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# A polynomial projection-type algorithm for linear programming

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## Abstract

We propose a simple  $O([n^5/\log n]L)$  algorithm for linear programming feasibility, that can be considered as a polynomial-time implementation of the relaxation method. Our work draws from Chubanov's "Divide-and-Conquer" algorithm [5], where the recursion is replaced by a simple and more efficient iterative method. A similar approach was used in a more recent paper of Chubanov [7].

*Keywords:* Linear Programming, polynomial-time algorithms, relaxation method.

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## 1. Introduction

In the *linear programming feasibility problem* we are given a matrix  $A \in \mathbb{Z}^{m \times n}$  and a vector  $b \in \mathbb{Z}^m$ , and we wish to compute a feasible solution to the system

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned} \tag{1}$$

or show that none exists. The first practical algorithm for linear programming was the *simplex method*, introduced by Dantzig in 1947 [8]; while efficient in practice, for most known pivoting rules the method has an exponential-time worst case complexity. The first polynomial-time algorithm, the *ellipsoid method*, was introduced by Khachiyan [12], followed a few years later by

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Karamarkar's first *interior point method* [11]. Between the 40s and the 70s several other algorithms with exponential worst case complexity were developed, such as von Neumann's method (see [9]) and the *relaxation method* by Agmon [2] and Motzkin and Shoenberg [14]. Betke [4] and Chubanov [5, 6] introduced polynomial-time rescaled versions of the relaxation method. A substantially simpler and improved algorithm was recently given by Chubanov [7]. Computational experiments of Chubanov's original algorithm, as well as a different treatment, were carried out by Basu, De Loera and Junod [3].

Here we present a polynomial time algorithm based on [5]. The engine behind our algorithm is the Bubble algorithm subroutine, which can be considered as an unfolding of the recursion in the Divide-and-Conquer algorithm described in [5]. Our algorithm is also related to the one in [7]; in particular, our Bubble algorithm is an analogue of the Basic algorithm in [7]. However, while our Bubble algorithm is a variant of the relaxation method (more precisely, its special case, the *projection method*), Chubanov's Basic algorithm is precisely von Neumann's algorithm (see Dantzig [9]).

The two algorithms proceed in a somewhat different manner. Chubanov's algorithm decides whether  $Ax = 0$  has a strictly positive solution, and reduces problems of the form (1) via an homogenization, whereas we work directly with the form (1). Also, the key updating step of the bounds on the feasible region after an iteration of the basic subroutine and the supporting argument substantially differs from ours. In particular, whereas [7] divides only one of the upper bounds on the variables by exactly two, our algorithm uses simultaneous updates of multiple components. Another difference is that instead of repeatedly changing the original system by a rescaling, we keep the same problem setting during the entire algorithm and modify a certain norm instead. This enables a clean understanding of the progress made by the algorithm.

If we denote by  $L$  the encoding size of the matrix  $(A, b)$ , our algorithm performs  $O([n^5/\log n]L)$  arithmetic operations. Chubanov's algorithm [7] has a better running time bound of  $O(n^4L)$ ; however, note that our algorithm is still a considerable improvement over  $O(n^{18+3\epsilon}L^{12+2\epsilon})$  in the previous version [6]. We get a better bound  $O([n/\log n]L)$  on the number of executions of the basic subroutine, as compared to  $O(nL)$  in [7]; on the other hand, [7] can use an argument bounding the overall number of elementary iterations of all executions of the basic subroutine, thus achieving a better running time estimation.

### 1.1. The LP algorithm

We highlight our polynomial-time algorithm to find a feasible solution of (1). We denote by  $P$  the feasible region of (1). Throughout the paper we will assume that  $A$  has full-row rank.

Let  $d^1, \dots, d^m$  be the  $m$  columns of  $(A, b)$  with largest Euclidean norm, and let  $\Delta = \|d^1\| \cdots \|d^m\|$ . It can be easily shown that  $\Delta < 2^L$  (see for example [10, Lemma 1.3.3]). By Hadamard's bound, for every square submatrix  $B$  of  $(A, b)$ ,  $|\det(B)| \leq \Delta$ . It follows that, for every basic feasible solution  $\bar{x}$  of (1), there exists  $q \in \mathbb{Z}$ ,  $1 \leq q \leq \Delta$ , such that, for  $j = 1, \dots, n$ ,  $\bar{x}_j = p_j/q$  for some integer  $p_j$ ,  $0 \leq p_j \leq \Delta$ . In particular,  $\bar{x}_j \leq \Delta$  for  $j = 1, \dots, n$ , and  $\bar{x}_j \geq \Delta^{-1}$  whenever  $\bar{x}_j > 0$ .

The algorithm maintains a vector  $u \in \mathbb{R}^n$ ,  $u > 0$ , such that every basic feasible solution of (1) is contained in the hypercube  $\{x : 0 \leq x \leq u\}$ . At the beginning, we set  $u_i := \Delta$ ,  $i = 1, \dots, n$ .

At every iteration, either the algorithm stops with a point in  $P$ , or it determines a vector  $u' \in \mathbb{R}^n$ ,  $0 < u' \leq u$ , such that every basic feasible solution of (1) satisfies  $x \leq u'$  and such that, for some index  $p \in \{1, \dots, n\}$ ,  $u'_p \leq u_p/2$ . For  $j = 1, \dots, n$ , if  $u'_j \leq \Delta^{-1}$  we reduce the number of variables by setting  $x_j := 0$ , and removing the  $j$ th column of the matrix  $A$ ; otherwise, we update  $u_j := u'_j$ .

The entire algorithm terminates either during an iteration when a feasible solution is found, or once the system  $Ax = b$  has a unique solution or is infeasible. If the unique solution is nonnegative, then it gives a point in  $P$ , otherwise the problem is infeasible.

Since at every iteration there exists some variable  $x_p$  such that  $u_p$  is at least halved, and since  $\Delta^{-1} \leq u_j \leq \Delta$  for every variable  $x_j$  that has not been set to 0, it follows that the algorithm terminates after at most  $n \log(\Delta^2) \in O(nL)$  iterations. The crux of the algorithm is the following theorem and the subsequent claim.

