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On asymptotic distributions of weighted sums of periodograms

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We establish asymptotic normality of weighted sums of periodograms of a stationary linear process where weights depend on the sample size. Such sums appear in numerous statistical applications and can be regarded as a discretized versions of quadratic forms involving integrals of weighted periodograms. Conditions for asymptotic normality of these weighted sums are simple, minimal, and resemble Lindeberg–Feller condition for weighted sums of independent and identically distributed random variables. Our results are applicable to a large class of short, long or negative memory processes. The proof is based on sharp bounds derived for Bartlett type approximation of these sums by the corresponding sums of weighted periodograms of independent and identically distributed random variables.

Keywords: Bartlett approximation; Lindeberg–Feller; linear process; quadratic forms

1. Introduction

Let X_j, $j = 0, \pm 1, \ldots,$ be a stationary process with a spectral density f_X and let $u_j = 2\pi j/n$, $j = 1, \ldots, [n/2]$, denote discrete Fourier frequencies. In this paper, we develop asymptotic distribution theory for the weighted sums

$$Q_{n,X} := \sum_{j=1}^{\nu} b_{n,j} I_X(u_j), \qquad \nu := [n/2] - 1, \qquad n \geq 1, \tag{1.1}$$

of periodograms $I_X(u_j) = (2\pi n)^{-1} |\sum_{t=1}^{n} \mathrm{e}^{\mathrm{i}tu_j} X_t|^2$, where $b_{n,j}$ are triangular arrays of real weights. When $b_{n,j} = b_n(u_j)$, where $b_n, n \geq 1$ is a sequence of real valued functions on $\Pi := [-\pi, \pi]$, $Q_{n,X}$ is an estimate of $\sum_{j=1}^{\nu} b_n(u_j) f_X(u_j)$ and can be viewed as a discretized version of the integral $\mathcal{I}_n := \int_0^{\pi} b_n(u) I_X(u)\,\mathrm{d}u$. Integrals $\mathcal{I}_n$ arise naturally in many situations in statistical inference. For example, the auto-covariance function of $\{X_j\}$ is

$$\operatorname{Cov}(X_k, X_0) = 2\int_0^{\pi} \cos(ku) f_X(u)\,\mathrm{d}u, \qquad k = 0, 1, 2, \ldots,$$

and the spectral distribution function can be written as $F(y) = \int_{-\pi}^{\pi} I(u \leq y) f_X(u)\,\mathrm{d}u$. In these two examples b does not depend on n. If one wishes to estimate $f_X(u_0)$ at a point $u_0 \in (0, \pi)$ by kernel smoothing method, then b will typically depend on n.

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Asymptotic distribution theory of $\mathcal{I}_n$ when b does not depend on n and $\{X_j\}$ is a stationary Gaussian or linear process is well understood and investigated both for short memory and long memory linear processes; for asymptotic normality results see Hannan [11], Fox and Taqqu [4], Giraitis and Surgailis [6] and Giraitis and Taqqu [7]; for non-Gaussian limits see Terrin and Taqqu [25] and Giraitis, Taqqu and Terrin [8]. Nualart and Peccati [19] give simple sufficient conditions for central limit theorem (CLT) of quadratic forms that can be written as a sequence of multiple stochastic integrals.

It is perhaps worth pointing out that even in the case when b does not depend on n, investigation of limit distribution of $\mathcal{I}_n$ is technically involved. As is evident from the works of Hannan [11] and Bhansali, Giraitis and Kokoszka [1], deriving asymptotic distribution of $\mathcal{I}_n$ in case of general weight sequences b_n depending on n will be prohibitively complicated, and conditions for asymptotic normality will lack desirable simplicity.

In comparison, the verification of asymptotic normality of weighted sums of periodograms is relatively simple. In Sections 2 and 3 below, we provide theoretical tools to establish the CLT for $Q_{n,X} - EQ_{n,X}$ and $\mathcal{D}_n := Q_{n,X} - \sum_{j=1}^{[n/2]} b_{n,j} f_X(u_j)$, and to evaluate the large sample behavior of $E\mathcal{D}_n$, $\operatorname{Var}(Q_{n,X})$ and the mean-squared error $E\mathcal{D}_n^2$, when $\{X_j\}$ is a stationary linear process with i.i.d. innovations, possibly having long memory. Our conditions for asymptotic normality of these weighted sums are formulated in terms of $\{b_{n,j}, f_X(u_j)\}$. They are simple and resemble Lindeberg–Feller type condition for weighted sums of i.i.d. r.v.'s, regardless of whether $\{X_j\}$ has short, long or negative memory.

A number of papers in the literature deal with more general quadratic forms (sums of weighted periodograms). Generalizations usually include relaxing assumption of linearity of $\{X_j\}$. Hsing and Wu [15] obtain asymptotic normality of a quadratic form $\sum_{t,s=1}^{n} b_{t-s} K(X_t, X_s)$ for a non-linear transform K of a linear process $\{X_j\}$ under a set of complex conditions that do not provide a direct answer in terms of $\{b_t\}$, K and $\{X_j\}$. Moreover, their weights b_t's are not allowed to depend on n. Wu and Shao [26] derive CLT for discrete Fourier transforms and spectral density estimates under some restrictions on dependence structure of $\{X_j\}$ based on conditional moments. Shao and Wu [23] establish the CLT for quadratic forms with weights depending on n using martingale approximation method. Liu and Wu [17] consider non-parametric estimation of spectral density of a stationary process using m-dependent approximation of X_j's. Generality of these papers requires verification of a number of complex technical conditions which impose a priori a rate condition in approximations, that must be verified in each specific case. For example, Wu and Shao [26] requires geometric-contraction condition, which implies exponential decay of the autocovariance function $\gamma_X(k)$ of $\{X_j\}$, whereas in Liu and Wu [17] the dependence is restricted assuming summability of $|\gamma_X(k)|$. Both papers also restrict the set of $b_{n,j}$'s to specific weights appearing in kernel estimation. Such structural assumptions may be easier to verify than verifying mixing conditions, but they are redundant, not informative and too restrictive in the case when $\{X_j\}$ is a linear process.

The present paper establishes the CLT for $Q_{n,X}$ in the latter case under minimal conditions, which allow for short, long or negative memory in $\{X_j\}$ and arbitrary weights $b_{n,j}$ as along as $f_X(u_j)b_{n,j}$'s satisfy condition (3.6) of uniform negligibility. The main tool of the proof is Bartlett type approximation for discrete Fourier transforms of X_j's which is essentially different from the methods of approximations used in the above works. Besides being simple and easy to verify, the obtained conditions are close to being necessary; see Remark 3.4 below.

Assumptions. *Accordingly, let* $\mathbb{Z} := \{0, \pm 1, \ldots\}$,

$$X_j = \sum_{k=0}^{\infty} a_k \zeta_{j-k}, \qquad j \in \mathbb{Z}, \ \sum_{k=0}^{\infty} a_k^2 < \infty, \tag{1.2}$$

be a linear process where $\{\zeta_j, j \in \mathbb{Z}\}$ *are i.i.d. standardized r.v.'s. Assume that the spectral density* f_X *of the process* X_j, $j \in \mathbb{Z}$, *satisfies*

$$f_X(u) = |u|^{-2d} g(u), \qquad |u| \leq \pi, \tag{1.3}$$

for some $|d| < 1/2$, *where* $g(u)$ *is a continuous function satisfying*

$$0 < C_1 \leq g(u) \leq C_2 < \infty, \qquad u \in \Pi \ (\exists 0 < C_1, C_2 < \infty).$$

Condition (1.3) allows to derive the mean square error bounds of estimates, which are given in Theorem 3.3. To derive asymptotic normality and some delicate Bartlett type approximations, we shall additionally need to assume that the transfer function $A_X(u) := \sum_{k=0}^{\infty} \mathrm{e}^{-\mathrm{i}ku} a_k$, $u \in \Pi$, is differentiable in $(0, \pi)$ and its derivative $\dot{A}_X$ satisfies

$$\left|\dot{A}_X(u)\right| \leq C|u|^{-1-d}, \qquad u \in \Pi. \tag{1.4}$$

Conditions (1.3) and (1.4) are formulated this way to cover long and negative memory models, with $|d| < 1/2$, $d \neq 0$. They allow spectral density to vanish or to have a singularity point at zero frequency. The short memory case where f_X and A_X are Lipshitz continuous and bounded away from 0 and ∞ is also discussed in Section 3.

To proceed further, define the *discrete Fourier transforms* (DFT) of $\{X_j\}$ and $\{\zeta_j\}$ computed at frequencies u_j's, $j = 0, \ldots, [n/2]$, to be, respectively,

$$w_{X,j} = \frac{1}{\sqrt{2\pi n}} \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}u_j k} X_k, \qquad w_{\zeta,j} = \frac{1}{\sqrt{2\pi n}} \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}u_j k} \zeta_k.$$

The corresponding periodograms, transfer functions and spectral densities of $\{X_j\}$ and $\{\zeta_j\}$ at frequency u_j are denoted by

$$I_{X,j} = |w_{X,j}|^2, \qquad I_{\zeta,j} = |w_{\zeta,j}|^2, \qquad A_{X,j} = A_X(u_j), \qquad A_{\zeta,j} = 1,$$
$$f_{X,j} := f_X(u_j), \qquad f_{\zeta,j} := f_\zeta(u_j) \equiv \frac{1}{2\pi}, \qquad j = 0, 1, \ldots, [n/2].$$

The goal of establishing asymptotic normality of $Q_{n,X}$ is facilitated by first developing asymptotic distribution theory for the sums

$$S_{n,X} := \sum_{j=1}^{\nu} b_{n,j} \frac{I_{X,j}}{f_{X,j}}.$$

Moreover, asymptotic analysis of these sums is more illustrative of the methodology used. The asymptotic normality of $S_{n,X}$ is discussed in Section 2.

The CLT for the quadratic forms $Q_{n,X}$ with weights not depending on n was investigated by Hannan [11]; see also Proposition 10.8.6. of Brockwell and Davis [3]. Their proof required restrictive condition $\sum_{k=0}^{\infty} k^{1/2}|a_k| < \infty$ on the coefficients a_k of the linear process $\{X_j\}$ of (1.2) and was based on Bartlett approximation of periodogram $I_{X,j}/f_{X,j}$ by periodogram $I_{\zeta,j}/f_{\zeta,j}$ of the noise. The idea for the theory and the proofs presented in this paper have their roots in Robinson [20].

We show that CLT's for $Q_{n,X}$ and $S_{n,X}$ hold under similar conditions as the classical CLT for weighted sums of i.i.d. r.v.'s. It requires Lindeberg–Feller type condition on weights $b_{n,j}$ and minimal restrictions on a linear process $\{X_j\}$ which may have short or long memory. For example, in short memory case it suffices to assume that a_k of (1.2) satisfy $\sum_{k=0}^{\infty} |a_k| < \infty$ and f_X is bounded away from 0 and ∞; see Section 3. Results below also show that weighted sums of rescaled periodogram $I_{X,j}/f_{X,j}$ of a linear process behave, to some extend, similarly as the weighted sums of i.i.d. r.v.'s.

