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How to gamble against all odds

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HOW TO GAMBLE AGAINST ALL ODDS

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ABSTRACT. We compare the prediction power of betting strategies (aka *martingales*) whose wagers take values in different sets of reals. A martingale whose wagers take values in a set A is called an A-martingale. A set of reals B *anticipates* a set A, if for every A-martingale there is a countable set of B-martingales, such that on every binary sequence on which the A-martingale gains an infinite amount at least one of the B-martingales gains an infinite amount, too.

We show that for a wide class of pairs of sets A and B, B anticipates A if and only if A is a subset of the closure of rB, for some $r > 0$, e.g., when B is well ordered (has no left-accumulation points). Our results answer a question posed by Chalcraft et al. (2012).

1. INTRODUCTION

Player 0, the cousin of the casino owner, is allowed to bet sums of money only within a set A (a subset of the real numbers). The regular casino players 1,2,3,... (countably many players) are allowed to make bets only within another set B. Player 0 announces her betting strategy first; then the regular players announce theirs. Now the casino owner wants to fix an infinite sequence of heads and tails, such that player 0 makes infinite gains, while every regular player gains only a finite amount. Can this be done when, e.g.,

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A is the set of all even integers, and B the set of all odds? We will present some sufficient and necessary conditions on the pair of sets A and B.

If it cannot be done, we say that B *(countably) anticipates* A. As it turns out, the even integers anticipate the odd integers, but not vice versa (see Theorem 7).

The origins of the countability condition lie in algorithmic randomness, where computability considerations are involved, as the set of computable strategies is countable (see also Peretz (2013)).

Chalcraft et al. (2012) showed that when A and B are finite, B anticipates A if and only if B contains a positive multiple of A. They asked whether this characterization extends to infinite sets (note that their framework deals only with computable strategies, while our definitions do not directly concern computability issues; see also Remark 1 below).

Theorem 8 implies a negative answer to this question. Nevertheless, we present some interesting classes of pairs A, B to which the characterization does extend. In Theorem 6, A is bounded and $B \setminus \{0\}$ is bounded away from 0; in Theorem 7, B is well ordered.[1]

Previous work on martingales with restricted wagers (Bienvenu et al. (2012), Teutsch (2013), Peretz (2013)) employed various solutions in various specific situations. The present paper proposes a systematic treatment that applies to most of the previously studied situations, as well as many others. Our proofs are elementary and constructive. The proposed construction is recursive: it does not rely on what the martingales do in the future; therefore we believe our method can be useful in many frameworks where computational or other constraints may be imposed.

The rest of the paper is organized as follows. Section 2 presents the definitions, results, and a few examples, as well as a discussion of related previous work. The next two sections contain proofs. Section 5 points to open problems and further directions.

2. Definitions and Results

A *martingale* is a gambling strategy that bets on bits of a binary sequence. Formally, it is a function $M : \{\mathsf{h}, \mathsf{t}\}^* \to \mathbb{R}$ that satisfies

$$M(\sigma) = \frac{M(\sigma\mathsf{h}) + M(\sigma\mathsf{t})}{2},$$

for every string $\sigma \in \{\mathsf{h}, \mathsf{t}\}^*$.

[1] We can in fact restrict our attention to closed sets (Lemma 3).

The *increment* of M at $\sigma \in \{\mathsf{h}, \mathsf{t}\}^*$ is defined as

$$M'(\sigma) = M(\sigma\mathsf{h}) - M(\sigma).$$

For $A \subset \mathbb{R}_+$, we say that M is an A-*martingale* if $|M'(\sigma)| \in A$, for every $\sigma \in \{\mathsf{h}, \mathsf{t}\}^*$.

The empty string is denoted ε and $M(\varepsilon)$ is called the *initial value* of M. Note that a martingale is determined by its initial value and its increments.

The initial sub-string of length n of a binary sequence, $X \in \{\mathsf{h}, \mathsf{t}\}^\infty$, is denoted $X \upharpoonright n$. A martingale M *succeeds* on X, if $\lim_{n\to\infty} M(X \upharpoonright n) = \infty$ and $M(X \upharpoonright n) \geq |M'(X \upharpoonright n)|$, for every n. The latter condition asserts that M doesn't bet on money it doesn't have. The set of sequences on which M succeeds is denoted $\mathrm{succ}(M)$. A martingale N *dominates* M if $\mathrm{succ}(N) \supseteq \mathrm{succ}(M)$, and a set of martingales $\mathcal{N}$ dominates M if $\bigcup_{N\in\mathcal{N}} \mathrm{succ}(N) \supseteq \mathrm{succ}(M)$.

The following are non-standard definitions.

Definition 1. A set $B \subseteq \mathbb{R}_+$ *singly anticipates* a set $A \subseteq \mathbb{R}_+$, if every A-martingale is dominated by some B-martingale. If A singly anticipates B and B singly anticipates A, we say that A and B are *strongly equivalent*.

Definition 2. A set $B \subseteq \mathbb{R}_+$ *countably anticipates* (*anticipates*, for short) a set $A \subseteq \mathbb{R}_+$, if every A-martingale is dominated by a countable set of B-martingales. If A anticipates B and B anticipates A, we say that A and B are *(weakly) equivalent*. If B does not anticipate A we say that A *evades* B.

Note that both "singly anticipates" and "anticipates" are reflexive and transitive relations (namely, they are preorders),[2] and that single anticipation implies anticipation. Also, if $A \subseteq B$ then B singly anticipates A.

