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Article (Published version)
(Refereed)

Original citation:
Dassios, Angelos and Lim, Jia Wei (2013) *Parisian option pricing: a recursive solution for the density of the Parisian stopping time*. SIAM Journal on Financial Mathematics, 4 (1). pp. 599-615. ISSN 1945-497X DOI: 10.1137/120875466

SIAM J. FINANCIAL MATH.
Vol. 4, pp. 599–615

Parisian Option Pricing: A Recursive Solution for the Density of the Parisian Stopping Time*

Angelos Dassios† and Jia Wei Lim†

Abstract. In this paper, we obtain the density function of the single barrier one-sided Parisian stopping time. The problem reduces to that of solving a Volterra integral equation of the first kind, where a recursive solution is consequently obtained. The advantage of this new method as compared to that in previous literature is that the recursions are easy to program as the resulting formula involves only a finite sum and does not require a numerical inversion of the Laplace transform. For long window periods, an explicit formula for the density of the stopping time can be obtained. For shorter window lengths, we derive a recursive equation from which numerical results are computed. From these results, we compute the prices of one-sided Parisian options.

Key words. Parisian option, Brownian excursion, Volterra equation

AMS subject classifications. Primary, 91G60; Secondary, 91G20, 60J65

DOI. 10.1137/120875466

1. Introduction.

Parisian options were first introduced by Chesney, Jeanblanc-Picque, and Yor [5]. They are path-dependent options whose payoff depends not only on the final value of the underlying asset, but also on the path trajectory of the underlying asset above or below a predetermined barrier L. The owner of a Parisian down-and-out call loses the option when the underlying asset price S reaches the level L and remains constantly below this level for a time interval longer than D, while for a Parisian down-and-in call, the same event gives the owner the right to exercise the option. Parisian options are a kind of barrier option. However, it has the advantage of not being as easily manipulated by an influential agent as a simple barrier option, and thus is a guarantee against easy arbitrage.

No explicit pricing formula is known for this type of option. Previous literature has largely focused on using Laplace transforms to price Parisian options. In [5, 7, 10], the problem is reduced to finding the Laplace transform of the Parisian stopping time, which is the first time the length of the excursion reaches level D. In [5], the Laplace transform of the stopping time was obtained using the Brownian meander and Azema martingale, while Dassios and Wu [7] introduced a perturbed Brownian motion and a semi-Markov model to obtain the Laplace transform. In both of these, an explicit form of the Laplace transform of the distribution of the Parisian stopping time and consequently that of the option price is found. Other methods of pricing Parisian options include the PDE method, studied by Haber, Schonbucher, and Wilmott [8]. There exist also other types of Parisian options. Cumulative Parisian options,

*Received by the editors April 30, 2012; accepted for publication (in revised form) June 3, 2013; published electronically August 15, 2013.
http://www.siam.org/journals/sifin/4/87546.html

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which are related to the total excursion time above (or below) a barrier, are studied in [5], while double-sided Parisian options are introduced in [7] and [2].

Several papers have also studied techniques to numerically invert the Laplace transforms of the option prices. Avellandea and Wu [3] used a lattice method. Labart and Lelong [9] used an inversion formula based on the Abate and Whitt [1] method, while Bernard, Courtois, and Quittard-Pinon [4] obtained numerical prices by approximating the Laplace transforms using a linear combination of fractional functions. In this paper, we used a different method to obtain the option price without numerically inverting its Laplace transform. Instead, we work directly with the Laplace transform of the stopping time and simply use it to obtain a recursive formula for the density. We always know that a recursive formula for the density function exists and is discontinuous in D, because if t is the first time the length of the excursion reaches D, and $kD < t < (k+1)D$, the excursion must start at $t - D$, which is between $(k-1)D < t - D < kD$, and there cannot be any excursions greater than length D before this. Hence, the density for the stopping time where t is between $kD < t < (k+1)D$ can be computed from the density of the previous step. Furthermore, to find the density for $kD < t < (k+1)D$, we will see later that we only need to compute a finite sum of k terms, allowing for a simple procedure that is fast to compute. For small time intervals, we give a direct intuitive proof of the formula for the density function. For larger time steps, we write the density function as a recursive equation which can be solved numerically. Furthermore, we also show how the prices of Parisian options can be computed from the density of the Parisian stopping time.

In section 2, we introduce the definitions, assumptions, and notation. Section 3 presents the main results concerning the density of the Parisian stopping time. Section 4 presents the applications of the results on the pricing of Parisian options. In section 5, we present the algorithm written in R and provide some numerical results.

2. Definitions.

We will use the same definitions for the excursions as in [5]. Let S be the underlying asset following a geometric Brownian motion, and let $\mathcal{Q}$ denote the risk neutral probability measure. We assume that the underlying asset S follows a geometric Brownian motion, and its dynamics under $\mathcal{Q}$ is

$$dS_t = S_t(rdt + \sigma dW_t), \qquad S_0 = x, \tag{2.1}$$

where W_t is a standard Brownian motion under $\mathcal{Q}$, and r and σ positive constants. We also introduce the notation

$$m = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right), \qquad b = \frac{1}{\sigma}\ln\left(\frac{L}{x}\right), \qquad k = \frac{1}{\sigma}\ln\left(\frac{K}{x}\right)$$

so that the asset price $S_t = xe^{\sigma(mt+W_t)}$. We define

$$g^S_{L,t} = \sup\{s \le t | S_s = L\}, \qquad d^S_{L,t} = \inf\{s \ge t | S_s = L\}$$

with the usual convention that $\sup\emptyset = 0$ and $\inf\emptyset = \infty$. The trajectory of S between $g^S_{L,t}$ and $d^S_{L,t}$ is the excursion which straddles time t. We are interested here in $t - g^S_{L,t}$, which is the age of the excursion at time t. For $D > 0$, we now define

$$\tau_{L,D}^{+}(S) = \inf\{t \geq 0 | \mathbf{1}_{S_t > L}(t - g_{L,t}^{S}) \geq D\}, \tag{2.2}$$

$$\tau_{L,D}^{-}(S) = \inf\{t \geq 0 | \mathbf{1}_{S_t < L}(t - g_{L,t}^{S}) \geq D\}. \tag{2.3}$$

$\tau_{L,D}^{+}(S)$ is thus the first time that the length of the excursion of process S above the barrier L reaches level D, while $\tau_{L,D}^{-}(S)$ corresponds to the excursion below level L. We also introduce the following notation for the stopping times where we refer to the standard Brownian motion W instead of S. Furthermore, without loss of generality since any time t of interest can be expressed in units of the window length D, we let $D = 1$ from now on and drop its notation.