We also investigate precision of Bartlett approximation of $Q_{n,X}$ and $S_{n,X}$ by sums of weighted periodograms $I_{\zeta,j}/f_{\zeta,j}$. Lemma 2.1 and Theorem 3.3 contain sharp bounds and are of independent interest. From these results, one sees that the above approximation is extremely precise, and the resulting error is small and can be effectively controlled by the weights $\{b_{n,j}\}$ alone. This type of approximation is a popular tool for establishing CLT for specific types of weights $b_{n,j}$, for example, for local Whittle estimators; see Robinson [20], Shao and Wu [24] and Shao [22]. In these papers, innovation sequence is allowed to be a martingale difference or an uncorrelated weakly dependent non-linear causal process. However, because of narrower focus, they deal with special weights and do not seek establishing a general CLT for $Q_{n,X}$ as such. In our setting, assumption of i.i.d. innovations is a secondary issue and also can be relaxed, while the major objective is obtaining the CLT for $Q_{n,X}$ with the most general feasible weighting scheme $b_{n,j}$.

Finally, in the present paper the spectral density f_X is allowed to take infinite or zero value only at the zero frequency restricting $|d| < 1/2$ to keep $\{X_j\}$ stationary. Establishing sufficient conditions for CLT for a differenced stationary process, as well as when the spectral density f_X may have singularity/zero at a frequencies away from zero is of definite interest, but needs further investigation.

In the sequel, $\mathrm{Cum}_k(Z)$ denotes the kth cummulant of the r.v. Z, IID(0, 1) denotes the class of i.i.d. standardized r.v.'s, $a \wedge b := \min(a, b)$, $a \vee b := \max(a, b)$, for any real numbers a, b, and all limits are taken s $n \to \infty$, unless specified otherwise.

2. Asymptotic normality of $S_{n,X}$

Important role in the asymptotic analysis of $S_{n,X}$ is played by Bartlett type approximation

$$(I_{X,j}/f_{X,j}) \sim (I_{\zeta,j}/f_{\zeta,j}) = 2\pi I_{\zeta,j}, \qquad j = 1, \ldots, \nu, \nu = [n/2] - 1.$$

Our first goal is to approximate $S_{n,X}$ by the weighted sum of $I_{\zeta,j}$,

$$S_{n,\zeta} = \sum_{j=1}^{\nu} b_{n,j}(I_{\zeta,j}/f_{\zeta,j}) \equiv \sum_{j=1}^{\nu} b_{n,j} 2\pi I_{\zeta,j}. \tag{2.1}$$

Let

$$\begin{aligned} R_n &:= S_{n,X} - S_{n,\zeta}, \qquad b_n := \max_{j=1,\dots,\nu} |b_{n,j}|, \qquad B_n := \left(\sum_{j=1}^{\nu} b_{n,j}^2\right)^{1/2}, \\ q_n^2 &:= B_n^2 + \mathrm{Cum}_4(\zeta_0)\frac{1}{n}\left(\sum_{j=1}^{\nu} b_{n,j}\right)^2. \end{aligned} \tag{2.2}$$

We show later that $\mathrm{Var}(S_{n,\zeta}) = q_n^2$; see (2.20)(b).

Lemma 2.1 below provides an upper bound of order $b_n \log^2(n)$ for $E R_n^2$ while Lemma 2.2 establishes the asymptotic normality of the approximating quadratic forms $S_{n,\zeta}$. The following theorem gives an approximation to $E S_{n,X}$, $\mathrm{Var}(S_{n,X})$, and proves asymptotic normality of $S_{n,X}$ under Lindeberg–Feller type condition (2.3) on the weights $b_{n,j}$.

Because of the invariance property $I_{X+\mu}(u_j) = I_X(u_j)$, $\mu \in \mathbb{R}$, $j = 1, \dots, n-1$, all results obtained below remain valid also for a process $\{X_j\}$ of (1.2) that has non-zero mean.

Theorem 2.1. *Suppose the linear process $\{X_j, j \in \mathbb{Z}\}$ of (1.2) satisfies assumptions (1.3) and (1.4), and $E\zeta_0^4 < \infty$. About the weights $b_{n,j}$'s assume*

$$\frac{b_n}{B_n} = \frac{\max_{j=1,\dots,\nu} |b_{n,j}|}{(\sum_{j=1}^{\nu} b_{n,j}^2)^{1/2}} \to 0. \tag{2.3}$$

Then

$$\begin{aligned} E S_{n,X} &= \sum_{j=1}^{\nu} b_{n,j} + \mathrm{o}(q_n), \qquad \mathrm{Var}(S_{n,X}) = q_n^2 + \mathrm{o}(q_n^2), \\ \mathrm{Var}(S_{n,X})^{-1/2}(S_{n,X} - E S_{n,X}) &\to_D \mathcal{N}(0,1), \qquad q_n^{-1}\left(S_{n,X} - \sum_{j=1}^{\nu} b_{n,j}\right) \to_D \mathcal{N}(0,1). \end{aligned} \tag{2.4}$$

Moreover,

$$\min(1, \mathrm{Var}(\zeta_0^2)/2) B_n^2 \le q_n^2 \le (1 + |\mathrm{Cum}_4(\zeta_0)|) B_n^2. \tag{2.5}$$

Proof. The proof uses Lemmas 2.1 and 2.2 given below. To prove (2.5), use definition of q_n and the Cauchy–Schwarz inequality to obtain the upper bound. The lower bound is derived in (2.21) of Lemma 2.2.

By (2.3), (2.5), (2.9)(b), and (2.20),

$$ES_{n,\zeta} = \sum_{j=1}^{\nu} b_{n,j}, \qquad E|R_n| \le \big(ER_n^2\big)^{1/2} = \mathrm{o}(B_n) = \mathrm{o}(q_n). \tag{2.6}$$

These facts in turn complete the proof of the first claim in (2.4).

To prove the second claim, note that by (2.20)(b), $\mathrm{Var}(S_{n,\zeta}) = q_n^2$, which together with (2.6) yields $\mathrm{Var}(R_n) \le ER_n^2 = \mathrm{o}(q_n^2)$, $|\mathrm{Cov}(S_{n,\zeta}, R_n)| = \mathrm{o}(q_n^2)$. These facts together with a routine argument complete the proof of the second claim in (2.4).

Finally, again in view of (2.6),

$$S_{n,X} - \sum_{j=1}^{\nu} b_{n,j} = S_{n,X} - ES_{n,\zeta} = S_{n,\zeta} - ES_{n,\zeta} + \mathrm{o}_p(q_n).$$

This and (2.20)(c) of Lemma 2.2 imply the first asymptotic normality result in (2.4), while the last claim follows from the first three claims in (2.4). □

Lemma 2.1 below provides the two types of sharp upper bounds for ER_n^2 that are useful in approximating $S_{n,X}$ by $S_{n,\zeta}$. The idea of using Bartlett type approximations to establish the asymptotic normality of an integrated weighted periodogram of a short memory linear process goes back to the work of Grenander and Rosenblatt [9], Hannan and Heyde [12] and Hannan [11], whereas for sums of weighted periodograms of an ARMA process it was used in Proposition 10.8.5 of Brockwell and Davis [3]. Their approximations were derived under the assumption that the weight function b did not depend on n, and the bounds they obtain have low-level of sharpness, though they are sufficient to show that the main term dominates the remainder. The sharp bounds for an integrated weighted periodogram established in Bhansali *et al*. [1] technically are more involved and harder to apply than those for sums in this lemma.

Lemma 2.1. *Assume that* $\{X_j\}$ *of* (1.2) *satisfies* (1.3) *and* (1.4), *and* $E\zeta_0^4 < \infty$. *Then*

$$E(R_n - ER_n)^2 \le Cb_n^2 \log^3(n) \quad \text{and} \quad E(R_n - ER_n)^2 \le Cb_n B_n, \tag{2.7}$$

$$|ER_n| \le Cb_n \log^2(n) \quad \text{and} \quad |ER_n| = \mathrm{o}(B_n) \qquad \text{if } b_n = \mathrm{o}(B_n). \tag{2.8}$$

In particular,

$$\begin{aligned} &\text{(a)} \quad E(S_{n,X} - S_{n,\zeta})^2 \le Cb_n^2 \log^4(n); \\ &\text{(b)} \quad E(S_{n,X} - S_{n,\zeta})^2 = o\big(B_n^2\big) \qquad \text{if } b_n = \mathrm{o}(B_n). \end{aligned} \tag{2.9}$$

The proof of this lemma is facilitated by the following two propositions.

Proposition 2.1. *Let* $\{Y_{n,j}^{(i)}, j = 1, \dots, n\}$, $i = 1, 2$, $n \ge 1$ *be the two sets of moving averages*

$$Y_{n,j}^{(i)} = \sum_{k\in\mathbb{Z}} b_{n,j}^{(i)}(k)\zeta_k, \qquad \sum_{k\in\mathbb{Z}}^{\infty} \big|b_{n,j}^{(i)}(k)\big|^2 < \infty, \qquad i = 1, 2,$$

where $\{b^{(i)}_{n,j}(k)\}$ *are possibly complex weights. Assume,* $\zeta_k \sim \mathrm{IID}(0,1)$, $E\zeta_0^4 < \infty$. *Then, for any real weights* $c_{n,j},\ j = 1, \dots, n$,

$$\begin{aligned} &\operatorname{Var}\left(\sum_{j=1}^{n} c_{n,j}\left\{\left|Y^{(1)}_{n,j}\right|^2 - \left|Y^{(2)}_{n,j}\right|^2\right\}\right) \\ &\quad\le \left(4 + 4\operatorname{Var}\left(\zeta_0^2\right)\right) \sum_{j,k=1}^{n} |c_{n,j}c_{n,k}| \left|\left|r^{11}_{n,jk}\right|^2 + \left|r^{22}_{n,jk}\right|^2 - 2\left|r^{12}_{n,jk}\right|^2\right|, \end{aligned} \tag{2.10}$$

where $r^{il}_{n,jk} := E[Y^{(i)}_{n,j}\overline{Y^{(l)}_{n,k}}] = \sum_{t\in\mathbb{Z}} b^{(i)}_{n,j}(t)\overline{b^{(l)}_{n,k}(t)},\ i,l = 1,2.$

Proof. Observe that

$$\begin{aligned} \mathcal{G}_n &:= \sum_{j=1}^{n} c_{n,j}\left\{\left|Y^{(1)}_{n,j}\right|^2 - \left|Y^{(2)}_{n,j}\right|^2\right\} \\ &= \sum_{t,s\in\mathbb{Z}} \left(\sum_{j=1}^{n} c_{n,j}\left\{b^{(1)}_{n,j}(t)\overline{b^{(1)}_{n,j}(s)} - b^{(2)}_{n,j}(t)\overline{b^{(2)}_{n,j}(s)}\right\}\right)\zeta_t\zeta_s =: \sum_{t,s\in\mathbb{Z}} B_n(t,s)\zeta_t\zeta_s. \end{aligned}$$