**Theorem 1.** *There exists a strongly polynomial time algorithm which, given  $A \in \mathbb{Z}^{m \times n}$ ,  $b \in \mathbb{Z}^m$ , and  $u \in \mathbb{R}^n$ ,  $u > 0$ , in  $O(n^4)$  arithmetic operations returns one of the following:*

1. *A feasible solution to (1);*
2. *A vector  $(v, w) \in \mathbb{R}^m \times \mathbb{R}_+^n$ ,  $w \neq 0$ , such that  $(v^\top A + w^\top)\bar{x} < v^\top b + \frac{1}{2n}w^\top u$  for every  $\bar{x} \in \{x \in \mathbb{R}^n : 0 \leq x \leq u\}$ .*

The algorithm referred to in Theorem 1 will be called the *Bubble algorithm*, described in Section 2.

**Claim 2.** *Let  $u \in \mathbb{R}^n$ ,  $u > 0$ , such that every basic feasible solution for (1) satisfies  $x \leq u$ , and let  $(v, w) \in \mathbb{R}^m \times \mathbb{R}_+^n$  be a vector as in point 2 of Theorem 1. For  $j = 1, \dots, n$ , let  $u'_j := \min \left\{ u_j, \frac{\sum_{i=1}^n u_i w_i}{2n w_j} \right\}$ . Then every basic feasible solution of (1) satisfies  $x \leq u'$ . Furthermore, if we let  $p := \arg \max_{j=1, \dots, n} \{u_j w_j\}$ , then  $u'_p \leq u_p/2$ .*

*Proof.* Since  $w \neq 0$ , up to re-scaling  $(v, w)$  we may assume that  $\sum_{i=1}^n u_i w_i = 2n$ , therefore  $u'_j = \min \{u_j, w_j^{-1}\}$ ,  $j = 1, \dots, n$ . Since every basic feasible solution  $\bar{x}$  for (1) satisfies  $(v^\top A + w^\top) \bar{x} < v^\top b + \frac{1}{2n} w^\top u$ , it follows that, for  $j = 1, \dots, n$ ,

$$0 > v^\top (A\bar{x} - b) + \sum_{i=1}^n w_i \left( \bar{x}_i - \frac{u_i}{2n} \right) = \sum_{i=1}^n w_i \bar{x}_i - 1 \geq w_j \bar{x}_j - 1.$$

It follows that  $\bar{x}_j < w_j^{-1}$ , thus  $\bar{x}_j \leq u'_j$ . Finally, by our choice of  $p$ ,  $u_p w_p \geq 2$ , therefore  $w_p^{-1} \leq u_p/2$ .  $\square$

Theorem 1 and Claim 2 imply that our algorithm runs in time  $O(n^5 L)$ . In Section 3 we will refine our analysis and show that the number of calls to the Bubble algorithm is actually  $O([n/\log n]L)$ . This gives an overall running time of  $O([n^5/\log n]L)$ .

### 1.2. Scalar products

We recall a few facts about scalar products that will be needed in the remainder. Given a symmetric positive definite matrix  $D$ , we denote by  $\langle x, y \rangle_D = x^\top D y$ . We let  $\|\cdot\|_D$  the norm defined by  $\|x\|_D = \sqrt{\langle x, x \rangle_D}$ , and refer to it as the *D-norm*. The *D-distance* between two points  $x$  and  $y$  is  $\|x - y\|_D$ . Given a point  $c \in \mathbb{R}^n$  and  $r > 0$ , we define  $B_D(c, r) := \{x : \|x - c\|_D \leq r\}$ , and refer to it as the *D-ball of radius  $r$  centered at  $c$* .

Given any system  $Cx = d$  of inequalities, we denote  $\langle Cx = d \rangle := \{x \in \mathbb{R}^n : Cx = d\}$ . We recall that, given a point  $\bar{x}$ , assuming w.l.o.g. that  $C$  has full row rank, the point  $y$  in  $\langle Cx = d \rangle$  at minimum  $D$ -distance from  $\bar{x}$  is given by the formula

$$y = \bar{x} + D^{-1} C^\top (C D^{-1} C^\top)^{-1} (d - C \bar{x}), \quad (2)$$

and thus the  $D$ -distance between  $\bar{x}$  and  $\langle Cx = d \rangle$  is

$$\|y - \bar{x}\|_D = \sqrt{(d - C\bar{x})^\top (CD^{-1}C^\top)^{-1} (d - C\bar{x})}. \quad (3)$$

**Remark 3.** If  $y$  is the point in  $\langle Cx = d \rangle$  at minimum  $D$ -distance from  $\bar{x}$ , then the vector  $\lambda := (CD^{-1}C^\top)^{-1}(d - C\bar{x})$  is the unique solution to  $y - \bar{x} = D^{-1}C^\top\lambda$  and  $\|y - \bar{x}\|_D^2 = (d - C\bar{x})^\top\lambda$ .

In particular, given  $\alpha \in \mathbb{R}^n \setminus \{0\}$  and  $\beta \in \mathbb{R}$ , and denoting by  $y$  the point in  $\langle \alpha^\top x = \beta \rangle$  at minimum  $D$ -distance from  $\bar{x}$ , we have

$$y = \bar{x} + \frac{D^{-1}\alpha(\beta - \alpha^\top\bar{x})}{\alpha^\top D^{-1}\alpha}, \quad \|y - \bar{x}\|_D = \frac{|\beta - \alpha^\top\bar{x}|}{\sqrt{\alpha^\top D^{-1}\alpha}}. \quad (4)$$

We recall the following fact.

**Lemma 4.** Let  $K \subseteq \mathbb{R}^n$  be a convex set, let  $\bar{x} \in \mathbb{R}^n \setminus K$ , and let  $z$  be the point in  $K$  at minimum  $D$ -distance from  $\bar{x}$ . Then  $\langle z - \bar{x}, x \rangle_D \geq \|z - \bar{x}\|_D^2$  for all  $x \in K$ , and  $\langle z - \bar{x}, x \rangle_D \leq \|z - \bar{x}\|_D^2$  for all  $x \in B_D(\bar{x}, \|z - \bar{x}\|_D)$ .

