Extremum Sieve Estimation in k-out-of-n Systems. [1]

Tatiana Komarova,[2]

This version: July 24 2013

ABSTRACT

The paper considers nonparametric estimation of absolutely continuous distribution functions of lifetimes of non-identical components in k-out-of-n systems from the observed "autopsy" data. In economics, ascending "button" or "clock" auctions with n heterogeneous bidders present 2-out-of-n systems. Classical competing risks models are examples of n-out-of-n systems. Under weak conditions on the underlying distributions the estimation problem is shown to be well-posed and the suggested extremum sieve estimator is proven to be consistent. The paper illustrates the suggested estimation method by using sieve spaces of Bernstein polynomials which allow an easy implementation of constraints on the monotonicity of estimated distribution functions.

Keywords: k-out-of-n systems, competing risks, sieve estimation, Bernstein polynomials

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[1] I am grateful to Xioahong Chen, Oliver Linton and Elie Tamer for their helpful comments and suggestions. I also thank seminar participants at Georgetown University, London School of Economics, Queen Mary University of London, and University of Rochester for their feedback.

[2] Department of Economics, London School of Economics and Political Science; e-mail: t.komarova@lse.ac.uk.

1 Introduction

The paper considers nonparametric estimation of absolutely continuous distribution functions of lifetimes in k-out-of-n systems. Such a system is "alive" if and only if at least k of its components are alive. k-out-of-n systems are often encountered in practice. In economics, ascending "button" or "clock" auctions with n bidders present 2-out-of-n systems. Classical competing risks models are examples of n-out-of-n systems. This paper considers general situations of heterogeneous components – that is, when the lifetimes of different components can have different distributions. They only available data are the "autopsy" data, which give information only on the lifetime of the system and the corresponding fatal set of $n-k+1$ components.

One way to approach the estimation problem would be to impose parametric assumptions on the underlying distributions of components' lifetimes. For instance, the assumption that these distributions are exponential would bring down the estimation task to the task of learning n scalar parameters for n exponential distributions. However, if the underlying distributions are not exponential, then the inference based on the obtained estimates would not be credible. Such a parametric approach is exploited, for instance, in Meilijson (1994).

This paper does not make any parametric assumptions and suggests consistent in the uniform metric nonparametric estimators of the CDFs of components' lifetimes. The first step in this approach is to write down the system of integral-differential equations that describes the relations between the underlying unknown CDFs of components' lifetimes and the observables. This system is given in (IDE) in section 2. In the second step the distributions of observables are estimated consistently from a given sample. The third step constructs an objective function that represents a distance between the left-hand side and the right-hand in (IDE). Finally, this objective function is minimized when unknown CDFs are represented as unknown functions from a chosen sieve space. It is proven that the operator that maps observable functions into underlying CDFs of components's lifetimes is continuous. This guarantees the well-posedness of the estimation problem and the consistency of described extremum sieve estimators. It is worth noting that this approach works for any $1 \leq k \leq n$ and is easy to implement in practice.

The paper considers spaces of Bernstein polynomials as sieve spaces. In these spaces it is easy to formulate and use the constraints that represent necessary and sufficient condition for the monotonicity of a function. Monotonicity is of course a desirable property for an estimator of CDF. For a detailed review of sieve estimation methods in econometrics see Chen (2007). Chen (2008) focuses specifically on extremum sieve estimation.

Nonparametric estimation methods of heterogeneous independent lifetimes from the autopsy data are considered in Watelet (1990) and Doss, Huffer and Lawson (1997). Watelet's approach is based on rewriting the mathematical model that describes the relation between the unknown underlying CDFs of components' lifetimes and the observable joint distribution of system's lifetime and the fatal set in a form that contains unknown distributions on the left-hand side and some integral expressions on the right-hand side. Then Watelet uses an iterative method to estimate unknown distributions. Importantly, Watelet explores such a procedure only in the simplest case of $k = n$, which is the case of classical competing risks. It is also worth noting that for $2 < k < n$ this procedure cannot work in general. Doss, Huffer and Lawson (1997) suggest a nonparametric Bayesian procedure, which uses mixtures of Dirichlets as priors on the the distributions of components' lifetimes.

The rest of this paper is organized as follows. Section 2 reviews nonparametric identification of the distributions of components' lifetimes in k-out-of-n from "autopsy" data. Section 3 establishes that when the space of underlying distributions of components' lifetimes and the space of distributions of observables are endowed with the uniform metric, the problem of estimating underlying distributions from observables is well-posed. The section also suggests and extremum sieve estimator and proves its consistency. Section 4 illustrates suggested sieve estimation in an ascending auction framework by performing estimation in two Monte Carlo experiments. Proofs of propositions, lemmas and theorems are collected in the Appendix.

2 Review of identifiability

Consider a system that consists of n components whose lifetimes are mutually independent random variables X_i with distribution functions F_i^*, $i = 1, \ldots, n$. The distribution of i's component is absolutely continuous with respect to the Lebesgue measure, that is,

$$F_i^* \text{ is absolutely continuous}, \quad i = 1, \ldots, n. \tag{C1}$$

Let t_0 denote the common lower support point for the distributions of the lifetimes: that is,

$$F_i^*(t_0) = 0 \text{ and } F_i^*(t) > 0 \text{ for } t > t_0, \quad i = 1, \ldots, n. \tag{C2}$$

Let t_i stand for the upper support point of the distribution of X_i: that is,

$$F_i^*(t_i) = 1 \text{ and } F_i^*(t) < 1 \text{ for } t < t_i, \quad i = 1, \ldots, n. \tag{C3}$$

Suppose this is a k-out-of-n system, that is, it works as long as at least k of its components are working. The lifetime of this system can be characterized by the so-called fatal sets. In reliability literature a *fatal set* is a subset of components such that the failure of all the components in the subset causes the failure of the system. For a k-out-of-n system, the collection of fatal sets is the collection of all $(n-k+1)$-element subsets of $\{1, \ldots, n\}$. Denote this collection as $\mathcal{A}$.

