Power indices as an aid to institutional design : the generalised apportionment problem

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The authors gratefully acknowledge that work on this paper was partly supported by the Leverhulme Trust (Grant F/07-004m).

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August 2002

Paper prepared for the workshop on Voting Power Analysis, LSE 9-11 August 2002.

Warwick Economic Research Papers Number 648

First Draft. Not to be quoted. Comments welcome.

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Abstract

A priori voting power analysis can be useful in helping to design a weighted voting system that has certain intended properties. Power indices can help determine how many weighted votes each member should be allocated and what the decision rule should be. These choices can be made in the light of a requirement that there be a given distribution of power and/or a desired division of powers between individual members and the collective institution. This paper focuses on the former problem: choosing the weights given that the power indices and the decision rule are fixed exogenously.

Introduction

In institutions where it is a fundamental constitutional principle that there should be differences in power between different voting members, this inequality is usually implemented by means of a system of weighted voting. Examples are the Bretton Woods institutions, the IMF and World Bank, and the European Union Council. However it is well known that in any system of weighted voting, the powers of voters who cast unequal numbers of votes are not proportional to their voting weights: that "weighted voting doesn't work", in the phrase that was the title of Banzhaf's famous paper of 1965. There are therefore serious issues of democratic legitimacy and accountability surrounding such institutional arrangements.

Power indices are a useful quantitative tool for modelling voting power in weighted voting and hence aiding understanding of the workings of the institution concerned. A power index measures a priori voting power which abstracts from voters' preferences and behaviour; power indices measure power which derives only from the constitution. All writers on power indices have emphasised their potentially important role in designing voting systems stemming from this Rawlsian "veil of ignorance".

This paper investigates the problem of using power indices as the basis of the choice of voting weights. There have been many studies of the distribution of a priori power in actual voting bodies where the decision rule and allocation of votes to voting members is given but relatively few where the approach has been used as a tool for designing weighted voting systems. Recent examples are Laruelle and Widgren (1998), Sutter (2000), Leech (2002b), Leech (forthcoming). This paper attempts to build on this applied empirical work by investigating its perspective and approach from a somewhat theoretical point of point of view.

The Problem

The question posed in this paper is: "how to find the voting weights assigned to the various voters in the institution in such a way that the powers of the voters are equal to given pre-assigned values". Under the constitutions of the IMF and World Bank, for example, the voting power to which a member state is entitled is related to its financial commitment, which is in turn related to the size of its economy. For example, the USA is entitled to over seventeen percent of the voting power in the IMF because that is the size of its financial commitment; by contrast India is entitled to about 2 percent of the voting power. In the implementation these figures are translated simply into shares of the votes without further thought. If the respective votes of member countries corresponded to numbers of representatives in a legislature, this method would have some legitimacy in its own terms (accepting for present purposes the principle of assigning votes to countries on the basis of economic not population criteria). But the votes are cast as blocs and the weighted voting problem arises.

In the bloc-voting system used by the EU Council each country has assigned a voting weight related to population. There is no explicit formula for doing this, the weights having been agreed at intergovernmental conferences among the members; however this method of weighting and reweighting has become impractical as a result of enlargement and a more systematic approach would be desirable.

Notationally, a voting decision rule is denoted $\{q; w_1, w_2, . . . , w_n\}$; there are n voters with weights w_i and a decision rule quota q. In general the weights and the quota are positive real numbers. The weights will also be denoted by the vector $w = (w_1, \dots ,w_n)$. The corresponding power indices are a vector $p = (p_1, \dots , p_n)$ where p_i is the number of swings for voter i relative to the number of voting outcomes. Letting the number of swings for i be η_i, then we can write $p_i = \eta_i/2^{n-1}$. If we make the conventional assumption that all voters other than i vote randomly, independently with equal probability of supporting or opposing i, then p_i is the probability of a swing.

Then power indices depend on all the weights and the quota, a relationship that can be written in general using functional notation, p = p(q, w). In this paper interest is focused on the choice of weights, so I will write simply p = p(w). In general we can think of w as a real-valued n-vector and p a vector of probabilities, $0 \le p_i \le 1$. Let $w = \Sigma\, w_i$, the total voting weight. If the voting weights are integers, it is equal to the total number of votes. However there is no requirement for them to be integers. For example in some institutions it may be more appropriate for voting weights and votes cast to be expressed as percentages in some cases (for example the British Labour Party leadership electoral college).

