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Debt and Incomplete Financial Markets: A Case for Nominal GDP Targeting

Kevin D. Sheedy

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Abstract
Financial markets are incomplete, thus for many agents borrowing is possible only by accepting a financial contract that specifies a fixed repayment. However, the future income that will repay this debt is uncertain, so risk can be inefficiently distributed. This paper argues that a monetary policy of nominal GDP targeting can improve the functioning of incomplete financial markets when incomplete contracts are written in terms of money. By insulating agents' nominal incomes from aggregate real shocks, this policy effectively completes the market by stabilizing the ratio of debt to income. The paper argues that the objective of nominal GDP should receive substantial weight even in an environment with other frictions that have been used to justify a policy of strict inflation targeting.

JEL Classifications: E21; E31; E44; E52.
Keywords: incomplete markets; heterogeneous agents; risk sharing; nominal GDP targeting

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Kevin Sheedy is an Associate of the Macro Programme at the Centre for Economic Performance and Lecturer in Economics, London School of Economics and Political Science.

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1 Introduction

Following the onset of the recent financial crisis, inflation targeting has increasingly found itself under attack. The frequent criticism is not that it has failed to achieve what it purports to do — to avoid a repeat of the inflationary 1970s or the deflationary 1930s — but that central banks have focused too much on price stability and too little on financial markets.[1] Such a view implicitly supposes there is a tension between the goals of price stability and financial stability when the economy is hit by shocks. However, it is not clear why this should be so, there being no widely accepted argument for why stabilizing prices in goods markets causes financial markets to malfunction.

The canonical justification for inflation targeting as optimal monetary policy rests on the presence of pricing frictions in goods markets (see, for example, Woodford, 2003). With infrequent price adjustment due to menu costs or other nominal rigidities, high or volatile inflation leads to relative price distortions that impair the efficient operation of markets, and which directly consumes time and resources in the process of setting prices. While there is a consensus on the importance of these frictions when analysing optimal monetary policy, it is increasingly argued that monetary policy must also take account of financial-market frictions such as collateral constraints or spreads between internal and external finance.[2] These frictions can magnify the effects of both shocks and monetary policy actions and make these effects more persistent. But the existence of a quantitatively important credit channel does not in and of itself imply that optimal monetary policy is necessarily so different from inflation targeting unless new types of shocks are introduced (Faia and Monacelli, 2007, Carlstrom, Fuerst and Paustian, 2010, De Fiore and Tristani, 2012).

This paper studies a simple and compelling friction in financial markets that immediately and straightforwardly leads to a stark conflict between the efficient operation of financial markets and price stability. The friction is a modest one: financial markets are assumed to be incomplete. Those who want to borrow can only do so through debt contracts that specify a fixed repayment (effectively issuing non-contingent bonds). The argument is that many agents, households in particular, will find it very difficult to issue liabilities with state-contingent repayments resembling equity or derivatives. Implicitly, it is assumed to be too costly to write lengthy contracts that spell out in advance different repayments conditional on each future state of the world.

The problem of non-contingent debt contracts for risk-averse households is that when borrowing for long periods, there will be considerable uncertainty about the future income from which fixed debt repayments must be made. The issue is not only idiosyncratic uncertainty — households do not know the future course the economy will take, which will affect their labour income. Will there be a

[1] White (2009b) and Christiano, Ilut, Motto and Rostagno (2010) argue that stable inflation is no guarantee of financial stability, and may even create conditions for financial instability. Christiano, Motto and Rostagno (2007) suggest that credit growth ought to have a role as an independent target of monetary policy. Contrary to these arguments, the conventional view that monetary policy should not react to asset prices is advocated in Bernanke and Gertler (2001). Woodford (2011) makes the point that flexible inflation targeting can be adapted to accommodate financial stability concerns, and that it would be unwise to discard inflation targeting's role in providing a clear nominal anchor.

[2] Starting from Bernanke, Gertler and Gilchrist (1999), there is now a substantial body of work that integrates credit frictions of the kind found in Carlstrom and Fuerst (1997) or Kiyotaki and Moore (1997) into monetary DSGE models. Recent work in this area includes Christiano, Motto and Rostagno (2010).

productivity slowdown, a deep and long-lasting recession, or even a 'lost decade' of poor economic performance to come? Or will unforeseen technological developments or terms-of-trade movements boost future incomes, and good economic management successfully steer the economy on a path of steady growth? Borrowers do not know what aggregate shocks are to come, but must fix their contractual repayments prior to this information being revealed.

