbol{o}$, hence $\boldsymbol{p}$ is in some facet of P. We apply Carathéodory's Theorem to the affine span of this facet, which is of dimension $< d$:

$$\boldsymbol{p} = \sum_{i\in J} \lambda_i \boldsymbol{x}_i \quad \text{where } J \subseteq [n],\ |J| \le d,\ \lambda_i \ge 0 \text{ for each } i \in J, \text{ and } \sum_{i \in J} \lambda_i = 1.$$

If $|J| = 1$ then $\boldsymbol{p} = \boldsymbol{x}_i$ for some $i \in [n]$ and $\|\boldsymbol{p}\| \ge 1 > 1/d^2$. Thus, without loss of generality we assume that $|J| \ge 2$. It follows that for each $j \in J$,

$$\boldsymbol{p} - \boldsymbol{x}_j = \sum_{i \in J\setminus\{j\}} \lambda_i (\boldsymbol{x}_i - \boldsymbol{x}_j)$$

and, again by the triangle inequality,

$$\begin{aligned} 1 - \|\boldsymbol{p}\| \le \|\boldsymbol{x}_j\| - \|\boldsymbol{p}\| \le \|\boldsymbol{p} - \boldsymbol{x}_j\| &\le \sum_{i \in J\setminus\{j\}} \lambda_i \|\boldsymbol{x}_i - \boldsymbol{x}_j\| \\ &\le \sum_{i \in J\setminus\{j\}} \lambda_i \operatorname{diam} P = (1-\lambda_j)\operatorname{diam} P. \end{aligned}$$

Sum over all $j \in J$ to obtain (since $|J| \ge 2$) that

$$(1 - \|\boldsymbol{p}\|)\,|J| \le (|J|-1)\operatorname{diam} P < (|J|-1)(1+1/d)$$

and

$$1 - \|\boldsymbol{p}\| < \frac{|J|-1}{|J|}(1+1/d) \le \frac{d-1}{d}(1+1/d) = 1 - 1/d^2.$$

It follows that $\|\boldsymbol{p}\| > 1/d^2$. □

The above proof in fact shows that if $\operatorname{diam}\{\boldsymbol{x}_i\} = 1 + 1/d - \varepsilon$ for some $\varepsilon > 0$, then $\left\|\frac{1}{n}\sum_{i=1}^n \boldsymbol{x}_i\right\| \ge 1/d^2 + (1-1/d)\varepsilon$. It can be shown that this inequality is sharp. It can also be shown that the right-hand side of (7) cannot be increased: There exist d-dimensional Minkowski spaces with $d+1$ unit vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{d+1}$ such that $\operatorname{diam}\{\boldsymbol{x}_i\} = 1 + 1/d$ although $\sum_{i=1}^{d+1} \boldsymbol{x}_i = \boldsymbol{o}$. For details see the extended version of this paper (`arXiv:1210.0366`).

Proof of Theorem 27. Suppose that $\mathcal{C}_k(X^d) \ge k+2$. Let $\{\boldsymbol{x}_i \mid i \in [k+2]\} \subset X^d$ be a k-collapsing collection of vectors of norm at least 1. We aim to show that $k = O(d^{d+2})$. We first bound $\left\|\sum_{i=1}^{k+2} \boldsymbol{x}_i\right\|$ as in (9) and each $\|\boldsymbol{x}_j\|$ as in (10). Then we define a graph by joining $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ if they are ε-close. We use the Brunn–Minkowski inequality and the k-collapsing condition to bound the number of isolated points of this graph in (11). We then use the k-collapsing condition and the triangle inequality to show that the non-isolated points have a small diameter, and, choosing ε appropriately, Lemma 37 bounds the norm of the sum of the non-isolated points in (13). Combined with the bound (9) on $\left\|\sum_{i=1}^{k+2} \boldsymbol{x}_i\right\|$ and (11) on the number of isolated points, we finally bound the number of vectors by an expression in $O(d^{d+2})$.

Let $\boldsymbol{s} := \sum_{i=1}^{k+2} \boldsymbol{x}_i$. The k-collapsing condition gives an upper bound to the norm of $\boldsymbol{s}$ as follows: Since

$$\sum_{S\in\binom{[k+2]}{k}} \sum_{i\in S} \boldsymbol{x}_i = \binom{k+1}{k-1} \sum_{i=1}^{k+2} \boldsymbol{x}_i = \binom{k+1}{k-1} \boldsymbol{s},$$

the triangle inequality gives

$$\binom{k+1}{k-1} \|\boldsymbol{s}\| \leq \sum_{S\in\binom{[k+2]}{k}} \Big\| \sum_{i\in S} \boldsymbol{x}_i \Big\| \leq \binom{k+2}{k},$$

and

$$\|\boldsymbol{s}\| \leq \binom{k+2}{k} \Big/ \binom{k+1}{k-1} = 1 + 2/k. \tag{9}$$

Without loss of generality, $\|\boldsymbol{x}_i\| = 1$ for some $i \in [k+2]$. For each $j \in [k+2]\setminus\{i\}$ the k-collapsing condition implies that $\|(\boldsymbol{s} - \boldsymbol{x}_i) - \boldsymbol{x}_j\| \leq 1$, and again by the triangle inequality,

$$\|\boldsymbol{x}_j\| \leq 1 + \|\boldsymbol{s}\| + \|\boldsymbol{x}_i\| \leq 3 + 2/k. \tag{10}$$

Let $\varepsilon > 0$ (to be fixed later). Define a graph G on $[k+2]$ by joining i and j whenever $\|\boldsymbol{x}_i - \boldsymbol{x}_j\| < \varepsilon$. Let $C \subseteq [k+2]$ be the set of all isolated vertices of G. Suppose for the moment that $|C| \geq 2$. Partition C into two parts as equally as possible: $C = C_1 \cup C_2$ with $C_1 \cap C_2 = \varnothing$ and $\big||C_1| - |C_2|\big| \leq 1$. Let

$$S_t := \bigcup_{j\in C_t} B(\boldsymbol{x}_j, \varepsilon/2) \quad \text{for } t = 1, 2.$$

Then

$$\operatorname{vol}(S_t) = |C_t| \, (\varepsilon/2)^d \operatorname{vol}(B_X).$$

By the k-collapsing condition, $S_1 + S_2 \subseteq B(\boldsymbol{s}, 1+\varepsilon)$, which gives

$$\operatorname{vol}\Big(\frac{1}{2}S_1 + \frac{1}{2}S_2\Big) \leq \Big(\frac{1+\varepsilon}{2}\Big)^d \operatorname{vol}(B_X).$$

By the Brunn–Minkowski inequality,

$$\operatorname{vol}\Big(\frac{1}{2}S_1 + \frac{1}{2}S_2\Big) \geq \operatorname{vol}(S_1)^{1/2} \operatorname{vol}(S_2)^{1/2} = \sqrt{|C_1| \cdot |C_2|} (\varepsilon/2)^d \operatorname{vol}(B_X).$$

It follows that

$$\frac{|C| - 1}{2} < \sqrt{|C_1| \cdot |C_2|} \leq \Big(1 + \frac{1}{\varepsilon}\Big)^d$$

and

$$|C| < 2\Big(1 + \frac{1}{\varepsilon}\Big)^d + 1. \tag{11}$$

This bound clearly also holds if $|C| < 2$.

Next consider the complement $C' := [k+2] \setminus C$, consisting of the vertices of G of degree at least 1. We claim that

$$\operatorname{diam}\{\boldsymbol{x}_i \mid i \in C'\} < 1 + \varepsilon. \tag{12}$$

Consider distinct $i, j \in C'$. There exist $i', j' \in C'$ such that $i' \neq i$, $j' \neq j$, $\|\boldsymbol{x}_i - \boldsymbol{x}_{i'}\| < \varepsilon$ and $\|\boldsymbol{x}_j - \boldsymbol{x}_{j'}\| < \varepsilon$. Then by the triangle inequality and the k-collapsing condition,

$$\begin{aligned}\|2\boldsymbol{x}_i - 2\boldsymbol{x}_j\| &= \|\boldsymbol{x}_i - \boldsymbol{x}_{i'} + \boldsymbol{x}_i + \boldsymbol{x}_{i'} - \boldsymbol{s} + \boldsymbol{s} - \boldsymbol{x}_j - \boldsymbol{x}_{j'} + \boldsymbol{x}_{j'} - \boldsymbol{x}_j\| \\ &\leq \|\boldsymbol{x}_i - \boldsymbol{x}_{i'}\| + \|\boldsymbol{x}_i + \boldsymbol{x}_{i'} - \boldsymbol{s}\| + \|\boldsymbol{s} - \boldsymbol{x}_j - \boldsymbol{x}_{j'}\| + \|\boldsymbol{x}_{j'} - \boldsymbol{x}_j\|\end{aligned}$$

$$< \varepsilon + 1 + 1 + \varepsilon,$$

which shows (12). In order to apply Lemma 37 to $\{\boldsymbol{x}_i \mid i \in C'\}$ we set $\varepsilon = 1/d$ and obtain that

$$\Big\| \sum_{i \in C'} \boldsymbol{x}_i \Big\| > \frac{|C'|}{d^2} = \frac{k+2-|C|}{d^2}. \tag{13}$$