Throughout the paper we denote by  $e^i$  the  $i$ th unit vector, where the dimension of the space will be clear from the context.

## 2. The Bubble algorithm

Given  $u \in \mathbb{R}^n$  such that  $u > 0$ , define the vector  $\ell := \frac{u}{2n}$  and let  $\tilde{P} := \{x : Ax = b, x \geq \ell\}$ . Denote by  $D = (d_{ij})$  the  $n \times n$  diagonal matrix whose  $i$ th diagonal element is  $d_{ii} = 4u_i^{-2}$ . Since  $u > 0$ ,  $D$  is positive definite. Furthermore,  $\{x \in \mathbb{R}^n : 0 \leq x \leq u\} \subset B_D(0, 2\sqrt{n})$ .

Throughout the rest of the paper we denote  $\mathcal{A} := \langle Ax = b \rangle$ . The Bubble algorithm is based on the following lemma; the proof is illustrated in Figure 1.

**Lemma 5.** Let  $z \in \mathcal{A}$  such that  $\langle z, x \rangle_D \geq \|z\|_D^2$  is valid for  $\tilde{P}$ . If  $z \notin P$ , let  $i \in \{1, \dots, n\}$  such that  $z_i < 0$ , and let  $K := \{x \in \mathcal{A} : \langle z, x \rangle_D \geq \|z\|_D^2, x_i \geq \ell_i\}$ . Assume that  $K \neq \emptyset$ , and let  $z'$  be the point in  $K$  at minimum  $D$ -distance from the origin. Then  $\langle z', x \rangle_D \geq \|z'\|_D^2$  is valid for  $\tilde{P}$  and  $\|z'\|_D^2 > \|z\|_D^2 + \frac{1}{n^2}$ .

*Proof.* Since  $K$  is a polyhedron containing  $\tilde{P}$ , it follows from Lemma 4 that  $\langle z', x \rangle_D \geq \|z'\|_D^2$  is valid for  $\tilde{P}$ . Applying (4) with  $\alpha = e^i$ ,  $\beta = \ell_i$ , and  $\bar{x} = z$ , we obtain that the  $D$ -distance between  $z$  and  $\langle x_i = \ell_i \rangle$  is equal to

$$\frac{|\ell_i - z_i|}{\sqrt{(e^i)^\top D^{-1} e^i}} > 2 \frac{\ell_i}{u_i} = \frac{1}{n}.$$

It follows that every point in  $B_D(z, \frac{1}{n})$  violates the inequality  $x_i \geq \ell_i$ . Next, we show that, for every  $x \in B_D(0, \sqrt{\|z\|_D^2 + \frac{1}{n^2}})$ , if  $x$  satisfies  $\langle z, x \rangle_D \geq \|z\|_D^2$  then  $x \in B_D(z, \frac{1}{n})$ . Indeed, any such point  $x$  satisfies  $\|z - x\|_D^2 = \|z\|_D^2 + \|x\|_D^2 - 2\langle z, x \rangle_D \leq 2\|z\|_D^2 + \frac{1}{n^2} - 2\|z\|_D^2 = \frac{1}{n^2}$ . Since every point in  $B_D(z, \frac{1}{n})$  violates  $x_i \geq \ell_i$ , it follows that  $B_D(0, \sqrt{\|z\|_D^2 + \frac{1}{n^2}})$  is disjoint from  $K$  thus, by definition of  $z'$ ,  $\|z'\|_D^2 > \|z\|_D^2 + \frac{1}{n^2}$ .  $\square$

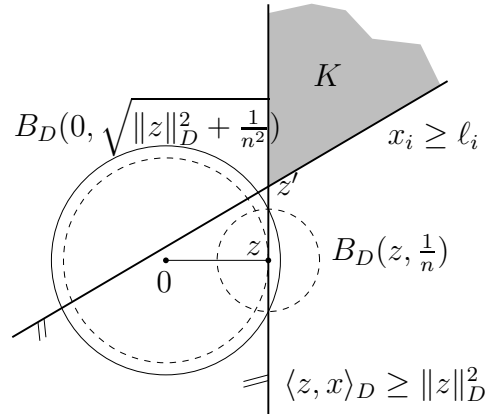


Figure 1: Bubbles merge: the  $D$ -ball  $B_D(0, \sqrt{\|z\|_D^2 + \frac{1}{n^2}})$  is disjoint from  $K$ .

Note that, if  $\langle z, x \rangle_D \geq \|z\|_D^2$  is a valid inequality for  $\tilde{P}$ , then there exists  $(v, w) \in \mathbb{R}^m \times \mathbb{R}_+^n$  such that  $Dz = A^\top v + w$  and  $\|z\|_D^2 = v^\top b + w^\top \ell$ , because  $\langle z, x \rangle_D = (Dz)^\top x$ . Also, whenever  $K$  in the statement of Lemma 5 is empty, there exists a vector  $(v, w) \in \mathbb{R}^m \times \mathbb{R}_+^n$  such that  $A^\top v + w = 0$  and  $v^\top b + w^\top \ell > 0$ . We will detail in Section 2.1 how these vectors  $(v, w)$  can be computed. Lemma 5 and the above considerations suggest the following algorithm.

### Bubble algorithm

**Input:** A system  $Ax = b$ ,  $x \geq 0$ , and a vector  $u > 0$ .

**Output:** Either:

1. A point in  $P$ , or;
2.  $(v, w) \in \mathbb{R}^m \times \mathbb{R}_+^n$  such that  
 $(v^\top A + w^\top)x < v^\top b + w^\top \ell \quad \forall x \in B_D(0, 2\sqrt{n})$ .

Initialize  $z$  as the point in  $\mathcal{A}$  at minimum  $D$ -distance from 0. Set  $\ell := \frac{u}{2n}$ .

**While**  $\|z\|_D \leq 2\sqrt{n}$ , do

**If**  $z \in P$ , **STOP**;

**Else**, Choose  $i$  such that  $z_i < 0$ ;

Let  $K := \{x \in \mathcal{A} : \langle z, x \rangle_D \geq \|z\|_D^2, x_i \geq \ell_i\}$ .