Hence,

$$\begin{aligned} &E|\mathcal{G}_n - E\mathcal{G}_n|^2 \\ &\quad\le 4\left(E\left|\sum_{t<s} B_n(t,s)\zeta_t\zeta_s\right|^2 + E\left|\sum_{s<t} B_n(t,s)\zeta_t\zeta_s\right|^2 + E\left|\sum_{t\in\mathbb{Z}} B_n(t,t)\left(\zeta_t^2 - E\zeta_t^2\right)\right|^2\right) \\ &\quad= 4\sum_{t<s}\left|B_n(t,s)\right|^2 + 4\sum_{s<t}\left|B_n(t,s)\right|^2 + 4\operatorname{Var}\left(\zeta_0^2\right)\sum_{t\in\mathbb{Z}}\left|B_n(t,t)\right|^2 \\ &\quad\le \left(4 + 4\operatorname{Var}\left(\zeta_0^2\right)\right)\sum_{t,s\in\mathbb{Z}}\left|B_n(t,s)\right|^2. \end{aligned}$$

But,

$$\begin{aligned} &\sum_{t,s\in\mathbb{Z}}\left|B_n(t,s)\right|^2 \\ &\quad= \sum_{j,k=1}^{n} c_{n,j}c_{n,k} \sum_{t,s\in\mathbb{Z}}\left\{b^{(1)}_{n,j}(t)\overline{b^{(1)}_{n,j}(s)} - b^{(2)}_{n,j}(t)\overline{b^{(2)}_{n,j}(s)}\right\}\left\{\overline{b^{(1)}_{n,k}(t)}b^{(1)}_{n,k}(s) - \overline{b^{(2)}_{n,k}(t)}b^{(2)}_{n,k}(s)\right\} \\ &\quad= \sum_{j,k=1}^{n} c_{n,j}c_{n,k}\left(\left|r^{11}_{n,jk}\right|^2 + \left|r^{22}_{n,jk}\right|^2 - \left|r^{12}_{n,jk}\right|^2 - \left|r^{12}_{n,kj}\right|^2\right). \end{aligned}$$

This completes the proof of (2.10). □

The next proposition describes some needed large sample properties of DFTs. Because $\sum_{t=1}^{n} e^{\mathrm{i}tu_m} = n\{I(m=0)+I(m=n)\}$, DFTs of a white noise process $\{\zeta_j\}$ are uncorrelated:

$$E[w_{\zeta,j}\overline{w_{\zeta,k}}] = \frac{1}{2\pi}, \qquad 1 \le k = j \le n,$$
$$= 0, \qquad 1 \le k < j \le n. \tag{2.11}$$

Consider now the two linear processes

$$X_j = \sum_{k=0}^{\infty} a_k \zeta_{j-k}, \qquad Y_j = \sum_{k=0}^{\infty} b_k \zeta_{j-k}, \qquad j \in \mathbb{Z}, \qquad \sum_{k=0}^{\infty} a_k^2 < \infty, \qquad \sum_{k=0}^{\infty} b_k^2 < \infty,$$

with the same white noise innovations $\{\zeta_j\} \sim WN(0,\sigma^2)$. Let $A_X(v) := \sum_{k=0}^{\infty} \mathrm{e}^{-\mathrm{i}kv} a_k$, $A_Y(v) := \sum_{k=0}^{\infty} \mathrm{e}^{-\mathrm{i}kv} b_k$, $f_X(v) = (\sigma^2/2\pi)|A_X(v)|^2$, $f_Y(v) = (\sigma^2/2\pi)|A_Y(v)|^2$, denote their respective transfer and spectral densities.

Let $f_{XY}(v)$ denote a (complex valued) cross-spectral density:

$$f_{XY}(v) := \frac{\sigma^2}{2\pi} A_X(v)\overline{A_Y(v)}, \qquad v \in \Pi,$$
$$E[X_j Y_{j-k}] = \int_{\Pi} \mathrm{e}^{\mathrm{i}kv} f_{XY}(v)\,\mathrm{d}v = \frac{\sigma^2}{2\pi} \sum_{l=0}^{\infty} a_{l+k} b_l, \qquad k \ge 0, j \in \mathbb{Z}. \tag{2.12}$$

If $Y_j = \zeta_j$, $j \in \mathbb{Z}$, then

$$f_{X\zeta}(v) := \frac{\sigma^2}{2\pi} A_X(v), \qquad v \in \Pi,$$
$$E[X_j \zeta_{j-k}] = \frac{\sigma^2}{2\pi} \int_{\Pi} \mathrm{e}^{\mathrm{i}kv} A_X(v)\,\mathrm{d}v = \sigma^2 a_k, \qquad k \ge 0.$$

Proposition 2.2 below summarizes asymptotic properties of cross-covariances $E[w_{X,j}\overline{w_{Y,k}}]$. It generalizes and extends Theorem 2 of Robinson [21] for short memory and long memory time series, which enable derivation of the upper bounds based on Bartlett approximation of this paper. Its proof is technical and appears in Giraitis, Koul and Surgailis [5].

In case when Fourier frequencies in covariances $E[w_{X,j}\overline{w_{Y,k}}]$ are from an interval $(-\Delta, \Delta)$, $\Delta < \pi$ (a neighborhood of 0), smoothness conditions on f_X, f_Y, A_X, A_Y are local, that is, they need to be imposed on an interval $[0, a]$, $a > \Delta$.

To proceed further, for any subset $A \subset \mathbb{R}$, let $\mathcal{C}(A)$ denote complex valued functions that are continuous on A, and $\Lambda_\beta(A)$ denote Lipschitz continuous functions on A with parameter $\beta \in (0,1]$. We write $h \in \mathcal{C}_{1,\alpha}[0,a]$, $|\alpha| < 1$, $a > 0$, if

$$|h(u)| \le C|u|^{-\alpha}, \qquad |\dot{h}(u)| \le C|u|^{-1-\alpha} \qquad \forall u \in [0,a].$$

Members of $\mathcal{C}_{1,\alpha}[0,a]$ can have an infinite peak and can be non-differentiabile at 0, whereas $\Lambda_\beta[0,a]$ covers continuous piecewise differentiable functions.

Note that for any $h \in \mathcal{C}[0,a]$, $\omega_h(\eta) := \sup_{u,v\in[0,a]:|u-v|\le\eta} |h(u)-h(v)| \to 0$, as $\eta \to 0$. Define $\delta_{n,\varepsilon}(h) := \omega_h(n^{-1}\log(n)) + (\log(n))^{-\varepsilon}$, $0<\varepsilon<1$. We also need to introduce

$$\begin{aligned}
\ell_n(\varepsilon;k) &:= \frac{\log(2+k)}{(2+k)^{1-\varepsilon}} + \frac{\log(2+n-k)}{(2+n-k)^{1-\varepsilon}}, \qquad 0\le k\le j\le n,\\
r_{n,jk}(g) &:= 0, \qquad g\in\Lambda_1[0,a],\ \beta=1,\\
&:= n^{-\beta}\ell_n(\beta;j-k), \qquad g\in\Lambda_\beta[0,a],\ 0<\beta<1,\\
&:= \delta_{n,\varepsilon}(g)\ell_n(\varepsilon;j-k), \qquad g\in\mathcal{C}[0,a],\ \varepsilon\in(0,1).
\end{aligned} \tag{2.13}$$

Proposition 2.2. *Let either* $\Delta < a < \pi$, *or* $\Delta = a = \pi$. *Then, the following facts* (i)–(iv) *hold for all* $0<|u_k|\le u_j<\Delta$,

(i) *If* $f_{XY}\in\Lambda_\beta[0,a]$, $0<\beta\le 1$, *then*

$$\begin{aligned}
\big|E[w_{X,j}\overline{w_{Y,j}}] - f_{XY}(u_j)\big| &\le Cn^{-1}\log(n), \qquad \beta=1,\\
&\le Cn^{-\beta}, \qquad 0<\beta<1.\\
\big|E[w_{X,j}\overline{w_{Y,k}}]\big| &\le Cn^{-1}\log(n), \qquad \beta=1,\\
&\le Cn^{-\beta}\ell_n(\beta;j-k), \qquad 0<\beta<1, k<j.
\end{aligned}$$

(ii) *If* $f_{XY}\in\mathcal{C}[0,a]$, *then*, $\forall\varepsilon\in(0,1)$,

$$\begin{aligned}
\big|E[w_{X,j}\overline{w_{Y,j}}] - f_{XY}(u_j)\big| &\le C\delta_{n,\varepsilon}(f_{XY}),\\
\big|E[w_{X,j}\overline{w_{Y,k}}]\big| &\le C\delta_{n,\varepsilon}(f_{XY})\ell_n(\varepsilon;j-k), \qquad k<j.
\end{aligned}$$

(iii) *If* $f_{XY}\in\mathcal{C}_{1,\alpha}[0,a]$, $|\alpha|<1$, *then*

$$\begin{aligned}
\big|E[w_{X,j}\overline{w_{Y,j}}] - f_{XY}(u_j)\big| &\le Cu_j^{-\alpha}j^{-1}\log(1+j),\\
\big|E[w_{X,j}\overline{w_{Y,k}}]\big| &\le C\big(|u_k|^{-\alpha}+u_j^{-\alpha}\big)j^{-1}\log j, \qquad k<j.
\end{aligned}$$

(iv) *Suppose* $f_{XY}=hg$, *where* $h\in\mathcal{C}_{1,\alpha}[0,a]$, $|\alpha|<1$, *and* $g\in\Lambda_\beta[0,a]\cup\mathcal{C}[0,a]$, $0<\beta\le 1$. *Then*

$$\begin{aligned}
&\big|E[w_{X,j}\overline{w_{Y,k}}] - f_{XY}(u_j)I(j=k)\big|\\
&\quad\le C\Big(\big(|u_k|^{-|\alpha|}+u_j^{-|\alpha|}\big)j^{-1}\log j + \big(|u_k|^{-|\alpha|}\wedge u_j^{-|\alpha|}\big)r_{n,jk}(g)\Big).
\end{aligned}$$

The constant C in the above (i)–(iv) *does not depend on k, j and n.*

Parts (i) and (ii) of the above proposition consider the case when f_X is continuous and bounded, whereas part (iv) covers the case when f_X satisfies (1.3) with a bounded and continuous g. The case when g has also bounded derivative is covered in part (iii). Obtaining upper bounds in the above proposition does not require the process $\{X_j\}$ to be linear. For convenience of applications, this proposition is formulated for a cross-spectral density of two stationary linear processes with the same underlying white noise innovations. This allows to express their cross spectral density via their transfer functions as indicated in (2.12). In general, the results of Proposition 2.2 are valid for any spectral density or cross-spectral density that satisfies the assumed smoothness condition.