This paper considers the case when the only data are observed after the failure of the system and pertain to the system's lifetime Z and a *diagnostic set*, which is the set of components that have failed by time Z and which is revealed during the autopsy. Clearly, $\mathcal{A}$ is the collection of all possible diagnostic sets. To summarize, the following $M \equiv \binom{n}{n-k+1}$ sub-distribution functions are observed: for each $A \in \mathcal{A}$,

$$G_A^*(t) = P(Z \le t,\ A - \text{diagnostic set}), \quad t \ge t_0.$$

A more detailed discussion of such systems (and coherent systems in general) can be found, for instance, in Barlow and Proschan (1972).

For convenience let us assign an order to sets in $\mathcal{A}$ and write this collection as

$$\mathcal{A} = \{A_1, A_2, \ldots, A_{M-1}, A_M\}.$$

Then the collection of observable functions can be written as

$$G_m^*(t) = P(Z \le t,\ A_m - \text{diagnostic set}), \quad t \ge t_0,$$

where $m = 1, \ldots, M$. Note that for $t \ge \max_{i \in A_m} t_i$ function G_m is constant:

$$G_m^*(t) = P(A_m - \text{diagnostic set}), \quad t \ge \max_{i \in A_m} t_i.$$

Unknown underlying distributions F_i^*, $i = 1, \ldots, n$, and observable sub-distributions G_m^*, $m = 1, \ldots, M$, are related by the following system of M integral-differential equations:

$$G_m^*(t) = \int_{t_0}^{t} \left(\prod_{i \in A_m} F_i^*(s) \right)' \prod_{i \in A_m^c} (1 - F_i^*(s))\, ds, \quad t \ge t_0, \quad m = 1, \ldots, M, \qquad (IDE)$$

where $A_m^c = \{1, \ldots, n\} \backslash A_m$. Indeed,

$$\begin{aligned} G_m^*(t) &= P\left(\max_{i \in A_m} X_i \le t, \max_{i \in A_m} X_i \le \min_{i \in A_m^c} X_i \right) \\ &= P\left(\max_{i \in A_m} X_i \le t, \min_{i \in A_m^c} X_i > t \right) + P\left(\max_{i \in A_m} X_i \le \min_{i \in A_m^c} X_i, \min_{i \in A_m^c} X_i \le t \right). \end{aligned}$$

Since the value of the density of $\min_{i \in A_m^c} X_i$ at t is equal to $-\left(\prod_{i \in A_m^c} (1 - F_i^*(t)) \right)'$, then

$$\begin{aligned} G_m^*(t) &= \prod_{i \in A_m} F_i^*(t) \prod_{i \in A_m^c} (1 - F_i^*(t)) - \int_{t_0}^{t} \prod_{i \in A_m} F_i^*(s) \left(\prod_{i \in A_m^c} (1 - F_i^*(s)) \right)' ds \\ &= \int_{t_0}^{t} \left(\prod_{i \in A_m} F_i^*(s) \right)' \prod_{i \in A_m^c} (1 - F_i^*(s))\, ds, \quad t \ge t_0. \end{aligned}$$

Applying the same techniques as the ones for 2-out-of-n systems in Komarova (2012), one can establish the following identifiability result.

Proposition 2.1. *Distributions F_i^*, $i = 1, \ldots, n$, that satisfy conditions (C1) and (C2) are identifiable on $[t_0, T]$, where T is the $(n-k+1)$-th order statistic of $\{t_1, \ldots, t_n\}$, from observable functions G_m^*, $m = 1, \ldots, M$, if the following condition holds:*

For each $m = 1, \ldots, M$, the function

$$\left(\sum_{A \in \mathcal{A}} \frac{1}{\prod_{i \in A} F_i^*(t)} \right) \cdot \left(\prod_{i \in A_m} F_i^*(t) \right)' \cdot \sum_{i \in A_m^c} F_i^*(t) \tag{C4}$$

has a finite Lebesgue integral in a neighborhood of t_0.

The mathematical technique of this identification result is based on establishing that if distribution functions F_i satisfy conditions (C1), (C2) and (C4), then the system of integral-differential equations (IDE) together with the initial conditions

$$F_i^*(t_0) = 0, \quad i = 1, \ldots, n, \tag{IC}$$

has a unique positive solution in a right-hand side neighborhood of t_0.

Let $\mathcal{C}_i$ denote the collection of fatal sets containing i and let $\mathcal{C}_{-i}$ stand for the collection of fatal sets not containing i:

$$\mathcal{C}_i = \{A \in \mathcal{A} \,|\, i \in A\},$$
$$\mathcal{C}_{-i} = \{A \in \mathcal{A} \,|\, i \in A^c\}.$$

Remark 2.2. *Applying techniques similar to those for 2-out-of-n systems in Komarova (2012), it can be shown that conditions (C1) and (C2) imply the following conditions on observable functions:*

1. *G_m^* is absolutely continuous, $m = 1, \ldots, M$.*

2. *$G_m^*(t_0) = 0$ and $G_m^*(t) > 0$ for $t > t_0$, $m = 1, \ldots, M$.*

3. *For any $i = 1, \ldots, n$,*

$$\lim_{t \downarrow t_0} \prod_{A \in \mathcal{C}_i} G_A^*(t) \cdot \prod_{A \in \mathcal{C}_{-i}} G_A^*(t)^{-\frac{n-k}{k-1}} = 0. \tag{2.1}$$

The first two of these conditions are obvious. As for the third one, (IDE) implies that for any $i = 1, \ldots, n$,

$$\lim_{t \downarrow t_0} \frac{\prod_{A \in \mathcal{C}_i} G_A^*(t) \cdot \prod_{A \in \mathcal{C}_{-i}} G_A^*(t)^{-\frac{n-k}{k-1}}}{F_i^*(t)^{\binom{n-1}{n-k}}} = 1.$$

Using (C1) and (C2), we obtain (2.1).

Also, it can be shown that condition (C4) can be equivalently written in terms of observable functions:

For each $m = 1, \ldots, M$, the function

$$\left(\sum_{A \in \mathcal{A}} \frac{1}{G_A^*(t)} \right) \cdot g_m(t) \cdot \sum_{i \in A_m^c} \left(\prod_{A \in \mathcal{C}_i} G_A^*(t) \cdot \prod_{A \in \mathcal{C}_{-i}} G_A^*(t)^{-\frac{n-k}{k-1}} \right)^{\binom{n-1}{n-k}^{-1}}$$

has a finite Lebesgue integral in a neighborhood of t_0.