**If**  $K = \emptyset$ , **then** output  $(v, w) \in \mathbb{R}^m \times \mathbb{R}_+^n$   
such that  $A^\top v + w = 0$  and  $v^\top b + v^\top \ell > 0$ ;

**Else** Reset  $z$  to be the point in  $K$   
at minimum  $D$ -distance from 0,

**Endwhile**;

Output  $(v, w) \in \mathbb{R}^m \times \mathbb{R}_+^n$  such that  
 $Dz = A^\top v + w$  and  $\|z\|_D^2 = v^\top b + w^\top \ell$ .

By Lemma 5, the value of  $\|z\|_D^2$  increases by at least  $\frac{1}{n^2}$  at every iteration. Therefore, after at most  $4n^3$  iterations,  $\|z\|_D > 2\sqrt{n}$ . In particular, if the algorithm terminates outside the “while” cycle, then the inequality  $\langle z, x \rangle_D \geq \|z\|_D^2$  is valid for  $\tilde{P}$  and it is violated by every point in  $B_D(0, 2\sqrt{n})$ , and therefore by every point in  $\{x : 0 \leq x \leq u\}$ . Note that this is the second outcome of Theorem 1.

To show that the Bubble algorithm is strongly polynomial, we need to address two issues. The first is how to compute, at any iteration, the new point  $z$  and a vector  $(v, w) \in \mathbb{R}^m \times \mathbb{R}_+^n$  such that  $Dz = A^\top v + w$  and  $\|z\|_D^2 = v^\top b + w^\top \ell$ , and how to compute a Farkas certificate of infeasibility for  $\tilde{P}$  if the Bubble algorithm stops with  $K = \emptyset$ . In Section 2.1, we show how this step can be implemented in  $O(n)$  time, and therefore Bubble algorithm performs  $O(n^4)$  arithmetic operations. The second issue is that, in the above algorithm, the encoding size of the vector  $z$  computed could grow exponentially. In Section 2.2 we show how this can be avoided by performing an appropriate rounding at every iteration.

Note that our Bubble algorithm terminates with either a point in  $P$ , or



with a separating hyperplane. The latter arises in two cases: either if  $K = \emptyset$  in a certain iteration, or at the end of the while cycle. The Divide-and-Conquer algorithm in Chubanov [5] and Basu et al. [3] has three possible outcomes, including a “failure” scenario, corresponding to  $K = \emptyset$ . Their reason for handling this scenario separately is due to an initial homogenization step; we do not have to make such a distinction as we do not perform such an homogenization.

### 2.1. Computing the closest point in $K$

Instead of maintaining  $(v, w) \in \mathbb{R}^m \times \mathbb{R}_+^n$  with  $Dz = A^\top v + w$  and  $\|z\|_D^2 = v^\top b + w^\top \ell$ , it is more convenient to work with a different representation inside the affine subspace  $\mathcal{A}$ , as detailed below.

Let us denote by  $r^0$  the point in  $\mathcal{A}$  at minimum  $D$ -distance from the origin. By (2) and Remark 3, using Gaussian elimination we can compute in strongly polynomial time  $\bar{v}^0 \in \mathbb{R}^m$  such that  $Dr^0 = A^\top \bar{v}^0$  and  $\|r^0\|_D^2 = b^\top \bar{v}^0$ .

**Remark 6.** *Observe that, for every  $x \in \mathcal{A}$ ,  $\langle x - r^0, r^0 \rangle_D = 0$ , thus  $\|x\|_D^2 = \|x - r^0\|_D^2 + \|r^0\|_D^2$ . It follows that, for any convex set  $C \subseteq \mathcal{A}$ , the point in  $C$  at minimum  $D$ -distance from the origin is the point in  $C$  at minimum  $D$ -distance from  $r^0$ .*

For  $j = 1, \dots, n$ , we may assume that  $\emptyset \neq \{x \in \mathcal{A} : x_j \geq \ell_j\} \subsetneq \mathcal{A}$ . Under this assumption, there exists  $\alpha^j \in \mathcal{A} - r^0$  and  $\beta_j \in \mathbb{R}$  such that  $\|\alpha^j\|_D = 1$  and  $\{x \in \mathcal{A} : x_j \geq \ell_j\} = \{x \in \mathcal{A} : \langle \alpha^j, x \rangle_D \geq \beta_j\}$ . Note that  $(\alpha^j, \beta_j)$  can be computed using Gaussian elimination, along with  $\bar{v}^j \in \mathbb{R}^m$  and  $\bar{w}_j \in \mathbb{R}_+$  such that  $D\alpha^j = A^\top \bar{v}^j + \bar{w}_j e^j$ ,  $\beta_j = b^\top \bar{v}^j + \bar{w}_j \ell_j$ . Observe that, if we denote by  $r^j$  the point in  $\{x \in \mathcal{A} : x_j = \ell_j\}$  at minimum  $D$ -distance from the origin, then  $|\beta_j| = \|r^j - r^0\|_D$  and  $r^j = r^0 + \beta_j \alpha^j$ .

We may assume that  $r^0 \notin P$ , otherwise the algorithm terminates immediately. It follows that at the first iteration  $z = r^t$  for some  $t$  such that  $\beta_t > 0$ . We can assume that at the first iteration, we choose  $t = \arg \max_j \beta_j$ . In particular,  $\|z\|_D \geq \|r^j\|_D$  for all  $j$  such that  $\beta_j > 0$ . By Lemma 5 and Remark 6,  $\|z - r^0\|_D \geq \frac{1}{n}$ .

At any subsequent iteration, we are given a point  $z$  in  $\mathcal{A} \setminus P$  such that  $\langle z, x \rangle_D \geq \|z\|_D^2$  is valid for  $\tilde{P}$ . Let  $\alpha = (z - r^0) / \|z - r^0\|_D$  and  $\beta = \|z - r^0\|_D$ . It follows that  $\{x \in \mathcal{A} : \langle z, x \rangle_D \geq \|z\|_D^2\} = \{x \in \mathcal{A} : \langle \alpha, x \rangle_D \geq \beta\}$ . We will

maintain a vector  $\lambda \in \mathbb{R}_+^n$  such that

$$(\alpha, \beta) = \sum_{j=1}^n \lambda_j (\alpha^j, \beta_j). \quad (5)$$

These provide the vectors  $(v, w)$  with  $Dz = A^\top v + w$ ,  $\|z\|_D^2 = b^\top v + \ell^\top w$ , and  $w \geq 0$ , by defining  $v := \bar{v}^0 + \beta \sum_{j=1}^n \lambda_j \bar{v}^j$  and  $w := \beta \sum_{j=1}^n \lambda_j \bar{w}_j e^j$ .