Remark 1. The motivation for the definition of countable anticipation comes from the study of algorithmic randomness (see, e.g., Downey and Riemann (2007)), where it is natural to consider the countable set of all B-martingales that are computable relative to M (see Peretz (2013)). Formally, a set $B \subseteq \mathbb{R}_+$ *effectively anticipates* a set $A \subseteq \mathbb{R}_+$, if for every A-martingale M and every sequence $X \in \mathrm{succ}(M)$, there is a B-martingale, computable relative to M, that succeeds on X. More generally, one could define in the same fashion a "$\mathcal{C}$-anticipation" relation with respect to any complexity class $\mathcal{C}$. Our main focus will be on countable anticipation, and specifically on sets A and B such that B does not countably anticipate A; and therefore B does not $\mathcal{C}$-anticipate A for any complexity class $\mathcal{C}$. When we present cases in which anticipation does hold, the dominating martingales

[2] The evasion relation is anti-reflexive and is not transitive.

will usually be fairly simple relative to the dominated martingales. We do not presume to rigorously address the computational complexity of those reductions, though.

The topological closure of a set $A \subseteq \mathbb{R}_+$ is denoted $\bar{A}$. The following lemma says that we can restrict our attention to closed subsets of $\mathbb{R}_+$.

Lemma 3. *Every subset of $\mathbb{R}_+$ is strongly equivalent to its closure.*

Proof. Let $A \subset \mathbb{R}_+$ and let M be an $\bar{A}$-martingale. Define an A-martingale S by

$$S(\varepsilon) = M(\varepsilon) + 2,$$
$$S'(\sigma) \in A \cap (M'(\sigma) - 2^{-|\sigma|}, M'(\sigma) + 2^{-|\sigma|}),$$

where $|\sigma|$ is the length of σ. Clearly, $S(\sigma) > M(\sigma)$, for every $\sigma \in \{\mathsf{h}, \mathsf{t}\}^*$; therefore $\operatorname{succ}(S) = \operatorname{succ}(M)$. □

Another simple observation is that for every $A \subseteq \mathbb{R}_+$ and $r > 0$, A is strongly equivalent to $rA := \{ra : a \in A\}$. This observation leads to the next definition.

Definition 4. Let $A, B \subseteq \mathbb{R}_+$. We say that A and B are *proportional*, if there exists $r > 0$ such that $rA = B$. If we only require that $rA \subseteq \bar{B}$, then A is proportional to a subset of the closure of B. In that case we say that A *scales* into B.

From the above and the fact that $A \subseteq B$ implies that B singly anticipates A, we have the following lemma.

Lemma 5. *If A scales into B, then B anticipates A, for every $A, B \subseteq \mathbb{R}_+$.*

The next two theorems provide conditions under which the converse of Lemma 5 also holds.

Theorem 6. *For every $A, B \subseteq \mathbb{R}_+$, if $\sup A < \infty$ and $0 \notin \overline{B \setminus \{0\}}$, then B anticipates A only if A scales into B.*

Theorem 7. *For every $A, B \subseteq \mathbb{R}_+$, if B is well ordered, namely, $\forall x \in \mathbb{R}_+\ x \notin \overline{B \setminus [0, x]}$, then B anticipates A only if A scales into B.*

Chalcraft et al. (2012) studied effective anticipation between finite subsets of $\mathbb{R}$. They showed that (effective) anticipation is equivalent to containing a proportional set on the domain of finite sets. They further asked whether their result extends to infinite sets. In particular, they asked if $\mathbb{Z}_+$ anticipates the set $V = \{0\} \cup [1, \infty)$. Peretz (2013) showed that the set $\{1 + \frac{1}{n}\}_{n=1}^{\infty} \subset V$ evades (i.e., is not anticipated by) $\mathbb{Z}_+$, and the set $\{\frac{1}{n}\}_{n=1}^{\infty}$

evades V. All of the above results follow immediately from Theorem 6. Furthermore, any set that contains two $\mathbb{Q}$-linearly independent numbers (e.g., $\{1, \pi\}$) evades $\mathbb{Z}_+$.

In light of Lemma 3, one may rephrase the above question of Chalcraft et al. (2012) and ask whether, for infinite sets, anticipation is equivalent to scaling. The answer is still negative (Theorem 8), although Theorems 6 and 7 above gave some conditions under which the equivalence does hold.

Theorem 7 says that if, for example, $A = \mathbb{Z}_+$ and B is a subset of $\mathbb{Z}_+$ whose density is zero, then A evades B. This is because B is well ordered, and A does not scale into B (by the zero density). Another example is when B is the set of all odd integers where, again, A evades B. Note that the set of even integers is proportional to $\mathbb{Z}_+$ and hence is equivalent to $\mathbb{Z}_+$. Therefore the even integers evade the odds, but not vice versa.

Furthermore, we can take any subset of $\mathbb{Z}_+$ that does not contain an "ideal" (i.e., all the integer multiples of some number, namely, a set proportional to $\mathbb{Z}_+$). If B is of the form $B = \mathbb{Z}_+ \setminus \{n \cdot \phi(n)\}_{n=1}^{\infty}$ for some function $\phi : \mathbb{N} \to \mathbb{N}$, then $\mathbb{Z}_+$ evades B even when, for example, the function ϕ grows very rapidly. In particular, the density of B could equal one.

The previous theorems gave some necessary conditions for anticipation. The next theorem gives a sufficient condition.

Theorem 8. *Let $A, B \subset \mathbb{R}_+$. For $x > 0$, let*

$$P(x) = \{t \geq 0 : t \cdot (A \cap [0, x]) \subseteq \bar{B} \cup \{0\}\}.$$

For any $M \geq 0$ let $q_M(x) = \max(P(x) \cap [0, M])$. If for some M, $\int_0^\infty q_M(x)dx = \infty$, then B singly anticipates A.

This is equivalent to the following seemingly stronger theorem.

Theorem 8*. Let $A, B \subset \mathbb{R}_+$. Suppose there is a non-increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$, such that

(1) $\bar{B} \supset f(x)\,(A \cap [0, x])$, for every $x \in \mathbb{R}_+$; and
(2) $\int_0^\infty f(x)\,\mathrm{d}x = \infty$.

Then, B singly anticipates A.