Lahiri [16] provides a characterization of asymptotic independence of the DFTs in terms of the distance between their arguments under both short- and long-range dependence of the underlying process. Nordman and Lahiri [18] contains some relevant results about Bartlett correction of the frequency domain empirical likelihood ratios.

Now rewrite

$$R_n = S_{n,X} - S_{n,\zeta} = \sum_{j=1}^{\nu} b_{n,j}\left(\frac{I_{X,j}}{f_{X,j}} - \frac{I_{\zeta,j}}{f_{\zeta,j}}\right) = \sum_{j=1}^{\nu} \frac{b_{n,j}}{f_{X,j}}\left\{I_{X,j} - f_{X,j}\frac{I_{\zeta,j}}{f_{\zeta,j}}\right\}. \tag{2.14}$$

The corollary below, which follows from Proposition 2.1, is useful in analyzing the sums of the types appearing in (2.14). Let $f_{X\zeta,j} := f_{X\zeta}(u_j)$.

Corollary 2.1. *Suppose that $\{X_j\}$ is a linear process as in (1.2) and $E\zeta_0^4 < \infty$. Then, for any real weights $c_{n,j}$, $j = 1, \dots, n$,*

$$\operatorname{Var}\left(\sum_{j=1}^{\nu} c_{n,j}\left\{I_{X,j} - f_{X,j}\frac{I_{\zeta,j}}{f_{\zeta,j}}\right\}\right) \le C(s_{n,1} + s_{n,2}), \tag{2.15}$$

where

$$\begin{aligned} s_{n,1} &:= C\sum_{j=1}^{\nu} c_{n,j}^2\big\{\big(E|w_{X,j}|^2 - f_{X,j}\big)^2 + f_{X,j}\big|E|w_{X,j}|^2 - f_{X,j}\big| \\ &\qquad + f_{X,j}\big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\big|^2 + f_{X,j}^{3/2}\big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\big|\big\}, \\ s_{n,2} &:= \sum_{1\le k<j\le\nu} |c_{n,j}c_{n,k}|\big\{\big|E[w_{X,j}\overline{w_{X,k}}]\big|^2 + f_{X,k}\big|E[w_{X,j}\overline{w_{\zeta,k}}]\big|^2\big\}. \end{aligned}$$

Proof. Observe that $f_{X,j}I_{\zeta,j}/f_{\zeta,j} = |A_{X,j}|^2 I_{\zeta,j} = |A_{X,j}w_{\zeta,j}|^2$, and that $Y_{n,j}^{(1)} := w_{X,j}$ and $Y_{n,j}^{(2)} := A_{X,j}w_{\zeta,j}$ are moving averages of ζ_j's with complex weights. Hence, by Proposition 2.1, the l.h.s. of (2.15) is bounded above by

$$\begin{aligned} &C\sum_{j,k=1}^{\nu} |c_{n,j}c_{n,k}|\big|\big|E[w_{X,j}\overline{w_{X,k}}]\big|^2 + |A_{X,j}|^2|A_{X,k}|^2\big|E[w_{\zeta,j}\overline{w_{\zeta,k}}]\big|^2 - 2|A_{X,k}|^2\big|E[w_{X,j}\overline{w_{\zeta,k}}]\big|^2\big| \\ &\quad = C\left(\sum_{j=k=1}^{n}[\cdots] + \sum_{k\ne j}[\cdots]\right) := C\big(s'_{n,1} + s'_{n,2}\big). \end{aligned}$$

By (2.11), $E|w_{\zeta,j}|^2 = 1/2\pi$, $E[w_{\zeta,j}\overline{w_{\zeta,k}}] = 0$, for $1 \le k < j \le \nu$. Recall also that $f_{X,j} = |A_{X,j}|^2/(2\pi)$. Therefore,

$$s'_{n,1} = \sum_{j,k=1}^{\nu} c_{n,j}^2 \big| \big(E|w_{X,j}|^2\big)^2 + f_{X,j}^2 - 4\pi f_{X,j} \big|E[w_{X,j}\overline{w_{\zeta,j}}]\big|^2 \big|,$$

$$s'_{n,2} = \sum_{1\le k<j\le\nu} |c_{n,j}c_{n,k}| \big(\big|E[w_{X,j}\overline{w_{X,k}}]\big|^2 + f_{X,k}\big|E[w_{X,j}\overline{w_{\zeta,k}}]\big|^2 \big) = s_{n,2}.$$

To bound $s'_{n,1}$, let $A := (E|w_{X,j}|^2)^2 - f_{X,j}^2$, and $B := |E[w_{X,j}\overline{w_{\zeta,j}}]|^2 - f_{X\zeta,j}$. Then use the fact that $4\pi f_{X,j}|f_{X\zeta,j}|^2 = 4\pi f_{X,j}|A_{X,j}|^2/(2\pi)^2 = 2f_{X,j}^2$ to rewrite the term within $|\cdots|$ in $s'_{n,1}$ as

$$\begin{aligned}
&\big(E|w_{X,j}|^2\big)^2 + f_{X,j}^2 - 4\pi f_{X,j}\big|E[w_{X,j}\overline{w_{\zeta,j}}]\big|^2 \\
&\quad = (A - 4\pi f_{X,j}B) + \big(2f_{X,j}^2 - 4\pi f_{X,j}|f_{X\zeta,j}|^2\big) = A - 4\pi f_{X,j}B.
\end{aligned}$$

Next, use the fact that $||z_1|^2 - |z_2|^2| \le |z_1 - z_2|^2 + 2|z_1 - z_2||z_2|$, for any complex numbers z_1, z_2, and that $|f_{X\zeta,j}| = |A_{X,j}|/(2\pi)^2 \le f_{X,j}^{1/2}$, to obtain

$$\begin{aligned}
|A - 4\pi f_{X,j}B| &\le |A| + 4\pi f_{X,j}|B| \\
&\le \big(E|w_{X,j}|^2 - f_{X,j}\big)^2 + 2f_{X,j}\big|E|w_{X,j}|^2 - f_{X,j}\big| \\
&\quad + 4\pi f_{X,j}\big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\big|^2 + 8\pi f_{X,j}^{3/2}\big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\big|,
\end{aligned}$$

which shows that $s'_{n,1} \le Cs_{n,1}$ and completes the proof of the corollary. □

Proof of Lemma 2.1. The proof uses Proposition 2.2. We shall prove (2.7) and (2.8). These two facts together imply (2.9) in a routine fashion.

Proof of (2.7). By (2.14), R_n is like the r.v. in the l.h.s. of (2.15) with $c_{n,j} = b_{n,j}/f_{X,j}$. Thus, $\mathrm{Var}(R_n) \le s_{n,1} + s_{n,2}$, where $s_{n,k}, k = 1, 2$ are the same in (2.15) with $c_{n,j} \equiv b_{n,j}/f_{X,j}$. It thus suffices to show that the sum $s_{n,1} + s_{n,2}$ is bounded from the above by the upper bounds given in (2.7).

Recall Proposition 2.2(iii). The spectral density f_X satisfies (1.3), whereas the cross-spectral density $f_{X\zeta}(u) = (2\pi)^{-1}A_X(u)$ has the property $|f_{X\zeta}(u)| \le C|u|^{-d}$, $|\dot{f}_{X\zeta}(u)| \le C|u|^{-1-d}$, $u \in \Pi$. Therefore, they satisfy conditions of this proposition, and hence

$$\begin{aligned}
\big|E|w_{X,j}|^2 - f_{X,j}\big| &\le C|u_j|^{-2d}j^{-1}\log(1+j), \\
\big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\big| &\le C|u_j|^{-d}j^{-1}\log(1+j),
\end{aligned} \tag{2.16}$$

where C does not depend on j and n. Since, by (1.3), $1/f_{X,j}(u) \le Cu_j^{2d}$, these bounds yield $s_{n,1} \le C\sum_{j=1}^{\nu} b_{n,j}^2(j^{-1}\log j)$. This bound, the Cauchy–Schwarz inequality, and the fact

$\sum_{j\ge 1} j^{-2}\log^2 j<\infty$, imply

$$s_{n,1}\le Cb_n^2\log(n)\sum_{j=1}^{\nu} j^{-1}\le Cb_n^2\log^2(n)\quad\text{and}\quad s_{n,1}\le Cb_n\sum_{j=1}^{\nu}|b_{n,j}|\bigl(j^{-1}\log j\bigr)\le Cb_nB_n.$$

This proves that $s_{n,1}$ satisfies both bounds of (2.7).

Next, again by Proposition 2.2(iii), for all $1\le k<j\le\nu$,

$$\bigl|E[w_{X,j}\overline{w_{X,k}}]\bigr|\le C\bigl(u_j^{-2d}+u_k^{-2d}\bigr)j^{-1}\log j,\qquad \bigl|E[w_{X,j}\overline{w_{\zeta,k}}]\bigr|\le C\bigl(u_j^{-d}+u_k^{-d}\bigr)j^{-1}\log j.$$

By (1.3),

$$(f_jf_k)^{-1}\bigl(u_j^{-2d}+u_k^{-2d}\bigr)^2\le C(u_ju_k)^{2d}\bigl(u_j^{-4d}+u_k^{-4d}\bigr)\le C(j/k)^{2|d|},$$
$$f_j^{-1}\bigl(u_j^{-d}+u_k^{-d}\bigr)^2\le Cu_j^{2d}\bigl(u_j^{-2d}+u_k^{-2d}\bigr)\le C(j/k)^{2|d|}.$$

These facts together imply

$$s_{n,2}\le C\sum_{1\le k<j\le\nu}|b_{n,j}b_{n,k}|\left(\frac{j}{k}\right)^{2|d|}\frac{\log^2 j}{j^2}. \tag{2.17}$$

Bound $|b_{n,j}b_{n,k}|$ by b_n^2 to obtain $s_{n,2}\le Cb_n^2\log^2(n)\sum_{1\le k<j\le\nu}k^{-2|d|}j^{2|d|-2}\le Cb_n^2\log^3(n)$, which implies the first estimate of (2.7). Next, bound $|b_{n,j}|$ by b_n in (2.17), to obtain

$$\begin{aligned}s_{n,2}&\le Cb_n\sum_{1\le k<j\le\nu}|b_{n,k}|\frac{\log^2 j}{k^{2|d|}j^{2-2|d|}}\le Cb_n\sum_{1\le k\le\nu}|b_{n,k}|\frac{\log^2 k}{k}\\&\le Cb_n\left(\sum_{1\le k\le\nu}b_{n,k}^2\right)^{1/2}\left(\sum_{1\le k\le\nu}\frac{\log^4 k}{k^2}\right)^{1/2}\le Cb_nB_n,\end{aligned}$$

which establishes the second bound of (2.7).

