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THE REVERSAL RATIO OF A POSET

GRAHAM BRIGHTWELL AND MITCHEL T. KELLER

ABSTRACT. Felsner and Reuter introduced the linear extension diameter of a partially ordered set $\mathbf{P}$, denoted led($\mathbf{P}$), as the maximum distance between two linear extensions of $\mathbf{P}$, where distance is defined to be the number of incomparable pairs appearing in opposite orders (reversed) in the linear extensions. In this paper, we introduce the reversal ratio $RR(\mathbf{P})$ of $\mathbf{P}$ as the ratio of the linear extension diameter to the number of (unordered) incomparable pairs. We use probabilistic techniques to provide a family of posets $\mathbf{P}_k$ on at most $k \log k$ elements for which the reversal ratio $RR(\mathbf{P}_k) \leq C/\log k$, where C is an absolute constant. We also examine the questions of bounding the reversal ratio in terms of order dimension and width.

1. INTRODUCTION

Let $\mathbf{P} = (X, P)$ be a finite poset that is not a chain, and let L_1 and L_2 be two linear extensions of $\mathbf{P}$. The *distance between L_1 and L_2*, denoted dist(L_1, L_2), is defined to be the number of pairs $(x, y) \in X \times X$ such that $x <_{L_1} y$ and $y <_{L_2} x$. Such an incomparable pair is said to be *reversed* by the linear extensions L_1 and L_2. In [6], Felsner and Reuter introduced the *linear extension diameter* of $\mathbf{P}$ as

$$\mathrm{led}(\mathbf{P}) = \max_{L_1, L_2} \ \mathrm{dist}(L_1, L_2),$$

where the maximum is taken over all pairs of linear extensions of $\mathbf{P}$. The linear extension diameter can also be viewed as the diameter of the *linear extension graph* of $\mathbf{P}$: this graph has a vertex corresponding to each linear extension of $\mathbf{P}$, with two vertices adjacent if the corresponding linear extensions differ only in the transposition of a single adjacent pair of elements.

If we let $\mathrm{inc}(\mathbf{P}) = |\{(x, y) \in X \times X : x \|_P y\}|/2$, the number of *unordered* incomparable pairs of $\mathbf{P}$, then it immediately follows that led($\mathbf{P}$) $\leq$ inc($\mathbf{P}$), with equality if and only if $\mathbf{P}$ has dimension 2. We define the *reversal ratio* of a pair (L_1, L_2) of linear extensions of $\mathbf{P}$ to be $RR(\mathbf{P}; L_1, L_2) = \mathrm{dist}(L_1, L_2)/\mathrm{inc}(\mathbf{P})$. The *reversal ratio of the poset* $\mathbf{P}$ is then $RR(\mathbf{P}) = \mathrm{led}(\mathbf{P})/\mathrm{inc}(\mathbf{P}) = \max_{L_1, L_2} RR(\mathbf{P}; L_1, L_2)$.

A question of interest about linear extension diameter is to determine if there exists a constant $c > 0$ such that led($\mathbf{P}$) $\geq c\,\mathrm{inc}(\mathbf{P})$, or equivalently if $RR(\mathbf{P}) \geq c$ for all $\mathbf{P}$. Let us briefly motivate this by examining cases where there is such a constant. First, if $\mathbf{P}$ has low dimension, then taking some pair of linear extensions from a small realizer reverses a large proportion of the incomparable pairs. Second, if the width of $\mathbf{P}$ is a positive proportion of the number of elements, then some pair of linear extensions inverting the large antichain automatically has a large reversal ratio. Thus, a poset $\mathbf{P}$ with small reversal ratio must have many incomparable pairs, low width, and unbounded dimension. One possible candidate with these properties is

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the d-dimensional Boolean lattice $\mathbf{Q}_d$, but Felsner and Reuter [6] found a pair of linear extensions of $\mathbf{Q}_d$ reversing approximately half of the incomparable pairs of $\mathbf{Q}_d$. Felsner and Massow [5] later showed that this pair is optimal, establishing that $\mathrm{led}(\mathbf{Q}_d) = 2^{2d-2} - (d+1)2^{d-1}$. More generally, in [5], Felsner and Massow give a polynomial-time algorithm for computing the linear extension diameter of downset lattices of 2-dimensional posets; an analysis of their proof shows that $RR(\mathbf{P}) \geq 1/2$ for all such posets. By contrast, Brightwell and Massow [4] showed that it is NP-complete to compute the linear extension diameter of an arbitrary poset.