On the other hand, by (9) and (10),

$$\begin{aligned} \Big\| \sum_{i \in C'} \boldsymbol{x}_i \Big\| &= \Big\| \boldsymbol{s} - \sum_{i \in C} \boldsymbol{x}_i \Big\| \le \|\boldsymbol{s}\| + \sum_{i \in C} \|\boldsymbol{x}_i\| \\ &\le 1 + \frac{2}{k} + |C| \left(3 + \frac{2}{k}\right). \end{aligned}$$

By (11) and the choice of ε, $|C| \le 2(d+1)^d$. Combined with (13), we obtain

$$\frac{k+2}{d^2} < 1 + \frac{2}{k} + |C| \left(3 + \frac{2}{k} + \frac{1}{d^2}\right) = O(d^d). \qquad \square$$

5. Reformulation in terms of matrices

In this section we reduce the existence of a d-dimensional Minkowski space admitting vectors satisfying the k-collapsing or strong balancing conditions to the existence of a matrix of rank at least d satisfying certain properties. As a consequence we show that there is no loss of generality in assuming that the vectors in the definitions of $\mathcal{C}(k,d)$ and $\mathcal{CB}(k,d)$ (Def. 13) are unit vectors. We also present a general version of a well-known lower bound for the rank of a square matrix in terms of its trace and Frobenius norm.

Lemma 38. *Let $2 \le k \le m-2$. Suppose that $\{\alpha_i \mid i \in [m]\} \subset \mathbb{R}$ is a k-collapsing family of real numbers. If $|\alpha_i| \ge 1$ for some $i \in [m]$, then $|\alpha_j| \le 2 - |\alpha_i| \le 1$ for all $j \ne i$.*

Proof. Without loss of generality, $\alpha_m \ge 1$. Let $j \in [m-1]$. Choose any $I, J \in \binom{[m]}{k}$ such that $I \setminus J = \{m\}$ and $J \setminus I = \{j\}$. By the k-collapsing condition, $\sum_{s \in I} \alpha_s \le 1$ and $\sum_{s \in J} \alpha_s \ge -1$. Subtract these two inequalities to obtain $\alpha_m - \alpha_j \le 2$, hence $\alpha_j \ge \alpha_m - 2$.

Before proving that $\alpha_j \le 2 - \alpha_m$, we first show that

$$S := \{s \in [m] \mid \alpha_i > 0\}$$

contains at most $k-1$ elements. Suppose this is false. Choose any $I \in \binom{S}{k}$ such that $m \in I$. By the k-collapsing condition,

$$0 < \sum_{s \in I \setminus \{m\}} \alpha_s \le 1 - \alpha_m \le 0,$$

a contradiction. Consequently,

$$|[m] \setminus (S \cup \{j\})| \ge m - k \ge 2,$$

and there exist two distinct indices $i', j' \in [m] \setminus \{j, m\}$ such that $\alpha_{i'} \le 0$ and $\alpha_{j'} \le 0$. Choose any $I, I' \in \binom{[m]}{k}$ such that $I \setminus I' = \{j, m\}$ and $I' \setminus I = \{i', j'\}$. By the k-collapsing condition, $\sum_{s \in I} \alpha_s \le 1$ and $\sum_{s \in I'} \alpha_s \ge -1$. Subtract these inequalities to obtain $\alpha_j + \alpha_m - \alpha_{i'} - \alpha_{j'} \le 2$. Therefore,

$$\alpha_j \le 2 - \alpha_m + \alpha_{i'} + \alpha_{j'} \le 2 - \alpha_m. \qquad \square$$

Lemma 38 does not hold if $k = m-1 \geq 4$, as shown by the family $\{\alpha_i \mid i \in [m]\}$, where

$$\alpha_1 = \dots = \alpha_{m-2} = \frac{-2}{m-3}, \; \alpha_{m-1} = \frac{m-1}{m-3}, \; \alpha_m = \frac{2m-4}{m-3}.$$

However, it is easily seen that Lemma 38 holds when $k = m-1 \in \{2,3\}$.

Lemma 39. *Let $2 \leq k < m$ and $d \geq 2$. Let X^d be a d-dimensional Minkowski space, $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in X^d$, and $\boldsymbol{x}_i^* \in (X^d)^*$ a dual unit vector of $\boldsymbol{x}_i$ for each $i \in [m]$. Then the $m \times m$ matrix $A = [a_{i,j}] := [\langle \boldsymbol{x}_i^*, \boldsymbol{x}_j \rangle]$ has rank at most d and satisfies the following properties:*

$$a_{i,i} \geq 1 \text{ for all } i \in [m] \;\; \text{if } \|\boldsymbol{x}_i\| \geq 1 \text{ for all } i \in [m], \tag{14}$$

$$\left.\begin{aligned} a_{i,i} = 1 \text{ for all } i \in [m] \text{ and } |a_{i,j}| \leq 1 \text{ for all distinct } i,j \in [m] \\ \text{if } \|\boldsymbol{x}_i\| = 1 \text{ for all } i \in [m], \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} \text{the family of numbers in each row of } A \text{ is } k\text{-collapsing (in } \mathbb{R}) \\ \text{if } \{\boldsymbol{x}_i \mid i \in [m]\} \text{ is } k\text{-collapsing,} \end{aligned}\right\} \tag{16}$$

$$\text{and the sum of each row of } A \text{ is } 0 \;\; \text{if } \sum_{i=1}^m \boldsymbol{x}_i = \boldsymbol{o}. \tag{17}$$

Conversely, given any $m \times m$ matrix $A = [a_{i,j}]$ with $\operatorname{rank}(A) \leq d$, there exists a d-dimensional Minkowski space X^d and a family $\{\boldsymbol{x}_i \mid i \in [m]\} \subset X^d$ such that

$$\|\boldsymbol{x}_i\| \geq 1 \text{ for all } i \in [m] \;\; \text{if } a_{i,i} \geq 1 \text{ for all } i \in [m], \tag{14$'$}$$

$$\left.\begin{aligned} \|\boldsymbol{x}_i\| = 1 \text{ for all } i \in [m] \;\; \text{if } a_{i,i} = 1 \text{ for all } i \in [m] \\ \text{and } |a_{i,j}| \leq 1 \text{ for all distinct } i,j \in [m], \end{aligned}\right\} \tag{15$'$}$$

$$\left.\begin{aligned} \{\boldsymbol{x}_i \mid i \in [m]\} \text{ is } k\text{-collapsing if the family of} \\ \text{numbers of each row of } A \text{ is } k\text{-collapsing (in } \mathbb{R}), \end{aligned}\right\} \tag{16$'$}$$

$$\sum_{i=1}^m \boldsymbol{x}_i = \boldsymbol{o} \;\; \text{if the sum of each row of } A \text{ is } 0. \tag{17$'$}$$

Proof. Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in X^d$ with dual unit vectors $\boldsymbol{x}_1^*, \dots, \boldsymbol{x}_m^* \in (X^d)^*$ be given, and let $A = [a_{i,j}] := [\langle \boldsymbol{x}_i^*, \boldsymbol{x}_j \rangle]$. The factorisation

$$A = [\langle \boldsymbol{x}_i^*, \boldsymbol{x}_j \rangle]_{i,j \in [m]} = [\boldsymbol{x}_1^*, \dots, \boldsymbol{x}_m^*]^{\mathsf{T}} [\boldsymbol{x}_1, \dots, \boldsymbol{x}_m]$$

of A into matrices of rank at most d shows that A has rank at most d.

Since $|a_{i,j}| = |\langle \boldsymbol{x}_i^*, \boldsymbol{x}_j \rangle| \leq \|\boldsymbol{x}_j\|$ and $a_{i,i} = \langle \boldsymbol{x}_i^*, \boldsymbol{x}_i \rangle = \|\boldsymbol{x}_i\|$, we obtain (14) and (15). Also, if $I \in \binom{[m]}{k}$ and $\left\|\sum_{j \in I} \boldsymbol{x}_j\right\| \leq 1$, then for any $i \in [m]$,

$$\Big|\sum_{j \in I} a_{i,j}\Big| = \Big|\sum_{j \in I} \langle \boldsymbol{x}_i^*, \boldsymbol{x}_j \rangle\Big| = \Big|\Big\langle \boldsymbol{x}_i^*, \sum_{j \in I} \boldsymbol{x}_j \Big\rangle\Big| \leq \Big\|\sum_{j \in I} \boldsymbol{x}_j\Big\| \leq 1,$$

which gives (16). Similarly, if $\sum_{j=1}^m \boldsymbol{x}_j = \boldsymbol{o}$, then for any $i \in [m]$,

$$\sum_{j \in I} a_{i,j} = \Big\langle \boldsymbol{x}_i^*, \sum_{j=1}^m \boldsymbol{x}_j \Big\rangle = \langle \boldsymbol{x}_i^*, \boldsymbol{o} \rangle = 0,$$

which is (17).