The problem is to find the weights such that the powers are pre-specified values. Thus, in general terms, if the chosen values for the powers are given by a vector d, we seek to solve the equations:

$$d = p(w) \qquad (1)$$

I will investigate aspects of this problem from three points of view. First, theoretically for small n; second, empirically, by means of examples in large real-world voting bodies: the IMF and EU Council; third, theoretically for large voting bodies, after making simplifying assumptions that appear reasonable, by exploring structural similarities with the theory of economic general equilibrium. But first, it is necessary to comment on the choice of power index.

The Choice of Power Index

Just as the limitations of weighted voting are well known, it is equally well known that power indices are a possible way to deal with them. However the development of methods of doing so has been limited by the existence of different, rival power indices that could claim equal theoretical validity. Moreover, the results obtained by applying them empirically have varied between suggesting great diversity and little difference, which has led to greater ambiguity. Many scholars have dismissed the power indices approach as of little use on the grounds that the results suggest that voters' powers are little different from being proportional to their weights; and moreover there is little variation among rival indices. For example many analyses of legislatures have found the powers calculated by the Shapley-Shubik index to be little different from those given by the (normalised) Banzhaf index; and both to be close to the weight shares. On the other hand other studies have found substantial differences both between the results for the different indices and between power and weight.

In this paper I take the view that there is now substantial evidence, both theoretical and empirical, to guide a choice of power index suitable for applied work. The index which has had the greatest use in applications is the Shapley-Shubik index but the evidence in comparative studies (such as Coleman (1971), Felsenthal and Machover (1998), Laruelle and Valenciano (2001) and Leech (2002a)) tends to find limitations in it. The empirical study in Leech (2002a) finds the index severely deficient. The normalised Banzhaf index also has deficiencies, deriving from the process of normalisation, that are not shared by the non-normalised or absolute Banzhaf index. Since this index was actually invented by Penrose in 1946, it seems appropriate to refer to it as the Banzhaf-Penrose index, or even the Penrose index. On occasion it might also be possible to refer to it simply as the power index.

A voter's power is measured by his ability to swing a decision, which gives rise to a power index defined in absolute terms. In this I follow the approach of Coleman, for whom voting power was the basis of a theory of social action, and who rejected game-theoretic ideas based on bargaining as inappropriate to voting over collective action. (Coleman's approach is discussed in my recent working paper Leech (2002c).) An absolute index can be used to measure the relative power of different voters, as a ratio, but does not require normalisation universally to be imposed at the level of the definition. Where appropriate use will be made of normalisation as a mathematical tool.

In order to study the relationships in (1), it is useful to begin by investigating them for small, finite, values of n. It is useful, initially, to consider the normalised versions of the weights and the powers: thus, let us assume that p, d and w are all normalised so that their elements are non-negative and sum to unity, and we can replace the quota q by q/w, then each w_i by w_i/w. All are points on the unit (n-1)-simplex in n-dimensional space: $X = \{x; x\in R^n, x \geq 0, \forall\ i, \sum x_i = 1\}$. Here p, d, w $\in$ X. This is illustrated in Figure 1.

Now d and w can be any points in X but p belongs to a subset; each element of the vector p, p_i , is a rational number in general since it is a ratio of positive integers. Obviously the problem becomes trivial if there is to be a dictator, where $d_i = 1$, and $w_i \geq q$ for some i.

Figure 2 shows all the possible normalised power indices for the case n = 3, and illustrates the nature of the problem. There are two stages: first, how to choose the desired values for the power indices p from the feasible set, given the desired values d; second, how to choose weights w given these desired power indices.