In every iteration, our algorithm terminates if  $z \geq 0$ , or it picks an index with  $z_i < 0$  and defines  $K := \{x \in \mathcal{A} : \langle z, x \rangle_D \geq \|z\|_D^2, x_i \geq \ell_i\}$ .

If  $K = \emptyset$  then the algorithm terminates; otherwise, the current  $z$  is replaced by the point  $z'$  in  $K$  at minimum  $D$ -distance from the origin.

In the rest of this section, we describe how to compute  $z'$  if  $K \neq \emptyset$ , or a Farkas certificate of infeasibility if  $K = \emptyset$ . Let

$$\bar{K} := \{x \in \mathbb{R}^n : \langle \alpha, x \rangle_D \geq \beta, \langle \alpha^i, x \rangle_D \geq \beta_i\}.$$

It follows that  $K = \mathcal{A} \cap \bar{K}$ .

**Claim 7.** *If  $\bar{K} \neq \emptyset$ , then the point in  $\bar{K}$  at minimum  $D$ -distance from  $r^0$  is equal to  $z'$ . In particular  $K \neq \emptyset$  if and only if  $\bar{K} \neq \emptyset$ .*

*Proof.* Let  $\bar{z}$  be the point in  $\bar{K}$  at minimum  $D$ -distance from  $r^0$ . Since  $\langle \bar{z} - r^0, x \rangle_D \geq \|\bar{z} - r^0\|_D^2$  is valid for  $\bar{K}$ , it follows that  $D(\bar{z} - r^0) = \mu_1 D\alpha^i + \mu_2 D\alpha$  for some  $\mu_1, \mu_2 \geq 0$ . From this, we get  $\bar{z} = r^0 + \mu_1 \alpha^i + \mu_2 \alpha$ , which implies  $A\bar{z} = Ar^0 + \mu_1 A\alpha^i + \mu_2 A\alpha = b$ . This shows that  $\bar{z} \in \mathcal{A}$ , and thus  $\bar{z} \in K$ . Since  $K \subseteq \bar{K}$ , it follows that  $\bar{z}$  is the point in  $K$  at minimum  $D$ -distance from  $r^0$ , and thus from the origin, i.e.  $\bar{z} = z'$ . In particular, if  $\bar{K} \neq \emptyset$  then also  $K \neq \emptyset$ . Conversely, if  $K \neq \emptyset$  then  $\bar{K} \neq \emptyset$  because  $K \subseteq \bar{K}$ .  $\square$

**Claim 8.**  *$K \neq \emptyset$  if and only if  $\alpha^i$  and  $\alpha$  are linearly independent. If  $K \neq \emptyset$ , then  $z'$  is the point in*

$$\mathcal{L} := \{\langle \alpha, x \rangle_D = \beta, \langle \alpha^i, x \rangle_D = \beta_i\}$$

*at minimum  $D$ -distance from  $r^0$ .*

*Proof.* Assume that  $K \neq \emptyset$ . By Claim 7,  $z'$  is the point in  $\bar{K}$  at minimum  $D$ -distance from  $r^0$ . We will show that  $z'$  is the point in  $\mathcal{L}$  at minimum  $D$ -distance from  $r^0$ . It suffices to show that  $z'$  satisfies  $\langle \alpha, z' \rangle_D = \beta$  and

$\langle \alpha^i, z' \rangle_D = \beta_i$ . If  $\langle \alpha^i, z' \rangle_D > \beta_i$ , then  $z'$  is the point in  $\{x : \langle \alpha, x \rangle_D \geq \beta\}$  at minimum  $D$ -distance from  $r^0$ , and thus  $z' = z$ , contradicting the fact that  $\|z'\|_D > \|z\|_D$ . If  $\langle \alpha, z' \rangle_D = \beta$ , then  $z'$  is the point in  $\{x : \langle \alpha^i, x \rangle_D \geq \beta_i\}$  at minimum  $D$ -distance from  $r^0$ . If  $\beta_i > 0$  then  $z' = r^i$ , contradicting the fact that  $\|z'\|_D > \|z\|_D \geq \max_j \beta_j$ , whereas if  $\beta_i \leq 0$  then  $z' = r^0$ , contradicting the fact that  $r^0 \notin \bar{K}$ .

For the first part of the statement, by Claim 7,  $K \neq \emptyset$  if and only if  $\bar{K} \neq \emptyset$ . Clearly  $\bar{K} \neq \emptyset$  if  $\alpha^i$  and  $\alpha$  are linearly independent. Conversely, assume  $\bar{K} \neq \emptyset$ . If  $\alpha^i$  and  $\alpha$  are linearly dependent, then, because  $z'$  is the point in  $\mathcal{L}$  at minimum  $D$ -distance from  $r^0$ , it follows that  $\mathcal{L} = \{\langle \alpha^i, x \rangle_D = \beta^i\} = \{\langle \alpha, x \rangle_D = \beta\}$  and that  $z' = z$ , a contradiction.  $\square$

Case  $K \neq \emptyset$ . By Claim 8  $z'$  is the closest point in  $\mathcal{L}$  from  $r^0$ , and  $\alpha^i$  and  $\alpha$  are linearly independent. According to Remark 3, we have that  $z' - r^0 = \mu^1 \alpha + \mu^2 \alpha^i$ , where  $(\mu_1, \mu_2)^\top = (CD^{-1}C^\top)^{-1}(d - Cr^0)$ ,  $C$  being the  $2 \times n$  matrix whose rows are  $(D\alpha^i)^\top$  and  $D\alpha^\top$ , and where  $d \in \mathbb{R}^2$  is defined by  $d_1 = \beta_i$ ,  $d_2 = \beta$ . A simple computation gives that

$$\mu_1 = \frac{\beta^i - \beta \langle \alpha^i, \alpha \rangle_D}{1 - \langle \alpha^i, \alpha \rangle_D^2} \quad \mu_2 = \frac{\beta - \beta^i \langle \alpha^i, \alpha \rangle_D}{1 - \langle \alpha^i, \alpha \rangle_D^2}. \quad (6)$$