However, both [5] and [6] reference personal communication from Brightwell that there is no such constant. Brightwell's idea was to take a random graph order $P_{n,p(n)}$ (see [1, 2, 3]) for an appropriately chosen $p(n)$, which is known to have the properties indicated above, and show that there is a function $r(n) = o(1)$ such that $RR(P_{n,p(n)}) \leq r(n)$ w.h.p. (i.e., with probability tending to 1 as the size of the poset tends to infinity).

It turns out to be awkward to analyze exactly how small the reversal ratio is for random graph orders. In this paper, we examine this problem further by considering a different model of random posets $\mathbf{P}_k$ and showing that $RR(\mathbf{P}_k) \leq 27/\log k$. (The poset $\mathbf{P}_k$ has width k and about $0.434k \log k$ elements.) We also consider the problems of bounding the linear extension diameter of a poset of fixed dimension and fixed width.

Readers unfamiliar with terminology and notation should consult Trotter's monograph [9].

2. POSETS WITH ARBITRARILY SMALL REVERSAL RATIO

In this section, we show there is no constant c such that $\mathrm{led}(\mathbf{P}) \geq c \, \mathrm{inc}(\mathbf{P})$ for all posets $\mathbf{P}$. To accomplish this, we define a family of posets whose linear extension diameter tends to zero.

Let k be a multiple of 3. We say that a bipartite graph with vertex classes A and B, each of size k, has the *doubling property* if (i) for every subset X of A of size at most $k/3$, the set $\Gamma(X)$ of neighbors of X has size greater than $2|X|$, and (ii) for every subset Y of B of size at most $k/3$, $|\Gamma(Y)| \geq 2|Y|$.

The doubling property is an instance of a *vertex expansion property*. A graph G is said to be an (α, ρ) *vertex-expander* if, for every subset X of $V(G)$ of size at most $\alpha|V(G)|$, we have $|\Gamma(X)| \geq \rho|X|$. (Here the only vertices of X included in $\Gamma(X)$ are those that have a neighbor in X.) If G is a $(1/3, 2)$ vertex-expander on k vertices, take two copies A, B of $V(G)$, and form a bipartite graph $H(G)$ with vertex classes A and B by joining a and b if and only if the corresponding vertices in G are adjacent. Thus the maximum degree of $H(G)$ is the same as in G, and $H(G)$ has the doubling property.

There are explicit constructions known of $(1/3, 2)$ vertex expanders: for instance, *Ramanujan graphs* ([8]) with degree at least 15 are $(1/3, 2)$ vertex expanders. To keep our proof self-contained, we shall give a short argument showing that a certain class of random bipartite graphs with maximum degree at most 10 has the doubling property w.h.p.

Fix an integer k which is a multiple of 3, and a real number $\varepsilon > 0$. Given a bipartite graph $G(k)$ on two vertex classes A and B, each of size k, with maximum degree at most 10, having the doubling property, we now describe how to construct our posets. Let $A^{k,\varepsilon}$ be the union of $m = \lfloor \varepsilon \log k \rfloor$ disjoint sets $A_1, \dots, A_m$, each

of size k. We form a poset $\mathbf{P}_{k,\varepsilon} = (A^{k,\varepsilon}, P^{k,\varepsilon})$ with vertex set $A^{k,\varepsilon}$ as follows. Let G_i be a bipartite graph isomorphic to $G(k)$ with vertex classes A_i and A_{i+1}. The comparabilities between A_i and A_{i+1} are defined so that $x_i \in A_i$ is less than $x_{i+1} \in A_{i+1}$ if and only if $x_i x_{i+1}$ is an edge of G_i. The partial order $P^{k,\varepsilon}$ is then formed by taking the transitive closure of the union of these sets of comparabilities. To be specific, $x_0 \in A_i$ is below $x_j \in A_{i+j}$ if and only if there are vertices $x_\ell \in A_{i+\ell}$ ($\ell = 1, \ldots, j-1$), such that each $x_\ell x_{\ell+1}$ ($\ell = 0, \ldots, j-1$) is an edge of the copy G_i of the graph $G(k)$.

A very similar construction was used by Fox [7] in a different context: he required an example of an n-element poset such that no element is comparable with more than n^ε others, while for each two sets A and B of order $Cn/\log n$, some element of A is comparable with some element of B.