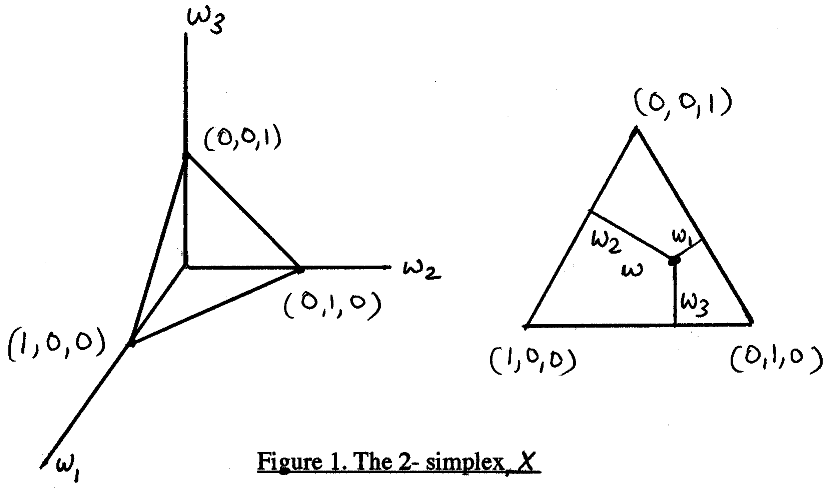

Figure 1. The 2- simplex, X

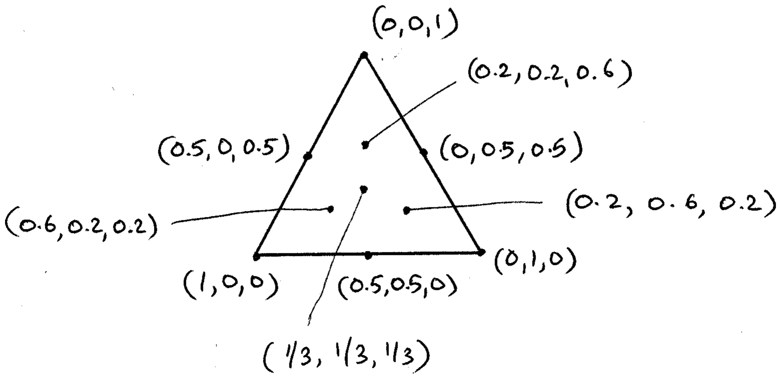

Figure 2. Possible Power Indices (Normalised Bz) for n=3

Consider the second problem first. The simplex X can be divided into regions in which each weight vector maps to a particular power vector. These regions are determined by hyperplanes that partition X. For $n = 3$ these consist of regions defined by inequalities depending on the quota q. (Only values of q greater than 1/2 are considered here.) There are 10 regions which are shown in Figure 3 and the corresponding power indices in Table 1.

Table 1. Regions and Corresponding Power Indices

Figure 3 shows the ten regions for two cases: $q < 2/3$ and $q > 2/3$. There are three dictator regions (I , II and III), three regions where one member is powerless (IV, V and VI), a central region (X) where all are equally powerful and three regions where power is unequal (VII, VIII and IX). In this example there are always two members with the same power. This is a result of the small dimensionality. It is not possible to represent a larger voting body than this geometrically. The effect of increasing the quota is to increase the size of region X which begins to resemble the entire simplex.

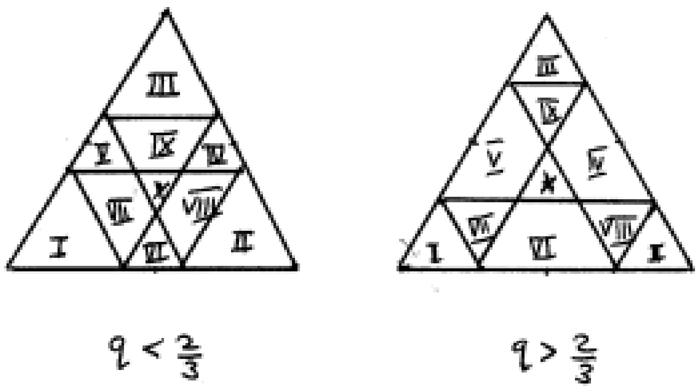

Figure 3. Shapes of the 10 Regions for q < 2/3 and q > 2/3

Region	I	II	III	IV	V
Power, p	1,0,0	0,1,0	0,0,1	0,0.5,0.5	0.5,0,0.5
Norm. Bz	1,0,0	0,1,0	0,0,1	0,0.5,0.5	0.5,0,0.5
Region	VI	VII	VIII	IX	X
Power, p	0.5,0.5,0	0.75,0.25,0.25	0.25,0.75,0.25	0.25,0.25,0.75	0.5,0.5,0.5
Norm. Bz	0.5,0.5,0	0.6,0.2,0.2	0.2,0.6,0.2	0.2,0.2,0.6	0.33,0.33,0.33

Table 1. Regions and Corresponding Power Indices

Thus the problem is to find a point representing weights in the appropriate region for the required value of the power index. Given the chosen vector p there is not a corresponding unique w, but this may not be a problem. We might wish to require that the set of w's for a given p be connected, however.