The simplicity of non-contingent debt contracts can be seen as coming at the price of bundling together two fundamentally different transfers: a transfer of consumption from the future to the present for borrowers, but also a transfer of aggregate risk to borrowers. The future consumption of borrowers is paid for from the difference between their uncertain future incomes and their fixed debt repayments. The more debt they have, the more their future income is effectively leveraged, leading to greater consumption risk. The flip-side of borrowers' leverage is that savers are able to hold a risk-free asset, reducing their consumption risk.

To see the sense in which this bundling together of a transfer of risk and borrowing is inefficient, consider what would happen in complete financial markets. Individuals would buy or sell state-contingent bonds (Arrow-Debreu securities) that make payoffs conditional on particular states of the world (or equivalently, write loan contracts with different repayments across all states of the world). Risk-averse borrowers would want to sell relatively few bonds paying off in future states of the world where GDP and thus incomes are low, and sell relatively more in good states of the world. As a result, prices of contingent bonds paying off in bad states would be relatively expensive and those paying off in good states relatively cheap. These price differences would entice savers to shift away from non-contingent bonds and take on more risk in their portfolios. Given that the economy has no risk-free technology for transferring goods over time, and as aggregate risk cannot be diversified away, the efficient outcome is for risk-averse individuals to share aggregate risk, and complete markets allow this to be unbundled from decisions about how much to borrow or save.

The efficient financial contract between risk-averse borrowers and savers in an economy subject to aggregate income risk (abstracting from idiosyncratic risk) turns out to have a close resemblance to an 'equity share' in GDP. In other words, borrowers' repayments should fall during recessions and rise during booms. This means the ratio of debt liabilities to GDP should be more stable than it would be in a world of incomplete financial markets where debt liabilities are fixed while GDP fluctuates.

With incomplete financial markets, monetary policy has a role to play in mitigating inefficiencies because private debt contracts are typically denominated in terms of money. Hence, the real degree of state-contingency in financial contracts is endogenous to monetary policy. If incomplete markets were the only source of inefficiency in the economy then the optimal monetary policy would aim to make nominally non-contingent debt contracts mimic through variation in the price level the efficient financial contract that would be chosen with complete financial markets.

Given that the efficient financial contract between borrowers and savers resembles an equity share in GDP, it follows that a goal of monetary policy should be to stabilize the ratio of debt liabilities to GDP. With non-contingent nominal debt liabilities, this can be achieved by having a non-contingent level of nominal income, in other words, a monetary policy that targets nominal

GDP. The intuition is that while the central bank cannot eliminate uncertainty about future real GDP, it can in principle make the level of future nominal GDP (and hence the nominal income of an average person) perfectly predictable. Removing uncertainty about future nominal income thus alleviates the problem of nominal debt repayments being non-contingent.

A policy of nominal GDP targeting generally deviates from inflation targeting because any fluctuations in real GDP would lead to fluctuations in inflation of the same size and in the opposite direction. Recessions would feature higher inflation and booms would feature lower inflation, or even deflation. These inflation fluctuations can be helpful because they induce variation in the real value of nominally non-contingent debt, making it behave more like equity, which promotes efficient risk sharing. A policy of strict inflation targeting would convert nominally non-contingent debt into real non-contingent debt, which would imply an uneven and generally inefficient distribution of risk.

The inflation fluctuations that occur with nominal GDP targeting would entail relative-price distortions if prices were sticky, so the benefit of efficient risk sharing is most likely not achieved without some cost. It is ultimately a quantitative question whether the inefficiency caused by incomplete financial markets is more important than the inefficiency caused by relative-price distortions, and thus whether nominal GDP targeting is preferable to inflation targeting.

This paper presents a model that allows optimal monetary policy to be studied analytically in an incomplete-markets economy with heterogeneous agents. The basic framework adopted is the life-cycle theory of consumption, which provides the simplest account of household borrowing and saving. The model contains overlapping generations of individuals: the young, the middle-aged, and the old. Individuals are risk averse, having an Epstein-Zin-Weil utility function. Individuals receive incomes equal to fixed age-specific shares of GDP (labour supply is exogenous, but this simplifying assumption can be relaxed). The age-profile of income is assumed to be hump shaped: the middle-aged receive the most income; the young receive less income; while the old receive the least. Real GDP is uncertain because of aggregate productivity shocks, but there are no idiosyncratic shocks.