*Proof of Theorem 8**. Let M be an A-martingale. By Lemma 3 we may assume that B is closed. Define a B-martingale, S, by

$$S(\varepsilon) = f(0)M(\varepsilon),$$
$$S'(\sigma) = f(M(\sigma))M'(\sigma).$$

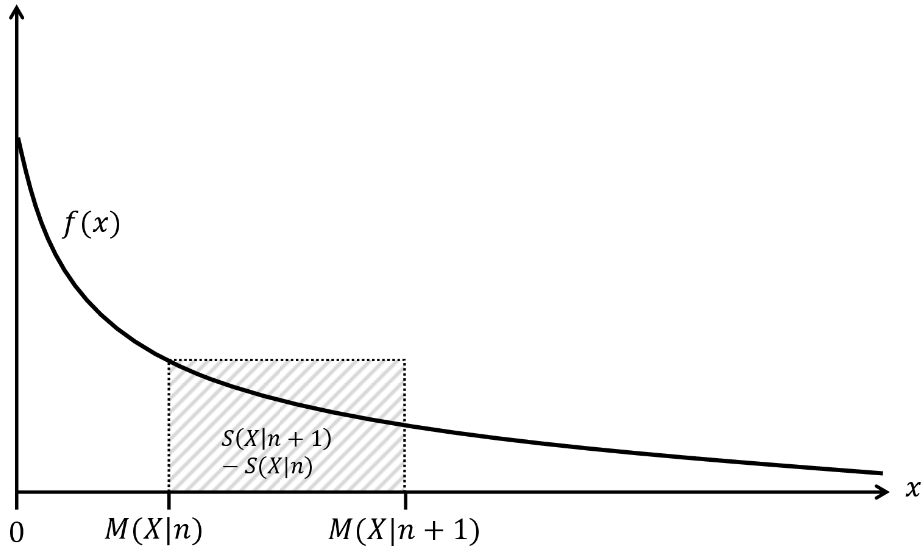

FIGURE 1. Inequality (1)

Let $X \in \{\mathsf{h}, \mathsf{t}\}^{\mathbb{N}}$ such that $X \in \operatorname{succ}(M)$, and let $n \in \mathbb{N}$. Since f is non-increasing,

$$S(X \restriction n+1) - S(X \restriction n) \geq \int_{M(X \restriction n)}^{M(X \restriction n+1)} f(x)\, \mathrm{d}x. \tag{1}$$

It follows by induction that

$$S(X \restriction n) \geq \int_{0}^{M(X \restriction n)} f(x)\, \mathrm{d}x,$$

which concludes the proof, since $M(X \restriction n) \to \infty$ and $\int_0^\infty f(x)\, \mathrm{d}x = \infty$. □

If, for example, A scales into B, namely, $rA \subseteq \bar{B}$, the function f in Theorem 8* can be taken to be simply $f(x) = r$. Also note that when A and B are finite, such a function f as in the theorem exists iff A scales into B.

The theorem tells us, for example, that although $\mathbb{R}_+$ does not scale into the interval $[0, 1]$, these two sets are (strongly) equivalent: to see that $[0, 1]$ singly anticipates $\mathbb{R}_+$, apply Theorem 8* with $f(x) = \min\{\frac{1}{x}, 1\}$.

Another example is the set $A = \{2^n\}_{n=-\infty}^{+\infty}$ being (strongly) equivalent to $B = \{2^n\}_{n=-\infty}^{0}$, although A does not scale into B. To see this, apply Theorem 8* with $f(x) = \min\{1/2^{\lfloor \log_2 x \rfloor}, 1\}$.

We previously saw, by Theorem 7, that $A = \mathbb{Z}_+$ evades $B = \mathbb{Z}_+ \setminus \{n \cdot \phi(n)\}_{n=1}^{\infty}$. Now look at this example in the context of Theorem 8, to see that there is no contradiction. Suppose $\phi(n)$ is increasing, and $\phi(n) \to \infty$. Then for any $x > 0$, the set $P(x)$ in the theorem is unbounded, i.e., $\sup P(x) = \infty$. Nevertheless, fix any choice of M, and note that for every x large enough, $P(x)$ does not contain any nonzero number smaller than M, i.e., $P(x) \cap [0, M] = \{0\}$. Thus, for any choice of M, $q_M(x) = 0$ for every x large enough. In particular, $\int_0^\infty q_M(x)dx < \infty$.

3. Proof of Theorem 6

Throughout this section $A, B \subset \mathbb{R}_+$ are two sets satisfying

$$\sup A < \infty,$$
$$0 \notin \overline{B \setminus \{0\}},$$
$$A \text{ does not scale into } B.$$

We must show A (countably) evades B.

Since A does not scale into B, one thing that B-martingales cannot do in general is to mimic A-martingales, not even up to a constant ratio. We use this idea in order to construct a sequence of heads and tails that will separate between the two types of martingales.

3.1. **Ratio minimization.** Let N and M be martingales with N non-negative. We say that $x \in \{\mathsf{h}, \mathsf{t}\}$ is the *N/M-ratio-minimizing outcome* at $\sigma \in \{\mathsf{h}, \mathsf{t}\}^{<\infty}$ (assuming $M(\sigma) > 0$) if either

(1) $\frac{N(\sigma)}{M(\sigma)} > \frac{N(\sigma x)}{M(\sigma x)}$, or
(2) $\frac{N(\sigma)}{M(\sigma)} = \frac{N(\sigma x)}{M(\sigma x)}$ and $M(\sigma x) > M(\sigma)$, or
(3) $M'(\sigma) = N'(\sigma) = 0$ and $x = \mathsf{h}$.

In words: our first priority is to make the ratio N/M decrease; if this is impossible (i.e., the increments of N and M are proportional to their value at σ, and so N/M doesn't change), then we want M to increase so as to insure that $M(\sigma x) > 0$; if that is impossible as well (i.e., both increments are 0), we set x to be h, for completeness of definition only.