3 Sieve estimation

In this section I present an approach to estimating functions F_i^* from a random sample. First, I define an operator B that maps F_i^* to observable functions G_m^*. I show that this operator is Lipschitz and that under weak conditions on the set of possible distributions $F = (F_1, \ldots, F_n)$, its inverse operator B^{-1} is continuous. I then derive sieve estimators of F_i^* and use the properties of B to show their consistency.

3.1 Operator B

As before, T stands for the $(n-k+1)$-th order statistic of $\{t_1, \ldots, t_n\}$.

For an absolutely continuous function $F = (F_1, \ldots, F_n)^{tr}$ with the domain $[t_0, T]$ define the M-dimensional vector function

$$B(F) = (B(F)_1, B(F)_2, \ldots, B(F)_{M-1}, B(F)_M)^{tr}$$

as follows:

$$B(F)_m(t) = \int_{t_0}^{t} \left(\prod_{i \in A_m} F_i(s) \right)' \prod_{i \in A_m^c} (1 - F_i(s)) \, ds, \quad t \in [t_0, T] \tag{3.1}$$

for $m = 1, \ldots, M$.

Let Λ be the set of vector functions $F = (F_1, \ldots, F_n)^{tr}$ with the domain $[t_0, T]$ satisfying the following conditions:

Conditions (I).

(i) F_i *is absolutely continuous on* $[t_0, T]$, $i = 1, \dots, n$.

(ii) F_i *is increasing on* $[t_0, T]$,

(iii) $F_i(t_0) = 0$ *and* $F_i(t) > 0$ *for* $t \in (t_0, T]$, $i = 1, \dots, n$.

(iv) $F_i(T) \le 1$, $i = 1, \dots, n$.

(v) *For each* $m = 1, \dots, M$, *the function*

$$\left(\sum_{A \in \mathcal{A}} \frac{1}{\prod_{i \in A} F_i(t)}\right) \cdot \left(\prod_{i \in A_m} F_i(t)\right)' \cdot \sum_{i \in A_m^c} F_i(t)$$

has a finite Lebesgue integral in a neighborhood of t_0.

Let B be defined on Λ. Properties of the image $B(\Lambda)$ are easily deduced from conditions (I): each function $B(F)_m$ is absolutely continuous and increasing on $[t_0, T]$, $B(F)_m(t_0) = 0$ and $B(F)_m(t) > 0$ for $t \in (t_0, T]$. The identification result in Proposition 2.1 means that there exists the inverse operator $B^{-1} : B(\Lambda) \to \Lambda$.

Endow the domain Λ and its image $B(\Lambda)$ with the following uniform metric:

$$d(F, \tilde{F}) = \sup_{t \in [t_0, T]} \sqrt{(F(t) - \tilde{F}(t))^{tr}(F(t) - \tilde{F}(t))}$$
$$d(B(F), B(\tilde{F})) = \sup_{t \in [t_0, T]} \sqrt{(B(F)(t) - B(\tilde{F})(t))^{tr}(B(F)(t) - B(\tilde{F})(t))}.$$

Properties of B are important for proving the consistency of the estimators introduced later in this section. Usually, it is easier to obtain desirable properties of B and establish consistency when the space of unknown functions is compact. Let us compactify Λ by bounding densities functions F_i' by the same Lebesgue integrable function:

Condition (II).

$$F_i'(t) \le \phi'(t) \quad a.e. \quad [t_0, T], \quad i = 1, \dots, n,$$

where ϕ *is some absolutely continuous function on* $[t_0, T]$.

Let Λ_ϕ be the subset of Λ such that all functions F from Λ_ϕ satisfy condition *(II)*. This condition guarantees that Λ_ϕ is relatively compact under the uniform metric. Indeed, for any $F \in \Lambda_\phi$ and $t, \tau \in [t_0, T]$,

$$|F_i(t) - F_i(\tau)| = \left| \int_\tau^t F_i'(s) ds \right| \le |\phi(t) - \phi(\tau)|, \quad i = 1, \dots, n.$$

Because ϕ is absolutely continuous, the last inequality implies that Λ_ϕ is equicontinuous. It is also uniformly bounded because the values of F_i do not exceed 1. According to the Arzela-Ascoli theorem, Λ_ϕ is relatively compact in metric $d(\cdot, \cdot)$.

Note that if $F \in \Lambda_\phi$, then each function $B(F)_m$, satisfies the following condition:

$$B(F)'_m(t) \leq k\phi'(t) \quad a.e. \quad [t_0, T], \quad m = 1, \dots, M.$$

Let $\overline{\Lambda}_\phi$ stand for the closure of Λ_ϕ under metric $d(\cdot,\cdot)$. Because Λ_ϕ is relatively compact, $\overline{\Lambda}_\phi$ is a compact set. To consider operator B on $\overline{\Lambda}_\phi$, we first need to show that B is defined for functions in $\overline{\Lambda}_\phi \backslash \Lambda_\phi$. The proposition below establishes that all functions in $\overline{\Lambda}_\phi$ satisfy conditions *(I)(i), (I)(ii), (I)(iv), (I)(v)* and a slightly modified condition *(I)(iii)*, and also satisfy condition *(II)*.

Proposition 3.1. *If $F = (F_1, \dots, F_n)^{tr} \in \overline{\Lambda}_\phi$, then each F_i, $i = 1, \dots, n$, is absolutely continuous, increasing on $[t_0, T]$, satisfies $F_i(t_0) = 0$, $F_i(T) \leq 1$ and is such that $F'_i(t) \leq \phi'(t)$ a.e. on $[t_0, T]$.*

Functions that are in $\overline{\Lambda}_\phi$ but not in Λ_ϕ are, for instance, those that are equal to 0 in a small right-hand side neighborhood of t_0.

Because all functions in $\overline{\Lambda}_\phi$ are absolutely continuous, operator B can be extended from Λ_ϕ to $\overline{\Lambda}_\phi \backslash \Lambda_\phi$ by applying (3.1) to each $F \in \overline{\Lambda}_\phi \backslash \Lambda_\phi$.

The next proposition implies that B is continuous in metric $d(\cdot,\cdot)$ on $\overline{\Lambda}_\phi$.