Next, assume that an $m \times m$ matrix $A = [a_{i,j}]$ of rank at most d is given. Let $\boldsymbol{x}_j$ be the j-th column of A, considered as an element of ℓ_∞^m. Let X^d be any d-dimensional subspace of ℓ_∞^m that contains $\operatorname{span}(\{\boldsymbol{x}_j \mid j \in [n]\})$. (If $d > m$, let $X^d = \ell_\infty^d$ be a superspace of ℓ_∞^m.) Keeping the definition of $\|\cdot\|_\infty$ in mind, it is easily seen that (14′), (15′), (16′), and (17′) all hold. □

Corollary 40. *Let $2 \leq k < m$ and $d \geq 2$. There exists a d-dimensional Minkowski space that contains a k-collapsing [and balancing] family of m vectors of norm ≥ 1 iff there exists a d-dimensional Minkowski space that contains a k-collapsing [and balancing] family of m unit vectors.*

Proof. The case $k = m-1$ is trivial, as there exist $k+1$ unit vectors that sum to $\boldsymbol{o}$ if $d \geq 2$. Thus, we assume that $k \leq m-2$. Suppose that there exists a d-dimensional Minkowski space that contains k-collapsing family of m vectors of norm ≥ 1 [that satisfies the balancing condition]. By the first part of Lemma 39 there exists an $m \times m$ matrix $A = [a_{i,j}]$ of rank at most d, such that each row is k-collapsing and $a_{i,i} \geq 1$ for each $i \in [m]$ [and each row sums to 0]. Crucially, by Lemma 38, $|a_{i,j}| \leq 1$ for all $j \neq i$. If we divide row i of A by $a_{i,i}$, for each i, we obtain a matrix $\tilde{A} = [\tilde{a}_{i,j}] := [a_{i,j}/a_{i,i}]$ of the same rank as A, with each row k-collapsing, $\tilde{a}_{i,i} = 1$, and $|\tilde{a}_{i,j}| \leq 1$ for all i, j [and each row sums to 0]. By the second part of Lemma 39, there exists a d-dimensional Minkowski space that contains a k-collapsing family of unit vectors [and also satisfies the balancing condition]. □

The next lemma has various combinatorial and geometric applications [1, 2, 3, 4, 6, 8, 28]. We omit the standard proof. See the extended version of this paper (`arXiv:1210.0366`) for a proof of a version for complex numbers and historical remarks.

Lemma 41. *Let $A = [a_{i,j}]$ be any $n \times n$ matrix with real entries. Then*

$$\Big|\sum_{i=1}^{n} a_{i,i}\Big|^2 \leq \operatorname{rank}(A)\bigg(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}^2\bigg). \tag{18}$$

Equality holds in (18) *if and only if A is symmetric and all its non-zero eigenvalues are equal.*

6. A tight upper bound for $\mathcal{CB}_k(X)$

In this section we prove Theorem 20 using the tools of Section 5. To show that $\mathcal{CB}_k(X^d) \leq \max\{k+1, 2d\}$ for all d-dimensional X^d, it is sufficient by Lemmas 38 and 39 to prove that for any $m \times m$ matrix $A = [a_{i,j}]$ of rank at most d, such that each row is k-collapsing and has sum 0, each entry $|a_{i,j}| \leq 1$, and each diagonal entry $a_{i,i} = 1$, we have that $m \leq 2d$ if $k \leq m-2$. By Lemma 41 it is sufficient to show that $\left|\sum_i a_{i,i}\right|^2 / \sum_{i,j} |a_{i,j}|^2 \geq m/2$. Since $\sum_i a_{i,i} = m$, this is equivalent to $\sum_{i,j} a_{i,j}^2 \leq 2m$. Also, it follows from $a_{i,i} = 1$ that it will be sufficient to show that

$$\sum_{\substack{j=1\\ j\neq i}}^{m} a_{i,j}^2 \leq 1 \text{ for each } i \in [m].$$

This is implied by the next lemma, which solves a convex maximisation problem with linear constraints.

Lemma 42. *Let $k, m \in \mathbb{N}$ such that $2 \leq k \leq m-2$. Then*

$$\max\left\{\sum_{i=1}^{m-1}\alpha_i^2 \;\middle|\; \sum_{i=1}^{m}\alpha_i = 0,\ \alpha_m = 1,\ \{\alpha_i \mid i \in [m]\} \text{ is } k\text{-collapsing}\right\} = 1.$$

The maximum value $\sum_{i=1}^{m-1}\alpha_i^2 = 1$ is attained under these constraints only if for some $j \in [m-1]$, $\alpha_j = -1$ and $\alpha_i = 0$ for all $i \in [m-1] \setminus \{j\}$.

Proof. Since $\sum_{i=1}^{m}\alpha_i = 0$, the family $\{\alpha_i \mid i \in [m]\}$ is k-collapsing iff it is $(m-k)$-collapsing. Thus, without loss of generality, $k \leq m/2$.

The k-collapsing and balancing conditions imply the following constraints in the variables $\alpha_1, \dots, \alpha_{m-1}$:

$$\sum_{i\in I}\alpha_i \leq 0 \text{ for all } I \in \binom{[m-1]}{k-1} \tag{19}$$

and

$$\sum_{i=1}^{m-1}\alpha_i = -1. \tag{20}$$

Since these constraints, as well as the objective function $f(\alpha_1, \dots, \alpha_{m-1}) := \sum_{i=1}^{m-1}\alpha_i^2$ are symmetric in the variables $\alpha_1, \dots, \alpha_{m-1}$, we may assume without loss of generality that

$$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{m-1}. \tag{21}$$

Then (19) becomes equivalent to the single inequality

$$\sum_{i=1}^{k-1}\alpha_i \leq 0. \tag{22}$$

By Lemma 38, all $|\alpha_i| \leq 1$, and it follows that the $m-1$ linear inequalities in (21) and (22) define a polytope P in the hyperplane H of $\mathbb{R}^{m-1}$ defined by (20). The convex function f attains its maximum on P at a vertex of P. Since the point in $(\alpha_1, \dots, \alpha_{m-1}) \in \mathbb{R}^{m-1}$ with coordinates

$$\alpha_i = \frac{-2i}{m(m-1)}, \quad i \in [m-1]$$

satisfies (20), as well as (21) and (22) with strict inequalities (as well as (20)), P has non-empty interior in H. It follows that P is an $(m-2)$-dimensional simplex, and it is easy to calculate its $m-1$ vertices, as follows.

Case I. If $\alpha_1 = \dots = \alpha_{m-1}$ then (20) gives

$$(\alpha_1, \dots, \alpha_{m-1}) = \Big(\underbrace{\frac{-1}{m-1}, \dots, \frac{-1}{m-1}}_{m-1 \text{ times}}\Big)$$

and $f(\alpha_1, \dots, \alpha_{m-1}) = 1/(m-1) < 1$.

Case II. If $\alpha_1 = \dots = \alpha_t$ and $\alpha_{t+1} = \dots = \alpha_{m-1}$ for some $t \in [m-2]$, and $\sum_{i=1}^{k-1}\alpha_i = 0$, we distinguish between two subcases:

Subcase II.i. $t \leq k-1$. Then solving these equations with (20) gives

$$(\alpha_1, \dots, \alpha_{m-1}) = \Big(\underbrace{\frac{k-1-t}{t(m-k)}, \dots, \frac{k-1-t}{t(m-k)}}_{t \text{ times}}, \underbrace{\frac{-1}{m-k}, \dots, \frac{-1}{m-k}}_{m-1-t \text{ times}}\Big)$$

and

$$\begin{aligned} f(\alpha_1, \dots, \alpha_{m-1}) &= \frac{1}{t}\frac{(k-1)^2}{(m-k)^2} + \frac{m-2k+1}{(m-k)^2} \\ &\leq \frac{(k-1)^2 + m - 2k + 1}{(m-k)^2} \quad \text{(since } t \geq 1\text{)} \\ &\leq \frac{(m/2-1)^2 + m - 2k + 1}{(m/2)^2} \quad \text{(since } 2k \leq m\text{)} \\ &= \frac{(m/2)^2 - 2k + 2}{(m/2)^2} < 1. \end{aligned}$$

Subcase II.ii. $t \geq k$. Then

$$(\alpha_1, \dots, \alpha_{m-1}) = \Big(\underbrace{0, \dots, 0}_{t \text{ times}}, \underbrace{\frac{-1}{m-1-t}, \dots, \frac{-1}{m-1-t}}_{m-1-t \text{ times}}\Big)$$

and

$$f(\alpha_1, \dots, \alpha_{m-1}) = \frac{1}{m-1-t} \leq 1$$

with equality if and only if $t = m-2$, and then

$$(\alpha_1, \dots, \alpha_{m-1}) = (0, \dots, 0, -1).$$

This shows that the maximum of f on P is 1, attained at only one point if the coordinates are in decreasing order. □

Proof of Theorem 20. We first show that if $m \geq k+2$, then a k-collapsing, strongly balancing family of vectors of norm at least 1 has size at most $2d$, and when it has size $2d$, it is indeed made up of a unit basis and its negative.