We also claim that  $\mu_1, \mu_2 \geq 0$ . Indeed,  $\langle z' - r^0, x \rangle_D \geq \|z' - r^0\|_D^2$  is a valid linear inequality for  $\bar{K}$ , and  $\mu_1$  and  $\mu_2$  are the unique coefficients satisfying  $D(z' - r^0) = \mu_1 \alpha^i + \mu_2 \alpha$  and  $\|z' - r^0\|_D^2 = \mu_1 \beta^i + \mu_2 \beta$ . Defining

$$\beta' = \|z' - r^0\|_D, \quad \alpha' = (z' - r^0)/\beta', \quad \lambda' := (\mu_1 e^i + \mu_2 \lambda)/\beta', \quad (7)$$

we have that  $\lambda' \geq 0$  and  $(\alpha', \beta') = \sum_{j=1}^n \lambda'_j (\alpha^j, \beta_j)$ . Therefore,  $z'$  and  $\lambda'$  can be computed by performing  $O(n)$  arithmetic operations at every iteration of the Bubble algorithm.

**Remark 9.** Since  $z' \in \{\langle \alpha^i, x \rangle_D = \beta_i\}$ , it follows that  $\|z'\|_D \geq \|r^i\|_D$ . Therefore, at every iteration of the algorithm,  $|\beta_j| \leq \beta$  whenever  $\lambda_j > 0$ .

Case  $K = \emptyset$ . By Claim 8,  $\bar{K} = \emptyset$  and the vectors  $\alpha_i$  and  $\alpha$  are linearly dependent. This implies that, for some  $\nu > 0$ ,  $\alpha^i = -\nu \alpha$  and  $\beta^i > -\nu \beta$ . Defining  $\lambda' := e^i + \nu \lambda$ , we obtain that  $\sum_{j=1}^n \lambda'_j \alpha^j = 0$  and  $\sum_{j=1}^n \lambda'_j \beta_j > 0$ .

A Farkas certificate of infeasibility  $(v', w')$  can be obtained by setting  $v' := \bar{v}^0 + \sum_{j=1}^n \lambda'_j \bar{v}^j$  and  $w' := \sum_{j=1}^n \lambda'_j \bar{w}_j e^j$ . We thus have  $A^\top v' + w' = 0$ ,  $w' \geq 0$  and  $b^\top v' + \ell^\top w' > 0$ , showing infeasibility of  $\tilde{P}$ .

## 2.2. Bounding the encoding sizes

Note that the encoding size of the vector  $z$  within the Bubble algorithm could grow exponentially, and also the size of the upper bound vector  $u$  maintained by the LP algorithm.

To maintain the size of  $u$  polynomially bounded, we will perform a rounding at every iteration as follows. Instead of the new bounds  $u'_j$  as in Claim 2, let us define  $\tilde{u}_j$  as the smallest integer multiple of  $1/(3n\Delta)$  with  $u'_j \leq \tilde{u}_j$ . We proceed to the next iteration of the Bubble algorithm with the input vector  $\tilde{u}$ . Clearly the encoding size of  $\tilde{u}$  is polynomially bounded in  $n$  and  $L$ , and we shall show in the next section that this rounding does not affect the asymptotic running time bound of  $O([n^5/\log n]L)$ .

In the rest of this section we show how a rounding step can be introduced in the Bubble algorithm in order to guarantee that the sizes of the numbers remain polynomially bounded. The rounding will be performed on the coefficients  $\lambda_j$  in (5).

In every iteration, after the new values of  $z$  and the  $\lambda_j$ 's are obtained, we replace them by  $\tilde{z}$  and  $\tilde{\lambda}_j$  satisfying (5), such that these values have polynomial encoding size. At the same time, we show that  $\|z\|_D^2 - \|\tilde{z}\|_D^2 \leq \frac{1}{2n^2}$  (Claim 10); since at every iteration of the Bubble algorithm the value of  $\|z\|_D^2$  increases by at least  $\frac{1}{n^2}$ , the number of iterations in Bubble algorithm may increase by at most a factor of 2, to  $8n^3$ . Let

$$q := \lceil 16n^3 \rceil.$$

For every number  $a \in \mathbb{R}$ , we denote by  $[a]_q$  the number of the form  $p/q$ ,  $p \in \mathbb{Z}$ , with  $|p/q - a|$  minimal. Given the current point  $z$  and  $\lambda \in \mathbb{R}_+^n$  satisfying (5) let

$$(\gamma, \delta) := \sum_{j=1}^n [\lambda_j]_q (\alpha^j, \beta^j).$$

It follows that  $\langle \gamma, x \rangle_D \geq \delta$  is a valid inequality for  $\tilde{P}$ . Let us define  $\tilde{z}$  as the closest point in  $\langle \langle \gamma, x \rangle_D = \delta \rangle$  to  $r^0$ . This can be obtained by

$$\tilde{\alpha} := \gamma / \|\gamma\|_D, \quad \tilde{\beta} = \delta / \|\gamma\|_D, \quad \tilde{z} := r_0 + \tilde{\alpha}\tilde{\beta}, \quad \tilde{\lambda}_j := [\lambda_j]_q / \|\gamma\|_D$$

Note that  $(\tilde{\alpha}, \tilde{\beta}) = \sum_{j=1}^n \tilde{\lambda}_j (\alpha^j, \beta^j)$  and  $\|\tilde{z} - r^0\|_D = |\tilde{\beta}|$  hold. The next claim will show that  $\tilde{\beta} > 0$ .