Before showing that a poset $\mathbf{P}_{k,\varepsilon}$ as above has small reversal ratio, we show the existence of the bipartite graphs $G(k)$. We describe the *configuration model* for random bipartite graphs. Given positive integers r and k (in our application, r will be equal to 10, and k will be sufficiently large and a a multiple of 3), we take two disjoint vertex sets A and B, each of size k. We now form two kr-element sets $\tilde{A}$ and $\tilde{B}$: for each vertex a of A, we place r elements $a_1, \ldots, a_r$ into $\tilde{A}$, and likewise for $\tilde{B}$. Next, we take a uniform random matching of the elements of $\tilde{A}$ and $\tilde{B}$: so each element a_i of $\tilde{A}$ is paired with exactly one element of $\tilde{B}$. Finally, we form our bipartite graph on $A \cup B$ by putting an edge between $a \in A$ and $b \in B$ whenever some a_i is paired with some b_j. We denote a random graph generated in this manner $G_r(A, B)$. The random graph $G_r(A, B)$ is bipartite, with maximum degree at most r. There may be instances of pairs (a, b) where two different a_i are paired with some $b_{j(i)}$, resulting in these vertices having degree less than r. However, if r is fixed and k is large, then there are w.h.p. few such instances.

The configuration model is the basis of the standard approach for analyzing random r-regular graphs. To get an r-regular graph, one has to avoid the problem of multiple edges, for instance by rejection sampling, but for our purposes there is no difficulty with using the random graph $G_r(A, B)$ directly.

We will work with expressions containing r to begin with for greater clarity. We then substitute $r = 10$ to obtain useful numerical estimates.

Lemma 1. *Let $G_r(A, B)$ be a random bipartite graph on vertex sets A and B of size k, chosen according to the configuration model. For k sufficiently large and $r = 10$, the probability that $G_r(A, B)$ does not have the doubling property is at most $2/k^5$.*

Proof. Take any natural number $x \leq k/3$, any subset X of A of size x, and any subset Y of B of size $2x$. Now the number of pairings in the configuration model such that all neighbors of X lie in Y is equal to

$$\binom{2xr}{xr}(xr)!(kr - xr)! = \frac{(2xr)!}{(xr)!}((k-x)r)!,$$

and so the probability that all neighbors of X lie in Y is equal to

$$\frac{\frac{(2xr)!}{(xr)!}((k-x)r)!}{(kr)!} \leq 2\frac{(2xr/e)^{2xr}}{(xr/e)^{xr}}\frac{((k-x)r/e)^{(k-x)r}}{(kr/e)^{kr}} = 2\left[\frac{(2x)^{2x}(k-x)^{k-x}}{x^x k^k}\right]^r.$$

To prove the lemma, we find an upper bound on the expected number e_x of pairs (X,Y) as above, for each $x \leq k/3$. This expectation e_x is at most

$$2\binom{k}{x}\binom{k}{2x}\left[\frac{(2x)^{2x}(k-x)^{k-x}}{x^x k^k}\right]^r \leq \left(\frac{ke}{x}\right)^x\left(\frac{ke}{2x}\right)^{2x}\left[\frac{4^x x^x}{k^x}\left(1-\frac{x}{k}\right)^{k-x}\right]^r$$

$$= \frac{x^{(r-3)x}}{k^{(r-3)x}}e^{3x}4^{(r-1)x}\left(1-\frac{x}{k}\right)^{(k-x)r}.$$

Since the last factor above is at most 1 and we are taking $r = 10$, e_x is at most

$$p(x) := \left[\left(\frac{x}{k}\right)^7 e^3 4^9\right]^x.$$

For $1 \leq x \leq k/11$, $p(x)$ is a convex function which is at most

$$\max(p(1), p(k/11)) = \max\left(\frac{e^3 4^9}{k^7}, \left[\frac{e^3 4^9}{11^7}\right]^{k/11}\right) \leq \frac{1}{k^6}$$

for sufficiently large k. Thus $e_x \leq 1/k^6$ for $1 \leq x \leq k/11$, provided k is sufficiently large.