Proposition 3.2. *For any $F, \tilde{F} \in \overline{\Lambda}_\phi$,*

$$d(B(F), B(\tilde{F})) \leq M\sqrt{n}\; d(F, \tilde{F}). \tag{3.2}$$

that is, operator B is Lipschitz on $\overline{\Lambda}_\phi$.

Finally, the continuity property of B are the compactness of $B(\Lambda_\phi)$ are used to establish the continuity of B^{-1} on $B(\Lambda_\phi)$.

Proposition 3.3. *B^{-1} is continuous on $B(\Lambda_\phi)$.*

3.2 Estimator

Let us now define sieve estimators of distribution functions F^*_i and prove their consistency.

Note that $G^* = B(F^*)$, where $F^* = (F^*_1, \dots, F^*_n)^{tr}$ and $G^* = (G^*_1, \dots, G^*_M)^{tr}$. Let us choose ϕ in such a way that $F^* \in \Lambda_\phi$.[3]

The next lemma introduces an objective function Q defined at each $F \in \overline{\Lambda}_\phi$ and uses the identification result from section 2 to show that it is uniquely minimized at $F = F^*$.

Lemma 3.4. *F^* is the unique minimizer of*

$$Q(F) = \int_{t_0}^{T} (G^*(t) - B(F)(t))^{tr} (G^*(t) - B(F)(t)) \, \frac{\sum_{m=1}^{M} {G^*}'_m(t)}{\sum_{m=1}^{M} G^*_m(T)} \, dt$$

[3]For instance, we can assume that $\phi'(t) \geq \sum_{i=1}^{n} {F^*_i}'(t)$.

on $\overline{\Lambda}_\phi$.

Note that $\frac{\sum_{m=1}^{M} G^{*\prime}_m(t)}{\sum_{m=1}^{M} G^*_m(T)}$ is the probability density function of the lifetime of the system on $[t_0, T]$.

The idea of sieve estimation here is to use the sample analog of Q and approximate $\overline{\Lambda}_\phi$ with finite-dimensional spaces. For instance, for each $r = 1, 2, \ldots$, choose base functions $p_{1,r}, \ldots, p_{\gamma(r),r}$ (for example, B-splines with uniform knots or basic Bernstein polynomials) and introduce the set of linear combinations of these functions:

$$\Gamma_r = \{(F_1, \ldots, F_n)^{tr} : F_i(t) = \sum_{l=1}^{\gamma(r)} \alpha_l^i p_{l,r}(t),\, t \in [t_0, T]\}.$$

In this set of functions, consider only those functions that are in Λ_ϕ:

$$\Sigma_r = \Lambda_\phi \cap \Gamma_r.$$

Set Σ_r consists of functions from Γ_r with certain restrictions on coefficients α_l^i. It is relatively compact and, hence, its closure $\overline{\Sigma}_r$ is compact, and $\overline{\Sigma}_r \subset \overline{\Lambda}_\phi$.

Consider a random sample of N observations $\{(z_j, a_j)\}_{j=1}^N$, where z_j is the observed lifetime of the system and a_j is the diagnostic set in j's round. Without a loss of generality, assume that $z_j \leq z_{j+1}$, $j = 1, \ldots, N-1$. From the sample, find consistent estimators $\widehat{G}_{m,N}$ of G_m, for instance, empirical sub-distribution functions on $[t_0, T]$:

$$\widehat{G}_{m,N}(t) = \frac{1}{N} \sum_{j=1}^{N} 1(z_j \leq t) 1(a_j = A_m), \quad m = 1, \ldots, M.$$

The sample objective function is

$$\widehat{Q}_N(F) = \frac{1}{N} \sum_{j=1}^{N} (\widehat{G}_N(z_j) - B(F)(z_j))^{tr} (\widehat{G}_N(z_j) - B(F)(z_j)),$$

where $\widehat{G}_N = (\widehat{G}_{1,N}, \ldots, \widehat{G}_{M,N})^{tr}$. Note that since the lifetime of the system cannot be strictly greater than T, then all z_j belong to $[t_0, T]$ and $\{(z_j)\}_{j=1}^N$ is a random sample from the distribution with density function $\frac{\sum_{m=1}^{M} G^{*\prime}_m(t)}{\sum_{m=1}^{M} G^*_m(T)}$.

Let $r = r(N)$, and define the following estimator of F^*:

$$\widehat{F}_N = \underset{F \in \overline{\Sigma}_{r(N)}}{\text{argmin}}\, \widehat{Q}_N(F).$$

The theorem below establishes the consistency of estimator $\widehat{F}_N$ when sets $\overline{\Sigma}_r$ well approximate set $\overline{\Lambda}_\phi$.

Theorem 3.5. *If*

$$\forall(F \in \overline{\Lambda}_\phi)\exists(\widetilde{F} \in \overline{\Sigma}_r) \quad d(F, \widetilde{F}) \xrightarrow{p} 0 \text{ as } r = r(N) \to \infty, \tag{3.3}$$

then estimator $\widehat{F}_N$ *is consistent:*

$$d(\widehat{F}_N, F^*) \xrightarrow{p} 0 \text{ as } N \to \infty.$$

Condition (3.3) holds if approximating sets are chosen properly – e.g., if base functions $p_{1,r}, \ldots, p_{\gamma(r),r}$ are B-splines with uniform knots, Bernstein polynomials or truncated power series.

4 Monte-Carlo experiment

This section illustrates the suggested sieve estimation method for 2-out-of-3 systems. Suppose that the lifetimes of all three components are distributed on the support $[0, 1]$. Consider two Monte-Carlo scenarios.

Scenario 1. For $t \in [0, 1]$,

$$F_1^*(t) = t^{\frac{1}{2}}, \quad F_2^*(t) = t^{\frac{2}{3}}, \quad F_3^*(t) = t^{\frac{3}{4}}.$$

All these functions are strictly concave and their derivatives approach ∞ as $t \downarrow 0$. In some sense these functions are not very different in their behavior around $t = 0$.