Young individuals would like to borrow to smooth consumption, repaying when they are middle-aged. The middle-aged would like to save, drawing on their savings when they are old. The economy is assumed to have no investment or storage technology, and is closed to international trade. There are no government bonds and no fiat money, and no taxes or fiscal transfers such as public pensions. In this world, consumption smoothing is facilitated by the young borrowing from the middle-aged, repaying when they themselves are middle-aged and their creditors are old. It is assumed the only financial contract available is a non-contingent nominal bond. The basic model contains no other frictions, and initially assumes that prices and wages are fully flexible.

The concept of a 'natural debt-to-GDP ratio' provides a useful benchmark for monetary policy. This is defined as the ratio of (state-contingent) debt liabilities to GDP that would prevail were financial markets complete, which is independent of monetary policy. The actual debt-to-GDP ratio in an economy with incomplete markets would coincide with the natural debt-to-GDP ratio if forecasts of future GDP were always correct ex post, but will in general fluctuate around it when the economy is hit by shocks. The natural debt-to-GDP ratio is thus analogous to concepts such as the natural rate of unemployment and the natural rate of interest.

If all movements in real GDP growth rates are unpredictable then the natural debt-to-GDP ratio turns out to be constant (or if utility functions are logarithmic, the ratio is constant irrespective of the statistical properties of GDP growth). Even when the natural debt-to-GDP ratio is not completely constant, plausible calibrations suggest it would have a low volatility relative to real GDP itself.

Since the equilibrium of an economy with complete financial markets would be Pareto efficient in the absence of other frictions, the natural debt-to-GDP ratio also has desirable welfare properties. A goal of monetary policy in an incomplete-markets economy is therefore to close the 'debt gap', defined as the difference between the actual and natural debt-to-GDP ratios. It is shown that doing this effectively 'completes the market' in the sense that the equilibrium with incomplete markets would then coincide with the hypothetical complete-markets equilibrium. Monetary policy can affect the actual debt-to-GDP ratio and thus the debt gap because that ratio is nominal debt liabilities (which are non-contingent with incomplete markets) divided by nominal GDP, where the latter is under the control of monetary policy.

When the natural debt-to-GDP ratio is constant, closing the debt gap can be achieved by adopting a fixed target for the level of nominal GDP. With this logic, the central bank uses nominal GDP as an intermediate target that achieves its ultimate goal of closing the debt gap. This turns out to be preferable to targeting the debt-to-GDP ratio directly because a monetary policy that targets only a real financial variable would leave the economy without a nominal anchor. Nominal GDP targeting uniquely pins down the nominal value of incomes and thus provides the economy with a well-defined nominal anchor.

It is important to note that in an incomplete-markets economy hit by shocks, whatever action a central bank takes or fails to take will have distributional consequences. Ex post, there will always be winners and losers. Creditors lose out when inflation is unexpectedly high, while debtors suffer when inflation is unexpectedly low. It might then be thought surprising that inflation fluctuations would ever be desirable. However, the inflation fluctuations implied by a nominal GDP target are not arbitrary fluctuations — they are perfectly correlated with the real GDP fluctuations that are the ultimate source of uncertainty in the economy, and which themselves have distributional consequences when individuals are heterogeneous. For individuals to share risk, it must be possible to make transfers ex post that act as insurance from an ex-ante perspective. The result of the paper is that ex-ante efficient insurance requires inflation fluctuations that are negatively correlated with real GDP (a countercylical price level) to generate the appropriate ex-post transfers between debtors and creditors.

It might be objected that there are infinitely many state-contingent consumption allocations that would also satisfy the criterion of ex-ante efficiency. However, only one of these — the hypothetical complete-markets equilibrium associated with the natural debt-to-GDP ratio — could ever be implemented through monetary policy. Thus for a policymaker solely interested in promoting efficiency, there is a unique optimal policy that does not require any explicit distributional preferences to be introduced.

The model also makes predictions for how different monetary policies will affect the volatility of financial-market variables such as credit and interest rates. It is shown that policies implying an inefficient distribution of risk, for example, inflation targeting, are associated with greater volatility in financial markets when compared to the nominal GDP targeting policy that allows the economy to mimic the hypothetical complete-markets equilibrium. Stabilizing inflation implies that new lending as fraction of GDP is excessively procyclical: credit expands too much during a boom and falls too much during a recession. Similarly, inflation targeting implies that real interest rates will be excessively countercyclical, permitting real interest rates to fall too much during an expansion. These findings allow the tension between price stability and efficient risk sharing to be seen in more familiar terms as a trade-off between price stability and financial stability.