$$\tau_{b}^{+} = \inf\{t \geq 0 | \mathbf{1}_{W_t > b}(t - g_{b,t}^{W}) \geq 1\}, \tag{2.4}$$

$$\tau_{b}^{-} = \inf\{t \geq 0 | \mathbf{1}_{W_t < b}(t - g_{b,t}^{W}) \geq 1\}. \tag{2.5}$$

The owner of a Parisian down-and-in option receives the payoff only if there is an excursion below the level L which is of length greater than D. This will be the case if $\tau_{L}^{-}(S) \leq T$. Denoting $C_i^d(x, T)$ as the price of a Parisian down-and-in call with initial underlying price x, maturity T, and parameters K, L, D, r fixed, we have the price formula

$$C_i^d(x, T) = E_{\mathcal{Q}}\left[e^{-rT}\mathbf{1}_{\{\tau_L^-(S) \leq T\}}(xe^{\sigma(mT + W_T)} - K)^+\right]. \tag{2.6}$$

We introduce a new probability measure $\mathcal{P}$, which makes $Z_t = W_t + mt$ a standard Brownian motion under $\mathcal{P}$. Applying Girsanov's theorem, we have

$$C_i^d(x, T) = E_{\mathcal{P}}\left[e^{-(r + \frac{1}{2}m^2)T}\mathbf{1}_{\{\tau_b^- \leq T\}}e^{mZ_T}\left(xe^{\sigma Z_T} - K\right)^+\right]. \tag{2.7}$$

To simplify things, we also let

$$^{*}C_i^d(x, T) = e^{(r + \frac{1}{2}m^2)T}C_i^d(x, T). \tag{2.8}$$

We denote by $\mathcal{F}_t = \sigma(Z_s, s \leq t)$ the natural filtration of the Brownian motion $(Z_t, t \geq 0)$. Then τ_b^- is an $\mathcal{F}_t$-stopping time, and by the strong Markov property of Brownian motion

$$^{*}C_i^d(x, T) = E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b^- \leq T\}}E\left[e^{mZ_T}\left(xe^{\sigma Z_T} - K\right)^+ | \mathcal{F}_{\tau_b^-}\right]\right] \tag{2.9}$$

$$= E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b^- \leq T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y} - K\right)^+ \frac{1}{\sqrt{2\pi(T - \tau_b^-)}}e^{-\frac{(y - Z_{\tau_b^-})^2}{2(T - \tau_b^-)}}\,dy\right]. \tag{2.10}$$

We will first look at the density function of τ_b^-, which we will denote by $f_b^-(t)$, and then show how it can be used to obtain the prices of a Parisian down-and-in call option.

3. Density of Parisian stopping time. In this section, we present the main result of this paper, which is to write the density function of τ_b^- as a recursive formula. We first give the intuitive proof for the first two steps of the recursion resulting in explicit formulas, and then use its Laplace transform to obtain a recursive equation for larger values of t.

Theorem 3.1. *For $b \leq 0$, we denote by $f_b^-(t)$ the probability density function of τ_b^-. Then*

$$f_b^-(t) = \sum_{k=0}^{n-1} (-1)^k L_k(t-1) \qquad \text{for } n < t \leq n+1,\ n = 1, 2, \dots, \tag{3.1}$$

for $t > 1$, where $L_k(t)$ is defined recursively as follows:

$$L_0(t) = \frac{1}{2\pi\sqrt{t}} e^{-\frac{b^2}{2t}} \qquad \text{for } t > 0, \tag{3.2}$$

$$L_{k+1}(t) = \int_1^{t-k} L_k(t-s) \frac{\sqrt{s-1}}{2\pi s} ds \qquad \text{for } t > k+1. \tag{3.3}$$

3.1. Intuitive proof for $1 < t < 3$. We look at the case $b = 0$, i.e., we start at the barrier, $S_0 = L$. We denote by T_x the first hitting time of level x of a standard Brownian motion, and recall the notation g_t as the last time the Brownian motion is at 0 before time t. We want to find the density of τ_0^-, which is the first time the excursion reaches length 1. The density of τ_0^- vanishes for $t < 1$. For $1 < t < 2$, the excursion must start at $0 < t - 1 < 1$. Now, we modify the problem slightly and find instead $P(\tau_0^- - 1 \in dt)$, the probability density for t being the start of the excursion greater than length 1. For $0 < t < 1$, we condition the value of the Brownian motion at time 1. At time 1, the probability that the start of an excursion of length 1 occurred at time t is equal to the probability that t is the time of the last exit time g_1, that the Brownian motion traveled to x between time t to time 1, and that the Brownian motion does not hit 0 before a further time period t, such that the total time spent above 0 is 1. The required probability is obtained by integrating over x.

$$\begin{aligned} P(\tau_0^- - 1 \in dt) &= \int_0^\infty P(g_1 \in dt, W_1 \in dx, T_x \geq t) dx \\ &= P(g_1 \in dt) \int_0^\infty P(W_1 \in dx | g_1 = t) P(T_x \geq t) dx, \end{aligned}$$

where we have conditioned on the value of the Brownian motion at time 1. The distribution of g_1 follows the arcsine law, and is

$$P(g_1 \in dt) = \frac{1}{\pi\sqrt{t}\sqrt{1-t}} dt.$$

$W_1 | g_1 = t$ has the same distribution as a Brownian meander of excursion length $1-t$ and has density (see [6])

$$P(W_1 \in dx | g_1 = t) = \frac{x}{2(1-t)} e^{-\frac{x^2}{2(1-t)}} dx.$$

Thus, we have

$$\begin{aligned}
P(\tau_0^- - 1 \in dt) &= \frac{1}{\pi\sqrt{t}\sqrt{1-t}}dt\frac{1}{2}\int_0^\infty \frac{x}{1-t}e^{-\frac{x^2}{2(1-t)}}\int_t^\infty \frac{x}{\sqrt{2\pi u^3}}e^{-\frac{x^2}{2u}}dudx \\
&= \frac{1}{\pi\sqrt{t}\sqrt{1-t}}\frac{1}{2}\sqrt{1-t}dt \\
&= \frac{1}{2\pi\sqrt{t}}dt.
\end{aligned}$$

We denote this by $L_0(t)$. For $0 < t < 1$, $L_0(t)$ is the probability that t is the start of one excursion greater than length 1. For $1 < t < 2$, however, there can be up to 2 excursions, and since we are only interested in the first excursion greater than length 1, we subtract the probability that there are indeed 2 excursions. We denote by $L_1(t)$ the probability density of t being the start of two excursions greater than length 1 for $1 < t < 2$. We break this probability up into 3 parts: the probability that the Brownian motion makes a first excursion of length 1, $L_0(s-1)$; that it traveled to x at time s, hits 0 again at time u, $s < u < t$; and that starting at 0 at time u, it will make a second excursion of length 1 at time t, $L_0(t-u)$. The required probability is then obtained by integrating over all s, x, and u.