Let $\{\boldsymbol{x}_i \mid i \in [m]\}$ be k-collapsing and strongly balancing with each $\|\boldsymbol{x}_i\| \geq 1$. For each $\boldsymbol{x}_i$, let $\boldsymbol{x}_i^* \in X^*$ be a dual unit vector. By Lemma 39, $A = [a_{ij}] := [\langle \boldsymbol{x}_i^*, \boldsymbol{x}_j \rangle]$ is an $m \times m$ matrix of rank at most d, each row is k-collapsing, each diagonal element is ≥ 1, and each row sum is 0. We will show that $\operatorname{rank}(A) \leq m/2$, with equality implying that, after some permutation of the $\boldsymbol{x}_i$,

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}. \tag{23}$$

By Lemma 38, $|a_{i,j}| \leq 1$ for all distinct i, j, and it follows that the matrix $\tilde{A} = [\tilde{a}_{i,j}] := [a_{i,j}/a_{i,i}]$ formed by dividing each row of A by $a_{i,i}$ has the same rank as

A, and its rows are still k-collapsing and sum to 0. By Lemma 42, $\sum_{j=1}^{m}\tilde{a}_{i,j}^2 \leq 2$ for all $i\in[m]$, and by Lemma 41,

$$d \geq \operatorname{rank}(A) = \operatorname{rank}(\tilde{A}) \geq \frac{m^2}{2m} = \frac{m}{2}.$$

This shows that $m \leq 2d$. Suppose now that $m = 2d$. Then $\operatorname{rank}(A) = \operatorname{rank}(\tilde{A}) = d$, by Lemma 41 $\tilde{A}$ is symmetric, and by Lemma 42 each row of $\tilde{A}$ has a 1 on the diagonal, a -1 at some non-diagonal entry, and 0s everywhere else. Thus $\tilde{A} = I - P$, where P is a symmetric permutation matrix. The associated permutation must be an involution. Therefore, after some permutation of the coordinates, $\tilde{A}$ is as in (23). Since $\tilde{A}$ has an off-diagonal entry of absolute value 1 in each column, each $a_{i,i} = 1$, hence $A = \tilde{A}$ and $\|\boldsymbol{x}_i\| = 1$ for all $i \in [m]$. Since $A = [\boldsymbol{x}_1^* \dots \boldsymbol{x}_{2d}^*]^{\mathsf{T}}[\boldsymbol{x}_1 \dots \boldsymbol{x}_{2d}]$ and the submatrix of A consisting of odd rows and columns is the $d\times d$ identity matrix, it follows that $\{\boldsymbol{x}_1, \boldsymbol{x}_3, \dots, \boldsymbol{x}_{2d-1}\}$ and $\{\boldsymbol{x}_1^*, \boldsymbol{x}_3^*, \dots, \boldsymbol{x}_{2d-1}^*\}$ are bases of X and X^*, respectively. Since $\langle \boldsymbol{x}_i^*, \boldsymbol{x}_1\rangle = \langle \boldsymbol{x}_i^*, \boldsymbol{x}_2\rangle = 0$ for all $i \geq 3$,

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in \bigcap_{j=2,\dots,d} \ker \boldsymbol{x}_{2j-1}^*,$$

which is a one-dimensional subspace of X. Therefore, $\boldsymbol{x}_1 = -\boldsymbol{x}_2$. Similarly, $\boldsymbol{x}_{2j-1} = -\boldsymbol{x}_{2j}$ for all $j \in [d]$. In particular, $\mathcal{CB}_k(X^d) \leq \max\{k+1, 2d\}$. Since by Proposition 19, $\mathcal{CB}_k(\ell_\infty^d) = \max\{k+1, 2d\}$, we obtain $\overline{\mathcal{CB}}(k,d) = \max\{k+1, 2d\}$.

We next show the last part of the theorem. Suppose that $k \leq 2d$. We have already shown that for any X^d, a k-collapsing family of $2d$ vectors of norm ≥ 1 is necessarily $\{\pm\boldsymbol{e}_i \mid i \in [d]\}$ for some unit basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_d\}$. Note that $\{\pm\boldsymbol{e}_i \mid i\in[d]\}$ is k-collapsing if $\sum_{i\in I}\boldsymbol{e}_i$ is contained in the unit ball for all $I \subseteq [d]$ with $|I| \leq k$. Any $\boldsymbol{o}$-symmetric convex body C that satisfies

$$P_k := \operatorname{conv}\left\{\pm\sum_{i\in I}\boldsymbol{e}_i \,\middle|\, I\subseteq[d], |I|\leq k\right\} \subseteq C \subseteq [-1,1]^d,$$

is the unit ball of a norm $\|\cdot\|_C$ such that $\{\pm\boldsymbol{e}_i \mid i\in[d]\}$ is k'-collapsing in the norm $\|\cdot\|_C$ for all $k' = 2,\dots,k$, with $\|\boldsymbol{e}_i\|_C = 1$. If $k < d$, P_k is a proper subset of $[-1,1]^d$ and we obtain infinitely many such unit balls C. If $k \geq d$, $P_k = [-1,1]^d$ and we obtain the unique norm $\|\cdot\|_\infty$ up to isometry. □

7. Tight and almost tight upper bounds for $\mathcal{C}_k(X)$

We now consider the k-collapsing condition without any balancing condition. As in the previous section we solve a convex optimisation problem. This case is more complicated and our results are only partial. Similar to the proof of Theorem 20, determining the maximum in the following lemma gives an upper bound for $\mathcal{C}_k(X)$ via Lemma 41.

Lemma 43. *Let $k, m \in \mathbb{N}$ be such that $2 \leq k \leq m-2$. Then*

$$\max\left\{\sum_{i=1}^{m-1}\alpha_i^2 \,\middle|\, \alpha_m = 1,\ \{\alpha_i \mid i\in[m]\} \text{ is } k\text{-collapsing}\right\}$$

$$\begin{cases} = \max\left\{\dfrac{m-1}{k^2}, 1, \dfrac{(k-2)^2+m-2}{k^2}\right\} \quad \textit{if } k < 2m/3, \\ \leq \max\left\{\dfrac{m-1}{k^2}, 1, \dfrac{(k-2)^2+m-2}{k^2}, \dfrac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)}\right\} \\ \qquad \textit{if } k \geq 2m/3, \end{cases}$$

$$\begin{cases} = \max\left\{\dfrac{m-1}{4}, 1\right\} & \textit{if } k = 2, \\ = \dfrac{(k-2)^2+m-2}{k^2} & \textit{if } 3 \leq k \leq \frac{m+2}{4}, \\ = 1 & \textit{if } \frac{m+2}{4} \leq k < \frac{2m}{3},\ k \geq 3, \\ \leq \max\left\{1, \dfrac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)}\right\} & \textit{if } k \geq 2m/3,\ k \geq 3. \end{cases}$$

Proof. Because the k-collapsing condition on $\{\alpha_i \mid i \in [m]\}$ and the objective function $f(\alpha_1, \ldots, \alpha_{m-1}) := \sum_{i=1}^{m-1} \alpha_i^2$ are symmetric in $\alpha_1, \ldots, \alpha_{m-1}$, we may assume without loss of generality that

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{m-1}. \tag{24}$$

Then the k-collapsing condition implies

$$-1 \leq \alpha_{m-k} + \alpha_{m-k+1} + \cdots + \alpha_{m-1} \tag{25}$$

and

$$\alpha_1 + \alpha_2 + \ldots \alpha_{k-1} \leq 0. \tag{26}$$

We find the maximum of f over the set Δ of points $(\alpha_1, \ldots, \alpha_{m-1})$ that satisfy (24), (25) and (26). In the cases of equality in the statement of the lemma, we will obtain points in Δ that also satisfy the k-collapsing condition. (In fact it can be shown that (25) and (26) are equivalent to the k-collapsing condition given that (24) holds.) By Lemma 38, $|\alpha_i| \leq 1$ for each $i \in [m-1]$, hence Δ is a polytope. Setting $\alpha_i = -i/km$ for $i \in [m-1]$, we see that (24) and (26) are obviously satisfied with strict inequalities, and (25) because

$$\sum_{i=m-k}^{m-1} \frac{-i}{km} = -1 + \frac{k(k+1)}{2km} > -1.$$

It follows that

$$\left(\frac{-1}{km}, \frac{-2}{km}, \ldots, \frac{-(m-1)}{km}\right) \in \mathbb{R}^{m-1}$$

is an interior point of Δ. Since (24), (25) and (26) are m inequalities in total, it follows that Δ is a simplex. The convex function f attains its maximum at one of the m vertices of Δ, which we calculate next. We distinguish between the following three cases:

Case I. Equality in (24) and (25):

$$\alpha_1 = \cdots = \alpha_{m-1} \quad \text{and} \quad -1 = \alpha_{m-k} + \cdots + \alpha_{m-1}.$$

The vertex is

$$(\alpha_1, \ldots, \alpha_{m-1}) = \Big(\underbrace{\frac{-1}{k}, \ldots, \frac{-1}{k}}_{m-1 \text{ times}}\Big),$$

and

$$f(\alpha_1, \dots, \alpha_{m-1}) = \frac{m-1}{k^2}.$$

Case II. Equality in (24) and (26):

$$\alpha_1 = \dots = \alpha_{m-1} \quad \text{and} \quad \alpha_1 + \dots + \alpha_{k-1} = 0.$$

Then $(\alpha_1, \dots, \alpha_{m-1}) = \boldsymbol{o}$ and $f(\alpha_1, \dots, \alpha_{m-1}) = 0 < (m-1)/k^2$.