The Determination of the Desired Power Indices

The first problem mentioned above is that of finding the appropriate desired values of the power indices in the vector p, given the design values vector d. This problem is one of finding a vector of discrete values, as approximations to continuous numbers, according to some criterion of fairness. This is formally equivalent to a well-known problem in political science, on which there is a large literature dating back many years: the apportionment problem.

The apportionment problem is that of how best to allocate seats in a legislature to different territories or different parties, given that the seats are constrained to be integers whereas the different entitlements of the territories or parties are not. It has to be addressed in the US House of Representatives every ten years following each population census, where the problem is how many seats should be allocated to each state, given that no seat can cross a state line. In the UK there is a somewhat similar problem where the parliamentary boundary commissioners periodically redraw constituency boundaries to reflect population changes but normally no seat can cross county boundaries. An exactly analogous problem is that of assigning seats to parties in a system of proportional representation.

Various apportionment methods have been proposed to solve this rounding problem in a manner consistent with democratic principles. They are mainly in three categories: Hamilton's method, Greatest Divisors and Minimum Distance. I do not intend to go into this literature here; the key reference is Balinski and Young (1982).

Several scholars have related the apportionment problem to power indices, among them Gambarelli (1999), Holler (1982, 1987), Nurmi (1978, 1982), Gambarelli and Holubiec (1990). Some of this literature is concerned with ensuring that the power indices of different groups of legislators correspond with the respective power indices of their constituencies (whether parties or territories) and the problem reduces to that described in the last section; here the vector d would be derived as the power indices of the constituencies and can be assumed to be identical to p. The perspective of the current paper is more general in that I am not specifically concerned about the derivation of the vector d. Thus, it might be the square roots of the electorates of the n member countries of the EU, or as the financial contributions of the member countries, as in the institutions of world economic governance created at Bretton Woods.

The problem is a slight generalisation of the apportionment problem because the power indices which are to be chosen are rational numbers rather than integers, but it is obvious that the same general mathematical methodology could be applied. This topic will not be pursued here, however, and the rest of the paper will consider large voting bodies where these rounding problems can be taken as of minor importance. For example, if n = 10, which might be a small institution compared with many encountered in the real-world, the difference in the values of successive (non-normalised) power indices (the power of a single swing) is $2^{1-n} = 2^{-9} = 0.00195$, or 0.2 percent. If n = 15, this becomes $2^{-14} = 0.00006$. Therefore it seems that a practical expedient of assuming the desired power indices to be effectively real numbers might work quite well.

The discussion in this section perhaps suggests that a suitable name for the general problem addressed in the paper might be "the generalised apportionment problem".

Numerical Determination of Voting Weights in Practice

In empirical work I have addressed the problem of solving the equations (1) directly as a set of n equations in n unknowns. The equations are not analytic and therefore powerful computational resources must be deployed, first to evaluate them and then to solve them. The general approach is an iterative one of successive approximations. First an initial guess is made, $w^{(0)}$, and power indices $p^{(0)}$ found by means of an appropriate numerical algorithm. If the power indices differ from the desired values, d, then the weights are adjusted and become $w^{(1)}$, the corresponding power indices $p^{(1)}$ computed, and again compared with d, and so on. This process of successive approximations continues until the distance between the power vector and the design vector is small enough to be acceptable, to the required accuracy, and we can write $d = p(w^*)$. The vector w* is then taken as a solution to equation (1) and the problem of the choice of weights has been solved. Given that the weights are such as to guarantee that the voting powers satisfy the desired criterion, these can be referred to as "fair" weights.

The adjustment procedure by which the weights are updated at each iteration must be such as to lead to convergence and therefore a solution to (1). Both examples described in this section used essentially the same adjustment procedure to update the weights, although they employed different algorithms to evaluate the power indices.