Our definition extends to finite/infinite extensions of σ by saying that X is the length $|X|$ (possibly $|X| = \infty$) *N/M-ratio-minimizing extension* of σ, if X_{t+1} is the N/M-ratio-minimizing outcome at $X \restriction t$, for every $|\sigma| \leq t < |X|$.

For any such N, M, and σ the infinite N/M-ratio-minimizing extension of σ, X, makes the ratio $N(X \restriction t)/M(X \restriction t)$ monotonically converging to a limit $L \in \mathbb{R}_+$, as $t \to \infty$. The next lemma will help us argue that the ratio between the increments $N'(X \restriction t)/M'(X \restriction t)$ also converges to L in a certain sense.

Lemma 9 (Discrete l'Hôpital rule). *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers. Assume that $a_n > 0$, for every n. If b_n/a_n monotonically converges to a limit $L \in \mathbb{R}$, and $\sup\{\frac{1}{n}\sum_{k=1}^{n}|a_{k+1}-a_k|\} < \infty$, then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|(b_{k+1}-b_k)-L(a_{k+1}-a_k)| = 0.$$

Proof. Since $\frac{1}{n}\sum_{k=1}^{n}|a_{k+1}-a_k|$ is bounded, $\frac{1}{n}\sum_{k=1}^{n}(a_{k+1}-a_k)$ is bounded, too; therefore

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(b_{k+1}-b_k)-L(a_{k+1}-a_k) = 0. \tag{2}$$

It remains to prove that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[b'_k - La'_k]_+ = 0, \tag{3}$$

where $a'_k = a_{k+1}-a_k$ and similarly $b'_k = b_{k+1}-b_k$.

We may assume w.l.o.g. that $\frac{b_n}{a_n} \searrow L$ (otherwise consider the sequence $(-b_n)_{n=1}^{\infty}$). Namely, $\frac{b_k}{a_k} \geq \frac{b_{k+1}}{a_{k+1}}$, which implies that $b'_k \leq \frac{b_k}{a_k}a'_k$; hence

$$[b'_k - La'_k]_+ \leq \left[\left(\frac{b_k}{a_k}-L\right)a'_k\right]_+ \leq \left(\frac{b_k}{a_k}-L\right)|a'_k|.$$

Now (3) follows since $\frac{b_k}{a_k}$ converges to L and $\frac{1}{n}\sum_{k=1}^{n}|a'_k|$ is bounded. □

Corollary 10. *Let M and N be a pair of martingales and $\sigma \in \{\mathsf{h},\mathsf{t}\}^*$. Assume that N is non-negative, $M(\sigma) > 0$, and M' is bounded.*[3] *Let X be the infinite N/M-ratio-minimizing extension of σ and $L = \lim_{t\to\infty}\frac{N(X\restriction t)}{M(X\restriction t)}$. For every $\epsilon > 0$ the set*

$$\{t : |N'(X\restriction t) - L\cdot M'(X\restriction t)| > \epsilon\}$$

has zero density.

Proof. Note that

$$|N'(X\restriction t) - L\cdot M'(X\restriction t)| = \\ |(N(X\restriction t+1)-N(X\restriction t)) - L\cdot(N(X\restriction t+1)-N(X\restriction t))|.$$

[3]The assumption that M' is bounded can be relaxed by assuming only that $\frac{1}{N}\sum_{t=0}^{N-1}|M'(X\restriction t)|$ is bounded.

By Lemma 9, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |N'(X \restriction t) - L \cdot M'(X \restriction t)| = 0,$$

which implies that the density of the set

$$\{t : |N'(X \restriction t) - L \cdot M'(X \restriction t)| > \epsilon\}$$

is zero, for every $\epsilon > 0$. □

The next step is to construct the history-independent A-martingale prescribed by Theorem 6. We formalize the properties of this martingale in the following lemma.

Lemma 11. *Let $A, B \subset \mathbb{R}_+$. Suppose that $\sup A < \infty$ and A does not scale into B; then there exists a history-independent A-martingale with positive initial value, M, such that for every non-negative B-martingale, N, and every $\sigma \in \{\mathtt{h}, \mathtt{t}\}^*$ such that $M(\sigma) > 0$, the infinite N/M-ratio-minimizing extension of σ, X, satisfies*

$$\lim_{t\to\infty} \frac{N(X \restriction t)}{M(X \restriction t)} = 0.$$

Note that the A-martingale, M, does not depend on N or σ, so the same M can be used against any N at any σ that leaves $M(\sigma)$ positive.

Proof of Lemma 11. Let $\{a_n\}_{n=0}^{\infty}$ be a countable dense subset of $A \setminus \{0\}$. For every positive integer t, let $n(t) \in \mathbb{Z}_+$ be the largest integer such that $2^{n(t)}$ divides t. Define $x_t := a_{n(t)}$. The sequence $\{x_t\}_{t=1}^{\infty}$ has the property that the set

$$\{t : |x_t - a| < \epsilon\} \tag{4}$$

has positive density, for every $\epsilon > 0$ and $a \in A$.

Let M be a history-independent martingale whose increment at time t is x_t, for every $t \in \mathbb{Z}_+$ (with an arbitrary positive initial value). Let N be an arbitrary non-negative B-martingale. Suppose that $M(\sigma) > 0$ and let X be the infinite N/M-ratio-minimizing extension of σ and let $L = \lim_{t\to\infty} \frac{N(X\restriction t)}{M(\restriction t)}$. Corollary 10 and (4) guarantee that $L \cdot A \subset \bar{B}$. By the assumption that A does not scale into B, we conclude that $L = 0$. □

3.2. **The casino sequence.** In the rest of this section we assume that $\sup A = \inf(B \setminus \{0\}) = 1$. This is w.l.o.g. since proportional sets are (strongly) equivalent.