Case III. For some $t \in [m-2]$,

$$\alpha_1 = \dots = \alpha_t =: a \quad \text{and} \quad \alpha_{t+1} = \dots = \alpha_{m-1} =: b$$

and equality in (25) and (26): Equality in (25) gives that

$$\text{if } m-k \geq t+1 \quad \text{then} \quad b = \frac{-1}{k}; \tag{25a}$$

$$\text{if } m-k \leq t \quad \text{then} \quad (k-m+1+t)a + (m-1-t)b = -1. \tag{25b}$$

Independent of these two cases, equality in (26) gives that

$$\text{if } k-1 \leq t \quad \text{then} \quad a = 0; \tag{26a}$$

$$\text{if } k-1 \geq t+1 \quad \text{then} \quad ta + (k-1-t)b = 0. \tag{26b}$$

This gives us four subcases, with some being empty, depending on k and m.

Subcase III.i. If $k-1 \leq t \leq m-k-1$, then by (25a) and (26a),

$$(\alpha_1, \dots, \alpha_{m-1}) = \Big(\underbrace{0, \dots, 0}_{t \text{ times}}, \underbrace{\frac{-1}{k}, \dots, \frac{-1}{k}}_{m-1-t \text{ times}}\Big)$$

and

$$f(\alpha_1, \dots, \alpha_{m-1}) = \frac{m-1-t}{k^2} \leq \frac{m-k}{k^2} < \frac{m-1}{k^2}.$$

This case occurs only if $2k \leq m$.

Subcase III.ii. If $\max\{k-1, m-k\} \leq t$, then by (25b) and (26a),

$$(\alpha_1, \dots, \alpha_{m-1}) = \Big(\underbrace{0, \dots, 0}_{t \text{ times}}, \underbrace{\frac{-1}{m-1-t}, \dots, \frac{-1}{m-1-t}}_{m-1-t \text{ times}}\Big)$$

and

$$f(\alpha_1, \dots, \alpha_{m-1}) = \frac{1}{m-1-t} \leq 1,$$

with equality if $t = m-2$. This case always occurs.

Subcase III.iii. If $t \le \min\{k-2, m-k-1\}$ (which occurs only if $k \ge 3$), then by (25a) and (26b),

$$(\alpha_1, \dots, \alpha_{m-1}) = \Big(\underbrace{\frac{k-1-t}{kt}, \dots, \frac{k-1-t}{kt}}_{t \text{ times}}, \underbrace{\frac{-1}{k}, \dots, \frac{-1}{k}}_{m-1-t \text{ times}}\Big)$$

and

$$\begin{aligned} f(\alpha_1, \dots, \alpha_{m-1}) &= \frac{1}{k^2}\left(\frac{(k-1)^2}{t} - 2k + 1 + m\right) \\ &\le \frac{1}{k^2}\big((k-1)^2 - 2k + 1 + m\big) \\ &= \frac{(k-2)^2 + m - 2}{k^2} =: g(k,m). \end{aligned}$$

Note that $g(k,m) \ge \frac{m-1}{k^2}$ (equality iff $k = 3$). Also, $g(k,m) \le 1$ iff $k \ge (m+2)/4$.

Subcase III.iv. If $m-k \le t \le k-2$ (which occurs only if $2k \ge m+2$ and $k \ge 4$), then we solve (25b) and (26b) to obtain

$$a = \frac{k-1-t}{t+(m-1-k)(k-1)} \quad \text{and} \quad b = \frac{-t}{t+(m-1-k)(k-1)}.$$

This gives the vertex as

$$(\alpha_1, \dots, \alpha_{m-1}) = \Big(\underbrace{\frac{k-1-t}{t+(m-1-k)(k-1)}}_{t \text{ times}}, \underbrace{\frac{-t}{t+(m-1-k)(k-1)}}_{m-1-t \text{ times}}\Big)$$

and

$$f(\alpha_1, \dots, \alpha_{m-1}) = \frac{(m-2k+1)t^2 + (k-1)^2 t}{(t+(m-1-k)(k-1))^2} =: s_{k,m}(t).$$

We now determine

$$h(k,m) := \max\left\{ s_{k,m}(t) \,\middle|\, t \in [m-k, k-2] \right\}.$$

Since this maximum could occur in the interior of the interval $[m-k, k-2]$, and the value of t where the maximum occurs might not be integral, we settle for determining the maximum of $s_{k,m}(t)$ over all real values of $t \in [m-k, k-2]$. Thus $h(k,m)$ will only be an upper bound for the maximum of $f(\alpha_1, \dots, \alpha_{m-1})$ on the vertices of Δ falling under this subcase. A calculation shows that $s'_{k,m}(t) \ge 0$ iff

$$t \le \frac{(k-1)^2(m-k-1)}{2(2k-m-1)(m-k-1)+k-1} =: t_0.$$

We next show that $m-k \le t_0$ unless $k = 4$ and $m = 6$. A calculation shows that

$$m-k \le t_0 \iff (k-1)^2 \le \frac{1}{2}(m-k)((k-1)^2 - 1 + (2m-3k)^2).$$

Since $m-k \ge 2$, this inequality clearly holds if $2m \ne 3k$, while if $2m = 3k$, it is equivalent to

$$(k-1)^2 \le \frac{1}{4}k((k-1)^2 - 1),$$

which holds if $k \ge 5$, but not if $k = 4$. However, in that case $(k,m) = (4,6)$ and $m-k = k-2$.

Next we show that if $k \geq 2m/3$ then $t_0 < k-2$, and if $k < 2m/3$ then $t_0 > k-2$. A calculation gives that

$$t_0 \leq k-2 \quad \iff \quad 0 \leq (k-2)(m-k)(3k-2m) + 2k - m - 1.$$

Since $2k - m - 1 > 0$, we obtain $t_0 < k-2$ if $k \geq 2m/3$. Otherwise $3k - 2m \leq -1$, and

$$\begin{aligned}&(k-2)(m-k)(3k-2m) + 2k - m - 1\\ &\leq -(k-2)(m-k) + 2k - m - 1 = -(k-1)(m-k-1) < 0.\end{aligned}$$

It follows that $t_0 > k-2$ if $k < 2m/3$.

In summary,

$$h(k,m) = \begin{cases} s_{k,m}(t_0) & \text{if } k \geq 2m/3 \text{ and } (k,m) \neq (4,6),\\ s_{k,m}(k-2) & \text{if } k < 2m/3 \text{ or } (k,m) = (4,6).\end{cases}$$

We next show that $s_{k,m}(k-2) < 1$, which means that this subcase is only relevant when $k \geq 2m/3$ and $(k,m) \neq (4,6)$. Since

$$s_{k,m}(k-2) = \frac{(m-2k+1)(k-2)^2 + (k-1)^2(k-2)}{(k-2+(m-1-k)(k-1))^2},$$

a calculation shows that

$$s_{k,m}(k-2) < 1 \iff m - 2k < (k-1)^2\left((m-k)(m-k-1) - 1\right),$$

which holds since $m - 2k < 0$ and $m - k \geq 2$. Finally we calculate

$$s_{k,m}(t_0) = \frac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)}.$$

This concludes estimating f at the vertices of Δ. To summarise the above case analysis, we have shown that

$$\max f(\Delta) = \max\left\{\frac{m-1}{k^2}, 1, \frac{(k-2)^2+m-2}{k^2}\right\} \quad \text{if } k < 2m/3,$$

and

$$\max f(\Delta) \leq \max\left\{\tfrac{m-1}{k^2}, 1, \tfrac{(k-2)^2+m-2}{k^2}, \tfrac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)}\right\} \text{ if } k \geq 2m/3.$$

The remaining claims of the lemma are now easily checked. ☐

Proof of Theorem 25. (1) Let $\sqrt{d} < k \leq (d+1)/2$. Suppose that there exist $m > 2d(1 + \frac{d-2k+1}{k^2-d})$ vectors of norm ≥ 1 satisfying the k-collapsing condition; equivalently, an $m \times m$ matrix A of rank $\leq d$ with 1s on the diagonal and such that each row satisfies the k-collapsing condition. Since $m > 2d \geq 2k-1$, we have $k < (m+2)/4$, and by Lemma 43 the sum of the squares of the entries in any row of A is $\leq 1 + \frac{(k-2)^2+m-2}{k^2} = 2 + \frac{m-4k+2}{k^2}$. By Lemma 41,

$$d \geq \operatorname{rank}(A) \geq \frac{m^2}{m\left(2 + \frac{m-4k+2}{k^2}\right)} = \frac{mk^2}{2k^2 + m - 4k + 2}.$$

Solving for m (and taking note that $k > \sqrt{d}$) we obtain

$$m \leq \frac{2d(k-1)^2}{k^2-d},$$

contradicting the assumption on m. This shows that $\overline{\mathcal{C}}(k,d) \leq \frac{2d(k-1)^2}{k^2-d}$.