Determining which of these objectives is the more quantitatively important requires introducing nominal rigidity into the model, allowing for there to be a cost associated with inflation fluctuations due to relative-price distortions. Nominal rigidity is introduced with a simple model of predetermined price-setting, but in a way that allows the welfare costs of inflation to be calibrated to match levels found in the existing literature. With both incomplete financial markets and sticky prices, optimal monetary policy is a convex combination of a nominal GDP target and a strict inflation target. After calibrating all the parameters of the model, the conclusion is that the nominal GDP target should receive approximately 95% of the weight.

This paper is related to a number of areas of the literature on monetary policy and financial markets. First, there is the empirical work of Bach and Stephenson (1974), Cukierman, Lennan and Papadia (1985), and more recently, Doepke and Schneider (2006), who document the effects of inflation in redistributing wealth between debtors and creditors. The novelty here is in studying the implications for optimal monetary policy in an environment where inflation fluctuations with such distributional effects may actually be desirable because financial markets are incomplete.

The most closely related theoretical paper is Pescatori (2007), who studies optimal monetary policy in an economy with rich and poor individuals, in the sense of there being an exogenously specified distribution of assets among otherwise identical individuals. In that environment, both inflation and interest rate fluctuations have redistributional effects on rich and poor individuals, and the central bank optimally chooses the mix between them (there is a need to change interest rates because prices are sticky, with deviations from the natural rate of interest leading to undesirable fluctuations in output). Another closely related paper is Lee (2010), who develops a model where heterogeneous individuals choose less than complete consumption insurance because of the presence of convex transaction costs in accessing financial markets. Inflation fluctuations expose households to idiosyncratic labour-income risk because households work in specific sectors of the economy, and sectoral relative prices are distorted by inflation when prices are sticky. This leads optimal monetary policy to put more weight on stabilizing inflation. Differently from those papers, the argument here is that inflation fluctuations can actually play a positive role in completing otherwise incomplete financial markets (and where debt arises endogenously owing to individual heterogeneity).[3]

[3]In other related work, Akyol (2004) analyses optimal monetary policy in an incomplete-markets economy where individuals hold fiat money for self insurance against idiosyncratic shocks. Kryvtsov, Shukayev and Ueberfeldt (2011) study an overlapping generations model with fiat money where monetary policy can improve upon the suboptimal level of saving by varying the expected inflation rate and thus the returns to holding money.

The idea that inflation fluctuations may have a positive role to play when financial markets are incomplete is now long-established in the literature on government debt (and has also been recently applied by Allen, Carletti and Gale (2011) in the context of the real value of the liquidity available to the banking system). Bohn (1988) developed the theory that nominal non-contingent government debt can be desirable because when combined with a suitable monetary policy, inflation will change the real value of the debt in response to fiscal shocks that would otherwise require fluctuations in distortionary tax rates.

Quantitative analysis of optimal monetary policy of this kind was developed in Chari, Christiano and Kehoe (1991) and expanded further in Chari and Kehoe (1999). One finding was that inflation needs to be extremely volatile to complete the market. As a result, Schmitt-Grohé and Uribe (2004) and Siu (2004) argued that once some nominal rigidity is considered so that inflation fluctuations have a cost, the optimal policy becomes very close to strict inflation targeting. This paper shares the focus of that literature on using inflation fluctuations to complete financial markets, but comes to a different conclusion regarding the magnitude of the required inflation fluctuations and whether the cost of those fluctuations outweighs the benefits. First, the benefits of completing the market in this paper are linked to the degree of household risk aversion, which is in general unrelated to the benefits of avoiding fluctuations in distortionary tax rates, and which proves to be large in the calibrated model. Second, the earlier results assumed government debt with a very short maturity. With longer maturity debt (household debt in this paper), the costs of the inflation fluctuations needed to complete the market are much reduced.[4]

This paper is also related to the literature on household debt. Iacoviello (2005) examines the consequences of household borrowing constraints in a DSGE model, while Guerrieri and Lorenzoni (2011) and Eggertsson and Krugman (2012) study how a tightening of borrowing constraints for indebted households can push the economy into a liquidity trap. Differently from those papers, the focus here is on the implications of household debt for optimal monetary policy. Furthermore, the finding here that the presence of household debt substantially changes optimal monetary policy does not depend on there being borrowing constraints, or even the feedback effects from debt to aggregate output stressed in those papers. Cúrdia and Woodford (2009) also study optimal monetary policy in an economy with household borrowing and saving, but the focus there is on spreads between interest rates for borrowers and savers, while their model assumes an insurance facility that rules out the risk-sharing considerations studied here. Finally, the paper is related to the literature on nominal GDP targeting (Meade, 1978, Bean, 1983, Hall and Mankiw, 1994) but proposes a different argument in favour of that policy.