For $x/k \geq 1/11$, we retain the factor of $(1-x/k)^{(k-x)r}$. Writing α for x/k, we then have

$$e_x \leq \left[\alpha^7 e^3 4^9 (1-\alpha)^{10(1/\alpha-1)}\right]^x.$$

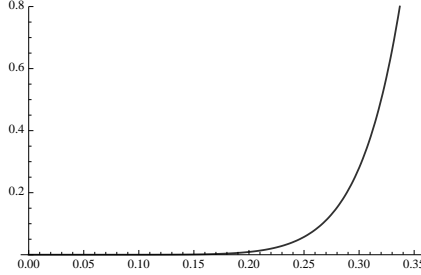For $1/11 \leq \alpha \leq 1/3$, the inner quantity is less than $3/4$, as shown by the plot in Figure 1.

FIGURE 1. A plot of $\alpha^7 e^3 4^9 (1-\alpha)^{10(1/\alpha-1)}$

Therefore, for $k/11 \leq x \leq k/3$, we have $e_x \leq (3/4)^{k/11} \leq 1/k^6$, for k sufficiently large. Summing over all values of x up to $k/3$, the probability that every set $X \subset A$ of size at most $k/3$ has a neighborhood of size greater than $2|X|$ is at least $1-2/k^5$. We have the same bound for the probability that every set $Y \subset B$ of size at most $k/3$ has a neighborhood of size greater than $2|Y|$, and thus we obtain the claimed result. $\square$

All we need from Lemma 1 is the existence, for all k sufficiently large, of a graph of maximum degree at most 10 with the doubling property, so that we may construct our example $\mathbf{P}_{k,\varepsilon}$ as stated.

Before bounding $\mathrm{led}(\mathbf{P}_{k,\varepsilon})$, we look at $\mathrm{inc}(\mathbf{P}_{k,\varepsilon})$.

Proposition 2. *For k sufficiently large and $\varepsilon \leq 1/\log r$, the number of incomparable pairs in $\mathbf{P}_{k,\varepsilon}$ is at least*

$$\frac{r-2}{2(r-1)}\varepsilon^2 k^2 \log^2 k.$$

Proof. The number of elements of $\mathbf{P}_{k,\varepsilon}$ is $n = km = k\lfloor \varepsilon \log k \rfloor$. Using just the fact that each graph G_i has maximum degree at most r, we see that each element is comparable to at most

$$1 + r + r^2 + \cdots + r^{m-1} = \frac{r^m - 1}{r - 1} \leq \frac{k}{r - 1}$$

elements, including itself. Therefore, the number of comparable pairs is at most $nk/(2(r-1)) \leq \binom{n}{2}/(r-1)$ for k sufficiently large. Now we see that for k sufficiently large, the number of incomparable pairs is at least

$$\frac{r-2}{r-1}\binom{n}{2} \geq \frac{r-2}{2(r-1)}\varepsilon^2 k^2 \log^2 k$$

as claimed. $\qquad\square$

We now turn to estimating the linear extension diameter of $\mathbf{P}_{k,\varepsilon}$.

Proposition 3. *Let k be a multiple of 3, and suppose $0 < \varepsilon \leq 1$. The linear extension diameter of $\mathbf{P}_{k,\varepsilon}$ is at most $(31/6)\varepsilon k^2 \log k$, for sufficiently large k.*

Proof. For any linear extension L of $\mathbf{P}_{k,\varepsilon}$, we define a second linear extension L'. For $x \in A_i$ and $y \in A_j$, we let $x <_{L'} y$ if $i < j$. If $i = j$, then $x <_{L'} y$ if and only if $x <_L y$. Our plan is to prove an upper bound on $\max_L \text{dist}(L, L')$. Given such a bound, we then note that, for any pair (L_1, L_2) of linear extensions, we have

$$\text{dist}(L_1, L_2) \leq \text{dist}(L_1, L_1') + \text{dist}(L_1', L_2') + \text{dist}(L_2', L_2).$$

If $x <_{L_1'} y$ and $y <_{L_2'} x$, then $x, y \in A_i$ for some i, so $\text{dist}(L_1', L_2') \leq m\binom{k}{2} \leq \frac{1}{2}\varepsilon k^2 \log k$. It thus suffices to prove that $\text{dist}(L, L') \leq (7/3)\varepsilon k^2 \log k$, for any linear extension L of $\mathbf{P}_{k,\varepsilon}$.

For the linear extension L, we need to consider, for each pair (i, j) with $i < j$, how many instances there are of an element x of A_i appearing above an element y of A_j in L – we say simply that (x, y) is *reversed in* (A_i, A_j).