(2) In particular we obtain that $\overline{\mathcal{C}}(k,d) \le 2d$ when $\sqrt{d} < k \le (d+1)/2$ if

$$\frac{2d(k-1)^2}{k^2-d} < 2d+1,$$

which is equivalent to $k \ge -2d + \sqrt{6d^2+3d+1}$. It remains to show that $\overline{\mathcal{C}}(k,d) \le 2d$ if $(d+1)/2 < k \le 2d - \sqrt{d/2}$. Suppose that there exists an $m \times m$ matrix A of rank $\le d$ with 1s on the diagonal and such that each row satisfies the k-collapsing condition, where $m = 2d+1$. It then follows from $k > (d+1)/2$ that $k > (m+2)/4$. If furthermore $k < 2m/3$ then by Lemmas 41 and 43, $d \ge \operatorname{rank}(A) \ge \frac{m^2}{m(1+1)}$ and $m \le 2d$, a contradiction. Therefore, $k \ge 2m/3$. We next show that

$$\frac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)} < 1, \tag{27}$$

which again gives the contradiction $m \le 2d$ by Lemmas 41 and 43.

Consider $f(x) = (m-x-1)(2x-m)(m-x)$, $2m/3 \le x \le m-2$. Then $f'(x) = (4m-6x)(m-x-1) - 2x + m < 0$, and it follows that the left-hand side of (27) increases with k. It is therefore sufficient to prove (27) for $k = 2d - \sqrt{d/2}$, that is

$$\frac{(2d-\sqrt{d/2}-1)^2}{4\sqrt{d/2}(2d-2\sqrt{d/2}-1)(\sqrt{d/2}+1)} < 1.$$

This is equivalent to $8d\sqrt{d/2} - 5d/2 - 6\sqrt{d/2} - 1 > 0$, which is easily seen to be true.

(3) Let $d \ge 3$ and $k > 2d - \sqrt{d/2}$. Suppose that there exists an $m \times m$ matrix of rank $\le d$ with 1s on its diagonal and each row k-collapsing, where $m > k + \frac{1+\sqrt{2d-3}}{2}$. As before, we aim to find a contradiction using Lemmas 41 and 43.

Writing $t = m-k$, we have $t > \frac{1+\sqrt{2d-3}}{2} > 1$. It follows that $d < 2t^2 - 2t + 2 < 2t^2$, hence $k > 2d - \sqrt{d/2} > 2d - t$ and $m = k + t \ge 2d+1$.

Now we may assume without loss of generality that

$$m = \left\lfloor k + \frac{1+\sqrt{2d-3}}{2} \right\rfloor + 1.$$

Since

$$3k > 3(2d - \sqrt{d/2}) = 3d + 3(d - \sqrt{d/2}) > 3d > 2d + \sqrt{d/2} \ge 4 + \sqrt{d/2},$$

we have $4k - 2 \ge k + 2 + \sqrt{d/2} > m$ and $k > (m+2)/4$. By Lemma 43, if $k < 2m/3$ or

$$\frac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)} \le 1,$$

then Lemma 41 would give $d \ge \frac{m^2}{m(1+1)}$ and $m \le 2d$, a contradiction. Therefore, $k \ge 2m/3$ and

$$\frac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)} > 1.$$

Lemma 41 now gives

$$d \ge \frac{m^2}{m\left(1 + \dfrac{(k-1)^2}{4(m-k-1)(m-k)(m-2k)}\right)} = \frac{m}{\left(1 + \dfrac{(k-1)^2}{4(t-1)t(k-t)}\right)},$$

which implies

$$k+t=m\le\left(1+\frac{(k-1)^2}{4(t-1)t(k-t)}\right)d. \tag{28}$$

If we set $f(x)=\left(1+\frac{(x-1)^2}{4(t-1)t(x-t)}\right)d-(x+t)$ for $x\ge 2d-t+1$, it follows (since $d<2t^2$, $t\ge 2$, and $k\ge 2m/3$) that

$$\begin{aligned} f'(x) &= \frac{d}{4(t-1)t}\left(1-\left(\frac{t-1}{x-t}\right)^2\right)-1 \\ &< \frac{2t^2}{4(t-1)t}-1=\frac{2-t}{2(t-1)}\le 0, \end{aligned}$$

and f is strictly decreasing. It follows that since (28) holds for some $k\ge 2d-t+1$, it remains true if we substitute $2d-t+1$ into k, that is,

$$2d+1\le\left(1+\frac{(2d-t)^2}{4(t-1)t(2d-2t+1)}\right)d, \tag{29}$$

which is equivalent to

$$4(d+1)(t-1)t(2d-2t+1)\le(2d-t)^2d. \tag{30}$$

We next show that the opposite inequality holds, which gives the required contradiction. Since $t=m-k=\lfloor\frac{1+\sqrt{2d-3}}{2}\rfloor+1$,

$$t-1\le\frac{1+\sqrt{2d-3}}{2}<t,$$

or equivalently,

$$2t^2-6t+6\le d\le 2t^2-2t+1. \tag{31}$$

It can be checked that

$$\begin{aligned} & 4(d+1)(t-1)t(2d-2t+1)-(2d-t)^2d \\ = \quad & (t-1)^3(6t+4)+(t-1)^2-1 \\ & +(2t^2-2t+1-d)\big((2d-t-2)^2+12d-4t^2-4t\big). \end{aligned} \tag{32}$$

By (31), since $t\ge 2$,

$$12d-4t^2-4t\ge 12(2t^2-6t+6)-4t^2-4t=(5t-9)(4t-8)\ge 0,$$

hence

$$(2t^2-2t+1-d)((2d-t-2)^2+12d-4t^2-4t)\ge 0.$$

Substitute this into (32) to obtain

$$\begin{aligned} & 4(d+1)(t-1)t(2d-2t+1)-(2d-t)^2d \\ & \ge(t-1)^3(6t+4)+(t-1)^2-1 \\ & >0, \end{aligned}$$

which contradicts (30). □

Proof of Theorem 26. Suppose that there exists an $m\times m$ matrix of rank $\le d$ with 1s on its diagonal and with each row k-collapsing. We first treat the case $k=2$. By Lemmas 41 and 43,

$$\frac{m^2}{m(1+\max\{1,(m-1)/4\})}\le d.$$

If the maximum in the denominator equals 1 then $m \leq 2d$. Otherwise, $m \leq (1+(m-1)/4)d$ and it follows that $(1-d/4)m \leq 3d/4$. If $d < 4$ then $m \leq 3d/(4-d)$. In particular, if $d = 2$ then $m \leq 3$, and if $d = 3$ then $m \leq 9$. This shows that $\overline{\mathcal{C}}(2,2) = 4$ and $\overline{\mathcal{C}}(2,3) \leq 9$.

Next assume that $k \geq 3$. Without loss of generality, $m = k+2 > 2d$. We aim for a contradiction. Clearly, $k = m-2 > (m+2)/4$. If the maximum in Lemma 43 equals 1, Lemma 41 gives $m \leq 2d$, a contradiction. Therefore, $k \geq 2m/3$, the maximum in Lemma 43 equals

$$\frac{(k-1)^2}{4(m-k-1)(2k-m)(m-k)} = \frac{(m-3)^2}{8(m-4)} > 1, \tag{33}$$

and by Lemma 41,

$$\frac{m^2}{m\left(1+\frac{(m-3)^2}{8(m-4)}\right)} \leq d. \tag{34}$$

By (33), $m \geq 10$ and $k \geq 8$. Solving for m in (34) gives

$$m \leq \frac{d+16+2\sqrt{6d^2-38d+64}}{8-d}$$

if we assume $d < 8$. Since $k = m-2$, we obtain

$$k \leq \frac{3d+2\sqrt{6d^2-38d+64}}{8-d}.$$

Keeping in mind that $m = k+2 > 2d$ and $m \geq 10$, we obtain a contradiction if $d \leq 5$ (and $k \geq 3$); or if $d = 6$ and $k \geq 17$; or if $d = 7$ and $k \geq 41$. This proves the theorem. □

8. Upper bounds using the ranks of Hadamard powers of a matrix

The following lemma, used by Alon in [1, 2], bounds the ranks of the integral Hadamard powers of a square matrix from above in terms of the rank of the matrix. It can be used to change a matrix to one that is sufficiently close to the identity matrix so that Lemma 41 can give a good bound.