We begin with an informal description of the casino sequence. Lemma 11 provides a history-independent A-martingale, M, that can be used against any B-martingale.

Given a sequence of B-martingales, $N_1, N_2, \ldots$, we start off by ratio-minimizing against N_1. When M becomes greater than N_1, we proceed to the next stage. We want to make sure that N_1 no longer makes any gains. This is done by playing adversarial to N_1 whenever he wagers a positive amount.

At times when N_1 wagers nothing (i.e., $N_1' = 0$), we are free to choose either h or t without risking our primary goal. At those times we turn to ratio-minimizing against N_2, while always considering the goal of keeping N_1 from making gains a higher priority. Since $\inf\{B \setminus \{0\}\} = 1 > 0$, it is guaranteed that at some point we will no longer need to concern N_1, and hence, at some even further point, M will become greater than $N_1 + N_2$.

The process continues recursively, where at each stage our highest priority is to prevent N_1 from making gains, then N_2, N_3, and so on until N_k; and if none of $N_1, \ldots, N_k$ wagers any positive amount, we ratio-minimize against N_{k+1}.

When a positive wager of some N_i, $i \in \{1, \ldots k\}$, is answered with an adversarial outcome, a new index k' must be calculated, so that M is sufficient to keep $N_1, \ldots, N_{k'}$ from making gains. That is, $M > N_1 + \cdots + N_{k'}$.

An inductive argument shows that for every fixed k, there is a point in time beyond which none of $N_1, \ldots, N_k$ will ever wager a positive amount; therefore, at some even further point, M becomes greater than $N_1 + \cdots + N_{k+1}$; hence the inductive step.

The above explains how the value of each N_i converges to some $L_i \in \mathbb{R}_+$, and the limit inferior of the value of M is at least $\sum_{i=1}^{\infty} L_i$. In order to make sure that M goes to infinity we include, among the N_is, infinitely many martingales of constant value 1.

We turn now to a formal description. As mentioned above, we assume without loss of generality that $\sup A = \inf\{B \setminus \{0\}\} = 1$. We additionally assume that $0 \in B$, and so we can convert arbitrary B-martingales to non-negative ones by making them stop betting at the moment they go bankrupt.

Let M be a history-independent A-martingale provided by Lemma 11. Let $N_1, N_2, \ldots$ be a sequence of non-negative B-martingales. Assume without loss of generality that infinitely many of the N_is are the constant 1 martingale.

We define a sequence $X \in \{\mathsf{h}, \mathsf{t}\}^\infty$ recursively. Assume $X \upharpoonright t$ is already defined.

First we introduce some notation. Denote the value of M at time t by $m(t) = M(X \restriction t)$, and similarly $n_i(t) = N_i(X \restriction t)$, for every $i \in \mathbb{N}$. Let

$$\begin{aligned} S_i(t) &= \sum_{j=1}^{i} n_j(t), \\ k(t) &= \max\{i : S_i(t) < m(t)\}, \text{ and} \\ S(t) &= S_{k(t)}(t). \end{aligned}$$

Note that the maximum is well defined, since the $n_i(t)$ include infinitely many 1s.

We are now ready to define X_{t+1}. We distinguish between two cases: Case I: there exists $1 \le j \le k(t)$ such that $N'_j(X \restriction t) \neq 0$; Case II: $N'_1(X \restriction t) = \cdots = N'_{k(t)}(X \restriction t) = 0$.

In Case I, let $i = \min\{j : N'_j(X \restriction t) \neq 0\}$ and define

$$X_{t+1} = \begin{cases} \mathsf{t} & \text{if } N'_i(X \restriction t) > 0, \\ \mathsf{h} & \text{if } N'_i(X \restriction t) < 0. \end{cases}$$

In Case II, X_{t+1} is the $\frac{N_{k(t)+1}}{M - S(t)}$-ratio-minimizing outcome at $X \restriction t$. Explicitly,

$$X_{t+1} = \begin{cases} \mathsf{t} & \text{if } \frac{N'_{k(t)+1}(X \restriction t)}{M'(X \restriction t)} > \frac{n_{k(t)+1}}{m(t) - S(t)}, \\ \mathsf{h} & \text{if } \frac{N'_{k(t)+1}(X \restriction t)}{M'(X \restriction t)} \le \frac{n_{k(t)+1}}{m(t) - S(t)}. \end{cases}$$

Consider the tuple $\alpha(t) = (\lfloor n_1(t) \rfloor, \ldots, \lfloor n_{k(t)}(t) \rfloor)$. In Case I, $\alpha(t+1)$ is strictly less that $\alpha(t)$ according to the lexicographic order. In Case II, $\alpha(t)$ is a prefix of $\alpha(t+1)$, and so under a convention in which a prefix of a tuple is greater than that tuple, we have that $\{\alpha(t)\}_{t=1}^{\infty}$ is a non-increasing sequence.[4] Let $k = \liminf_{t \to \infty} k(t)$. It follows that from some point in time, α consists of at least k elements; therefore the first k elements of α must stabilize at some further point in time. Namely, for t large enough we have $\lfloor n_i(t) \rfloor = \lim_{t' \to \infty} \lfloor n_i(t') \rfloor < \infty$, for every $i \le k$. Since the increments of $n_i(t)$ are bounded below by 1, $n_1(t), \ldots, n_k(t)$ stabilize, too. Also, since $m(t) > S(t)$, we have $\liminf_{t \to \infty} m(t) \ge \lim_{t \to \infty} S_i(t)$, for every $i \le \liminf_{t \to \infty} k(t)$. Since there are infinitely many i's for which $n_i(t)$ is constantly 1, the proof of Theorem 6 is concluded by showing that $\liminf_{t \to \infty} k(t) = \infty$.

[4]Alternatively, one can use the standard lexicographic order where $\alpha(t)$ is appended with an infinite sequence of ∞ elements.