Let the weights after r iterations be denoted by the vector $w^{(r)}$, and corresponding power indices by the vector of functions $p(w^{(r)})$. The iterative procedure consists of an intial guess $w^{(0)}$ and an updating rule:

$$w^{(r+1)} = w^{(r)} + \lambda(d - p(w^{(r)})) \qquad (2)$$

for some appropriate scalar $\lambda>0$. (More generally we might think of replacing λ by a matrix of adjustment coefficients.).

It is necessary to define convergence in terms of some measure of the distance between the desired and the actual power indices at the r^{th} iteration. Then, given this, convergence can be defined relative to a suitable stopping criterion in terms of the level of accuracy that is either feasible or desireable. In both the examples described here the simple sum of squares measure $\sum(p_i^{(r)} - d_i)^2$ with an appropriate stopping criterion has been found to be satisfactory.

Application 1: The International Monetary Fund Board of Governors

The first application is to the governing body of the IMF which has n = 178 (that was in 1999 – the number of members has increased slightly since then). The calculations have been made for 2 decision rules, $q = 0.5$ and $q = 0.85$, both of which are used, according to the institution's constitution, for different kinds of decision. (Other decision rules also exist but simple majority and the 85% supermajority are the most important ones. The very high 85% rule is designed to institutionalise a veto for the USA while allowing that country's financial contribution to fall below 20%.) For details see Leech (2002b).

	Banzhaf Power β_i	Weight w_i* q=50%	q=85%
USA	17.55	14.06	69.78
Japan	6.30	6.53	2.20
Germany	6.15	6.38	2.16
France	5.08	5.27	1.82
UK	5.08	5.27	1.82
Italy	3.34	3.48	1.23
Saudi Arabia	3.31	3.45	1.21
Canada	3.02	3.15	1.11
Russia	2.82	2.94	1.04
Netherlands	2.45	2.56	0.91
China	2.22	2.32	0.82
India	1.97	2.06	0.73
Switzerland	1.64	1.72	0.61
Australia	1.54	1.61	0.57
Belgium	1.48	1.54	0.55
Spain	1.45	1.52	0.54
Brazil	1.45	1.51	0.54
Venezuela	1.27	1.32	0.47
Mexico	1.23	1.29	0.46
Sweden	1.14	1.19	0.43
Argentina	1.01	1.06	0.38
Indonesia	0.99	1.04	0.37
Austria	0.90	0.94	0.33
...	...	...	...
…	…	…	…
…	…	…	…

Table adapted from Leech (2002b).
Figures are percentages.

Table 2 "Fair" Weights in the IMF Board of Governors (n=178)

Table 2 shows that analysis for the top 23 member countries in terms of voting weight and voting power.. The desired voting powers are those referred to as simply "powers" in the IMF jargon, which are based on their financial contributions and "quotas". The "fair" weights w_i* are found accordingly for the 2 separate decision rules. The results show that, in order that the USA should have a normalised power index of 17.55% (in line with its IMF "quota"), its weight should be *reduced* to 14.06% (and those of all the other members increased correspondingly). By contrast, in order to achieve the same voting power with a decision rule with q = 0.85, its weight should be *increased* to 69.78% and those of all others reduced. This is because a high supermajority decision rule is relatively egalitarian, being closer to unanimity than a simple majority rule, but the desired powers are not, and therefore the "fair" weights must be very unequal to achieve this desired degree of inequality.

The computational details are as follows. The algorithm used to find the power indices was the modified Owen approximation method - necessary given the large value of n. In every set of calculations, full convergence was easily achieved using a stopping rule requiring the sum of squares function to be less than 10^{-15}. This corresponded to an accuracy of the order of 10^{-9} which was considered sufficient for all purposes and no investigations were carried out with smaller convergence criteria.

Application 2: The Council of the EU

The second application is taken from Leech (forthcoming). Other studies that have employed the same approach are Laruelle and Widgren (1998) and Sutter (2000). In this exercise it is assumed that the voting body has n = 15. The decision rule is the treble majority rule as agreed at the Nice IGC: a decision of the council requires a qualified majority of the voting weight (at least 169 votes out of a total of 237, about 71%), plus 62% of the population and a majority of the states. (The latter criterion is dominated by the other two in this example so the requirement is effectively only for a double majority.) In this exercise the desired powers are taken to be proportional to the square roots of the electorates (or populations) of the member countries in order to equalise the voting powers of citizens in all countries.