Lemma 44 (Alon [1, Lemma 9.2])**.** *Let $A = [a_{i,j}]$ be an $n \times n$ matrix of rank d (over any field), and let $p \geq 1$ be an integer. Then the rank of the p-th Hadamard power $A^{\odot p}$ satisfies*

$$\operatorname{rank}(A^{\odot p}) = \operatorname{rank}([a_{i,j}^p]) \leq \binom{p+d-1}{p}.$$

In order to use the above lemma in combination with Lemma 41 as before, we need to maximise $\sum_i x_i^{2p}$ on the simplex Δ from the proof of Lemma 43. Here we restrict the range of k to avoid the difficulties in Case III.iv in the proof of Lemma 43.

Lemma 45. *Let $p,k,m \in \mathbb{N}$ be such that $2 \leq k \leq (m+1)/2$. Then*

$$\max\left\{\sum_{i=1}^{m-1} \alpha_i^{2p} \,\middle|\, \alpha_m = 1,\ \{\alpha_i \mid i \in [m]\} \text{ is } k\text{-collapsing}\right\}$$

$$= \begin{cases} \max\left\{1, \dfrac{m-1}{k^{2p}}\right\} & \textit{if } k=2, \\ \max\left\{1, \dfrac{(k-2)^{2p}+m-2}{k^{2p}}\right\} & \textit{if } k \geq 3. \end{cases}$$

Proof. As in the proof of Lemma 43 we have to maximise the new objective function $f_p(\alpha_1,\dots,\alpha_{m-1}) = \sum_{i=1}^{m-1} x_i^{2p}$ over the same simplex Δ defined by (24), (25) and (26) as before. Since f_p is convex, it is again sufficient to calculate the values of f_p on the vertices of Δ. Using the same case numbering as in the proof of Lemma 43, we obtain the following values:

Case I. $f_p(\alpha_1,\dots,\alpha_{m-1}) = \dfrac{m-1}{k^{2p}}$.

Case II. $f_p(\alpha_1,\dots,\alpha_{m-1}) = 0 < \dfrac{m-1}{k^{2p}}$.

Subcase III.i. $f_p(\alpha_1,\dots,\alpha_{m-1}) = \dfrac{m-1-t}{k^{2p}} \leq \dfrac{m-k}{k^{2p}} < \dfrac{m-1}{k^{2p}}$.

Subcase III.ii. $f_p(\alpha_1,\dots,\alpha_{m-1}) = \dfrac{1}{(m-1-t)^{2p-1}} \leq 1$ with equality iff $t = m-2$.

Subcase III.iii.

$$\begin{aligned} f_p(\alpha_1,\dots,\alpha_{m-1}) &= \frac{1}{k^{2p}}\left(t\left(\left(\tfrac{k-1}{t}-1\right)^{2p}-1\right)+m-1\right) =: g_p(t) \\ &\leq g_p(1) = \frac{1}{k^{2p}}\left((k-2)^{2p}+m-2\right) \end{aligned}$$

since $g_p(t)$ is decreasing for $0 < t < k-1$. This case occurs only if $k \geq 3$.

Subcase III.iv. The case $m-k \leq t \leq k-2$ occurs only if $2k \geq m+2$, which we have assumed to be false. □

Lemma 46. *If* $p \in \mathbb{N}$ *and* $k > \binom{d+p-1}{p}^{\frac{1}{2p}}$ *then*

$$\overline{C}(k,d) < \max\left\{\frac{2k^{2p}\binom{d+p-1}{p}}{k^{2p}-\binom{d+p-1}{p}}, 2k-1\right\}.$$

Proof. By Lemmas 38 and 39, there exists an $m \times m$ matrix $A = [a_{i,j}]$ of rank at most d, with 1s on its diagonal, and with each row k-collapsing, where $m = \overline{C}(k,d)$. Without loss of generality, $m \geq 2k-1$. By Lemma 45, for any row $i \in [m]$ of $A^{\odot 2p}$,

$$\sum_{j=1}^{m} a_{i,j}^{2p} < 2 + \frac{m}{k^{2p}}$$

and by Lemmas 41 and 44,

$$\binom{p+d-1}{p} \geq \operatorname{rank}([a_{i,j}^{2p}]) > \frac{m^2}{m\left(2+\frac{m}{k^{2p}}\right)},$$

from which follows

$$m < \frac{2k^{2p}\binom{d+p-1}{p}}{k^{2p}-\binom{d+p-1}{p}}.$$ □

Proof of Theorem 24. This is just a calculation from Lemma 46. Since

$$\frac{k^{2p}}{\binom{d+p-1}{p}} > \frac{((p!)^{-1/2p}+\varepsilon)^{2p}d^p}{\binom{d+p-1}{p}} \xrightarrow{d\to\infty} (1+(p!)^{1/2p}\varepsilon)^{2p} > 1+2p(p!)^{1/2p}\varepsilon,$$

it follows that if d is sufficiently large depending on p and ε, then

$$\frac{k^{2p}}{\binom{d+p-1}{p}} > 1+p(p!)^{1/2p}\varepsilon =: 1+\delta,$$

where $\delta > 0$ depends only on p and ε. Then

$$\binom{d+p-1}{p}^{-1} - k^{-2p} > \frac{\delta}{k^{2p}},$$

and by Lemma 46, (since $\overline{C}(k,d) \geq 2d \geq 2k^2 > 2k-1$)

$$\overline{C}(k,d) < \frac{2}{\binom{d+p-1}{p}^{-1} - k^{-2p}} < \frac{2k^{2p}}{\delta} \leq \frac{2d^p}{\delta}. \qquad \square$$

Lemma 47. *Let $n > k \geq 1$ be integers and $\varepsilon = k/n$. Then*

$$\binom{n}{k} < \frac{(\varepsilon^{-\varepsilon}(1-\varepsilon)^{-(1-\varepsilon)})^n}{\sqrt{2\pi\varepsilon(1-\varepsilon)n}}.$$

Proof. Substitute the Stirling formula in the form $m! = \mathrm{e}^{\delta_m}(\frac{m}{\mathrm{e}})^m\sqrt{2\pi m}$, where $\frac{1}{12m+1} < \delta_m < \frac{1}{12m}$ [27] into $\frac{n!}{k!(n-k)!}$ to obtain

$$\binom{n}{k} < \frac{(\varepsilon^{-\varepsilon}(1-\varepsilon)^{-(1-\varepsilon)})^n}{\sqrt{2\pi\varepsilon(1-\varepsilon)n}}\mathrm{e}^{\frac{1}{12n}-\frac{1}{12k+1}-\frac{1}{12(n-k)+1}}.$$

It is easily seen that $\frac{1}{a+b} < \frac{1}{a+1}+\frac{1}{b+1}$ for all $a,b \geq 1$. In particular, $\frac{1}{12n} < \frac{1}{12k+1}+\frac{1}{12(n-k)+1}$ and the lemma follows. $\square$

Proof of Theorem 23. The function $f(x) = (1+x)^{1/x}(1+1/x)$ is strictly decreasing on $(0,1]$ with $\lim_{x\to 0+} f(x) = \infty$ and $f(1) = 4$. Therefore, $\gamma_2 = 1$ and (γ_k) is strictly decreasing. Since $f(x) < \mathrm{e}\cdot(1+1/x)$, we have $f(\mathrm{e}/(k^2-\mathrm{e})) < k^2$ and $\gamma_k < \mathrm{e}/(k^2-\mathrm{e})$. Also, since

$$\frac{x}{x+1} = 1-\frac{1}{x+1} < \mathrm{e}^{-1/(x+1)},$$

it follows that $(1+1/x)^{x+1} > \mathrm{e}$. Set $x = k^2/\mathrm{e}$ to obtain that $f(\mathrm{e}/k^2) > k^2$ and $\mathrm{e}/k^2 < \gamma_k$.