$$\begin{aligned}
L_1(t) &= \int_1^t L_0(s-1)\int_0^\infty P(W_s \in dx | g_s = s-1)\int_s^t P(T_x \in du)L_0(t-u) \\
&= \int_1^t L_0(s-1)\int_0^\infty xe^{-\frac{x^2}{2}}\int_s^t \frac{x}{\sqrt{2\pi(u-s)^3}}e^{-\frac{x^2}{2(u-s)}}\frac{1}{2\pi\sqrt{t-u}}dudsdx \\
&= \int_1^t L_0(s-1)\frac{1}{2\pi}\frac{\sqrt{t-s}}{t-s+1}ds \\
&= \int_1^t L_0(t-s)\frac{1}{2\pi}\frac{\sqrt{s-1}}{s}ds,
\end{aligned}$$

where we have conditioned on $s-1$ the start of the first excursion greater than length 1, the value of the Brownian motion at the end of this excursion W_s, and u, the first time the Brownian motion comes back to zero again after that. $L_0(t-u)$ is the probability that t is the start of an excursion with length larger than 1, given that we start from 0 at u. For $2 < t < 3$, the density of τ_0^- is $L_0(t-1) - L_1(t-1)$. The same argument follows by induction for $t > 3$ and we obtain the recursion.

3.2. General case ($b \leq 0$). Below we give the formal proof for the recursive formula.

Proof. For simplicity, we define the following function:

$$\Psi(x) = 1 + x\sqrt{2\pi}e^{\frac{x^2}{2}}\mathcal{N}(x), \tag{3.4}$$

where $\mathcal{N}(x)$ is the cumulative distribution function for the standard normal distribution. The Laplace transform $\hat{h}(\beta)$ of a function $h(t)$ on the positive real line is defined by

$$\mathcal{L}(h(t)) = \hat{h}(\beta) = \int_0^\infty e^{-\beta t}h(t)dt.$$

For $b \leq 0$, the Laplace transform of the density $f_b^-(t)$ of the stopping time (with $D = 1$) is (see [5] for more detail)

$$\hat{f}_b^-(\beta) = \frac{e^{\sqrt{2\beta}b}}{\Psi(\sqrt{2\beta})}. \tag{3.5}$$

Instead of inverting this numerically, we find a direct formula for $f_b^-(t)$ by writing the above equation as a renewal equation, which can then be solved recursively. First, we rewrite $\Psi(\sqrt{2\beta})$ as

$$\begin{aligned}
\frac{1}{\beta}e^{-\beta}\Psi\left(\sqrt{2\beta}\right) &= \frac{e^{-\beta}}{\beta} + 2\sqrt{\frac{\pi}{\beta}}\int_{-\infty}^{\sqrt{2\beta}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= \frac{e^{-\beta}}{\beta} + \sqrt{\frac{\pi}{\beta}}\left(1 + 2\int_0^{\sqrt{2\beta}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\right) \\
&= \sqrt{\frac{\pi}{\beta}} + \frac{e^{-\beta}}{\beta} + \int_0^1 \frac{e^{-\beta s}}{\sqrt{s}}ds \\
&= \sqrt{\frac{\pi}{\beta}} + \int_1^\infty e^{-\beta s}ds + \left(\int_0^\infty \frac{e^{-\beta s}}{\sqrt{s}}ds - \int_1^\infty \frac{e^{-\beta s}}{\sqrt{s}}ds\right) \\
&= 2\sqrt{\frac{\pi}{\beta}} + \frac{1}{\beta}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds \\
&= 2\sqrt{\frac{\pi}{\beta}}\left(1 + \frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right).
\end{aligned}$$

So we have

$$\begin{aligned}
\hat{f}_b^-(\beta) &= \frac{e^{-\beta}e^{\sqrt{2\beta}b}}{2\sqrt{\pi\beta}\left(1 + \frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right)} \\
&= e^{-\beta}\sum_{k=0}^{\infty}(-1)^k \frac{e^{\sqrt{2\beta}b}}{2\sqrt{\pi\beta}}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k.
\end{aligned}$$

Since $\hat{L}_1(\beta) = 0$ as $\beta \to \infty$, and $\hat{L}_k(\beta)$ is continuous and decreasing in β, there exists $\beta^* > 0$ such that the above expansion is valid for all $\beta > \beta^*$. We denote

$$\hat{L}_k(\beta) = \frac{e^{\sqrt{2\beta}b}}{2\sqrt{\pi\beta}}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k.$$

Furthermore, we have the following Laplace inversions:

$$\mathcal{L}^{-1}\left(\frac{e^{\sqrt{2\beta}b}}{2\sqrt{\pi\beta}}\right) = \frac{1}{2\pi\sqrt{t}}e^{-\frac{b^2}{2t}}, \tag{3.6}$$

$$\mathcal{L}^{-1}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right) = \frac{\sqrt{t-1}}{2\pi t}\mathbf{1}_{\{t>1\}}. \tag{3.7}$$

Equation (3.6) can be checked by integrating

$$\begin{aligned}
&\int_0^\infty e^{-\beta t}\frac{1}{2\pi\sqrt{t}}e^{-\frac{b^2}{2t}}dt \\
&= \frac{e^{\sqrt{2\beta}b}}{2\sqrt{2\beta}}\int_0^\infty \frac{\sqrt{2\beta}-\frac{b}{t}}{2\pi\sqrt{t}}e^{-\frac{(b+\sqrt{2\beta}t)^2}{2t}}dt + \frac{e^{-\sqrt{2\beta}b}}{2\sqrt{2\beta}}\int_0^\infty \frac{\sqrt{2\beta}+\frac{b}{t}}{2\pi\sqrt{t}}e^{-\frac{(b-\sqrt{2\beta}t)^2}{2t}}dt \\
&= \frac{e^{\sqrt{2\beta}b}}{2\sqrt{\pi\beta}},
\end{aligned}$$

where both integrals are evaluated using a change of variable $x = \frac{b\pm\sqrt{2\beta}t}{\sqrt{t}}$ and the second term turns out to be zero. The left-hand side of (3.7) is the product of two functions whose inversion is known, so by taking their convolution we get

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right) &= \int_0^t \frac{1}{2\pi\sqrt{t-s}}\frac{1}{2s^{3/2}}\mathbf{1}_{\{s>1\}}ds \\
&= \left[-\frac{\sqrt{t-s}}{2\pi t\sqrt{s}}\right]_1^t = \frac{\sqrt{t-1}}{2\pi t}\mathbf{1}_{\{t>1\}}.
\end{aligned}$$

So L_k is the kth convolution of (3.7), and L_0 is the right-hand side of (3.6). Finally, we note that for $n < t < n+1$, $L_k(t)$ is zero for $k > n$, so we only need a finite sum up to n, where the series expansion is valid for $\beta > \beta^*$. ■

3.3. General case ($b > 0$). We let T_b be the first hitting time of the Brownian motion of level b. For $b > 0$, we are only concerned with the case where $T_b < D = 1$. If $T_b \geq 1$, the Parisian stopping time $\tau_b^- = 1$ since we are already below the barrier, and the problem simplifies.