Scenario 2. For $t \in [0, 1]$,

$$F_1^*(t) = \begin{cases} t^{\frac{1}{2}} & \text{if } t \in \left[0, \frac{1}{2}\right], \\ 1 - (\sqrt{2} - 1)(1 - t)^{\frac{1}{2}} & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

$$F_2^*(t) = \frac{e^t - 1}{e - 1}, \quad F_3^*(t) = t.$$

The behavior of F_1^* on $[0, 1]$ is different from that of F_2^* or F_3^* around $t = 0$ and $t = 1$. The derivative of F_1^* approaches ∞ as $t \downarrow 0$ or $t \uparrow 1$. Functions F_2^* and F_3^* are infinitely differentiable on $[0, 1]$.

Bernstein polynomials. As sieve spaces, I consider spaces of Bernstein polynomials. Namely, I consider linear sieve spaces with the basic Bernstein polynomials as the base functions. The basic Bernstein polynomials of power r on $[0, 1]$ are the following $r + 1$ functions:

$$p_{l,r}(t) = \binom{r}{l} t^l (1 - t)^{r-l}, \quad l = 0, \ldots, r.$$

The corresponding sieve space is

$$\Gamma_r = \left\{ (F_1, F_2, F_3)^{tr} : \; F_i(t) = \sum_{l=0}^{r} \alpha_l^i \, p_{l,r}(t), \; t \in [0,1] \right\}.$$

An important property of Bernstein polynomials[4] says that for a continuous on $[0,1]$ function f, the relation

$$\lim_{r\to\infty} \sum_{l=0}^{r} f\left(\frac{l}{r}\right) \binom{r}{l} t^l (1-t)^{r-l} = f(t)$$

holds uniformly on $[0,1]$. This property implies that the constraints

$$\alpha_0^i \leq \alpha_1^i \leq \ldots \leq \alpha_{r-1}^i \leq \alpha_r^i$$

imposed for each $i = 1,2,3$ guarantee that functions in Γ_r are increasing. Conditions

$$\alpha_0^i = 0 \quad \text{and} \quad \alpha_r^i = 1$$

guarantee that $F_i(0) = 0$ and $F_i(1) = 1$, respectively.

Scenario 1. N = 500.

Table 1 is constructed based on the simulations of outcomes in 500 runs of the system. It shows how often each of the subsets $A_1 = \{2,3\}$ $A_2 = \{1,3\}$ and $A_3 = \{1,2\}$ happens to be the set responsible for the failure of system, or in other words, is the diagnostic set which is discovered during the autopsy. The table also shows the minimum, the maximum and the average lifetime of the system in each of these situations.

	diagnostic	$\min Z$	$\max Z$	Z_{av}
$A_1 = \{2,3\}$	125 (25%)	0.0146	0.8756	0.3867
$A_2 = \{1,3\}$	181 (36.6%)	0.0033	0.9438	0.3665
$A_3 = \{1,2\}$	194 (38.8%)	0.0114	0.9461	0.3553

Table 1. Monte Carlo experiment for Scenario 1 with $N = 500$ rounds. Number of rounds in which each A_m is the diagnostic set discovered during autopsy (*diagnostic*), the minimum lifetime ($\min Z$), the maximum lifetime ($\max Z$), the average lifetime (Z_{av}).

We can think about our 2-out-of-3 system as an observed open ascending auction with three bidders having independent private values. In this auction, bidders hold down a

[4] See Lorentz (1986).

button as the auctioneer raises the price. When the price gets too high for a bidder, she drops out by releasing the button. The auction ends when only one bidder remains. This person wins the object and pays the price at which the auction stopped. The distribution of the lifetime of component i corresponds to the distribution of bidder i's private value. Observing $A_1 = \{2, 3\}$ as a diagnostic set after the failure of the system is equivalent to the case of bidder 3 winning the auction. Analogously, observing $A_2 = \{1, 3\}$ as a diagnostic set is equivalent to the case of bidder 2 winning the auction, and observing $A_1 = \{2, 3\}$ as a diagnostic set is equivalent to the case of bidder 1 winning the auction. The observed lifetime of the system corresponds to the observed transaction price.

From Table 1, the highest observed transaction price in the simulated data is approximately 0.9461 (when bidder 3 wins the auction) and the lowest observed transaction price is 0.0033 (when bidder 2 wins the auction). As we see, bidder 3 wins the auction most often which stems from the fact that the distribution of private value of bidder 3 first-order stochastically dominates that of bidder 1 and that of bidder 2. Bidder 1 wins the auction least often because the distribution of private value of bidder 1 is first-order stochastically dominated by that of bidder 2 and that of bidder 3.

Sieve estimation uses Bernstein polynomials of order 4 with the constraints on the coefficients that guarantee the monotonicity of sieve estimators for F_1^*, F_2^* and F_3^*. These estimators are depicted in Figure 1.

Scenario 2. N = 500.

	diagnostic	$\min Z$	$\max Z$	Z_{av}
$A_1 = \{2, 3\}$	98 (19.6%)	0.0857	0.9635	0.5832
$A_2 = \{1, 3\}$	213 (42.6%)	0.0216	0.9848	0.4467
$A_3 = \{1, 2\}$	189 (37.8%)	0.0288	0.9187	0.4599

Table 2. Monte Carlo experiment for Scenario 2 with $N = 500$ rounds. Number of rounds in which each A_m is the diagnostic set discovered during autopsy (*diagnostic*), the minimum lifetime ($\min Z$), the maximum lifetime ($\max Z$), the average lifetime (Z_{av}).

The distribution of private value of bidder 2 first-order stochastically dominates that of bidder 2. Also, there is no first-order stochastic between the the distribution of private value of bidder 1 and those of bidders 2 and 3.

Figure 2 shows the results of sieve estimation of F_i^*, $i = 1, 2, 3$, by Bernstein polynomials of order 4 with monotonicity constraints. As can be seen, the sieve estimator for F_1^* provides a worse approximation of this function around $t = 0$ and $t = 1$ than on the rest of the support because F_1^* has the infinite derivative from the right at $t = 0$ and the infinite derivative from the left at $t = 1$.

5 Appendix

Proof of Proposition 3.1. Let us start by establishing absolute continuity. Because $F \in \overline{\Lambda_\phi}$, then there exists a sequence $F_q \in \Lambda_\phi$ such that $d(F_q, F) \to 0$ as $q \to \infty$. Take any two points $t_1, t_2 \in [t_0, T]$. Convergence in metric $d(\cdot, \cdot)$ implies point-wise convergence. Therefore, for any $i = 1, \dots, n$,

$$|F_i(t_1) - F_i(t_2)| = \lim_{q\to\infty} |F_{q,i}(t_1) - F_{q,i}(t_2)| \leq |\phi(t_1) - \phi(t_2)|.$$

The last inequality and the absolute continuity of ϕ imply that each F_i is absolutely continuous.