Assume by negation that $\liminf_{t\to\infty} k(t) = k < \infty$. There is a time T_0 such that $n_k(t) = \lim_{t'\to\infty} n_k(t')$ and $k(t) \geq k$, for every $t > T_0$. There cannot be a $t > T_0$, for which $k(t) > k$ and $k(t+1) = k$. That would mean a Case I transition from time t to $t+1$ and we would have $n_i(t+1) < n_i(t)$, for some $i \leq k$. It follows that $k(t) = k$, for every $t > T_0$.

From time T_0 ratio-minimization against n_{k+1} takes place. Let $l = \liminf_{t\to\infty} n_{k+1}(t)$. If $l = 0$, then $n_{k+1}(t) < 1$, for some $t > T_0$; at this point n_{k+1} stabilizes (otherwise N_{k+1} would go bankrupt); therefore $n_{k+1}(t) = 0$; therefore $k(t) > k$, which is not possible. If $l > 0$, then by Lemma 11, there must be some time $t > T_0$ in which $m(t) > n_{k+1}(t) + S_k(T_0) = n_{k+1}(t) + S(t)$, which contradicts the definition of $S(t)$.

4. PROOF OF THEOREM 7

To show that if B is well ordered and A does not scale into B, then A evades B, we construct an A-martingale M, s.t. for any B-martingales $N_1, N_2, \ldots$ we construct a sequence X on which M succeeds, while every N_i does not.

We begin with a rough outline of the proof ideas. M always bets on "heads." Before tackling every N_i, we first gain some money and "put it aside." Then we ratio-minimize against N_i. It will eventually make M sufficiently richer than N_i, so that we can declare N_i to be "fragile" now. This means that from now on the casino can make N_i lose whenever it is "active" (i.e., makes a non-zero bet), since M can afford losses until N_i is bankrupt. When N_i is not active, we can start tackling N_{i+1}, while constantly making sure that we have enough money kept aside for containing the fragile opponents. An important point is to show that once some N_i becomes fragile, it remains fragile unless a lower-index martingale becomes active.

Let (a_n) be a sequence that is dense within $A \setminus \{0\}$, and such that each number in the sequence appears infinitely many times. For example, given a dense sequence (x_n) in $A \setminus \{0\}$, the sequence

$$x_1;\ x_1, x_2;\ x_1, x_2, x_3;\ x_1, x_2, x_3, x_4; \ldots$$

can be used.

Since multiplying A or B by a positive constant does not make a difference, we may assume w.l.o.g. that $a_1 = 1$, and that $\inf(B \setminus \{0\}) = 1$ (B is well ordered; hence in particular $B \setminus \{0\}$ is bounded away from 0).

We construct M and X as follows. Denote $m(t) = M(X \restriction t)$, and similarly $m'(t) = M'(X \restriction t)$. Take integers

$$f(t,k) \geq \max\{m(t), |(a_{k+1} - a_k)/a_k|\}.$$

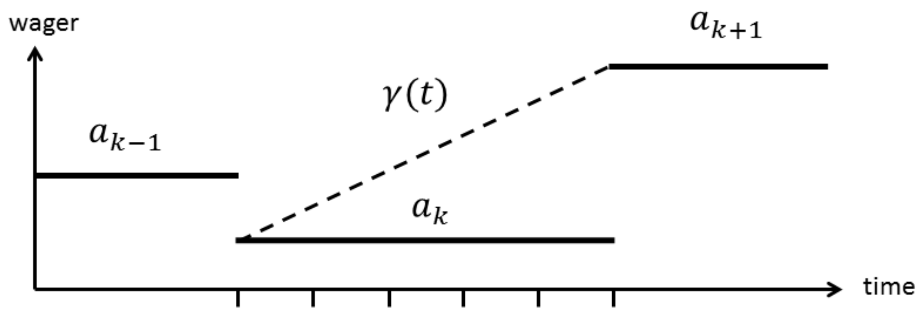

FIGURE 2. γ and the wagers

We take $M(\varepsilon) = a_1$. For the first $f(0,1)$ stages, $M' = a_1$. Then, after stage $t = f(0,1)$, $M' = a_2$ for the next $f(t,2)$ periods, and after stage $t' = t + f(t,2)$, $M' = a_3$ for $f(t',3)$ stages, and so on. But this goes on only as long as no "tails" appears. Whenever a "tails" appears, namely, at a stage t where $X_t = \mathsf{t}$, we revert to playing from the beginning of the sequence, i.e., $M' = a_1$ for the next $f(t,1)$ stages (or until another "tails" appears), then $M' = a_2$, etc.

For $t \geq 0$ we define the function $\gamma(t)$, which is similar to $m'(t)$, but modifies sudden increases of m' into more gradual ones. At the beginning of a block of stages where a_k is wagered (i.e., where $m'(\cdot) = a_k$), γ equals a_k. If $a_{k+1} \leq a_k$, then γ remains a_k throughout this block. Otherwise, it linearly increases until reaching a_{k+1} exactly at the beginning of the next block. I.e., suppose $m'(t) = a_k$, and let $s \leq t$ be the beginning of the block (of length $f(s,k)$) of a_k wagers. If $a_{k+1} \leq a_k$ then $\gamma(t) = a_k$. Otherwise,

$$\gamma(t) = \frac{(s + f(s,k) - t)\, a_k + (t - s) a_{k+1}}{f(s,k)}.$$

Note: $(i)\, \gamma(t) \geq m'(t)$,

(ii) If $X_{t+1} = \mathsf{t}$ then $\gamma(t+1) = a_1 = 1$,

(iii) If $X_{t+1} = \mathsf{h}$ then $\gamma(t+1) - \gamma(t) \leq m'(t)$.

The last one follows from $m'(t) = a_k$, $\gamma(t+1) - \gamma(t) \leq (a_{k+1} - a_k)/f(s,k)$, and $f(s,k) \geq |(a_{k+1} - a_k)/a_k|$, by the definition of f.