Theorem 3.2. *For $b > 0$, we let $f_b^-(t, T_b < 1)$ be the probability density function of τ_b^- on the set $\{T_b < 1\}$. Then*

$$f_b^-(t, T_b < 1) = \sum_{k=0}^{n-1}(-1)^k L_k(t-1) \qquad \text{for } n < t \leq n+1,\ n = 1, 2, \ldots, \tag{3.8}$$

for $t > 0$, where $L_k(t)$ is defined recursively as follows:

$$L_0(t) = \mathbf{1}_{\{0<t\leq 1\}}\frac{1}{2\pi\sqrt{t}}e^{-\frac{b^2}{2t}} + \mathbf{1}_{\{t>1\}}\frac{1}{\pi\sqrt{t}}e^{-\frac{b^2}{2t}}\mathcal{N}\left(-b\sqrt{\frac{t-1}{t}}\right), \tag{3.9}$$

$$L_{k+1}(t) = \int_1^{t-k} L_k(t-s)\frac{\sqrt{s-1}}{2\pi s}ds \qquad \text{for } t > k+1. \tag{3.10}$$

Proof. In this case, we have

$$\begin{aligned}
E\left[e^{-\beta\tau_b^-(t)}\mathbf{1}_{\{T_b<1\}}\right] &= E\left[e^{-\beta(T_b+\tau_0^-)}\mathbf{1}_{\{T_b<1\}}\right] \\
&= E\left[e^{-\beta T_b}\mathbf{1}_{\{T_b<1\}}\right]\frac{1}{\Psi\left(\sqrt{2\beta}\right)} \\
&= e^{-\beta}\sum_{k=0}^{\infty}(-1)^k\frac{E\left[e^{-\beta T_b}\mathbf{1}_{\{T_b<1\}}\right]}{2\sqrt{\pi\beta}}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k.
\end{aligned}$$

As in the previous case, there exists some β^* such that the series expansion is valid for $\beta > \beta^*$. We have

$$\begin{aligned} L_0(t) = \mathcal{L}^{-1}\left(\frac{E\left[e^{-\beta T_b}\mathbf{1}_{\{T_b<1\}}\right]}{2\sqrt{\pi\beta}}\right) &= \mathbf{1}_{\{0<t\le 1\}}\int_0^t \frac{b}{\sqrt{2\pi s^3}}e^{-\frac{b^2}{2s}}\frac{1}{2\pi\sqrt{t-s}}ds \\ &\quad + \mathbf{1}_{\{t>1\}}\int_0^1 \frac{b}{\sqrt{2\pi s^3}}e^{-\frac{b^2}{2s}}\frac{1}{2\pi\sqrt{t-s}}ds \\ &= \mathbf{1}_{\{0<t\le 1\}}\frac{1}{2\pi\sqrt{t}}e^{-\frac{b^2}{2t}} + \mathbf{1}_{\{t>1\}}\frac{1}{\pi\sqrt{t}}e^{-\frac{b^2}{2t}}\mathcal{N}\left(-b\sqrt{\frac{t-1}{t}}\right). \end{aligned}$$

L_k for $k = 1, 2, \ldots$ is the same as in the previous case. ∎

4. Pricing a down-and-in Parisian call. We focus on the case of a down-and-in option. Let S be the underlying asset price in (2.1), L the barrier level, and m, b, l defined as in section 2.

$$m = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right), \qquad b = \frac{1}{\sigma}\ln\left(\frac{L}{x}\right), \qquad k = \frac{1}{\sigma}\ln\left(\frac{K}{x}.\right)$$

We denote by $Z(.)$ the probability density function of a standard normal random variable, and by $\mathcal{N}_\rho(.,.)$ the joint cumulative function for a pair of bivariate standard normal random variables with correlation coefficient ρ. For ${}^*C_i^d(x,T)$ defined as in (2.8), we present the following explicit formulas for the price.

Theorem 4.1. *For $b < 0$, the price of a down-and-in Parisian option on the underlying S with barrier $L < x$ and maturity time $T > 1$ is given by*

$$(4.1) \qquad {}^*C_i^d(x,T) = \sqrt{2\pi}\int_0^T f_b^-(t)\left(x\psi(\sigma+m,h) - K\psi(m,h')\right)dt,$$

where $f_b^-(t)$ is the density function of the Parisian stopping time with barrier b as in (3.11), *and we define the function*

(4.2)

$$\psi(x,y) = e^{\frac{x^2(1+T-t)+2bx}{2}}\left(Z(-x)\mathcal{N}\left(\frac{-x\rho - y}{\sqrt{1-\rho^2}}\right) - \rho Z(y)\mathcal{N}\left(\frac{-x-\rho y}{\sqrt{1-\rho^2}}\right) - x\left(\mathcal{N}(-x) - \mathcal{N}_\rho(-x,y)\right)\right)$$

and

$$(4.3) \qquad h = \frac{1}{\sqrt{1+T-t}}\left(k - b - (\sigma+m)(1+T-t)\right),$$

$$(4.4) \qquad h' = \frac{1}{\sqrt{1+T-t}}\left(k - b - m(1+T-t)\right),$$

$$(4.5) \qquad \rho = \frac{1}{\sqrt{1+T-t}}.$$

Proof. As in (2.10), we change to measure $\mathcal{P}$ under which Z_t is a standard Brownian motion. Furthermore, since τ_b^- is an $\mathcal{F}_t$-stopping time, by the strong Markov property of

Brownian motion, we have

$$
\begin{aligned}
{}^*C_i^d(x,T) &= E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b^-\le T\}}E\left[e^{mZ_T}\left(xe^{\sigma Z_T}-K\right)^+|\mathcal{F}_{\tau_b^-}\right]\right]\\
&= E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b^-\le T\}}\int_{-\infty}^{\infty}e^{my}\left(xe^{\sigma y}-K\right)^+\frac{1}{\sqrt{2\pi(T-\tau_b^-)}}e^{-\frac{(y-Z_{\tau_b^-})^2}{2(T-\tau_b^-)}}dy\right].
\end{aligned}
$$