To show (2.8), recall that $f_{X,j}E|w_{\zeta,j}|^2/f_{\zeta,j}=f_{X,j}$. Therefore,

$$ER_n=\sum_{j=1}^{\nu}\frac{b_{n,j}}{f_{X,j}}\left(E|w_{X,j}|^2-\frac{f_{X,j}}{f_{\zeta,j}}E|w_{\zeta,j}|^2\right)=\sum_{j=1}^{\nu}\frac{b_{n,j}}{f_{X,j}}\bigl(E|w_{X,j}|^2-f_{X,j}\bigr).$$

Then, by (2.16) and (1.3),

$$|ER_n|\le C\sum_{j=1}^{\nu}\frac{|b_{n,j}|}{f_{X,j}}u_j^{-2d}j^{-1}\log j\le C\sum_{j=1}^{\nu}|b_{n,j}|j^{-1}\log j\le Cb_n\log^2(n),$$

which implies the first bound in (2.8).

To establish the second bound, let $K = (B_n/b_n)^{1/2}$. Because of (2.3), $K \to \infty$, $b_n K = (b_n/B_n)^{1/2} B_n = \mathrm{o}(B_n)$. Thus,

$$
\begin{aligned}
|ER_n| &\le C\left(\sum_{j=1}^{K-1} |b_{n,j}| j^{-1} \log j + \sum_{j=K}^{\nu} |b_{n,j}| j^{-1} \log j\right) \\
&\le C\left\{ b_n K + \left(\sum_{j=K}^{\nu} b_{n,j}^2\right)^{1/2} \left(\sum_{j=K}^{\infty} j^{-2} \log^2 j\right)^{1/2}\right\} = \mathrm{o}(B_n).
\end{aligned} \tag{2.18}
$$

This completes proof of the second estimate in (2.8). □

Now we establish the asymptotic normality of the weighted quadratic forms $S_{n,\zeta}$. The CLT for quadratic forms in i.i.d. r.v.'s is well investigated; see Guttorp and Lockhart [10]. The following theorem summarizes a useful criterion for asymptotic normality, given in Theorem 2.1 in Bhansali *et al.* [2]. Let $C_n = \{c_{n,ts}, t, s = 1, \dots, n\}$ be a symmetric $n \times n$ matrix of real numbers $c_{n,ts}$, and define the quadratic form

$$
Q_n := \sum_{t,s=1}^{n} c_{n,ts} \zeta_t \zeta_s.
$$

Let $\|C_n\| := (\sum_{t,s=1}^n c_{n,ts}^2)^{1/2}$ and $\|C_n\|_{\mathrm{sp}} := \max_{\|x\|=1} \|C_n x\|$ denote Euclidean and spectral norms, respectively, of C_n.

Theorem 2.2. *Suppose $\zeta_j \sim \mathrm{IID}(0, 1)$ and $E\zeta_0^4 < \infty$. Then*

$$
\frac{\|C_n\|_{\mathrm{sp}}}{\|C_n\|} \to 0 \tag{2.19}
$$

implies $(\mathrm{Var}(Q_n))^{-1/2}(Q_n - EQ_n) \to_D \mathcal{N}(0, 1)$. In addition, if $\sum_{t=1}^n c_{n;tt}^2 = \mathrm{o}(\|C_n\|^2)$, then $\mathrm{Var}(Q_n) \sim 2\|C_n\|^2$. Furthermore, in this case, if $E\zeta_0^4 < \infty$ is replaced by $E|\zeta_0|^{2+\delta} < \infty$, for some $\delta > 0$, then $(2\|C_n\|^2)^{-1/2}(Q_n - EQ_n) \to_D \mathcal{N}(0, 1)$.

Next lemma derives asymptotic distribution of the sum $S_{n,\zeta}$ of (2.1). Its proof uses Theorem 2.2 and some ideas of the proof of Theorem 2, Robinson [24].

Lemma 2.2. *Suppose $\zeta_j \sim \mathrm{IID}(0, 1)$, $E\zeta_0^4 < \infty$, and $b_{n,j}$ satisfy (2.3). Then*

$$
\begin{aligned}
&\text{(a)}\quad ES_{n,\zeta} = \sum_{j=1}^{\nu} b_{n,j}, \qquad \text{(b)}\quad \mathrm{Var}(S_{n,\zeta}) = q_n^2, \\
&\text{(c)}\quad q_n^{-1}(S_{n,\zeta} - ES_{n,\zeta}) \to_D \mathcal{N}(0, 1).
\end{aligned} \tag{2.20}
$$

Moreover,

$$q_n^2 \geq \min\bigl(1, \operatorname{Var}\bigl(\zeta_0^2\bigr)/2\bigr) B_n^2. \tag{2.21}$$

Proof. Let $c_n(t) := n^{-1} \sum_{j=1}^{\nu} b_{n,j} \cos(tu_j)$, $t = 1, 2, \ldots$. Note that

$$S_{n,\zeta} = \frac{1}{n} \sum_{t,s=1}^{n} \sum_{j=1}^{\nu} \mathrm{e}^{\mathrm{i}(t-s)u_j} b_{n,j} \zeta_s \zeta_t = \sum_{t,s=1}^{n} c_n(t-s) \zeta_s \zeta_t.$$

The matrix $C_n = (c_n(t-s))_{t,s=1,\ldots,n}$ is a symmetric $n \times n$ matrix with real entries. Hence, (2.20)(a) and (2.20)(b) follow because ζ_j's are IID(0, 1). For the same reason, and because $\operatorname{Var}(\zeta_0^2) - 2 = E\zeta_0^4 - 3 = \operatorname{Cum}_4(\zeta_0)$, and $c_n(0) = n^{-1} \sum_{j=1}^{\nu} b_{n,j}$,

$$\begin{aligned} \operatorname{Var}(S_{n,\zeta}) &= 2 \sum_{s,t=1:t\neq s}^{n} c_n^2(t-s) + \operatorname{Var}\bigl(\zeta_0^2\bigr) \sum_{t=1}^{n} c_n^2(t-t) \\ &= 2\|C_n\|^2 + \operatorname{Cum}_4(\zeta_0) n^{-1} \Biggl(\sum_{j=1}^{\nu} b_{n,j}\Biggr)^2 \geq \min\bigl(2, \operatorname{Var}\bigl(\zeta_0^2\bigr)\bigr) \|C_n\|^2. \end{aligned} \tag{2.22}$$

We shall show below that

$$\text{(a)} \quad \|C_n\|^2 = 2^{-1} B_n^2, \qquad \text{(b)} \quad \|C_n\|_{\mathrm{sp}} = \mathrm{o}(\|C_n\|). \tag{2.23}$$

Then (2.22) and (2.23)(a) imply (2.21), whereas by Theorem 2.2, (2.23)(b) implies

$$\bigl(\operatorname{Var}(S_{n,\zeta})\bigr)^{-1/2} \bigl(S_{n,\zeta} - E[S_{n,\zeta}]\bigr) \to_D \mathcal{N}(0, 1),$$

$$\operatorname{Var}(S_{n,\zeta}) \;=\; B_n^2 + \operatorname{Cum}_4(\zeta_0) n^{-1} \Biggl(\sum_{j=1}^{\nu} b_{n,j}\Biggr)^2,$$

which proves (2.20)(c). It remains to show (2.23).

To prove (2.23)(a), recall that for all $1 \leq j, k \leq m$, $j + k < n$ and $a, b \in \mathbb{R}$,

$$\sum_{t=1}^{n} \cos(tu_j + a) \cos(tu_k + b) = \frac{n}{2} \cos(a - b) I(j = k). \tag{2.24}$$

This fact and the definition of $c_n(t)$ imply (2.23)(a), because

$$\|C_n\|^2 = \sum_{t,s=1}^{n} c_n^2(t-s) = n^{-2} \sum_{j,k=1,\, j+k<n}^{\nu} b_{n,j} b_{n,k} \sum_{s,t=1}^{n} \cos\bigl((t-s)u_j\bigr) \cos\bigl((t-s)u_k\bigr) = 2^{-1} B_n^2.$$

To establish (2.23)(b), note that by (2.24), $\sum_{t=1}^{n} c_n(t-s)c_n(t-v) = (2n)^{-1} \times \sum_{j=1}^{\nu} b_{n,j}^2 \cos((s-v)u_j)$. Hence, for any $x \in \mathbb{R}^n$, such that $\|x\| = 1$,

$$\begin{aligned}\|C_n x\|^2 &= \sum_{t=1}^{n}\left(\sum_{s=1}^{n} c_n(t-s)x_s\right)^2 = \sum_{s,v=1}^{n} x_s x_v \left(\sum_{t=1}^{n} c_n(t-s)c_n(t-v)\right) \\ &= \frac{1}{2n}\sum_{j=1}^{\nu} b_{n,j}^2 \sum_{s,v=1}^{n} \cos\bigl((s-v)u_j\bigr)x_s x_v \le \frac{1}{2n} b_n^2 \sum_{j=1}^{n}\left|\sum_{s=1}^{n} e^{\mathrm{i}su_j} x_s\right|^2.\end{aligned}$$

Expand the last quadratic and use the fact $\sum_{j=1}^{n} e^{\mathrm{i}(t-s)u_j} = nI(t=s)$, to obtain

$$\|C_n x\|^2 \le \frac{1}{2} b_n^2 \sum_{t=s=1}^{n} x_t^2 = \frac{1}{2} b_n^2 \|x\|^2, \qquad \|C_n\|_{\mathrm{sp}} \le (1/\sqrt{2})b_n.$$

Since $b_n = \mathrm{o}(B_n)$, and $B_n = \sqrt{2}\|C_n\|$ by (2.23)(a), this proves (2.23)(b), and also completes the proof of the lemma. $\square$

3. A general case of sums of weighted periodogram

We now focus on the sums $Q_{n,X}$ of (1.1). Bartlett approximation $I_{X,j} \sim f_{X,j}(I_{\zeta,j}/f_{\zeta,j})$ suggests to approximate $Q_{n,X}$ by the sum

$$Q_{n,\zeta} := \sum_{j=1}^{\nu}(b_{n,j} f_{X,j})\left(\frac{I_{\zeta,j}}{f_{\zeta,j}}\right) = \sum_{j=1}^{\nu} b_{n,j} f_{X,j}(2\pi) I_{\zeta,j}.$$

In Theorem 2.1 above, f_X can be unbounded at 0, but differentiable on $(0,\pi)$. Then the asymptotic normality of the sums $S_{n,X} = \sum_{j=1}^{\nu} b_{n,j}(I_{X,j}/f_{X,j})$ holds under (2.3).

Now we turn to the case when f_X is continuous on Π and satisfies

$$0 < C_1 \le f_X(u) \le C_2 < \infty, \qquad u \in \Pi \ (\exists 0 < C_1, C_2 < \infty). \tag{3.1}$$

Theorems 3.1 and 3.3 below show that under (2.3), continuity of f_X, or more precisely, continuity of the transfer function A_X, suffices for asymptotic normality of the centered sums $Q_{n,X} - EQ_{n,X}$ and for obtaining an upper bound on the variance $\mathrm{Var}(Q_{n,X})$, whereas satisfactory asymptotics of $EQ_{n,X}$ requires f_X to be Lipshitz(β), $\beta > 1/2$.