Let $p := \lceil \gamma_k d\rceil$ and $\gamma := p/d$. Then $\gamma \geq \gamma_k$ and it follows that

$$(1+\gamma)^{1/\gamma}\left(1+\frac{1}{\gamma}\right) \leq k^2. \qquad (35)$$

We estimate $\binom{p+d-1}{p}$ as follows:

$$\binom{p+d-1}{p} = \binom{(1+\gamma)d-1}{\gamma d} = \frac{1}{1+\gamma}\binom{(1+\gamma)d}{\gamma d}$$
$$< \frac{\left((1+1/\gamma)^\gamma(1+\gamma)\right)^d}{\sqrt{2\pi\gamma(1+\gamma)d}} \quad \text{by Lemma 47}$$

$$\leq \frac{k^{2\gamma d}}{\sqrt{2\pi\gamma(1+\gamma)d}} \quad \text{by (35)}$$
$$= \frac{k^{2p}}{\sqrt{2\pi\gamma(1+\gamma)d}}.$$

In particular, $\binom{p+d-1}{p} < k^{2p}$ since

$$\sqrt{2\pi\gamma(1+\gamma)d} > \sqrt{2\pi\gamma d} = \sqrt{2\pi p} \geq \sqrt{2\pi} > 1.$$

By Lemma 46, either $\overline{\mathcal{C}}(k,d) < 2k-1$ or

$$\overline{\mathcal{C}}(k,d) < \frac{2k^{2p}k^{2p}}{\sqrt{2\pi\gamma(1+\gamma)d}\left(k^{2p} - \frac{k^{2p}}{\sqrt{2\pi\gamma(1+\gamma)d}}\right)}$$
$$= \frac{2k^{2p}}{\sqrt{2\pi\gamma(1+\gamma)d}-1}.$$

This gives

$$\overline{\mathcal{C}}(k,d) < \max\left\{\frac{2}{\sqrt{2\pi}-1}k^{2p}, 2k-1\right\} < 1.33k^{2\gamma_k d+2}.$$

We now assume that $k < \sqrt{d}$. Then $\overline{\mathcal{C}}(k,d) \geq 2d > 2k-2$ and

$$\overline{\mathcal{C}}(k,d) < \frac{2k^{2p}}{\sqrt{2\pi\gamma(1+\gamma)d}-1}$$
$$< \frac{2k^{2\gamma_k+2}}{\sqrt{2\pi\gamma_k d}-1}$$
$$< \frac{2k^{2\gamma_k+2}}{\sqrt{2\pi(\mathrm{e}/k^2)d}-1}.$$

Since $\sqrt{2\pi\mathrm{e}} > 3$ and $d/k^2 > 1$, it follows that $\sqrt{2\pi(\mathrm{e}/k^2)d}-1 > 2\sqrt{d/k^2}$ and $\overline{\mathcal{C}}(k,d) < k^{3+2\gamma_k d}/\sqrt{d}$. □

9. Lower bounds

Lemma 48. *Let $k \geq 2$. Suppose there exist at least m unit vectors $\boldsymbol{u}_i \in \ell_2^{d-1}$ such that*

$$|\langle \boldsymbol{u}_i, \boldsymbol{u}_j\rangle| \leq \frac{1}{2k+1} \quad \text{for all distinct } i,j.$$

Then there exists a d-dimensional Minkowski space X^d such that $\mathcal{C}_k(X^d) \geq m$. If $|\langle \boldsymbol{u}_i, \boldsymbol{u}_j\rangle| < 1/(2k+1)$ for all distinct i,j, then X^d can be chosen to be strictly convex and C^∞.

Proof. The construction is similar to the construction in [13] of a strictly convex d-dimensional space X^d such that $C_2(X^d) \geq 1.02^d$. The main difference is that we define the unit ball as an intersection of half spaces instead of a convex hull of a finite set of points.

Consider ℓ_2^{d-1} to be a hyperplane of ℓ_2^d with unit normal $\boldsymbol{e}$. Let $\boldsymbol{x}_i = \boldsymbol{u}_i + \boldsymbol{e}$ and $\boldsymbol{y}_i = (1+\frac{1}{2k})\boldsymbol{u}_i - \frac{1}{2k}\boldsymbol{e}$ for each $i \in [m]$. Let

$$B := \left\{\boldsymbol{x} \in \ell_2^d \,\middle|\, |\langle \boldsymbol{x}, \boldsymbol{y}_i\rangle| \leq 1 \text{ for all } i \in [m]\right\}.$$

If $\operatorname{span}(\{\boldsymbol{y}_i\}) = \mathbb{R}^d$ then B is bounded and is the unit ball of some norm $\|\cdot\|_B$. Otherwise $\{\boldsymbol{y}_i\}$ spans a hyperplane with normal $\boldsymbol{e}'$, say. In this case B as defined above is unbounded, so we have to modify it. Before doing that, we show that $\boldsymbol{x}_i \in \partial B$ and

$$\sum_{i\in I} \boldsymbol{x}_i \in B \quad \text{for all } I \in \binom{[m]}{k}.$$

Let $i, j \in [m]$. Then

$$\langle \boldsymbol{x}_i, \boldsymbol{y}_j \rangle = \left(1 + \frac{1}{2k}\right) \langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle - \frac{1}{2k}.$$

In particular, $\langle \boldsymbol{x}_i, \boldsymbol{y}_i \rangle = 1$, and since $-\frac{1}{2k+1} \leq \langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle \leq \frac{1}{2k+1}$ for distinct i, j, we obtain

$$-\frac{1}{k} \leq \langle \boldsymbol{x}_i, \boldsymbol{y}_j \rangle \leq 0 \text{ for distinct } i, j \in [m], \tag{36}$$

and it follows that $\boldsymbol{x}_i \in \partial B$.

Next let $I \in \binom{[m]}{k}$ and $i \in [m]$. We distinguish between two cases, depending on whether $i \in I$ or not.

If $i \notin I$, then by (36),

$$-1 \leq \left\langle \sum_{j\in I} \boldsymbol{x}_j, \boldsymbol{y}_i \right\rangle \leq 0.$$

If $i \in I$, then again by (36),

$$\frac{1}{k} = 1 - \frac{k-1}{k} \leq \left\langle \sum_{j\in I} \boldsymbol{x}_j, \boldsymbol{y}_i \right\rangle \leq 1.$$

In both cases we have $\left|\left\langle \sum_{j\in I} \boldsymbol{x}_j, \boldsymbol{y}_i \right\rangle\right| \leq 1$ for all i, and it follows that $\sum_{j\in I} \boldsymbol{x}_j \in B$ for all I. If $\operatorname{span}(\{\boldsymbol{y}_i\}) = \mathbb{R}^d$, then we have shown that B is the unit ball of a norm $\|\cdot\|_B$ such that $\{\boldsymbol{x}_i\}$ is a k-collapsing family of unit vectors in $\mathbb{R}^d, \|\cdot\|_B)$. In the case where $\operatorname{span}(\{\boldsymbol{y}_i\})$ is a hyperplane with normal $\boldsymbol{e}'$, we choose $\lambda > 0$ sufficiently large so that $|\langle \boldsymbol{x}_i, \boldsymbol{e}' \rangle| < \lambda$ for all i and $\left|\left\langle \sum_{i\in I} \boldsymbol{x}_i, \boldsymbol{e}' \right\rangle\right| < \lambda$ for all $I \in \binom{[m]}{k}$, and define the required unit ball to be

$$B := \left\{ \boldsymbol{x} \in \ell_2^d \,\middle|\, |\langle \boldsymbol{x}, \boldsymbol{y}_i \rangle| \leq 1 \text{ for all } i \in [m] \text{ and } |\langle \boldsymbol{x}, \boldsymbol{e}' \rangle| \leq \lambda \right\}.$$

If $|\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle| < 1/(2k+1)$ for distinct i, j, then $\left|\left\langle \sum_{j\in I} \boldsymbol{x}_j, \boldsymbol{y}_i \right\rangle\right| < 1$ for all i, and $\sum_{j\in I} \boldsymbol{x}_j \in \operatorname{int} B$ for all I. Also note that no $\boldsymbol{x}_j$, $j \neq i$, is on any of the hyperplanes

$$\left\{ \boldsymbol{x} \in \ell_2^d \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{y}_i \rangle = \pm 1 \right\} \text{ or } \left\{ \boldsymbol{x} \in \ell_2^d \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{e}' \rangle = \pm \lambda \right\}.$$

Then a strictly convex and C^∞ norm can be found with unit ball between $\operatorname{conv}\{\boldsymbol{x}_i\}$ and B [14]. □

For a detailed proof of the following lemma, see [34]. It uses a greedy construction.

Lemma 49. *Let $\delta > 0$. For sufficiently large d depending on δ, there exist $m \geq \left(1 + \frac{\delta^2}{2}\right)^d$ unit vectors $\boldsymbol{u}_i$ in ℓ_2^{d-1} such that $|\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle| < \delta$ for all distinct i, j.*

Proof of Theorem 32. Immediate from Lemmas 48 and 49. □

The following well-known lemma can be traced back to Kashin [18] (see also [10, Lemmas 3.2 and 3.3]). We omit the proof, which can be found in the extended version of this paper (`arXiv:1210.0366`).

Lemma 50. *Let q be a prime power and $s \in \mathbb{N}$ with $s < q$. Then there exist q^{s+1} unit vectors in $\ell_2^{q^2-q}$ such that the inner product of any two vectors is in the interval $[-\frac{1}{q-1}, \frac{s-1}{q-1}]$.*

Proof of Theorem 33. Set $s = c + 1$ in Lemma 50 and then apply Lemma 48. □

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