The power indices were computed with complete accuracy, since it was easy to do so for a voting body with such a small value of n. Convergence was achieved with a stopping criterion of 10^{-8} but it was not possible to get convergence to a better accuracy than that. This empirical finding is consistent with the theoretical level of accuracy that is achievable for this number of voters. This level of accuracy is just about adequate to obtain the weights to within an accuracy of one hundredth of one percent.

The same exercise was also carried out assuming that all twelve current candidate countries have joined, with n = 27. Here the iterative algorithm was very much slower but the accuracy achievable was much greater. The stopping rule used here was 10^{-10}.

The conclusion of this section is that empirical approaches to solving equation (1) work well.

N15	q_1=169	q_2=62%			
Weighted Votes		$w^{(0)}$ Weight%	$p(w^{(0)})$ Bz Index %	d=p(w*) √Pop%	w* "Fair" Weight %
29	Germany	12.24	12.11	13.97	15.12
29	UK	12.24	11.99	11.87	12.06
29	France	12.24	11.99	11.84	12.05
29	Italy	12.24	11.99	11.70	11.99
27	Spain	11.39	11.11	9.68	9.34
13	Netherlands	5.49	5.50	6.12	5.98
12	Greece	5.06	5.16	5.00	4.64
12	Belgium	5.06	5.16	4.93	4.61
12	Portugal	5.06	5.16	4.87	4.58
10	Sweden	4.22	4.30	4.59	4.47
10	Austria	4.22	4.30	4.38	4.41
7	Denmark	2.95	3.09	3.55	3.22
7	Finland	2.95	3.09	3.50	3.20
7	Ireland	2.95	3.09	2.98	3.03
4	Luxembourg	1.69	1.96	1.01	1.29
237		100.00	100.00	100.00	100.00

Adapted from Leech (forthcoming). Bz: Normalised Banzhaf index.
q_1 threshold in terms of weighted votes; q_2= the population condition.

Table 3. "Fair" Weights for EU QMV under the Nice Treaty (n=15)

The Choice of Weights in Large Voting Bodies: Existence and Uniqueness

It is - potentially at least – to consider the problem of finding the weights by analogy with that of finding equilibrium prices in economic Walrasian general equilibrium. The motivation for this is that the structure of the problem embodied in equation (1) is at least sufficiently similar to it to suggest it is worthy of investigation and there is a large literature, both theoretical and empirical. The key reference on the topic is Arrow and Hahn (1971).

General equilibrium modelling is concerned with an economy characterised by a system of n excess demand (or equivalently supply) functions for n goods which vary according to the values of their prices. The parallels are between the power indices of the n voters and quantities of the goods supplied, and between voting weights and prices. At the most general the power indices can be thought of as probabilities (non-normalised Banzhaf-Penrose indices) and the weights are positive real numbers; voters with zero weight can be ruled out. The institution is assumed large enough for the power indices to be continuously differentiable functions. In order to discuss the questions of existence and uniqueness of a solution to equation (1), w^* - corresponding to an equilibrium price vector in general equilibrium theory - , it is convenient to make certain assumptions.

First, it is relative weights that are important in determining power. The power index functions are homogeneous of degree zero:

$$p(w) = p(kw) \text{ for any scalar } k > 0.$$

Powers do not depend on the absolute voting weights but the relative weights of the voters. This means that, in general, the set from which the weights are chosen can be taken to be the unit (n-1)-simplex, X, already defined. This requires that the quota is adjusted accordingly, q being replaced by kq. This condition seems innocuous in the context of voting power theory.

The second assumption is that the power index functions are continuous over their domain X. This can reasonably be taken to hold if n is large enough.

The "fair" weight vector can be shown to exist using a fixed-point theorem. Let the "excess power demand" functions be a vector

$$z(w) = d - p(w). \tag{3}$$

Consider an arbitrary point $w \in X$. Now define the following routine to update w and find another point in X.

First, normalise d and p(w) to ensure that they belong to X. Then adjust the weights by amounts equal to m_i where

$$m_i = \lambda(d_i - p_i(w)) \tag{4}$$

and $\lambda > 0$ is some scalar. Then the new vector becomes $w_i + m_i$. By construction this new vector belongs to X since $\Sigma\, m_i = 0$ always, and the possibilities that $w_i + m_i \leq 0$ and $w_i + m_i \geq 1$, for any i, can be excluded by suitable choice of λ. If either of these cases occurs, then we choose a smaller value for λ in (4).