By Lemma 2.2, $EQ_{n,\zeta} = \sum_{j=1}^{\nu} b_{n,j} f_{X,j}$ and $\mathrm{Var}(Q_{n,\zeta}) = v_n^2$, where

$$v_n^2 := \sum_{j=1}^{\nu}(b_{n,j} f_{X,j})^2 + \mathrm{Cum}_4(\zeta_0)\frac{1}{n}\left(\sum_{j=1}^{\nu} b_{n,j} f_{X,j}\right)^2.$$

Let $b_{f,n} = \max_{j=1,\dots,\nu} |b_{n,j}| f_{X,j}$ and $B_{f,n}^2 = \sum_{j=1}^{\nu} (b_{n,j} f_{X,j})^2$. Similarly as in (2.5), one can show that for some $C_1, C_2 > 0$,

$$C_1 B_{f,n}^2 \le v_n^2 \le C_2 B_{f,n}^2 \quad \text{and} \quad C_1 B_n^2 \le v_n^2 \le C_2 B_n^2, \qquad \text{under (3.1)}. \tag{3.2}$$

The folowing theorem describes the asymptotic behavior of bias, variance, and asymptotic normality of $Q_{n,X}$ when f_X is continuous and bounded.

Theorem 3.1. *Suppose the linear process* $\{X_j, j \in \mathbb{Z}\}$ *of* (1.2) *is such that* $E\zeta_0^4 < \infty$, *and the real weights* $b_{n,j}$*'s satisfy* (2.3).

In addition, if f_X *satisfies* (3.1) *and* $A_X \in \mathcal{C}(\Pi)$, *then*

$$\operatorname{Var}(Q_{n,X}) = v_n^2 + o\big(v_n^2\big), \qquad v_n^{-1}(Q_{n,X} - EQ_{n,X}) \to_D \mathcal{N}(0,1). \tag{3.3}$$

In addition, if $f_X \in \Lambda_\beta(\Pi)$, *with* $\beta > 1/2$, *then*

$$EQ_{n,X} = \sum_{j=1}^{\nu} b_{n,j} f_{X,j} + \mathrm{o}(v_n), \qquad v_n^{-1}\Bigg(Q_{n,X} - \sum_{j=1}^{\nu} b_{n,j} f_{X,j}\Bigg) \to_D \mathcal{N}(0,1). \tag{3.4}$$

The next theorem covers the case when the f_X is not bounded in the neighborhood of 0, that is, $d > 0$, or is not bounded away from 0, that is, $d < 0$. Then the second bound of (3.2) does not hold. Assumption (2.3) now has to be formulated using the weights $b_{n,j} f_{X,j}$ and we need to impose some additional smoothness conditions on A_X in a small neighborhood of 0. We assume that A_X can be factored into a product $A_X = hG$ of a differentiable function h, which may have a pole at 0, and a continuous bounded function G. In particular, if A_X satisfies (1.4), we take $G \equiv 1$.

Theorem 3.2. *Suppose* $\{X_j, j \in \mathbb{Z}\}$ *is the linear process* (1.2) *with* $E\zeta_0^4 < \infty$. *Assume that* f_X *satisfies* (1.3) *with* $|d| < 1/2$, *the transfer function* A_X *can be factored as* $A_X = hG$, *where* G *is continuous and bounded away from* 0 *and* ∞, *and* h *is differentiable having derivative* $\dot{h}$ *and satisfying*

$$C_1|u|^{-d} \le \big|h(u)\big| \le C_2|u|^{-d}, \qquad \big|\dot{h}(u)\big| \le C|u|^{-1-d}, \qquad 0 < |u| \le \pi, \tag{3.5}$$

for some $0 < C, C_1, C_2 < \infty$. *Then, for any real weights* $b_{n,j}$*'s satisfying*

$$\frac{b_{f,n}}{B_{f,n}} \equiv \frac{\max_{j=1,\dots,\nu} |b_{n,j} f_{X,j}|}{(\sum_{j=1}^{\nu} (b_{n,j} f_{X,j})^2)^{1/2}} \to 0, \tag{3.6}$$

(3.3) *continues to hold.*

If, in addition, $G \in \Lambda_\beta(\Pi)$, *with* $\beta > 1/2$, *then also* (3.4) *holds.*

Proofs of Theorems 3.1 and 3.2. The proof of both theorems follows from Lemmas 2.2 and 3.1. The latter lemma will be proved shortly.

Let $r_n := Q_{n,X} - Q_{n,\zeta} - E[Q_{n,X} - EQ_{n,\zeta}]$. In Lemmas 3.1(i) and 3.1(ii), it is shown that $Er_n^2 = \mathrm{o}(v_n^2)$ under the assumptions of Theorems 3.1 and 3.2. Therefore, the claim (3.3) made in these two theorems follows, noticing that, by Lemma 2.2, under assumption (3.6), $v_n^{-1}(Q_{n,X} - Q_{n,\zeta}) \to_D \mathcal{N}(0,1)$. The second claim (3.4) of these theorems is shown in (3.15) of Theorem 3.3 below. □

Lemma 3.1 below shows that the order of approximation of $Q_{n,X} - EQ_{n,X}$ by $Q_{n,\zeta} - EQ_{n,\zeta}$ is determined by the smoothness of the transfer function A_X. For example, by Lemma 3.1(i), if A_X is a bounded continuous function, then

$$Q_{n,X} - EQ_{n,X} = Q_{n,\zeta} - EQ_{n,\zeta} + \mathrm{o}_p(v_n). \tag{3.7}$$

If, in addition, A_X has a bounded derivative, then the order improves to $\mathrm{o}_p(n^{-1/2}\log(n)v_n)$ without requiring any additional assumptions on $b_{n,j}$. Lemma 3.1(ii) shows that if A_X is discontinuous at 0, then approximation (3.7) is valid under additional regularity behavior of A_X in a neighborhood of 0, as long as the weights $b_{n,j}$ satisfy (3.6).

To state the lemma, we need the following notation. For a complex valued function $h(u)$, $u \in \Pi$, define

$$\begin{aligned}\varepsilon_{n,h} &:= n^{-1}\log^2(n), \qquad h \in \Lambda_1[\Pi],\\ &:= n^{-\beta}, \qquad h \in \Lambda_\beta[\Pi],\ 0<\beta<1,\\ &:= \delta_n,\ \delta_n \to 0, \qquad h \in \mathcal{C}[\Pi].\end{aligned}$$

Lemma 3.1. *Assume that $\{X_j\}$ is as in (1.2) and $E\zeta_0^4 < \infty$. Then the following hold.*

(i) *If $A_X \in \Lambda_\beta[\Pi]$, $0<\beta\le 1$, or $A_X \in \mathcal{C}[\Pi]$, then*

$$Er_n^2 \le C\varepsilon_{n,A_X}B_n^2 = \mathrm{o}(v_n^2). \tag{3.8}$$

(ii) *If $A_X = hG$, where h satisfies (3.5) and either $G \in \mathcal{C}(\Pi)$ or $G \in \Lambda_\beta(\Pi)$, $0<\beta\le 1$, then*

$$\begin{aligned}Er_n^2 &\le C\big(\min\big(b_{f,n}^2\log^3 n, b_{f,n}B_{f,n}\big) + \varepsilon_{n,G}B_{f,n}^2\big),\\ &\le C\min\big(b_{f,n}^2\log^3 n, b_{f,n}B_{f,n}\big), \qquad G \in \Lambda_1(\Pi).\end{aligned} \tag{3.9}$$

If, in addition, (3.6) holds, then

$$Er_n^2 = \mathrm{o}(v_n^2). \tag{3.10}$$

Proof. Rewrite $r_n = D_n - ED_n$, where $D_n = Q_{n,X} - Q_{n,\zeta} = \sum_{j=1}^{\nu} b_{n,j}\{I_{X,j} - (f_{X,j}/f_{\zeta,j}) \times I_{\zeta,j}\}$. Let $t_{n,i}$, $i=1,2$ denote the $s_{n,i}$, $i=1,2$, of Corollary 2.1 with $c_{n,j} \equiv b_{n,j}$. By Corollary 2.1,

$$\operatorname{Var}(D_n) \le C(t_{n,1} + t_{n,2}). \tag{3.11}$$

Proof of (i). Arguing as in the proof of Lemma 2.1, one can show that

$$E\big(r_n^2\big) \le C\varepsilon_{n,A_X} B_n^2, \tag{3.12}$$

which, in view of (3.2), proves (3.8). We need to verify (3.12) in the following three cases.

Case (1). $A_X \in \Lambda_1[\Pi]$. Then, by Proposition 2.2(i),

$$\big|E|w_{X,j}|^2 - f_{X,j}\big| \vee \big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\big| \le Cn^{-1}\log n,$$
$$\big|E[w_{X,j}\overline{w_{X,k}}]\big| \vee \big|E[w_{X,j}\overline{w_{\zeta,k}}]\big| \le Cn^{-1}\log n, \qquad 1 \le k < j \le \nu.$$

Therefore, $t_{n,1} \le Cn^{-1}\log n \sum_{j=1}^{\nu} b_{n,j}^2 = Cn^{-1}\log n B_n^2$, and

$$t_{n,2} \le Cn^{-2}\log^2 n \sum_{1\le k<j\le\nu} |b_{n,j}b_{n,k}| \le Cn^{-1}\log^2 n B_n^2,$$

which proves (3.12).

Case (2). $A_X \in \Lambda_\beta[\Pi]$, $0 < \beta < 1$. Then by Proposition 2.2(i),

$$\big|E|w_{X,j}|^2 - f_{X,j}\big| \vee \big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\big| \le Cn^{-\beta},$$
$$\big|E[w_{X,j}\overline{w_{X,k}}]\big| \vee \big|E[w_{X,j}\overline{w_{\zeta,k}}]\big| \le Cn^{-\beta}\ell_n(\beta; j-k), \qquad k < j.$$

Note that for $1 \le k < j \le \nu < n/2$, $j - k \le n - j + k$, and hence bound

$$\ell_n(\beta; j-k) \le C\frac{\log(2+j-k)}{(2+j-k)^{1-\beta}}, \qquad \big(n^{-\beta}\ell_n(\beta; j-k)\big)^2 \le C\frac{\log^2(2+j-k)}{n^\beta(2+j-k)^{2-\beta}}.$$

Apply this fact, to obtain, that for $0 < \beta < 1$, $t_{n,1} \le Cn^{-\beta}\sum_{j=1}^{\nu} b_{n,j}^2 = Cn^{-\beta}B_n^2$,

$$\begin{aligned} t_{n,2} &\le C \sum_{1\le k<j\le\nu} |b_{n,j}b_{n,k}|\big(n^{-\beta}\ell_n(\beta; j-k)\big)^2 \\ &\le Cn^{-\beta} \sum_{1\le k<j\le\nu} |b_{n,j}b_{n,k}|\frac{\log^2(2+j-k)}{(2+j-k)^{2-\beta}} \le Cn^{-\beta}B_n^2, \end{aligned}$$

which proves (3.12).