The plan of the paper is as follows. Section 2 sets out the basic model and derives the equilibrium conditions. The main optimal monetary policy results are given in section 3. Section 4 introduces some extensions of the basic model and studies the observable consequences of following a suboptimal monetary policy. Section 5 introduces sticky prices and hence a trade-off between incomplete markets and price stability. Finally, section 6 draws some conclusions.

[4] This point is made by Lustig, Sleet and Yeltekin (2008) in the context of government debt.

2 A model of a pure credit economy

The population of an economy comprises overlapping generations of individuals. Time is discrete and is indexed by t. A new generation of individuals is born in each time period and each individual lives for three periods. During their three periods of life, individuals are referred to as the 'young' (y), the 'middle-aged' (m), and the 'old' (o), respectively. An individual derives utility from consumption of a composite good at each point in his life. There is no intergenerational altruism. At time t, per-person consumption of the young, middle-aged, and old is denoted by $C_{\mathrm{y},t}$, $C_{\mathrm{m},t}$, and $C_{\mathrm{o},t}$.

Individuals have identical lifetime utility functions, which have the Epstein-Zin-Weil functional form (Epstein and Zin, 1989, Weil, 1990). Future utility is discounted by subjective discount factor δ ($0 < \delta < \infty$), the intertemporal elasticity of substitution is σ ($0 < \sigma < \infty$), and α is the coefficient of relative risk aversion ($0 < \alpha < \infty$). The utility $\mathscr{U}_t$ of the generation born at time t is

$$\mathscr{U}_t = \frac{V_{\mathrm{y},t}^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \quad \text{where } V_{\mathrm{y},t} = \left(C_{\mathrm{y},t}^{1-\frac{1}{\sigma}} + \delta\left\{\mathbb{E}_t\left[V_{\mathrm{m},t+1}^{1-\alpha}\right]^{\frac{1}{1-\alpha}}\right\}^{1-\frac{1}{\sigma}}\right)^{\frac{1}{1-\frac{1}{\sigma}}},$$

$$V_{\mathrm{m},t} = \left(C_{\mathrm{m},t}^{1-\frac{1}{\sigma}} + \delta\left\{\mathbb{E}_t\left[V_{\mathrm{o},t+1}^{1-\alpha}\right]^{\frac{1}{1-\alpha}}\right\}^{1-\frac{1}{\sigma}}\right)^{\frac{1}{1-\frac{1}{\sigma}}}, \quad \text{and} \quad V_{\mathrm{o},t} = C_{\mathrm{o},t}. \qquad [2.1]$$

The utility function is written in a recursive form with $V_{\mathrm{y},t}$, $V_{\mathrm{m},t}$, and $V_{\mathrm{o},t}$ denoting the continuation values of the young, middle-aged, and old in terms of current consumption equivalents.[5]

The number of young individuals born in any time period is exactly equal to the number of old individuals alive in the previous period who now die. The economy thus has no population growth and a balanced age structure. Assuming that the population of individuals currently alive has measure one, each generation of individuals has measure one third. Aggregate consumption at time t is denoted by C_t:

$$C_t = \frac{1}{3}C_{\mathrm{y},t} + \frac{1}{3}C_{\mathrm{m},t} + \frac{1}{3}C_{\mathrm{o},t}. \qquad [2.2]$$

All individuals of the same age at the same time receive the same income, with $Y_{\mathrm{y},t}$, $Y_{\mathrm{m},t}$, and $Y_{\mathrm{o},t}$ denoting the per-person incomes (in terms of the composite good) of the young, middle-aged, and old, respectively, at time t. Age-specific incomes are assumed to be time-invariant multiples of aggregate income Y_t, with Θ_{y}, Θ_{m}, and Θ_{o} denoting the multiples for the young, middle-aged, and old:

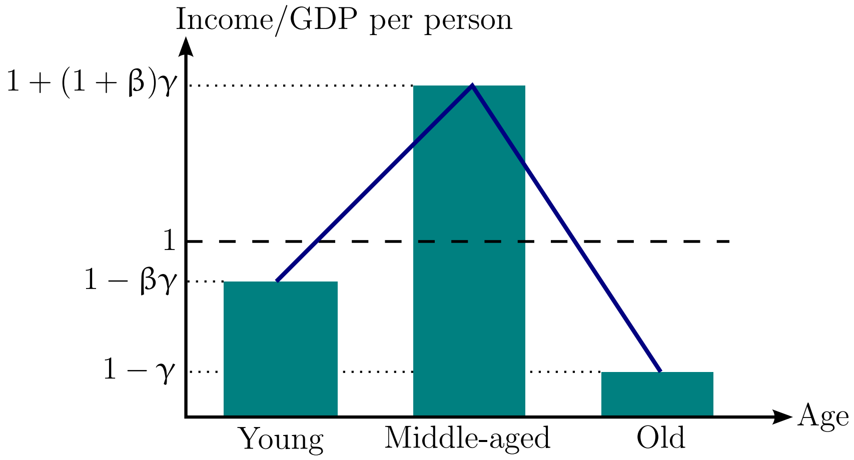
$$Y_{\mathrm{y},t} = \Theta_{\mathrm{y}}Y_t, \quad Y_{\mathrm{m},t} = \Theta_{\mathrm{m}}Y_t, \quad Y_{\mathrm{o},t} = \Theta_{\mathrm{o}}Y_t, \quad \text{where } \Theta_{\mathrm{y}}, \Theta_{\mathrm{m}}, \Theta_{\mathrm{o}} \in (0,3) \text{ and } \frac{1}{3}\Theta_{\mathrm{y}} + \frac{1}{3}\Theta_{\mathrm{m}} + \frac{1}{3}\Theta_{\mathrm{o}} = 1. \qquad [2.3]$$

Real GDP is specified as an exogenous stochastic process. This assumption turns out not to affect the main results of the paper, but is relaxed later.[6] The growth rate of real GDP between period

[5]The functional form reduces to the special case of time-separable isoelastic utility when the coefficient of relative risk aversion is equal to the reciprocal of the intertemporal elasticity of substitution ($\alpha = 1/\sigma$).

[6]The introduction of an endogenous labour supply decision need not affect the results unless prices or wages are sticky.

Figure 1: *Age profile of non-financial income*



$t-1$ and t, denoted by $g_t \equiv (Y_t - Y_{t-1})/Y_{t-1}$, is given by

$$g_t = \bar{g} + \varsigma x_t, \quad \text{where} \ \ \mathbb{E}x_t = 0, \quad \mathbb{E}x_t^2 = 1, \quad \text{and} \ \ x_t \in [\underline{x}, \overline{x}], \tag{2.4}$$

with x_t being an exogenous stationary stochastic process with bounded support. The growth rate g_t has mean $\bar{g}$ and standard deviation ς. Defining β in terms of the parameters δ, σ, α, $\bar{g}$, and ς (and the stochastic process of x_t), the following parameter restriction is imposed:

$$0 < \beta < 1, \quad \text{where} \ \ \beta \equiv \delta\mathbb{E}\left[(1+g_t)^{1-\alpha}\right]^{\frac{1-1/\sigma}{1-\alpha}}. \tag{2.5}$$

The income multiples Θ_{y}, Θ_{m}, and Θ_{o} for each generation are parameterized to specify a hump-shaped age profile of income in terms of β and a single new parameter γ:

$$\Theta_{\text{y}} = 1 - \beta\gamma, \quad \Theta_{\text{m}} = 1 + (1+\beta)\gamma, \quad \text{and} \ \ \Theta_{\text{o}} = 1 - \gamma. \tag{2.6}$$

The income multiples are all well-defined and strictly positive for any $0 < \gamma < 1$. The general pattern is depicted in Figure 1. As $\gamma \to 0$, the economy approaches the special case where all individuals alive at the same time receive the same income irrespective of age, while as $\gamma \to 1$, the differences in income between individuals of different ages are at their maximum with old individuals receiving a zero income. Intermediate values of γ imply age profiles that lie between these extremes, thus the parameter γ can be interpreted as the gradient of the age profile of income over the life cycle. The presence of the coefficient β in the specification [2.6] implies that the income gradient from young to middle-aged is less than the gradient from middle-aged to old.[7]