We start with the case where $j = i + 2s + 1$, i.e., there are an even number of levels between A_i and A_j. Consider first $s = 0$, and let $T \subset A_i$ be the set consisting of the top $k/3$ elements of A_i in L, and let $T' \subset A_{i+1}$ be the set consisting of the bottom $k/3$ elements of A_{i+1} in L. Suppose that some element of $A_i \setminus T$ is above some element y of $A_{i+1} \setminus T'$ in L. Then all the elements of T are above all the elements of $T' \cup \{y\}$, which means there are no edges between T and $T' \cup \{y\}$ in G_i, contradicting the doubling property. Thus every element of $A_i \setminus T$ is below every element of $A_{i+1} \setminus T'$ in L, and therefore the number of reversed pairs (x, y) in (A_i, A_{i+1}) is at most

$$|T||A_{i+1}| + |T'||A_i| \leq 2k^2/3.$$

For $0 < s \leq \lfloor \varepsilon \log k \rfloor / 2$, let $T \subset A_i$ be the set consisting of the top $\lceil k/(2^s \cdot 3) \rceil$ elements of A_i in L, and let $T' \subset A_{i+2s+1}$ be the set consisting of the bottom $\lceil k/(2^s \cdot 3) \rceil$ elements of A_{i+2s+1} in L. Again suppose that some element of $A_i \setminus T$ is above some element y of $A_{i+2s+1} \setminus T'$ in L. Then all the elements of T are above all the elements of $T' \cup \{y\}$ in L. By the doubling property, there are at least $2^s|T| \geq k/3$

elements of A_{i+s} above some element of T in L, and at least $2^s(|T'|+1) > k/3$ elements of A_{i+s+1} below some element of $T' \cup \{y\}$ in L, and again by the doubling property applied to $(k/3)$-element subsets of T and $T' \cup \{y\}$ this is not possible. Thus, we see that the number of pairs reversed in (A_i, A_{i+2s+1}) is at most $2k\lceil k/(2^s \cdot 3)\rceil \le k^2/(2^{s-1} \cdot 3) + 2k$. For fixed i, summing over all possible values of s, we obtain an upper bound of

$$\sum_{s=0}^{\lfloor \varepsilon \log k\rfloor/2} \left(\frac{k^2}{2^{s-1} \cdot 3} + 2k\right) \le \frac{4k^2}{3} - \frac{2k^2}{3} 2^{-\varepsilon \log k/2} + k\varepsilon \log k \le \frac{4k^2}{3}$$

on the number of reversed pairs involving an element of A_i and an element an odd number of layers above A_i. Accounting for all values of i gives that we reverse at most $4\varepsilon k^2/3 \log k$ pairs in pairs with an even number of layers between them (including zero).

One approach for counting pairs reversed in (A_i, A_{i+2s+2}) is to proceed as above, taking $T \subset A_i$ of size $\lceil k/(2^s \cdot 3)\rceil$ and $T' \subset A_{i+2s+2}$ of size $\lceil k/(2^{s+1} \cdot 3)\rceil$. As above, every reversed pair involves an element of either T or T', and so we reverse at most

$$k\left\lceil \frac{k}{2^s \cdot 3}\right\rceil + k\left\lceil \frac{k}{2^{s+1} \cdot 3}\right\rceil \le \frac{k^2}{2^{s+1}} + 2k$$

pairs in (A_i, A_{i+2s+2}). For fixed i, summing over all s with $0 \le s \le \lfloor \varepsilon \log k\rfloor/2$, we have that at most k^2 pairs involving an element of A_i and an element an even number of layers above A_i. Multiplying this by $\varepsilon \log k$ gives that we reverse at most $\varepsilon k^2 \log k$ pairs with an odd number of layers between them.

Therefore,

$$\text{dist}(L, L') \le \frac{4\varepsilon k^2 \log k}{3} + \varepsilon k^2 \log k = \frac{7}{3}\varepsilon k^2 \log k$$

for any linear extension L of $\mathbf{P}_{k,\varepsilon}$, as required. We are now able to conclude that $\text{led}(\mathbf{P}_{k,\varepsilon}) \le (31/6)\varepsilon k^2 \log k$. $\qquad \square$

As a consequence of Propositions 2 (with $r = 10$) and 3, we have the following corollary.