Let $N_1, N_2, \ldots$ be B-martingales (and assume these martingales never bet on money that they do not have). Denote $n_i(t) = N_i(X \restriction t)$. To define the sequence X, denote $\nu_k(t) = k + n_1(t) + \ldots + n_k(t)$, and let $p = p(t) \geq 0$

be the largest integer such that

$$m(t) - (\gamma(t) - 1) > \nu_p(t).$$

$N_1, \ldots, N_{p(t)}$ are the "fragile" martingales at time t. Define

$$\mu(t) = m(t) - (\nu_p(t) + 1)$$

and consider two cases. (i) If there exists some index $1 \leq j \leq p(t)$ s.t. $n'_j(t) \neq 0$, let i be the smallest such index, and X_{t+1} is chosen adversely to n'_i. (ii) Otherwise, X_{t+1} is chosen by μ/N_{p+1}-ratio-minimizing, i.e., if $\mu(t) > 0$ and $n'_{p+1}(t)/m'(t) > n_{p+1}(t)/\mu(t)$ then $X_{t+1} = \mathsf{t}$, and otherwise $X_{t+1} = \mathsf{h}$.

We now show that these M and X indeed work.

Lemma 12. *For any t, if $p(t) \geq i$ and $n'_j(t) = 0$ for every $j < i$, then $p(t+1) \geq i$.*

Proof of Lemma 12. We prove the following equivalent claim:
(I) If $i \leq p(t)$ is the smallest index such that $n'_i(t) \neq 0$, then $i \leq p(t+1)$.
(II) If $n'_j(t) = 0$ for any $j \leq p(t)$, then $p(t) \leq p(t+1)$.

In case (II), denote $i = p(t)$. Then in both cases

$$m(t) - (\gamma(t) - 1) > \nu_i(t)$$

is known. Let $\mathcal{L}(t)$ designate the LHS of this inequality. We need to show that $\mathcal{L}(t+1) > \nu_i(t+1)$. Note that for any $j < i$, $n_j(t+1) = n_j(t)$, and that if $X_{t+1} = \mathsf{h}$ then $\mathcal{L}(t+1) \geq \mathcal{L}(t)$, since $m(t+1) = m(t) + m'(t)$ and $\gamma(t+1) \leq \gamma(t) + m'(t)$.

(I) In this case the casino makes i lose, hence $n_i(t+1) \leq n_i(t) - 1$ (recall that $\inf(B \setminus \{0\}) = 1$); therefore $\nu_i(t+1) \leq \nu_i(t) - 1$. If $X_{t+1} = \mathsf{h}$ we are done. If $X_{t+1} = \mathsf{t}$ then $m(t+1) = m(t) - m'(t)$, and $\gamma(t+1) = 1$; therefore, $\mathcal{L}(t+1) = \mathcal{L}(t) - m'(t) + (\gamma(t) - 1) \geq \mathcal{L}(t) - \gamma(t) + (\gamma(t) - 1) = \mathcal{L}(t) - 1$.

(II) In this case $n_i(t+1) = n_i(t)$; hence $\nu_i(t+1) = \nu_i(t)$. If $X_{t+1} = \mathsf{h}$ we are done. If $X_{t+1} = \mathsf{t}$ then $m(t+1) = m(t) - m'(t)$ and $\gamma(t+1) = 1$. But $X_{t+1} = \mathsf{t}$ also implies (by the definition of X) that $\mu(t) > 0$ and $n'_{i+1}(t)/m'(t) > n_{i+1}(t)/\mu(t)$. Since $n'_{i+1}(t)$ is always $\leq n_{i+1}(t)$, we get that $\mu(t) > m'(t)$. Now, $\mathcal{L}(t+1) = (m(t) - m'(t)) - (1 - 1) = m(t) - m'(t) > m(t) - \mu(t) = \nu_i(t) + 1$, because $\mu(t)$ is $m(t) - (\nu_i(t) + 1)$. Thus, $\mathcal{L}(t+1) > \nu_i(t) + 1 > \nu_i(t) = \nu_i(t+1)$. □

Remark: The above argument also proves that M is never bankrupt, i.e., $m(t) \geq m'(t)$, and moreover $m(t) \geq \gamma(t)$, as follows.

In the beginning $1 = m(0) \geq \gamma(0) = 1$. As long as $p(t) = 0$ we are in case (II). In this case, if $X_{t+1} = \mathsf{h}$ then $\mathcal{L}(t)$ does not decrease; hence

$m(t) - \gamma(t)$ does not decrease. And if $X_{t+1} = \mathtt{t}$ we just saw that actually $\mathcal{L}(t+1) > 1 + \nu_i(t+1)$, which implies that $m(t) - \gamma(t) > 0$.

Once $p(t) > 0$, then $\nu_i(t) \geq 1$, hence $\mathcal{L}(t) > \nu_i(t)$ implies that $m(t) - \gamma(t) > 0$; and Lemma 12 implies that $p(t)$ remains > 0.

Lemma 13. *For any i there exists a stage T_i s.t. for any $t > T_i$, $n'_1(t) = n'_2(t) = \ldots = n'_i(t) = 0$, and $p(t) \geq i$.*

Proof of Lemma 13. We proceed by induction over $i \geq 0$; namely, the induction hypothesis is that the lemma holds for $i-1$. Note that the induction base case $i = 0$ holds vacuously.[5]

If $p(t_0) \geq i$ for some stage $t_0 > T_{i-1}$, then $p(t) \geq i$ for every $t \geq t_0$, by lemma 12. From this stage on, the casino chooses adversely to i whenever i is active (because the lower-index players are not active). Therefore, i will be active at no more than $n_i(t_0)$ stages after t_0, since afterwards i has nothing to wager, and we are done.