Case (3). $A_X \in \mathcal{C}[\Pi]$. By Proposition 2.2(ii), for any $0 < \varepsilon < 1/2$,

$$\big|E|w_{X,j}|^2 - f_{X,j}\big| \vee \big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\big| \le C\delta_n,$$
$$\big|E[w_{X,j}\overline{w_{X,k}}]\big| \vee \big|E[w_{X,j}\overline{w_{\zeta,k}}]\big| \le C\delta_n\ell_n(\varepsilon; j-k), \qquad k < j,$$

with some $\delta_n \to 0$, that does not depend on k, j and n, and (3.12) follows by the same argument as in the case (2) above. This completes the proof of (i) of the lemma.

Proof of (ii). First, we prove (3.9). As above, for that we need to bound $t_{n,1}$ and $t_{n,2}$ of (3.11). Recall that $f_X = |A_X|^2/(2\pi)$, $f_{X\zeta} = A_X/(2\pi)$, $A_X = h(u)G(u)$, where h satisfies (3.5), which together with (1.3) implies that G is bounded away from infinity and zero. For $1 \le k \le j \le \nu$, define

$$
\begin{aligned}
\tilde{r}_{n,jk} &:= 0, \qquad G \in \Lambda_1(\Pi), \\
&:= n^{-\beta} \frac{\log(2+j-k)}{(2+j-k)^{1-\beta}}, \qquad G \in \Lambda_\beta(\Pi), 0 < \beta < 1, \\
&:= \delta_n \frac{\log(2+j-k)}{(2+j-k)^{1-\varepsilon}}, \qquad G \in \mathcal{C}(\Pi), 0 < \varepsilon < 1/2, \delta_n \to 0.
\end{aligned}
$$

By Proposition 2.2(iv), for $1 \le k \le j$,

$$
\begin{aligned}
\bigl|E[w_{X,j}\overline{w_{X,k}}] - f_{X,j} I(j=k)\bigr| &\le C\bigl\{\bigl(u_k^{-2d} + u_j^{-2d}\bigr) j^{-1}\log j + \bigl(u_k^{-2d} \wedge u_j^{-2d}\bigr)\tilde{r}_{n,jk}\bigr\} \\
\bigl|E[w_{X,j}\overline{w_{\zeta,k}}] - f_{X\zeta,j} I(j=k)\bigr| &\le C\bigl\{\bigl(u_k^{-d} + u_j^{-d}\bigr) j^{-1}\log j + \bigl(u_k^{-d} \wedge u_j^{-d}\bigr)\tilde{r}_{n,jk}\bigr\}.
\end{aligned}
$$

Since $f_X = |A_X|^2/(2\pi) = |hG|^2/(2\pi)$, assumptions on h and G here imply that for all $u \in \Pi$,

$$
f_X(u) \le C|u|^{-2d}, \qquad f_X^{-1}(u) \le C|u|^{2d}, \qquad \bigl|f_{X\zeta}(u)\bigr| \le C|u|^{-d}, \qquad \bigl|f_{X\zeta}^{-1}(u)\bigr| \le C|u|^{d}.
$$

Therefore, for $1 \le k \le j$,

$$
\begin{aligned}
(f_{X,j} f_{X,k})^{-1}\bigl(u_k^{-2d} + u_j^{-2d}\bigr)^2 &\le C|j/k|^{2|d|}, \qquad (f_{X,j} f_{X,k})^{-1}\bigl(u_k^{-2d} \wedge u_j^{-2d}\bigr)^2 \le C, \\
(f_{X,j})^{-1}\bigl(u_k^{-d} + u_j^{-d}\bigr)^2 &\le C|j/k|^{2|d|}, \qquad (f_{X,j})^{-1}\bigl(u_k^{-d} \wedge u_j^{-d}\bigr)^2 \le C.
\end{aligned}
$$

Recall the bound (3.11). It suffices to show that $t_{n,1} + t_{n,2}$ can be bounded above by the r.h.s. of (3.9). The above bounds readily yield that

$$
\begin{aligned}
t_{n,1} &\le C \sum_{j=1}^{\nu} (b_{n,j} f_{X,j})^2 \bigl(j^{-1}\log j + \tilde{r}_{n,jj}\bigr), \\
t_{n,2} &\le C \sum_{1 \le k < j \le \nu} |b_{n,j} f_{X,j}||b_{n,k} f_{X,k}| \biggl(\Bigl(\frac{j}{k}\Bigr)^{2|d|} \frac{\log^2 j}{j^2} + \tilde{r}_{n,jk}^2\biggr).
\end{aligned}
$$

The arguments analogous to one used in evaluating $s_{n,1}$ and $s_{n,2}$ in Lemma 2.1 yield

$$
\begin{aligned}
&\sum_{j=1}^{\nu} (b_{n,j} f_{X,j})^2 \frac{\log j}{j} + \sum_{1 \le k < j \le \nu} |b_{n,j} f_{X,j}||b_{n,k} f_{X,k}| \Bigl(\frac{j}{k}\Bigr)^{2|d|} \frac{\log^2 j}{j^2} \\
&\quad \le C \min\bigl(b_{f,n}^2 \log^3(n), b_{f,n} B_{f,n}\bigr), \\
&\sum_{j=1}^{\nu} (b_{n,j} f_{X,j})^2 \tilde{r}_{n,jk} + \sum_{1 \le k < j \le \nu} |b_{n,j} f_{X,j}||b_{n,k} f_{X,k}| \tilde{r}_{n,jk}^2 \le C \varepsilon_{n,G} B_{f,n}^2.
\end{aligned}
$$

Therefore, $t_{n,1} + t_{n,2} \le C(\min(b_{f,n}^2 \log^3(n), b_{f,n} B_{f,n}) + \varepsilon_{n,G} B_{f,n}^2)$, which proves (3.9).

Observe that $\varepsilon_{n,G} \to 0$. Therefore, (3.9), (3.6) and (3.2) imply (3.10). This completes the proof of the lemma. $\square$

As seen above, proving CLT for $v_n^{-1}(Q_{n,X} - \sum_{j=1}^{\nu} b_{n,j} f_{X,j})$ requires some smoothness of the spectral density f_X and the transfer function A_X. Conditions on A_X can be relaxed if one wishes to establish only an upper bound for the mean square error of the estimator $Q_{n,X}$ of $\sum_{j=1}^{\nu} b_{n,j} f_{X,j}$ as is shown in the next theorem. The results of Theorem 3.3 also remain valid for $\nu = [n/2]$.

Theorem 3.3. *Let $\{X_j\}$ be as in (1.2) with $E\zeta_0^4 < \infty$ and f_X satisfying (1.3).*

(i) *Then*

$$E(Q_{n,X} - EQ_{n,X})^2 \le CB_{f,n}^2. \tag{3.13}$$

(ii) *In addition,*

$$E\Biggl(Q_{n,X} - \sum_{j=1}^{\nu} b_{n,j} f_{X,j}\Biggr)^2 \le CB_{f,n}^2, \tag{3.14}$$

in each of the following three cases.
(c1) $d = 0$, $g \in \Lambda_\beta[\Pi]$, $1/2 < \beta \le 1$;
(c2) $d \ne 0$, $g \in \Lambda_\beta[\Pi]$, $1/2 < \beta \le 1$;
(c3) $|\dot{f}_X(u)| \le Cu^{-1-2d}$, $0 < u \le \pi$.
Moreover, in case (c1),

$$EQ_{n,X} - \sum_{j=1}^{\nu} b_{n,j} f_j = \mathrm{o}(B_{f,n}). \tag{3.15}$$

If $b_{n,j}$'s satisfy (3.6), *then* (3.15) *holds also in cases* (c2) *and* (c3).

Proof. (i) Recall $I_{X,j} = |w_{X,j}|^2$. By Proposition 2.1,

$$E(Q_{n,X} - EQ_{n,X})^2 = \mathrm{Var}\Biggl(\sum_{j=1}^{\nu} b_{n,j} I_{X,j}\Biggr) \le C \sum_{j,k=1}^{\nu} |b_{n,j} b_{n,k}| \bigl|E[w_{X,j}\overline{w_{X,k}}]\bigr|^2.$$

For $j = k$ bounding $(E|w_{X,j}|^2)^2 \le 2(E|w_{X,j}|^2 - f_{X,j})^2 + 2f_{X,j}^2$, and letting

$$s'_{n,1} := \sum_{j=1}^{\nu} b_{n,j}^2 \bigl(E|w_{X,j}|^2 - f_{X,j}\bigr)^2,$$

$$s'_{n,2} := \sum_{1 \le k < j \le \nu} |b_{n,j} b_{n,k}| \bigl|E[w_{X,j}\overline{w_{X,k}}]\bigr|^2,$$

one obtains $E(Q_{n,X} - EQ_{n,X})^2 \le C(s'_{n,1} + s'_{n,2} + B^2_{f,n})$. Under the current assumptions, by Proposition 2.2(iv), for $1 \le k < j \le \nu$ $(0 < \varepsilon < 1/2)$,

$$\big|E|w_{X,j}|^2 - f_{X,j}\big| \le Cu_j^{-2d}\big(j^{-1}\log j + \delta_n\big),$$
$$\big|E[w_{X,j}\overline{w_{X,k}}]\big| \le C\big(\big(u_k^{-2d} + u_j^{-2d}\big)j^{-1}\log j + \big(u_k^{-2d} \wedge u_j^{-2d}\big)\delta_n \ell(\varepsilon, j-k)\big),$$

where $\delta_n \to 0$. Observe that $s'_{n,i} \le t_{n,i}, i = 1, 2$, where $t_{n,1}$ and $t_{n,2}$ are as in the proof of Lemma 3.1. Therefore, the same argument as used in proving (3.9) implies that $s'_{n,1} + s'_{n,2}$ satisfies the bound (3.9), which in turn yields $s'_{n,1} + s'_{n,2} \le C(b_{f,n}B_{f,n} + \varepsilon_{n,G}B^2_{f,n}) \le CB^2_{f,n}$, since $b_{f,n} \le B_{f,n}$. This completes proof of (3.13).

(ii) By parts (i), (iv) and (iii) of Proposition 2.2, we respectively obtain

$$\begin{aligned}\big|E|w_{X,j}|^2 - f_{X,j}\big| &\le Cu_j^{-2d}n^{-\beta}, &&\text{in case (c1)},\\ &\le Cu_j^{-2d}\big(j^{-1}\log j + n^{-\beta}\big), &&\text{in case (c2)},\\ &\le Cu_j^{-2d}\big(j^{-1}\log j\big), &&\text{in case (c3)}.\end{aligned}$$

Let $D_n := |EQ_{n,X} - \sum_{j=1}^{\nu} b_{n,j}f_{X,j}| = |\sum_{j=1}^{\nu} b_{n,j}(E|w_{X,j}|^2 - f_{X,j})|$. Under the current assumptions, $f_{X,j}^{-1} \le Cu_j^{2d}$, $0 < u \le \pi$. Thus, in case (c1),

$$D_n \le C\sum_{j=1}^{\nu}|b_{n,j}f_{X,j}|n^{-\beta} \le Cn^{1/2-\beta}\left(\sum_{j=1}^{\nu}(b_{n,j}f_{X,j})^2\right)^{1/2} = \mathrm{o}(B_{f,n}), \tag{3.16}$$

which proves (3.14) and (3.15).