**Claim 10.**  $\|z\|_D^2 - \|\tilde{z}\|_D^2 \leq \frac{1}{2n^2}$  and  $\tilde{\beta} > 0$ .

*Proof.* We first show that  $\|\alpha - \gamma\|_D \leq \frac{n}{2q}$  and  $|\beta - \delta| \leq \frac{n\beta}{2q}$ . Indeed,  $\|\alpha - \gamma\|_D^2 = \sum_{j,h=1}^n (\lambda_j - [\lambda_j]_q)(\lambda_h - [\lambda_h]_q) \langle \alpha^j, \alpha^h \rangle_D \leq \frac{n^2}{4q^2}$ , because  $\langle \alpha^j, \alpha^h \rangle_D \leq 1$  for  $j, h = 1, \dots, n$ . Also,  $|\beta - \delta| = |\sum_{j=1}^n (\lambda_j - [\lambda_j]_q) \beta^j| \leq \frac{n\beta}{2q}$ , because by Remark 9  $|\beta^j| \leq \beta$  whenever  $\lambda_j > 0$ . Note that  $\delta \geq \beta \left(1 - \frac{n}{2q}\right) > 0$ , thus  $\tilde{\beta} > 0$ , proving the second claim.

We assume that  $\|z\|_D \geq \|\tilde{z}\|_D$ , otherwise the first claim is trivial. Note that  $\|z\|_D - \|\tilde{z}\|_D = \beta - \frac{\delta}{\|\gamma\|_D} \leq |\beta - \delta| + \frac{\delta}{\|\gamma\|_D} \|\gamma\|_D - 1 \leq \frac{\beta n}{q}$ , where the last inequality follows from  $|\beta - \delta| \leq \frac{n\beta}{2q}$ ,  $\beta \leq \beta$  and from  $\|\gamma\|_D - 1 = \|\gamma\|_D - \|\alpha\|_D \leq \|\alpha - \gamma\|_D \leq \frac{n}{2q}$ . Finally  $\|z\|_D^2 - \|\tilde{z}\|_D^2 = (\|z\|_D - \|\tilde{z}\|_D)(\|z\|_D + \|\tilde{z}\|_D) \leq \frac{2\beta\|z\|_D n}{q} \leq \frac{8n^2}{q} \leq \frac{1}{2n^2}$ . The second inequality follows since  $\beta \leq \|z\|_D \leq 2\sqrt{n}$  by the termination criterion of the Bubble algorithm.  $\square$

We claim that the encoding size of the  $\tilde{\lambda}_j$ 's remains polynomially bounded. Note that, if at every iteration we guarantee that  $\lambda_j$  ( $j = 1, \dots, n$ ) is bounded from above by a number of polynomial size, then the encoding sizes of  $[\lambda_j]_q$  is polynomial as well, and therefore so is the encoding size of  $\tilde{\lambda}$ .

Let  $z$  and  $\lambda$  satisfy (5), and let  $z'$  denote the next point, with  $\lambda'$  defined by (7); note that  $z'$  and  $\lambda'$  also satisfy (5).

**Claim 11.** *If  $\|z'\|_D \leq 2\sqrt{n}$ , then  $\lambda'_j \leq 8n^3(\lambda_j + 1)$  for  $j = 1, \dots, n$ .*

*Proof.* Let  $\mu_1, \mu_2 \geq 0$  be defined as in (6); recall that these satisfy  $z' - r^0 = \mu_1 \alpha^i + \mu_2 \alpha$  and  $\|z' - r^0\|_D^2 = \mu_1 \beta_i + \mu_2 \beta$ . We first show that  $\mu_1, \mu_2 \leq 8n^3 \beta$ . It follows from (6) that

$$\begin{aligned} \|z' - r^0\|_D^2 &= \frac{\beta_i^2 + \beta^2 - 2\beta\beta_i \langle \alpha, \alpha^i \rangle_D}{1 - \langle \alpha^i, \alpha \rangle_D^2} = \\ &= \frac{\|r^i - r^0\|_D^2 + \|z - r^0\|_D^2 - 2\langle r^i - r^0, z - r^0 \rangle_D}{1 - \langle \alpha^i, \alpha \rangle_D^2} = \frac{\|z - r^i\|_D^2}{1 - \langle \alpha^i, \alpha \rangle_D^2}. \end{aligned}$$

In the second equality we use  $r^i - r^0 = \alpha^i \beta^i$ ,  $z - r^0 = \alpha \beta$ , and  $\|\alpha^i\|_D = \|\alpha\|_D = 1$ . Since  $z$  has distance at least  $1/n$  from the hyperplane  $\langle \langle r^i, x \rangle_D = \|r^i\|_D^2 \rangle$ , it follows that  $\|z - r^i\|_D \geq 1/n$ . Since  $\|z' - r^0\|_D \leq 2\sqrt{n}$ , we obtain  $1 - \langle \alpha^i, \alpha \rangle_D^2 \geq 1/4n^3$ . Further, by Remark 9 we have  $|\beta_i| \leq \beta$ . These together with (6) and  $|\langle \alpha^i, \alpha \rangle_D| \leq 1$ , imply  $\mu_1, \mu_2 \leq 8n^3 \beta$ .

Since (7) defines  $\lambda' = (\mu_1 e^i + \mu_2 \lambda) / \|z' - r^0\|$ , using that  $\|z' - r^0\| > \beta$ , it follows that  $\lambda'_j \leq 8n^3(\lambda_j + 1)$  for  $j = 1, \dots, n$ .  $\square$

Since at the first iteration  $\lambda_j \leq 1$ ,  $j = 1, \dots, n$ , it follows from the above claim that after  $k$  iterations of the Bubble algorithm, we have  $\lambda_j \leq k(8n^3)^k$ . As argued above, the rounded Bubble algorithm terminates in at most  $8n^3$  iterations, therefore  $\lambda_j \in O(n^3(8n^3)^{8n^3})$  in all iterations. Consequently, the  $\lambda_j$ 's encoding sizes are polynomially bounded.

Since every iteration of the Bubble algorithm can be carried out in  $O(n)$  arithmetic operations and the encoding size of the numbers remains polynomially bound it follows that the Bubble algorithm is strongly polynomial, with running time  $O(n^4)$ .