Because functions $F_{q,i}$ are increasing and converge to F_i point-wise, then F_i is increasing.

Because the values of $F_{q,i}(t_0)$ converge to $F_i(t_0)$, then $F_i(t_0) = 0$.

Because $F_{q,i}(T) \leq 1$ and $F_{q,i}$ converge to F_i point-wise, then $F_i(T) \leq 1$.

Because F_i is absolutely continuous, it can differentiated a.e. on $[t_0, T]$. Let t be a point at which both F_i and ϕ have derivatives. For any fixed h,

$$\frac{F_i(t+h) - F_i(t)}{h} = \lim_{q\to\infty} \frac{F_{q,i}(t+h) - F_{q,i}(t)}{h} \leq \frac{\phi(t+h) - \phi(t)}{h}.$$

Taking the limit as $h \to 0$, we obtain that $F_i'(t) \leq \phi'(t)$.

Proof of Proposition 3.2. Let F, $\tilde{F} \in \overline{\Lambda_\phi}$. For convenience, let us temporarily use the following metric:

$$d_1(F, \tilde{F}) = \sup_{t\in[t_0,T]} \sum_{i=1}^{n} |F_i(t) - \tilde{F}_i(t)|,$$
$$d_1(B(F), B(\tilde{F})) = \sup_{t\in[t_0,T]} \sum_{m=1}^{M} |B(F)_m(t) - B(\tilde{F})_m(t)|.$$

From the definition of B,

$$\begin{aligned} B(F)_m(t) - B(\tilde{F})_m(t) = &\int_{t_0}^{t} \left(\prod_{i\in A_m} F_i(s) - \prod_{i\in A_m} \tilde{F}_i(s) \right)' \prod_{i\in A_m^c} (1 - F_i(s))\, ds \\ &+ \int_{t_0}^{t} \left(\prod_{i\in A_m} \tilde{F}_i(s) \right)' \left(\prod_{i\in A_m^c} (1 - F_i(s)) - \prod_{i\in A_m^c} \left(1 - \tilde{F}_i(s)\right) \right) ds. \end{aligned}$$

Integration by parts gives that

$$\begin{aligned} \int_{t_0}^{t} \left(\prod_{i\in A_m} F_i(s) - \prod_{i\in A_m} \tilde{F}_i(s) \right)' \prod_{i\in A_m^c} (1 - F_i(s))\, ds = &\left(\prod_{i\in A_m} F_i(t) - \prod_{i\in A_m} \tilde{F}_i(t) \right) \prod_{i\in A_m^c} (1 - F_i(t)) \\ &+ \int_{t_0}^{t} \left(\prod_{i\in A_m} F_i(s) - \prod_{i\in A_m} \tilde{F}_i(s) \right) \left(- \prod_{i\in A_m^c} (1 - F_i(s)) \right)' ds, \end{aligned}$$

and thus,

$$\left|\int_{t_0}^{t}\left(\prod_{i\in A_m}F_i(s)-\prod_{i\in A_m}\tilde{F}_i(s)\right)'\prod_{i\in A_m^c}(1-F_i(s))\,ds\right| \le \sup_{t\in[t_0,T]}\left|\prod_{i\in A_m}F_i(t)-\prod_{i\in A_m}\tilde{F}_i(t)\right|$$
$$\le \sup_{t\in[t_0,T]}\sum_{i\in A_m}|F_i(t)-\tilde{F}_i(t)|.$$

Also note that

$$\left|\int_{t_0}^{t}\left(\prod_{i\in A_m}\tilde{F}_i(s)\right)'\left(\prod_{i\in A_m^c}(1-F_i(s))-\prod_{i\in A_m^c}\left(1-\tilde{F}_i(s)\right)\right)ds\right| \le \sup_{t\in[t_0,T]}\left|\prod_{i\in A_m^c}(1-F_i(t))-\prod_{i\in A_m^c}\left(1-\tilde{F}_i(t)\right)\right|$$
$$\le \sup_{t\in[t_0,T]}\sum_{i\in A_m^c}|F_i(t)-\tilde{F}_i(t)|.$$

To summarize,

$$\left|B(F)_m(t)-B(\tilde{F})_m(t)\right| \le \sup_{t\in[t_0,T]}\sum_{i=1}^{n}|F_i(t)-\tilde{F}_i(t)| = d_1(F,\tilde{F}),$$

which implies that

$$d_1(B(F),B(\tilde{F})) \le M\ d_1(F,\tilde{F}).$$

Because

$$d_1(F,\tilde{F}) \le \sqrt{n}\ d(F,\tilde{F}) \quad \text{and} \quad d_1(B(F),B(\tilde{F})) \ge d(B(F),B(\tilde{F})), \tag{5.1}$$

then

$$d(B(F),B(\tilde{F})) \le M\sqrt{n}\ d(F,\tilde{F}).$$

Proof of Proposition 3.3. Essentially, the statement of this proposition follows from the fact that if a continuous operator is defined on a compact set and the inverse operator is defined on the image of that set, then the inverse operator is continuous. This result cannot be applied here directly however because even though the inverse operator B^{-1} is clearly defined on $B(\Lambda_\phi)$ it is not defined on the larger set $B(\overline{\Lambda}_\phi)$.