In case (c2), $D_n \le C\sum_{j=1}^{\nu}|b_{n,j}f_{X,j}|(j^{-1}\log(n) + n^{-\beta})$. Arguing as for (2.18), one can show that $\sum_{j=1}^{\nu}|b_{n,j}f_{X,j}|j^{-1}\log j = \mathrm{o}(B_{f,n})$, if (3.6) holds, and $\sum_{j=1}^{\nu}|b_{n,j}f_{X,j}|j^{-1}\log j = \mathrm{O}(B_{f,n})$, otherwise, which together with (3.16) yields (3.14) and (3.15). The proof of (3.14) and (3.15) in case (c3) is the same as in case (c2). This completes the proof of the theorem. □

Remark 3.1. Consider now the sum

$$Q_{n,X} = \sum_{j=1}^{\theta n} b_{n,j}I_{X,j}, \qquad (0 < \theta < 1/2), \tag{3.17}$$

where summation is taken over a fraction $\{1, \dots, \theta n\}$ of the set $\{1, \dots, \nu\}$, and periodograms $I_{X,j}$ used in $Q_{n,X}$ are based on frequencies u_j from the zero neighborhood $[0, 2\pi\theta]$, sub-interval of $[0, \pi]$. In this case, the smoothness conditions on f_X and A_X are required only to obtain upper bounds on the covariances $E[w_{X,j}\overline{w_{X,k}}]$ and $E[w_{X,j}\overline{w_{\zeta,k}}]$ in Proposition 2.2. Therefore, in order for these bounds to be valid at frequencies $u_j \in [0, 2\pi\theta]$ it suffices to impose smoothness conditions on f_X and A_X on a slightly larger interval $[0, a]$, $a > 2\pi\theta$, covering $[0, 2\pi\theta]$. Hence, for the sum $Q_{n,X}$ of (3.17), all of the above results derived in this section remain valid if conditions on f_X and A_X are satisfied on some interval $[0, a]$, with $a > \Delta$, instead of on $[0, \pi]$.

Remark 3.2. To highlight the main method of establishing the asymptotic normality of the weighted sums of the periodograms, we focused mainly on a linear process with an i.i.d. noise $\{\zeta_j\}$. Since by the Wold decomposition most of stationary processes can be written as a linear process with white noise innovations, it is of interest to extend the above results to a linear processes with martingale-difference innovations. Without assuming that the first conditional moments of ζ_j are constant, such extension requires substantial effort which includes deriving the general CLT for quadratic forms in martingale-differences and justification of the Bartlett approximation, by proving the bound of Proposition 2.1. Such extension, although non-standard, can be established for a wide class of martingale difference innovations under tractable conditions and is currently under our consideration.

Remark 3.3. In the proof of the asymptotic normality of the local Whittle estimator of the memory parameter d in (1.3), Robinson [20] established the CLT

$$\frac{m^{-1/2}}{g(0)}(S_{n,X} - ES_{n,X}) \to N(0,1), \qquad S_{n,X} = \sum_{j=1}^{m} \nu_{n,j} \frac{I_{X,j}}{|u_j|^{-2d}} = \sum_{j=1}^{m} b_{n,j} \frac{I_{X,j}}{f(u_j)}$$

for special weights $b_{n,j} = g(u_j)\nu_{n,j}$, where $g(u) \to g(0) > 0$, and $\nu_{n,j} := \log(j/m) - m^{-1}\sum_{k=1}^m \log(k/m)$, and $m = o(n)$, $m \to \infty$. Since $\nu_{n,j} := \log(j/m) + 1 + o(1)$ and $\sum_{j=1}^m b_{n,j}^2 \sim g^2(0)\sum_{j=1}^m \nu_{n,j}^2 \sim g^2(0)m$, they satisfy (2.3) of Theorem 2.1 which implies the above CLT. This fact is also apparent upon examining the Robinson's proof. Additional restrictions on m in that work were required to show that the bias term $m^{-1/2}ES_{n,X}$ of the local Whittle estimator is negligible.

Remark 3.4. Here, we provide an example where the weights $b_{n,j}$ in $S_{n,X}$ do not satisfy Lindeberg–Feller type condition (2.3) and the corresponding $S_{n,X}$ does not satisfied the CLT. Suppose $\{X_j\}$ is a stationary Gaussian zero mean long memory process, with $f_X(u) = |u|^{-2d}$, $1/4 < d < 1/2$.

Let $\bar{X} = n^{-1}\sum_{j=1}^n X_j$ and $\hat{\gamma}(0) := n^{-1}\sum_{j=1}^n (X_j - \bar{X})^2$. Recall the identity

$$2\pi \sum_{j=1}^{n} I_X(u_j) = \sum_{j=1}^{n} X_j^2 = \sum_{j=1}^{n} (X_j - \bar{X})^2 + n\bar{X}^2 = n\hat{\gamma}(0) + n\bar{X}^2.$$

Suppose n is even and $\nu = n/2 - 1$. Since $I_X(u_j) = I_X(u_{n-j})$, $1 \le j \le n$, and $2\pi I_X(u_0) = n\bar{X}^2$, we obtain $2\pi\sum_{j=1}^n I_X(u_j) = 4\pi\sum_{j=1}^{\nu} I_X(u_j) + 2\pi\{I_X(u_0) + I_X(u_{n/2})\}$, and

$$4\pi \sum_{j=1}^{\nu} I_X(u_j) = n\hat{\gamma}(0) - 2\pi I_X(u_{n/2}).$$

Now, let $b_{n,j} := n^{-2d} 4\pi f_X(u_j) = 4\pi(2\pi j)^{-2d}$. Then

$$S_{n,X} = \sum_{j=1}^{\nu} b_{n,j} \frac{I_X(u_j)}{f_X(u_j)} = n^{-2d} 4\pi \sum_{j=1}^{\nu} I_X(u_j).$$

By Hosking ([14], Theorem 4), under the assumed set up here, $n^{1-2d}(\hat{\gamma}(0) - E\hat{\gamma}(0)) \to_D Y$, where Y is a non-Gaussian r.v. Arguing as in the proof of Theorem 3.3(i), one can verify that $\mathrm{Var}(I_X(u_{n/2})) = \mathrm{O}(1)$. Hence, $S_{n,X} - ES_{n,X} \to_D Y$ does not satisfy the CLT. It remains to show that for $d > 1/4$, the weights $b_{n,j}$ do not satisfy (2.3):

$$\frac{\max_{j=1,\dots,\nu} |b_{n,j}|^2}{\sum_{j=1}^{\nu} b_{n,j}^2} = \frac{\max_{j=1,\dots,\nu} |j|^{-4d}}{\sum_{j=1}^{\nu} j^{-4d}} \to \frac{1}{\sum_{j=1}^{\infty} j^{-4d}} > 0.$$

Moreover, Theorem 2.1 does not provide the asymptotic of $\mathrm{Var}(S_{n,X})$ and approximations of Lemma 2.1 break down. To see that, we now have $b_n = 2(2\pi)^{1-2d}$, $B_n^2 = q_n^2 = \sum_{j=1}^{n} b_{n,j}^2 \to 4(2\pi)^{2-4d} \sum_{j=1}^{\infty} j^{-4d}$. Since the error of approximation $E(S_{n,X} - S_{n,\zeta})^2 \le C \log^4(n)$ in (2.9) is no more negligible compared to B_n^2, the claim that $\mathrm{Var}(S_{n,X}) \sim q_n^2$ of Theorem 2.1 does not hold. On the other hand, by Theorem 3 of Hosking [14], $\mathrm{Var}(n^{1-2d}\hat{\gamma}(0)) \to C > 0$, so that $\mathrm{Var}(S_{n,X}) \to C$.

Example 3.1. Consider the stationary ARFIMA(p, d, q) model

$$\phi(B)X_j = (1-B)^{-d}\theta(B)\zeta_j, \qquad j \in \mathbb{Z}, \qquad \{\zeta_j\} \sim \mathrm{IID}\big(0, \sigma_\zeta^2\big), \qquad |d| < 1/2.$$

Hosking [13] has shown that the spectral density f_X of this model satisfies (1.3). We shall show it also satisfies (1.4). Let $h(u) = (1 - \mathrm{e}^{-\mathrm{i}u})^{-d}$ and $A_Y(u) = \theta(\mathrm{e}^{-\mathrm{i}u})/\phi(\mathrm{e}^{-\mathrm{i}u})$. The transfer function A_X can be written as

$$A_X(u) = h(u)A_Y(u), \qquad f_X(u) = \big|A_X(u)\big|^2. \tag{3.18}$$

Now observe that h is differentiable and satisfies $|h(u)| \le C|u|^{-2d}$, $|\dot{h}(u)| \le C|u|^{-1-2d}$, for all $u \in [0, \pi]$, and $|h(u)| \sim |u|^{-2d}$, as $u \to 0$. Thus, for all $0 < |u| < \pi$,

$$\big|\dot{A}_X(u)\big| \le C\big(\big|\dot{h}(u)\big|\big|A_Y(u)\big| + \big|h(u)\big|\big|\dot{A}_Y(u)\big|\big) \le C\big|1 - \mathrm{e}^{-\mathrm{i}u}\big|^{-d-1} \le C|u|^{-d-1},$$

and hence A_X satisfies (1.4). Note also that $A_X = hA_Y$ is naturally factored into a differentiable component h and continuous component A_Y as required in Theorem 3.2. Thus, Theorems 2.1, 3.1–3.3 are applicable.

Example 3.2. Now consider a more general process $\{X_j\}$,

$$X_j = (1-B)^{-d}Y_j, \qquad j \in \mathbb{Z},\ |d| < 1/2,$$

where $Y_j = \sum_{k=0}^{\infty} b_k \zeta_{j-k}$, $\{\zeta_j\} \sim \mathrm{IID}(0, 1)$, $\sum_{k=0}^{\infty} |b_k| < \infty$, is a short memory process. Because, letting $A_Y(u) = \sum_{k=0}^{\infty} b_k e^{-\mathrm{i}uk}$, f_X and A_X are the same as in (3.18), the same argument as used in Example 3.1 shows that f_X satisfies (1.3) with parameter $|d| < 1/2$. Although A_X may not satisfy (1.4), because A_Y is only continuous, but A_X is factored as required in Theorems 3.2 and 3.3. Hence, these two theorems are applicable.

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