It is easy to see that τ_b^- and $Z_{\tau_b^-}$ are independent (see [5] for more detail). We denote the density functions of τ_b^- and $Z_{\tau_b^-}$ by $f_b^-(t)$ and $v(dz)$, respectively. The density of $Z_{\tau_b^-}$ is associated to the Brownian meander and for window length $D=1$ is

$$
v(dz)=P(Z_{\tau_b^-}\in dz)=(b-z)e^{-\frac{(z-b)^2}{2}}\mathbf{1}_{\{z<b\}}dz.
$$

So we have

$$
\begin{aligned}
{}^*C_i^d(x,T) &= \int_0^T\int_{-\infty}^{\infty}f_b^-(t)v(dz)\int_{-\infty}^{\infty}e^{my}\left(xe^{\sigma y}-K\right)^+\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(y-z)^2}{2(T-t)}}dydt\\
&= \sqrt{2\pi}\int_0^T f_b^-(t)\int_{-\infty}^{b}\int_k^{\infty}(b-z)e^{-\frac{(z-b)^2}{2}}e^{my}\left(xe^{\sigma y}-K\right)\frac{1}{2\pi\sqrt{T-t}}e^{-\frac{(y-z)^2}{2(T-t)}}dydzdt.
\end{aligned}
$$

We are interested in evaluating the double integral with respect to y and z.

$$
\frac{1}{2\pi\sqrt{T-t}}\int_{-\infty}^{b}\int_k^{\infty}e^{my}(xe^{\sigma y}-K)(b-z)e^{-\frac{(z-b)^2}{2}}e^{-\frac{(y-z)^2}{2(T-t)}}dzdy \tag{4.6}
$$

$$
=\frac{1}{2\pi\sqrt{T-t}}\int_{-\infty}^{b}\int_k^{\infty}xe^{(\sigma+m)y}(b-z)e^{-\frac{(z-b)^2}{2}}e^{-\frac{(y-z)^2}{2(T-t)}}dzdy \tag{4.7}
$$

$$
-\frac{1}{2\pi\sqrt{T-t}}\int_{-\infty}^{b}\int_k^{\infty}Ke^{my}(b-z)e^{-\frac{(z-b)^2}{2}}e^{-\frac{(y-z)^2}{2(T-t)}}dzdy. \tag{4.8}
$$

We look at the first integral. The integrand can be written as the joint density function of a bivariate normal distribution.

$$
x\frac{1}{2\pi\sqrt{T-t}}\int_{-\infty}^{b}\int_k^{\infty}e^{(\sigma+m)y}(b-z)e^{-\frac{(z-b)^2}{2}}e^{-\frac{(y-z)^2}{2(T-t)}}dzdy \tag{4.9}
$$

$$
=x\exp\left\{\frac{(\sigma+m)^2(1+T-t)+2b(\sigma+m)}{2}\right\}\frac{1}{2\pi\sqrt{T-t}} \tag{4.10}
$$

$$
\cdot\int_{-\infty}^{b}\int_k^{\infty}(b-z)\exp\left\{-\frac{(y-(b+(\sigma+m)(1+T-t)))^2}{2(T-t)}\right\}\exp\left\{-\frac{(z-(b+(\sigma+m)))^2}{2(T-t)/(1+T-t)}\right\} \tag{4.11}
$$

$$
\cdot\exp\left\{\frac{2(y-(b+(\sigma+m)(1+T-t)))(z-(b+(\sigma+m)))}{2(T-t)}\right\}dzdy \tag{4.12}
$$

$$
=x\exp\left\{\frac{(\sigma+m)^2(1+T-t)+2b(\sigma+m)}{2}\right\} \tag{4.13}
$$

$$
\cdot\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{-(\sigma+m)}\int_h^{\infty}(-v-(\sigma+m))\exp\left\{-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\right\}dudv, \tag{4.14}
$$

where we have used the transformation $u = \frac{y-(b+(\sigma+m)(1+T-t)}{\sqrt{1+T-t}}$ and $v = z-(b+(\sigma+m))$, h and ρ as defined in (4.3) and (4.5). Now, we have the following result for (U, V) bivariate normal with mean 0, variance 1, and correlation coefficient ρ:

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{1-\rho^2}}\int_h^\infty \int_{-\infty}^{-(\sigma+m)} ve^{-\frac{(u^2-2\rho uv+v^2)}{2(1-\rho^2)}}dudv\\
&=\frac{1}{2\pi}\int_h^\infty \int_{-\infty}^{\frac{-(\sigma+m)-\rho u}{\sqrt{1-\rho^2}}} \left(\sqrt{1-\rho^2}w+\rho u\right)e^{-\frac{1}{2}(u^2+w^2)}dudw,
\end{aligned}$$

where we used the transformation $v-\rho u = w\sqrt{1-\rho^2}$. Now applying integration by parts, we obtain

$$\begin{aligned}
&\frac{\sqrt{1-\rho^2}}{2\pi}\int_h^\infty \left[-e^{-\frac{1}{2}(u^2+w^2)}\right]_{-\infty}^{\frac{-(\sigma+m)-\rho u}{\sqrt{1-\rho^2}}} + \frac{\rho}{2\pi}\int_h^\infty ue^{-\frac{1}{2}u^2}\int_{-\infty}^{\frac{-(\sigma+m)-\rho u}{\sqrt{1-\rho^2}}} e^{-\frac{1}{2}w^2}dwdu\\
&=\frac{\sqrt{1-\rho^2}}{2\pi}\int_h^\infty -e^{-\frac{((\sigma+m)^2+2\rho(\sigma+m)u+u^2)}{2(1-\rho^2)}}du\\
&+\frac{\rho}{2\pi}\left\{\left[-e^{-\frac{1}{2}u^2}\int_{-\infty}^{\frac{-(\sigma+m)-\rho u}{\sqrt{1-\rho^2}}} e^{-\frac{1}{2}w^2}dw\right]_h^\infty + \int_h^\infty e^{-\frac{1}{2}u^2}e^{-\frac{1}{2}\left(\frac{-(\sigma+m)-\rho u}{\sqrt{1-\rho^2}}\right)^2}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)du\right\}\\
&=-\frac{\sqrt{1-\rho^2}}{2\pi}\int_h^\infty e^{-\frac{1}{2}((\sigma+m)^2+2\rho(\sigma+m)u+u^2)}du\left(1+\frac{\rho^2}{1-\rho^2}\right)+\frac{\rho}{2\pi}e^{-\frac{1}{2}h^2}\int_{-\infty}^{\frac{-(\sigma+m)-\rho h}{\sqrt{1-\rho^2}}} e^{-\frac{1}{2}w^2}dw.
\end{aligned}$$