Let $G_0 \in B(\Lambda_\phi)$ and $d(G_q,G_0) \to 0$ as $q\to\infty$ for $G_q \in B(\Lambda_\phi)$. Denote $F_0 = B^{-1}(G_0)$, $F_q = B^{-1}(G_q)$. Clearly, F_0, $F_q \in \Lambda_\phi$. I want to show that $d(F_q,F_0)\to 0$ as $q\to\infty$. Suppose this is not so and for some $\varepsilon>0$ there exists a subsequence F_{q_l} such that

$$d\left(F_{q_l},F_0\right) > \varepsilon \quad \text{for all } l=1,2,\ldots. \tag{5.2}$$

Notice that the subsequence F_{q_l} is equicontinuous because all functions in it are bounded and

$$|F_{q_l}(t_1)-F_{q_l}(t_2)| \le |\phi(t_1)-\phi(t_2)|$$

for any $t_1, t_2 \in [t_0, T]$. According to the Arzela-Ascoli theorem, there is a convergent subsequence $F_{q_{l_j}}$. Let $\widetilde{F}$ be the limit of $F_{q_{l_j}}$. Because $\widetilde{F} \in \overline{\Lambda}_\phi$ and B is continuous on $\overline{\Lambda}_\phi$, then

$$d(B(F_{q_{l_j}}), B(\widetilde{F})) \to 0.$$

Thus, $B(\widetilde{F}) = G_0$. Given that on $B(\Lambda_\phi)$ the inverse operator B^{-1} is defined, conclude that $\widetilde{F} = F_0$. Thus, we obtain that $d(F_{q_{l_j}}, F_0) \to 0$, contradicting (5.2). Therefore, $d(F_q, F_0) \to 0$.

Proof of Lemma 3.4. Note that $Q(F^*) = 0$. Because the inverse operator B^{-1} exists on $B(\Lambda_\phi)$, then $B(F) \neq G^*$ and, hence, $Q(F) > 0$ for any $F \in \Lambda_\phi$ such that $F \neq F^*$.

Now consider $F \in \overline{\Lambda}_\phi \backslash \Lambda_\phi$. Since $F \notin \Lambda_\phi$, then some F_i takes value 0 in a right-hand side neighborhood of t_0. without a loss of generality assume that $F_1(t) = 0$, $t \in [t_0, t_0 + \omega)$. Then for every $m = 1, \ldots, M$, such that $1 \in A_m^c$, we have $B(F)_m(t) = 0$, $t \in [t_0, t_0 + \omega)$. Because $G_m^*(t) > 0$ for $t > t_0$, $m = 1, \ldots, M$ (see Remark 2.2), then obviously $B(F) \neq G^*$.

Proof of Theorem 3.5. To prove this theorem, I use lemmas A1 and A2 from Newey and Powell (2003).[5] Consistency will hold if all conditions in Lemma A1 are satisfied. I divide these conditions into three groups, as in Newey and Powell (2003).

(i) According to Lemma 3.4, F^* is the unique minimizer of Q on $\overline{\Lambda}_\phi$.

(ii) Set $\overline{\Lambda}_\phi$ is compact. Let me show that Q and $\widehat{Q}_N$ are continuous on $\overline{\Lambda}_\phi$ and

$$\sup_{F \in \overline{\Lambda}_\phi} |\widehat{Q}_N(F) - Q(F)| \xrightarrow{p} 0. \tag{5.3}$$

The continuity of Q and $\widehat{Q}_N$ will follow from the properties of B on $\overline{\Lambda}_\phi$. First, consider Q. For any F, $\widetilde{F} \in \overline{\Lambda}_\phi$

$$\begin{aligned} |Q(F) - Q(\widetilde{F})| &= \left| \int_{t_0}^{T} \left[(G^* - B(F))^{tr}(G^* - B(F)) - (G^* - B(\widetilde{F}))^{tr}(G^* - B(\widetilde{F})) \right] \frac{\sum_{m=1}^M {G^*}'_m(t)}{\sum_{m=1}^M G_m^*(T)} \, dt \right| = \\ &= \left| \int_{t_0}^{T} \left[\sum_{m=1}^{M} (B(\tilde{F})_m - B(F)_m)(2G_m^* - B(F)_m - B(\widetilde{F})_m) \right] \frac{\sum_{m=1}^M {G^*}'_m(t)}{\sum_{m=1}^M G_m^*(T)} \, dt \right|. \end{aligned}$$

For any $t \in [t_0, T]$, $B(F)_m(t) \leq 1$ and $G_m^*(t) \leq 1$, $m = 1, \ldots, M$, therefore

$$|Q(F) - Q(\widetilde{F})| \leq 4 \int_{t_0}^{T} \left[\sum_{m=1}^{M} |B(\widetilde{F})_m - B(F)_m| \right] \frac{\sum_{m=1}^M {G^*}'_m(t)}{\sum_{m=1}^M G_m^*(T)} \, dt.$$

Applying the Cauchy-Schwartz inequality and (3.2),

$$\begin{aligned} |Q(F) - Q(\widetilde{F})| &\leq 4\sqrt{M} \int_{t_0}^{T} \left[\sqrt{(B(\widetilde{F}) - B(F))^{tr}(B(\widetilde{F}) - B(F))} \right] \frac{\sum_{m=1}^M {G^*}'_m(t)}{\sum_{m=1}^M G_m^*(T)} \, dt \\ &\leq 4\sqrt{M} \; d(B(F), B(\widetilde{F})) \leq 4M\sqrt{Mn} \; d(F, \widetilde{F}). \end{aligned}$$

[5] Some theorems from Chen (2007) can also be used to prove this result.

Thus, function Q is Lipschitz and therefore continuous.

Now consider function $\widehat{Q}_N$. Similar to the methods described above,

$$
\begin{aligned}
|\widehat{Q}_N(F) - \widehat{Q}_N(\widetilde{F})| &\leq \frac{1}{N}\sum_{j=1}^{N}\sum_{m=1}^{M} |(\widehat{G}_{N,m}(z_j) - B(F)_m(z_j))^2 - (\widehat{G}_{N,m}(z_j) - B(\widetilde{F})_m(z_j))^2| = \\
&= \frac{1}{N}\sum_{j=1}^{N}\sum_{m=1}^{M} |(B(\widetilde{F})_m(z_j) - B(F)_m(z_j))(2\widehat{G}_{N,m}(z_j) - B(\widetilde{F})_m(z_j) - B(F)_m(z_j))| \leq \\
&\leq \frac{4\sqrt{M}}{N}\sum_{j=1}^{N}\sqrt{(B(\widetilde{F})(z_j) - B(F)(z_j))^{tr}(B(\widetilde{F})(z_j) - B(F)(z_j))} \leq \\
&\leq 4\sqrt{M}\ d(B(F), B(\widetilde{F})) \leq 4M\sqrt{Mn}\ d(F, \widetilde{F}). \qquad (5.4)
\end{aligned}
$$

Property (5.3) will follow from Lemma A2 in Newey and Powell (2003). Indeed, it is clear that

$$
\forall (F \in \overline{\Lambda}_\phi) \quad \widehat{Q}_N(F) \xrightarrow{p} Q(F).
$$

This fact combined with (5.4) implies (5.3).