Corollary 4. *Whenever k is a sufficiently large multiple of 3, and $0 < \varepsilon \le 1/\log 10$, we have*

$$RR(\mathbf{P}_{k,\varepsilon}) \le \frac{93}{8\varepsilon}\frac{1}{\log k}.$$

In particular, setting $\varepsilon = 1/\log 10$ gives us examples of posets $\mathbf{P}_k$, for arbitrarily large k, with $n = k\lfloor \log k/\log 10\rfloor$ elements, and

$$RR(\mathbf{P}_k) \le \frac{93\log 10}{8}\frac{1}{\log k} \le \frac{27}{\log n},$$

where the last inequality holds for k sufficiently large.

It is natural to ask if, for all posets $\mathbf{P}$, we have $RR(\mathbf{P}) \ge f(n)$, where n is some reasonable parameter of the poset—such as number of elements or width—and $f(n)$ tends to zero fairly slowly with n. In this section, we have shown that if such an f exists, it must be $O(1/\log n)$.

3. POSETS OF FIXED DIMENSION

Because both order dimension and linear extension diameter focus on linear extensions and reversing incomparable pairs, it is natural to ask about the minimum value of $RR(\mathbf{P})$ on a poset of dimension d. An easy lower bound on the reversal ratio in terms of d can be obtained as follows. Let $\mathscr{R}$ be a realizer of size d and consider the sum

$$\sum_{\substack{L_i, L_j \in \mathscr{R} \\ i < j}} \operatorname{dist}(L_i, L_j).$$

Each term is at most $\operatorname{led}(\mathbf{P})$, and thus the sum is at most $\binom{d}{2} \operatorname{led}(\mathbf{P})$. On the other hand, each incomparable pair must contribute 1 to at least $d - 1$ terms in the sum, and so the sum is at least $(d-1)\operatorname{inc}(\mathbf{P})$. Combining the two inequalities gives $RR(\mathbf{P}) \geq 2/d$.

We define

$$DRR(d) = \inf\{RR(\mathbf{P}) : \dim(\mathbf{P}) \leq d\}.$$

The argument above shows that $DRR(d) \geq 2/d$ for each $d \geq 2$. A poset $\mathbf{P}$ with dimension $d \leq 2$ is exactly one in which there is a pair of linear extensions whose distance is equal to the number of incomparable pairs, and so $RR(\mathbf{P}) = 1$ for all posets $\mathbf{P}$ of dimension at most 2, and hence $DRR(2) = 1$.

In seeking upper bounds on $DRR(d)$ for $d \geq 3$, we need a d-dimensional poset with small reversal ratio, and we turn to the d-dimensional grid $\mathbf{n}^d$. This poset is the downset lattice of a disjoint union of chains. In [5], Felsner and Massow give an exact formula for the linear extension diameter of such posets, from which we conclude that $\operatorname{led}(\mathbf{n}^d) \sim n^{2d}/4$. On the other hand, a straightforward calculation gives that

$$\operatorname{inc}(\mathbf{n}^d) \sim \frac{1}{2} n^{2d} \left(1 - \frac{1}{2^{d-1}}\right).$$

Hence, $RR(\mathbf{n}^d) \to 1/(2(1 - 1/2^{d-1}))$ as $n \to \infty$. This limit in turn tends to $1/2$ as $d \to \infty$, so these are not good examples for large values of d.

For $d = 3$, however, we deduce that $RR(\mathbf{n}^3) \to 2/3$ as $n \to \infty$, matching the lower bound. Thus we have $DRR(3) = 2/3$. There is already a gap between our upper and lower bounds for $d = 4$, and we don't know the value of $DRR(d)$ for any $d \geq 4$. For $d = 4$, the bounds above give $1/2 \leq DRR(4) \leq 4/7$, and we know of no improvements.

Perhaps of greater interest is the behavior of $DRR(d)$ as d increases. To move away from posets of small dimension, we recall the posets $\mathbf{P}_k = \mathbf{P}_{k, 1/\log 10}$ developed in Section 2. It is easy to see that $\operatorname{width}(\mathbf{P}_k) = k$ and so $\dim(\mathbf{P}_k) \leq k$. Therefore $DRR(d) \leq RR(\mathbf{P}_d) \leq 27/\log d$, whenever d is a sufficiently large multiple of 3. (It is easy to deduce the same bound when d is not a multiple of 3.) Our belief is that this upper bound is closer to the true value of $DRR(d)$ than the lower bound of $2/d$. It is worth noting that, for any $\varepsilon > 0$, one can show $\dim(\mathbf{P}_{k,\varepsilon}) \geq k^\delta$ for some $0 < \delta < 1$, so having a more precise value for the dimension of $\mathbf{P}_k$ would not significantly improve the bounds on $DRR(d)$.