Here, we apply another transformation, $v = \frac{u+(\sigma+m)\rho}{\sqrt{1-\rho^2}}$, to the first integral to get

$$\begin{aligned}
&= -\frac{1}{2\pi}\int_{\frac{h+(\sigma+m)\rho}{\sqrt{1-\rho^2}}}^\infty e^{-\frac{1}{2}((\sigma+m)^2+v^2)}dv + \rho Z(h)\mathcal{N}\left(\frac{-(\sigma+m)-\rho h}{\sqrt{1-\rho^2}}\right)\\
&= -Z(-(\sigma+m))\mathcal{N}\left(\frac{-(\sigma+m)\rho-h}{\sqrt{1-\rho^2}}\right)+\rho Z(h)\mathcal{N}\left(\frac{-(\sigma+m)-\rho h}{\sqrt{1-\rho^2}}\right).
\end{aligned}$$

Substituting this back into (4.14), we obtain $x\psi(\sigma+m, h)$. Doing the same for the second integral (4.8), we get the result. ∎

Theorem 4.2. *For $b > 0$, the price of a down-and-in Parisian option on the underlying S with barrier $L < x$ and maturity time $T > 1$ is given by*

(4.15)
$$^*C_i^d(x,T) = x\phi(\sigma+m) - K\phi(m) + \sqrt{2\pi}\int_0^T f_b^-(t; T_b < 1)\left(x\psi(\sigma+m,h) - K\psi(m,h')\right)dt,$$

where $f_b^-(t)$ is the density function of the Parisian stopping time with barrier b as in (3.12), and ψ, h, h', ρ defined as in Theorem 4.1, and we also used the function

$$\phi(x) = e^{\frac{x^2T}{2}}\left(\mathcal{N}(b-x)-\mathcal{N}_\rho\left(b-x,\frac{k-xT}{\sqrt{T}}\right)\right) \tag{4.16}$$

$$-e^{\frac{x^2T+4bx}{2}}\left(\mathcal{N}(-b-x)-\mathcal{N}_\rho\left(-b-x,\frac{k-2b-xT}{\sqrt{T}}\right)\right), \tag{4.17}$$

where, as before, $\mathcal{N}_\rho$ is the joint cumulative distribution of a bivariate standard normal distribution but with correlation coefficient $\rho = \frac{1}{\sqrt{T}}$.

Proof. For $b > 0$, we split into the cases when $T_b > 1$ and $T_b < 1$.

$$\begin{aligned}{}^*C_i^d(x,T) &= E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b>1\}}\mathbf{1}_{\{\tau_b^-\le T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^+ \frac{1}{\sqrt{2\pi(T-\tau_b^-)}}e^{-\frac{(y-Z_{\tau_b^-})^2}{2(T-\tau_b^-)}}\,dy\right] \\ &+ E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b<1\}}\mathbf{1}_{\{\tau_b^-\le T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^+ \frac{1}{\sqrt{2\pi(T-\tau_b^-)}}e^{-\frac{(y-Z_{\tau_b^-})^2}{2(T-\tau_b^-)}}\,dy\right].\end{aligned}$$

The law of Z_1 on the set $\{T_b > 1\}$ is

$$\begin{aligned}P(Z_1\in dz, T_b>1) &= P(Z_1\in dz) - P(Z_1\in dz, T_b<1)\\ &= \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{z^2}{2}} - e^{-\frac{(z-2b)^2}{2}}\right)dz.\end{aligned}$$

Since we start below the barrier, $\tau_b^- = 1$ if $T_b > 1$. So we have

$$E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b>1\}}\mathbf{1}_{\{\tau_b^-\le T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^+ \frac{1}{\sqrt{2\pi(T-\tau_b^-)}}e^{-\frac{(y-Z_{\tau_b^-})^2}{2(T-\tau_b^-)}}\,dy\right] \tag{4.18}$$

$$= E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b>1\}}\mathbf{1}_{\{1\le T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^+ \frac{1}{\sqrt{2\pi(T-1)}}e^{-\frac{(y-Z_1)^2}{2(T-1)}}\,dy\right] \tag{4.19}$$

$$= \frac{1}{\sqrt{2\pi(T-1)}}\int_{-\infty}^{b}\int_{k}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)e^{-\frac{(y-z)^2}{2(T-1)}}\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{z^2}{2}} - e^{-\frac{(z-2b)^2}{2}}\right)dzdy \tag{4.20}$$

$$= x\phi(\sigma+m) - K\phi(m), \tag{4.21}$$

where the last step involves writing the integrand as the density function of a pair of bivariate normal random variables as before to obtain a joint cumulative distribution function. On the set $\{T_b < 1\}$, $Z_{\tau_b^-}$ is again independent of τ_b^-, so we have

$$\begin{aligned}&E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b<1\}}\mathbf{1}_{\{\tau_b^-\le T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^+ \frac{1}{\sqrt{2\pi(T-\tau_b^-)}}e^{-\frac{(y-Z_{\tau_b^-})^2}{2(T-\tau_b^-)}}\,dy\right]\\ &= \int_0^T\int_{-\infty}^{\infty} f_b^-(t;T_b<1)v(dz)\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^+\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(y-z)^2}{2(T-t)}}dydt\\ &= \sqrt{2\pi}\int_0^T f_b^-(t;T_b<1)\left(x\psi(\sigma+m,h) - K\psi(m,h')\right)dt,\end{aligned}$$