(iii) This condition follows from assumption (3.3).

Conditions (i)-(iii) imply the consistency property (3.5).

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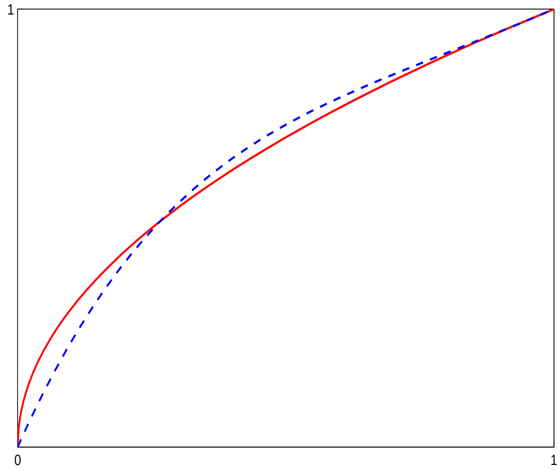
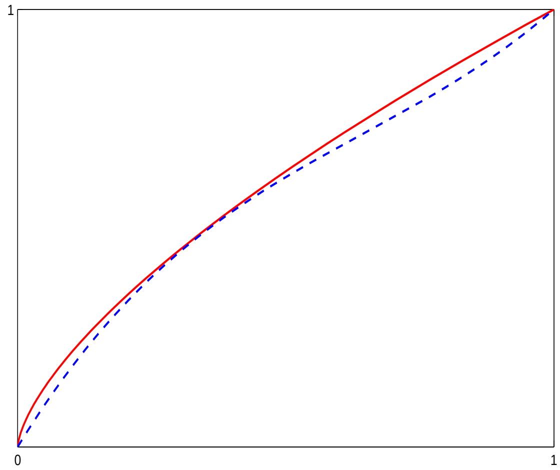
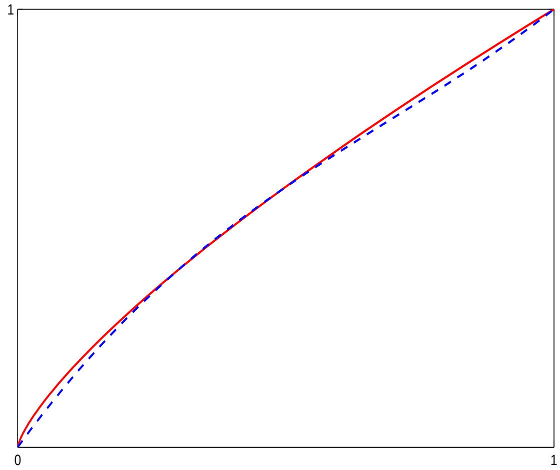


Scenario 1: F_1^* (solid line) and its sieve estimator (dashed line).

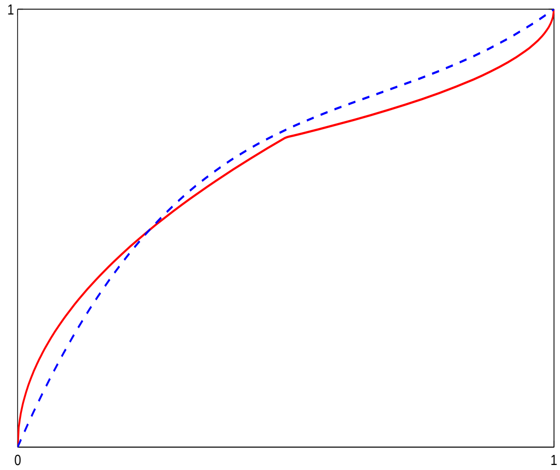


Scenario 1: F_2^* (solid line) and its sieve estimator (dashed line).

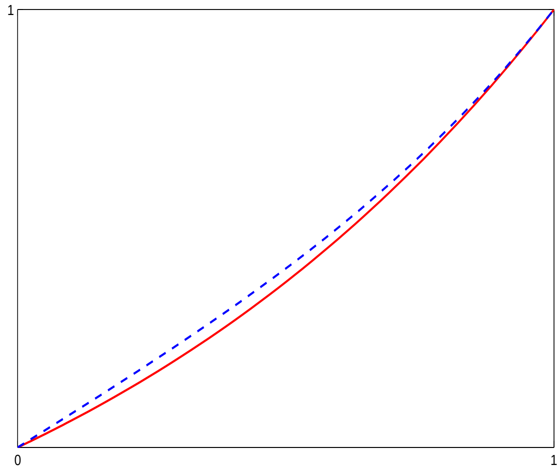


Scenario 1: F_3^* (solid line) and its sieve estimator (dashed line).

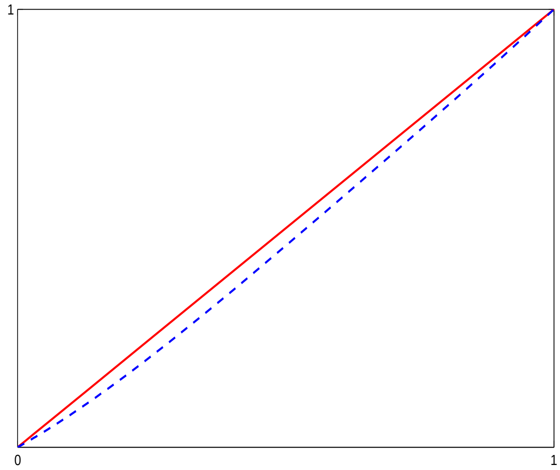
Figure 1. Scenario 1: CDFs F_i^*, $i = 1, 2, 3$, and their sieve estimators by Bernstein polynomials.



Scenario 2: F_1^* (solid line) and its sieve estimator (dashed line).



Scenario 2: F_2^* (solid line) and its sieve estimator (dashed line).



Scenario 2: F_3^* (solid line) and its sieve estimator (dashed line).

Figure 2. Scenario 2: CDFs F_i^*, $i = 1, 2, 3$, and their sieve estimators by Bernstein polynomials.