4. POSETS OF FIXED WIDTH

Finally, we consider the minimum reversal ratio for posets of width w. In a similar manner to the previous section, we define

$$WRR(w) = \inf\{RR(\mathbf{P}) : \text{width}(\mathbf{P}) \leq w\}.$$

Since $\dim(\mathbf{P}) \leq \text{width}(\mathbf{P})$ for every poset $\mathbf{P}$, we have $WRR(w) \geq g_{RR(w)}$. In particular, $WRR(2) = 1$, and the first interesting case is that of width 3; we know that $WRR(3) \geq DRR(3) = 2/3$.

A family of examples demonstrating that $WRR(3) \leq 5/6$ is easy to come by. Let $\mathbf{W}_k$ denote a "stack of k standard examples": we take antichains $A_i = \{a_i, b_i, c_i\}$, for $i = 0, \ldots, k$, and, for each $i = 1, \ldots, k$, put all of A_{i-1} below all of A_i except that the pairs (a_{i-1}, b_i), (b_{i-1}, c_i) and (c_{i-1}, a_i) are incomparable. For each i, no pair of linear extensions can reverse all of the 6 incomparable pairs (a_i, b_i), (a_i, c_i), (b_i, c_i), (a_{i-1}, b_i), (b_{i-1}, c_i), (c_{i-1}, a_i). Thus, of the $6k + 3$ incomparable pairs in $\mathbf{W}_k$, a pair of linear extensions reverses at most $5k + 3$.

However, we believe that neither the upper nor the lower bound is correct, and that in fact $WRR(3)$ is equal to $3/4$. We conclude by explaining the basis of this belief.

We construct a family of posets $\mathbf{Z}_{t,k} = (Z, P)$, with $t \geq 3$ and $k \geq 3$. The set Z consists of t disjoint blocks $A_1, A_2, \ldots, A_t$, with each block A_i containing k elements arranged as a chain $a_{i,1} < \cdots < a_{i,k}$. For $j \geq i + 2$, each element of A_j is above each element of A_i. We also have comparabilities $a_{i,j} < a_{i+1,j}$, for each $i = 1, \ldots t - 1$ and $1 \leq j \leq k$. The remaining relations are those induced by transitivity.

The poset $\mathbf{Z}_{t,k}$ has width 3, with chains given by $A_1 < A_4 < A_7 < \cdots$, $A_2 < A_5 < \cdots$ and $A_3 < A_6 < \cdots$. The only incomparabilities are between pairs of blocks A_i, A_ℓ, with $\ell = i + 1$ or $\ell = i + 2$. For such a pair (i, ℓ), we have $a_{i,j} < a_{\ell,m}$ if and only if $j \leq m$. Therefore there are $(2t - 3)\binom{k}{2}$ incomparable pairs.

The poset $\mathbf{Z}_{9,4}$ is shown in Figure 2.

Theorem 5. *The linear extension diameter of $\mathbf{Z}_{t,k}$ is at least*

$$\left(t - 1 + \left\lceil \frac{t-2}{2} \right\rceil\right)\binom{k}{2}.$$

Proof. To show this lower bound on the linear extension diameter, we construct two linear extensions L_1 and L_2 of $\mathbf{Z}_{t,k}$.

For a triple $A_\ell, A_{\ell+1}, A_{\ell+2}$ of three consecutive blocks of $\mathbf{Z}_{t,k}$, we write $A_\ell / A_{\ell+1} / A_{\ell+2}$ for the following linear extension of the poset restricted to $A_\ell \cup A_{\ell+1} \cup A_{\ell+2}$:

$$a_{\ell,1} < a_{\ell+1,1} < a_{\ell+2,1} < a_{\ell,2} < a_{\ell+1,2} < a_{\ell+2,2} < a_{\ell,3} < \cdots < a_{\ell,k} < a_{\ell+1,k} < a_{\ell+2,k}.$$

For a consecutive pair of blocks, we define $A_\ell / A_{\ell+1}$ similarly, omitting the elements $a_{\ell+2,j}$ from the order described above.