So assume by way of contradiction that $p(t) = i-1$ for every $t > T_{i-1}$. Then X_{t+1} is μ/N_i-ratio-minimizing. As long as $\mu(t) \leq 0$ we get "heads"; therefore, from some stage on, $\mu > 0$ (as every a_k appears infinitely many times, the sum of the wagers will not converge). Denote $q(t) = n_i(t)/\mu(t)$. $q(t) \geq 0$ is non-increasing and therefore converges to a limit L. Denote $0 \leq r(t) = n_i(t) - L\mu(t)$.

Suppose there exists some k s.t. the wagers $m'(t)$ never reach beyond $a_1, \ldots, a_k$. Hence, there are infinitely many stages t where i over-bets, i.e., $n'_i(t) > q(t)m'(t)$. For $1 \leq j \leq k$, let $x_j = L\,a_j$. Since B is well ordered, there exists some $\delta_j > 0$ s.t. $(x_j, x_j + \delta_j) \cap B = \emptyset$. Since $q(t) \to L$, $q(t) < L + \min_{1 \leq j \leq k}(\delta_j/a_j)$ for t large enough.

When i over-bets and $m'(t) = a_j$, then $r(t+1) = n_i(t+1) - L\mu(t+1) \leq n_i(t) - (La_j + \delta_j) - L(\mu(t) - a_j) = r(t) - \delta_j$. When i does not over-bet, then $n'_i(t) \leq x_j = La_j$, and $r(t+1) = n_i(t+1) - L\mu(t+1) \leq (n_i(t) + La_j) - L(\mu(t) + a_j) = r(t)$. Therefore $r(t)$ does not increase, and infinitely many times it decreases by at least $\delta = \min_{1 \leq j \leq k} \delta_j > 0$; hence eventually $r(t) < 0$, which is a contradiction.

Therefore, there does not exist an index k as above. This implies that for any j, there is a stage t after which a_j is wagered $f(t,j)$ consecutive times, and N_i does not over-bet (otherwise a_{j+1} cannot be reached). Now suppose that $L > 0$. Let $A_0 = \{a_1, a_2, \ldots\}$ be the set of all the values that the sequence (a_n) takes. A_0 is dense in A, and $L \cdot A \not\subseteq \bar{B}$ (since A does not

[5] Incidentally, the inequality defining fragility always holds for $p(t) = 0$, as $m(t) \geq \gamma(t)$ and $\nu_0(t) = 0$ imply that $m(t) - (\gamma(t) - 1) > \nu_0(t)$.

scale into B); therefore, also $L \cdot A_0 \nsubseteq \bar{B}$. Hence, there exists an $a \in A_0$ s.t. the distance between La and $\bar{B}$ is $\delta > 0$.

Let $\Delta = \min\{\delta/a, \delta\}$. For t larger than some T_Δ, $q(t) < L + \Delta$. Since a appears infinitely many times in the sequence (a_n), there exist $T_a > T_\Delta$ and an index j, s.t. $a_j = a$ is wagered $f(T_a, j)$ consecutive times, and at each of these times $n_i'(t) \le La - \delta$, as otherwise $n_i'(t) \ge La + \delta$, but that is over-betting since $(La + \delta)/a = L + \delta/a > q(t)$. Hence, $r(t+1) \le n_i(t) + La - \delta - L(\mu(t) + a) = r(t) - \delta$. But $q(T_a) < L + \delta$; therefore $n_i(T_a) < (L + \delta)\,\mu(T_a)$; hence $r(T_a) < \delta\mu(T_a)$. By the definition of f, $f(T_a, j) \ge m(T_a) \ge \mu(T_a)$; therefore after those $f(T_a, j)$ times, $r < 0$. This cannot be; therefore $L = 0$.

As $q(t) \to 0$, surely $q(t) < 1$ for large enough t, namely, $\mu(t) > n_i(t)$. Since $\mu(t) = m(t) - (\nu_{i-1}(t) + 1)$, we get $m(t) > n_i(t) + \nu_{i-1}(t) + 1 = n_i(t) + ((i-1) + n_1(t) + \ldots + n_{i-1}(t)) + 1 = i + n_1(t) + \ldots + n_i(t) = \nu_i(t)$. At some stage t, M starts wagering 1. For this t, $\gamma(t) = m'(t) = 1$; hence $m(t) - (\gamma(t) - 1) = m(t) > \nu_i(t)$, contradicting our assumption that i is not fragile. □

Lemma 13 states that any N_i is only active a finite number of times, and therefore it is bounded; it also states that for any i and large enough t, $m(t) - (\gamma(t) - 1) > \nu_i(t)$, hence $m(t) > \nu_i(t) + (\gamma(t) - 1) > \nu_i(t) - 1 \ge i - 1$, and therefore $m(t) \to \infty$.

5. Further research

It seems possible that our proof of Theorem 7 could be modified so as to avoid the assumption that B is well ordered. We conjecture that Theorems 6 and 7 could be unified as the following statement.

Conjecture 14. *Let $A, B \subset \mathbb{R}_+$. If $0 \notin \overline{B \setminus \{0\}}$, then B anticipates A only if A scales into B.*

The present paper strove to understand the effect of restricting the wager sets on the prediction power of martingales. As is often the case, understanding one thing brings up many new questions. We list just a few.

- Under the assumptions of Theorem 6, A can evade B through a history-independent martingale. Is it also the case under the assumptions of Theorem 7?
- Are single anticipation and countable anticipation different? That is, are there sets $A, B \subset \mathbb{R}_+$, such that B countably, but not singly, anticipates A?

- What can be said about the anticipation relation between sets that *do* include 0 as an accumulation point, for example, $\{2^{-n}\}_{n=1}^{\infty}$, $\{\frac{1}{n}\}_{n=1}^{\infty}$, and $\mathbb{R}_{+}$?
- Buss and Minnes (2013) introduce martingales defined by probabilistic strategies. How do these martingales behave in our framework?

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