### 3. Improving the running time by a $1/\log(n)$ factor

In this section, we show that the total number of calls of the Bubble algorithm can be bounded by  $O\left(\frac{n}{\log n}L\right)$ . This will be achieved through an amortized runtime analysis by means of a potential. For simplicity, we first present the analysis for the version where the updated vector  $u$  is not rounded, and then explain the necessary modifications when rounding is used.

A main event in the LP algorithm is when, for some coordinate  $j$ , we obtain  $u_j < \Delta^{-1}$  and therefore we may conclude  $x_j = 0$ . This reduces the number of variables by one. The algorithm terminates once the system  $Ax = b$  has a unique solution or is infeasible; assume this happens after eliminating  $f \leq n - 1$  variables. For simplicity of notation, let us assume that the variables are set to zero in the order  $x_n, x_{n-1}, \dots, x_{n-f+1}$ , breaking ties arbitrarily. For  $k = 2, \dots, f$ , the  $k$ -th phase of the algorithm starts with the iteration after the one when  $x_{n+2-k}$  is set to zero and terminates when  $x_{n+1-k}$  is set to zero; the first phase consists of all iterations until  $x_n$  is set to 0. A phase can be empty if multiple variables are set to 0 simultaneously. In the  $k$ -th phase there are  $n - k + 1$  variables in the problem. We analyze the potential

$$\Psi := \sum_{j=1}^n \log \max \left\{ u_j, \frac{1}{\Delta} \right\}.$$

Note that the initial value of  $\Psi$  is  $n \log \Delta$ , and it is monotone decreasing during the entire algorithm. Let  $p_k$  denote the decrease in the potential value in phase  $k$ . Since  $u_j$  decreases from  $\Delta$  to at most  $\frac{1}{\Delta}$  for every  $j = n + 1 - k, \dots, n$  in the first  $k$  phases, we have that the value of  $\Psi$  at the end

of the  $k$ -th phase is at most  $(n - 2k) \log \Delta$ , or equivalently,

$$\sum_{i=1}^k p_i \geq 2k \log \Delta. \quad (8)$$

**Claim 12.** *In the  $k$ -th phase of the algorithm,  $\Psi$  decreases by at least  $\log(n - k + 2)$  in every iteration, with the possible exception of the last one.*

*Proof.* Note that in the  $k$ -th phase the values  $u_n, u_{n-1}, \dots, u_{n-k+2}$  do not change anymore. Recall from Claim 2 that, for  $j = 1, \dots, n - k + 1$ , the new value of  $u_j$  is set as  $u'_j := \min\{u_j, w_j^{-1}\}$ , where  $(v, w)$  is a vector as in Theorem 1 normalized to  $\sum_{i=1}^{n-k+1} u_i w_i = 2(n - k + 1)$ , since  $n - k + 1$  is the number of variables not yet fixed to 0. In particular,  $u_j/u'_j = \max\{1, u_j w_j\}$  for  $j = 1, \dots, n - k + 1$ . The new value of the potential is  $\Psi' = \sum_{j=1}^n \log \min\{u'_j, \frac{1}{\Delta}\}$ . In every iteration of the  $k$ -th phase except for the last one we must have  $u_j \geq u'_j > \frac{1}{\Delta}$  for every  $j = 1, \dots, n - k + 1$ , hence

$$\Psi - \Psi' = \sum_{j=1}^{n-k+1} \log \frac{u_j}{u'_j} = \log \prod_{j=1}^{n-k+1} \max\{1, u_j w_j\} \geq \log(n - k + 2). \quad (9)$$

The last inequality follows from the fact that, for any positive integer  $t$ , we have

$$\min \left\{ \prod_{j=1}^t \max\{1, \alpha_j\} : \sum_{j=1}^t \alpha_j = 2t, \alpha \in \mathbb{R}_+^t \right\} = t + 1,$$

the minimum being achieved when  $\alpha_j = t + 1$  for a single value of  $j$  and  $\alpha_j = 1$  for all other values.  $\square$

Let  $r_k$  denote the number of iterations in phase  $k$ . The claim implies the following upper bound:

$$r_k \leq \frac{p_k}{\log(n - k + 2)} + 1 \quad (10)$$

Together with (8), it follows that the total number of iterations  $\sum_{k=1}^f r_k$  is bounded by the optimum of the following LP

$$\begin{aligned} \max \quad & (n - 1) + \sum_{i=1}^n \frac{1}{\log(n-i+2)} p_i \\ \sum_{i=1}^k p_i & \geq 2k \log \Delta \quad k = 1, \dots, n - 1 \\ \sum_{i=1}^n p_i & \leq 2n \log \Delta \\ p & \in \mathbb{R}_+^n \end{aligned} \quad (11)$$

It is straightforward that the optimum solution is  $p_i = 2 \log \Delta$  for  $i = 1, \dots, n$ . One concludes that the number of iterations is at most

$$(n-1) + 2 \log \Delta \sum_{i=1}^n \frac{1}{\log(n-i+2)} \leq (n-1) + 2 \log \Delta \int_1^{n+1} \frac{dt}{\log t},$$

where the inequality holds because the function  $\frac{1}{\log x}$  is decreasing. The function  $\text{li}(x) := \int_0^x \frac{dt}{\ln t}$  (defined for  $x > 1$ ) is the *logarithmic integral* [1], and it is known that  $\text{li}(x) = O\left(\frac{x}{\log x}\right)$ . This gives the bound  $O\left(\frac{n}{\log n} L\right)$  on the total number of iterations, using that  $\Delta \leq 2^L$ .

Let us now turn to the version of the algorithm where  $u'_j$  is rounded up to  $\tilde{u}_j$ , an integer multiple of  $1/(3n\Delta)$ . In all but the last iteration of the  $k$ 'th phase  $u'_j \geq 1/\Delta$  holds, and therefore  $\frac{u_j}{\tilde{u}_j} \geq \frac{u_j}{u'_j} \cdot \frac{1}{1+1/(3n)}$ . Hence, from (9), in the  $k$ 'th phase we have  $\Psi - \Psi' \geq \log(n-k+2) - (n-k+1) \log \frac{3n+1}{3n} > \frac{1}{2} \log(n-k+2)$ . This ensures at least half of the drop in potential guaranteed in Claim 12, giving the same asymptotic running time bound.

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