We will consider linear extensions L of $\mathbf{Z}_{t,k}$ in which blocks appear either consecutively or in pairs and triples $A_\ell / A_{\ell+1}$ or $A_\ell / A_{\ell+1} / A_{\ell+2}$. If, in some such linear extension L, A_i and A_j ($j = i + 1$ or $j = i + 2$) appear together within an order $A_\ell / A_{\ell+1} / A_{\ell+2}$, we say that the pair (A_i, A_j) is *entangled* in L.

Observe that, if the pair (A_i, A_j) appears entangled in one linear extension, and in order in the other (i.e., with all of A_i below all of A_j), then every pair of incomparable pairs between A_i and A_j is reversed by the pair of linear extensions.
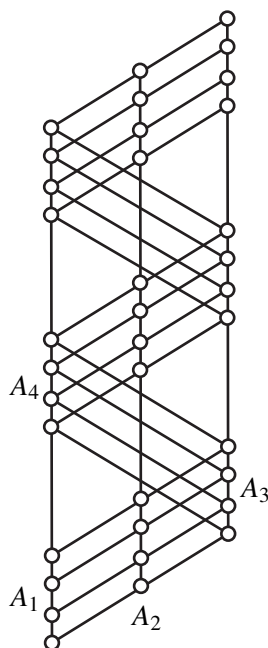
FIGURE 2. The poset $\mathbf{Z}_{9,4}$

We now form specific linear extensions of the form described that achieve the required lower bound. We start L_1 with $A_1/A_2/A_3$ and L_2 with A_1 in its entirety followed by A_2 and then $A_3/A_4/A_5$. The continuation of this pattern through the blocks appearing in $\mathbf{Z}_{12,k}$ is shown below.

L_1	L_2
A_{12}	A_{11}/A_{12}
$A_9/A_{10}/A_{11}$	A_{10}
A_8	$A_7/A_8/A_9$
$A_5/A_6/A_7$	A_6
A_4	$A_3/A_4/A_5$
$A_1/A_2/A_3$	A_2
	A_1

More formally, the linear extension L_1 contains each linear order $A_{4i-3}/A_{4i-2}/A_{4i-1}$, for $i \geq 1$, and the blocks A_{4i} ($i \geq 1$) appear separately, above all blocks with lower index and below all blocks with higher index. If t is congruent to 1 or 2 modulo 4, so that some designated triple $A_{4i-3}/A_{4i-2}/A_{4i-1}$ includes A_t and supposedly some non-existent blocks with higher index, then the blocks beyond A_t are omitted. Similarly, L_2 includes each linear order $A_{4i-1}/A_{4i}/A_{4i+1}$ ($i \geq 1$), with the blocks A_{4i+2} appearing singly: again, blocks with index outside $\{1,\ldots,t\}$ are omitted.

To determine the number of incomparable pairs reversed between L_1 and L_2, we first observe that, for blocks A_i and A_j, with $j = i+1$ or $j = i+2$, the pair (A_i, A_j) is entangled in at most one of the linear extensions. Moreover, each of the $t-1$ consecutive pairs (A_i, A_{i+1}) is entangled in exactly one of the two linear extensions, so the incomparable pairs between such a pair of blocks are all reversed. For the pairs at distance 2, the $\lceil (t-2)/2 \rceil$ pairs (A_i, A_{i+2}) with i odd are entangled in one

of the two linear extensions, but the pairs with i even are not. Therefore,

$$\text{dist}(L_1, L_2) = \left(t - 1 + \left\lceil \frac{t-2}{2} \right\rceil \right) \binom{k}{2},$$

and consequently this is a lower bound on $\text{led}(\mathbf{Z}_{t,k})$, as claimed. $\qquad\square$

Since $\text{inc}(\mathbf{Z}_{t,k}) = (2t - 3)\binom{k}{2}$, Theorem 5 implies that $RR(\mathbf{Z}_{t,k}) \geq 3/4$. We believe the inequality of Theorem 5 to be an equality for k and t sufficiently large, which would imply that $WRR(3) \geq 3/4$. Moreover, we suspect that this family of examples is (essentially) extremal.

Conjecture. $WRR(3) = 3/4$.

We can form similar families of posets for higher values of the width, and it is possible that these remain extremal.

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