This defines a routine which can be written:

$$T(w) = w + m(w) \tag{5}$$

Therefore we can say that T(w) is a continuous mapping which takes points in X into points in X. It is mapping of X into itself. It follows that if, for some w* we have T(w*) = w*, then w* is a fixed point of the mapping.

This is an application of Brouwer's fixed point theorem which states that every continuous mapping of a compact convex set into itself has a fixed point.

The fixed point is obviously the required weight vector since then $m(w^*) = 0$ and $d = p(w^*)$.

This establishes the existence of a solution. The key assumption on which this result depends is that the power indices can be taken as continuous functions of the weights. It remains to find conditions for uniqueness.

In one sense it matters little, if at all, whether the weights are unique. All that really matters is that they have the property of giving rise to the required power indices. If there are multiple solutions to equation (1), then any will suffice since they all give the same power distribution among the voters. However, this may be an undesirable feature if non-uniqueness implied non-monotonicity, and it is of some interest to know under what conditions this can be avoided.

In general equilibrium theory, an important case where an equilibrium can be shown to be unique is that where all goods can be assumed to be gross substitutes. The excess demand functions satisfy the properties that the direct partial derivatives are negative, and all cross-partial derivatives are positive. By analogy we write $z_{ii}(w) < 0$ and $z_{ij}(w) > 0$ for all i and $j \neq i$, where z_{ij} is the partial derivative of z_i with respect to w_j. The first property follows from monotonicity of the (non-normalised) Banzhaf-Penrose index, but the second condition – the equivalent of the assumption of gross substitutes – is not true in general.

In one important case, however, it is true: where the decision rule is based on a simple majority with q = 0.5. In this case we know, from a well-known result, that the probability that voter i is in the majority can be written as a combination of power and luck:

Pr[voter i is on the winning side] = $0.5p_i + 0.5$,

and therefore, p_i = 2Pr[i is on the winning side] – 1.

An increase in the weight of any other voter j will unambiguously reduce (or at least not increase) the probability that i is on the winning side and therefore the power index p_i.But this cannot be assumed to hold in general.

A weaker condition for uniqueness is diagonal dominance. This occurs in general equilibrium theory where we can in effect assume that the own price effect dominates all the effects of the other prices, that the effect on the excess demand function for good i of a change in the price of good i dominates all the effects of changes in prices of other goods j. In our context the parallel condition would be that the effect of an increase in the weight of voter i dominates all the effects of changes in the weights of other voters in some sense.

Diagonal dominance requires that there is a vector of strictly positive numbers h(w) such that

$$h_i(w)\left|z_{ii}(w)\right| > \sum_{j\neq i} h_j(w)\left|z_{ij}(w)\right| \qquad i = 1, \ldots, n$$

Arrow and Hahn argue that diagonal dominance is a reasonable, and weak, condition under which equilibrium can be shown to be unique, although not one which is well founded in economic theory. It would be interesting if it could be shown to hold under fairly general conditions in the somewhat different context of voting power theory.

In this section I have attempted to draw some lessons from the extensive literature on economic general equilibrium theory, on the basis of a superficial similarity of the structure of the problem. If this line of research can be developed further then it might offer the prospect of being able to exploit the large empirical literature on computable general equilibrium theory for voting power analysis. There is one important difference however in that the problem of finding the "fair" weights is not that of finding an equilibrium and therefore there is no role for Walras law in the former as in the latter. The real parallel is perhaps rather in the mathematical structure in that both problems involve the use of fixed point theorems.

Conclusions

This paper hasinvestigated the problem of using power indices as a tool for the design of a voting institution which makes decisions by weighted voting. I have considered the problem from two general perspectives: where the number of voters is small and the discreteness of the mathematics is paramount, and where thenumber of voters is large, making it possible to use continuous mathematics. I have discussed some of the issues involved including the relationship with the apportionment problem, empirical solutions in the context of real-world institutions and the matter of proving existence and uniqueness. There are suggestions for further theoretical and empirical research but the overall conclusion is that power indices can be a useful normative tool.

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