where the proof is as before. ■

5. Numerical results. The code is written in R. First we compute the density $f_b^-(t)$ for different values of t, using n number of steps and h for the size of each time step. Using the numerical values of $f_b^-(t)$, we can do a numerical integration and use formula (4.1) to obtain the price of the down-and-in Parisian call. We note that since we have chosen the window length D as the unit of time, all parameters (r, σ) are correspondingly normalized depending on the window length. Using the same parameters as in [4], $\sigma = 0.2$, $r = 0.05$, $T = 1$ year, $K = 95$, and $L = 90$, we obtain similar results. Below is the code, using the above parameters, number of time steps $n = 1000$, $d = 3$ months, and initial price $S_0 = 92$.

```
# load package
library(mnormt)

#parameters
n <- 1000
t <- 4
r <- 0.05
sigma <- 0.2
S0 <- 92
L <- 90
K <- 95

t<-t-1
h<-1/n
r<-r/(t+1)
sigma<-sigma/sqrt(t+1)
b<- 1/sigma*log(L/S0)
m<- 1/sigma*(r-sigma^2/2)

f<-mat.or.vec(t*n,1) #vector of densities for tau

L1<-1/sqrt(pi*(1:(t*n)-0.5)*h)*exp(-b^2/(2*(1:(t*n)-0.5)*h)) #vector of Lk starting at L0
L2<-mat.or.vec(t*n,1) #vector of Lk's
x<-sqrt(((1:(t*n)-0.5)*h))/(pi*(1+((1:(t*n)-0.5)*h)))

f<-L1

for(i in 1:(t-1)) {
y<-convolve(L1[((i-1)*n+1):((t-1)*n)],rev(x[1:((t-i)*n)]),type = "open")
L2[(i*n+1):(t*n)]<-y[1:((t-i)*n)]*h
f<-f+L2*(-1/2)^i
L1<-L2
L2<-mat.or.vec(t*n,1)
}

f<-f/(2*sqrt(pi)) #we obtain the density for \tau_b^-. Next part is to price the option.

rho<-1/sqrt(1+((t*n):1-0.5)*h)
c1<-S0*exp(((sigma+m)^2*(1+((t*n):1-0.5)*h)+2*b*(sigma+m))/2)
c2<-K*exp((m^2*(1+((t*n):1-0.5)*h)+2*b*m)/2)
```

```
k1<-1/sqrt(1+((t*n):1-0.5)*h)*(1/sigma*log(K/S0)-b-(sigma+m)*(1+((t*n):1-0.5)*h))
l1<-rep(-(sigma+m),times=(t*n))
k2<-1/sqrt(1+((t*n):1-0.5)*h)*(1/sigma*log(K/S0)-b-m*(1+((t*n):1-0.5)*h))
l2<-rep(-m,times=(t*n))

mnorm1<-mat.or.vec(t*n,1) #cdf of bivariate normal computed at (l,k)
for(i in 1:(t*n)) {
varcov<-matrix(c(1,rho[i],rho[i],1),2,2)
mnorm1[i]<-pmnorm(c(l1[i],k1[i]),c(0,0),varcov)
}

mnorm1<-pnorm(l1)-mnorm1

mnorm2<-mat.or.vec(t*n,1) #cdf of bivariate normal computed at (l',k')
for(i in 1:(t*n)) {
varcov<-matrix(c(1,rho[i],rho[i],1),2,2)
mnorm2[i]<-pmnorm(c(l2[i],k2[i]),c(0,0),varcov)
}

mnorm2<-pnorm(l2)-mnorm2

q<-c1*(dnorm(l1)*pnorm((l1*rho-k1)/sqrt(1-rho^2))
   -rho*dnorm(k1)*pnorm((l1-rho*k1)/sqrt(1-rho^2))
   -(sigma+m)*mnorm1)-c2*(dnorm(l2)*pnorm((l2*rho-k2)/sqrt(1-rho^2))
   -rho*dnorm(k2)*pnorm((l2-rho*k2)/sqrt(1-rho^2))-m*mnorm2)
q<-sqrt(2*pi)*q

price<-f%*%q*h*exp(-(r+0.5*m^2)*(t+1))
```

For $b > 0$, there is an extra term for $T_b > 1$. The code for $b > 0$ follows:

```
# load package
library(mnormt)

#parameters
n <- 1000
t <- 4
r <- 0.05
sigma <- 0.2
S0 <- 80
L <- 90
K <- 95

t<-t-1
h<-1/n
r<-r/(t+1)
sigma<-sigma/sqrt(t+1)
b<- 1/sigma*log(L/S0)
m<- 1/sigma*(r-sigma^2/2)
k<- 1/sigma*log(K/S0)
```

```
f<-mat.or.vec(t*n,1) #vector of densities for tau

L1<-mat.or.vec(t*n,1) #vector of Lk starting at L0
L1[1:n]<-1/sqrt(pi*(1:n-0.5)*h)*exp(-b^2/(2*(1:n-0.5)*h))
L1[(n+1):(t*n)]<-2/sqrt((pi*((n+1):(t*n)-0.5)*h))*
                 pnorm(-b*sqrt(1-1/(((n+1):(t*n)-0.5)*h)))*
                 exp(-b^2/(2*(((n+1):(t*n)-0.5)*h)))
L2<-mat.or.vec(t*n,1) #vector of Lk's
x<-sqrt(((1:(t*n)-0.5)*h))/(pi*(1+((1:(t*n)-0.5)*h)))

f<-L1

for(i in 1:(t-1)) {
y<-convolve(L1[((i-1)*n+1):((t-1)*n)],rev(x[1:((t-i)*n)]),type = "open")
L2[(i*n+1):(t*n)]<-y[1:((t-i)*n)]*h
f<-f+L2*(-1/2)^i
L1<-L2
L2<-mat.or.vec(t*n,1)
}

f<-f/(2*sqrt(pi))

rho<-1/sqrt(1+((t*n):1-0.5)*h)
c1<-S0*exp(((sigma+m)^2*(1+((t*n):1-0.5)*h)+2*b*(sigma+m))/2)
c2<-K*exp((m^2*(1+((t*n):1-0.5)*h)+2*b*m)/2)
k1<-1/sqrt(1+((t*n):1-0.5)*h)*(1/sigma*log(K/S0)-b-(sigma+m)*(1+((t*n):1-0.5)*h))
l1<-rep(-(sigma+m),times=(t*n))
k2<-1/sqrt(1+((t*n):1-0.5)*h)*(1/sigma*log(K/S0)-b-m*(1+((t*n):1-0.5)*h))
l2<-rep(-m,times=(t*n))

mnorm1<-mat.or.vec(t*n,1) #cdf of bivariate normal computed at (l,k)
for(i in 1:(t*n)) {
varcov<-matrix(c(1,rho[i],rho[i],1),2,2)
mnorm1[i]<-pmnorm(c(l1[i],k1[i]),c(0,0),varcov)
}

mnorm1<-pnorm(l1)-mnorm1

mnorm2<-mat.or.vec(t*n,1) #cdf of bivariate normal computed at (l',k')
for(i in 1:(t*n)) {
varcov<-matrix(c(1,rho[i],rho[i],1),2,2)
mnorm2[i]<-pmnorm(c(l2[i],k2[i]),c(0,0),varcov)
}

mnorm2<-pnorm(l2)-mnorm2

q<-c1*(dnorm(l1)*pnorm((l1*rho-k1)/sqrt(1-rho^2))
   -rho*dnorm(k1)*pnorm((l1-rho*k1)/sqrt(1-rho^2))
   -(sigma+m)*mnorm1)-c2*(dnorm(l2)*pnorm((l2*rho-k2)/sqrt(1-rho^2))
```

```
   -rho*dnorm(k2)*pnorm((l2-rho*k2)/sqrt(1-rho^2))-m*mnorm2)
q<-sqrt(2*pi)*q

price<-f%*%q*h

rhot<-1/sqrt(t+1)
varcov<-matrix(c(1,rhot,rhot,1),2,2)
phi1<-exp((sigma+m)^2*(t+1)/2)*(pnorm(b-(sigma+m))
       -pmnorm(c(b-(sigma+m),(k-(sigma+m)*(t+1))/sqrt(t+1)),c(0,0),varcov))-
       exp((sigma+m)^2*(t+1)/2+4*b*(sigma+m)/2)*((pnorm(-b-(sigma+m))
       -pmnorm(c(-b-(sigma+m),(k-2*b-(sigma+m)*(t+1))/sqrt(t+1)),c(0,0),varcov)))
phi2<-exp(m^2*(t+1)/2)*(pnorm(b-m)
       -pmnorm(c(b-m,(k-m*(t+1))/sqrt(t+1)),c(0,0),varcov))-
       exp(m^2*(t+1)/2+4*b*m/2)*((pnorm(-b-m)
       -pmnorm(c(-b-m,(k-2*b-m*(t+1))/sqrt(t+1)),c(0,0),varcov)))

price<-price+S0*phi1-K*phi2

price<-price*exp(-(r+0.5*m^2)*(t+1))
```

Table 1 shows the density and cumulative function for $b = 0$ at intervals of 0.5, computed using a time step of $h = 0.001$.

We plot the tail in Figure 1 using a logarithmic scale.

Table 2 shows the prices of Parisian down-and-in calls, valued using parameters $\sigma = 0.2$, $r = 0.05$, $T = 1$ year, $K = 95$, and $L = 90$, and at different window lengths D and initial stock price S_0.

Table 3 gives a comparison of the CPU times for our algorithm and that using the Laplace inversion technique in [9], computed using the above parameters and $S_0 = 90$. Due to the increasing number of recursions required, the computation times increase rapidly as the window length decreases. As we can see in Table 3, our algorithm is very efficient for long window lengths relative to the time to maturity. For window lengths of 2 months and above, the CPU time required for this algorithm is less than a second. However, for window length of 1 month, our algorithm is slower because of the large number of recursions.

Table 1

Density $f_0^-(t)$ for $0 < t \leq 10$.

t	$f_0^-(t)$	$F_0^-(t)$	t	$f_0^-(t)$	$F_0^-(t)$
1.5	0.225192	0.224967	6.0	0.032951	0.596578
2.0	0.159195	0.318230	6.5	0.029312	0.612044
2.5	0.115597	0.385764	7.0	0.026296	0.625858
3.0	0.089488	0.436448	7.5	0.023763	0.638296
3.5	0.071858	0.476398	8.0	0.021613	0.649571
4.0	0.059334	0.508918	8.5	0.019768	0.659854
4.5	0.050062	0.536056	9.0	0.018171	0.669282
5.0	0.042972	0.559146	9.5	0.016778	0.677967
5.5	0.037410	0.579104	10.0	0.015554	0.686003

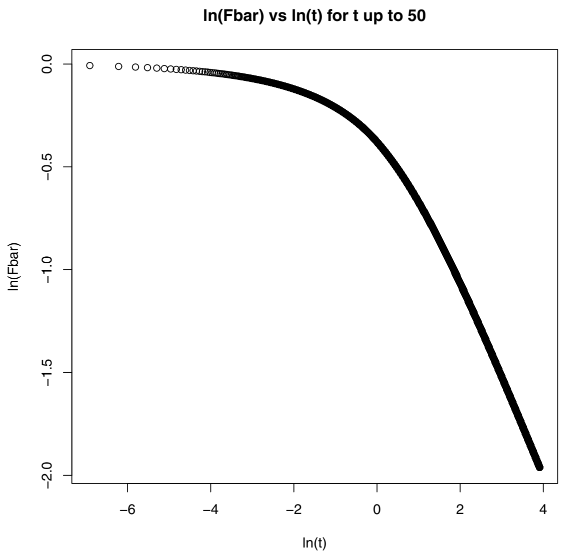

Figure 1. *Graph of* $\ln(\bar{F}_0^-(t))$ *vs.* $\ln(t)$ *for* $0 < t \leq 50$.

Table 2
Price of Parisian down-and-in call.

S_0	$D = 1$ month	$D = 2$ months	$D = 3$ months	$D = 4$ months
80	2.599144	1.917126	1.325256	0.894224
82	2.915856	1.951244	1.278959	0.833757
84	3.024509	1.850371	1.158805	0.732982
86	2.862540	1.630234	0.983769	0.605966
88	2.466282	1.337957	0.783991	0.471268
90	1.969965	1.034182	0.589757	0.345171
92	1.558517	0.798889	0.445765	0.255106
94	1.223695	0.610794	0.332570	0.185555
96	0.957872	0.465487	0.247251	0.134427
98	0.747613	0.353669	0.183210	0.097016
100	0.581894	0.267937	0.135329	0.069763

Table 3
Computation times (s).

D	Recursion formula	Laplace inversion
1 month	2.56	1.06
2 months	0.98	1.24
3 months	0.70	1.48
4 months	0.58	1.66

Acknowledgment. We thank the anonymous referees for their useful comments and suggestions.

REFERENCES

[1] J. Abate and G. Whitt, *Numerical inversion of Laplace transforms of probability distributions*, ORSA J. Comput., 7 (1995), pp. 36–43.
[2] J. H. M. Anderluh and J. A. M. Van der Weide, *Double-sided Parisian option pricing*, Finance Stoch., 13 (2009), pp. 205–238.
[3] M. Avellandea and L. Wu, *Pricing Parisian-style options with a lattice method*, Int. J. Theor. Appl. Finance, 2 (1999), pp. 1–16.
[4] C. Bernard, O. Courtois, and F. Quittard-Pinon, *A new procedure for pricing Parisian options*, J. Derivatives, 12 (2005), pp. 45–53.
[5] M. Chesney, M. Jeanblanc-Picque, and M. Yor, *Brownian excursions and Parisian barrier options*, Adv. Appl. Probab., 29 (1997), pp. 165–184.
[6] K. L. Chung, *Excursions in Brownian motion*, Ark. Mat., 14 (1976), pp. 155–177.
[7] A. Dassios and S. Wu, *Perturbed Brownian motion and its application to Parisian option pricing*, Finance Stoch., 14 (2009), pp. 473–494.
[8] R. J. Haber, P. J. Schonbucher, and P. Wilmott, *Pricing Parisian options*, J. Derivatives, 6 (1999), pp. 71–79.
[9] C. Labart and J. Lelong, *Pricing double barrier Parisian options using Laplace transforms*, Int. J. Theor. Appl. Finance, 12 (2009), pp. 19–44.
[10] M. Schroder, *Brownian excursions and Parisian barrier options: A note*, J. Appl. Probab., 40 (2003), pp. 855–864.