There is assumed to be no government spending and no international trade, and the composite good is not storable, hence the goods-market clearing condition is

$$C_t = Y_t. \tag{2.7}$$

The economy has a central bank that defines a reserve asset, referred to as 'money'. Reserves

[7] Introducing this feature implies that the steady state of the model will have some convenient properties. See section 2.3 for details.

held between period t and $t+1$ are remunerated at a nominal interest rate i_t known in advance at time t. The economy is 'cash-less' in that money is not required for transactions, but money is used by agents as a unit of account in writing financial contracts and in pricing goods. One unit of the composite good costs P_t units of money at time t, and $\pi_t \equiv (P_t - P_{t-1})/P_{t-1}$ denotes the inflation rate between period $t-1$ and t. Monetary policy is specified as a rule for setting the nominal interest rate i_t. Finally, in equilibrium, the central bank will maintain a supply of reserves equal to zero.

2.1 Incomplete financial markets

Asset markets are assumed to be incomplete. No individual can sell state-contingent bonds (Arrow-Debreu securities), and hence in equilibrium in this economy, no such securities will be available to buy. The only asset that can be traded is a one-period, nominal, non-contingent bond. Individuals can take positive or negative positions in this bond (save or borrow), and there is no limit on borrowing other than being able to repay in all states of the world given non-negativity constraints on consumption. With this restriction, no default will occur, and thus bonds are risk free in nominal terms.[8]

Bonds that have a nominal face value of 1 paying off at time $t+1$ trade at price Q_t in terms of money at time t. These bonds are perfect substitutes for the reserve asset defined by the central bank, so the absence of arbitrage opportunities requires that

$$Q_t = \frac{1}{1+i_t}. \qquad [2.8]$$

The central bank's interest-rate policy thus sets the nominal price of the bonds.

Let $B_{\mathrm{y},t}$ and $B_{\mathrm{m},t}$ denote the net bond positions per person of the young and middle-aged at the end of time t (positive denotes saving, negative denotes borrowing). The absence of intergenerational altruism implies that the old will make no bequests ($B_{\mathrm{o},t} = 0$) and the young will begin life with no assets. The budget identities of the young, middle-aged, and old are respectively:

$$C_{\mathrm{y},t} + \frac{Q_t}{P_t} B_{\mathrm{y},t} = Y_{\mathrm{y},t}, \quad C_{\mathrm{m},t} + \frac{Q_t}{P_t} B_{\mathrm{m,t}} = Y_{\mathrm{m},t} + \frac{1}{P_t} B_{\mathrm{y},t-1}, \quad \text{and} \quad C_{\mathrm{o},t} = Y_{\mathrm{o},t} + \frac{1}{P_t} B_{\mathrm{m},t-1}. \qquad [2.9]$$

Maximizing the lifetime utility function [2.1] for each generation with respect to its bond holdings, subject to the budget identities [2.9], implies the Euler equations:

$$\delta \mathbb{E}_t \left[\frac{P_t}{P_{t+1}} \left\{ \frac{V_{\mathrm{m},t+1}}{\mathbb{E}_t \left[V_{\mathrm{m},t+1}^{1-\alpha} \right]^{\frac{1}{1-\alpha}}} \right\}^{\frac{1}{\sigma}-\alpha} \left(\frac{C_{\mathrm{m},t+1}}{C_{\mathrm{y},t}} \right)^{-\frac{1}{\sigma}} \right] = Q_t$$
$$= \delta \mathbb{E}_t \left[\frac{P_t}{P_{t+1}} \left\{ \frac{V_{\mathrm{o},t+1}}{\mathbb{E}_t \left[V_{\mathrm{o},t+1}^{1-\alpha} \right]^{\frac{1}{1-\alpha}}} \right\}^{\frac{1}{\sigma}-\alpha} \left(\frac{C_{\mathrm{o},t+1}}{C_{\mathrm{m},t}} \right)^{-\frac{1}{\sigma}} \right]. \qquad [2.10]$$

[8] With the utility function [2.1], marginal utility tends to infinity as consumption tends to zero. Thus, individuals would not choose borrowing that led to zero consumption in some positive-probability set of states of the world, so this constraint will not bind. Furthermore, given that the stochastic process for GDP growth in [2.4] has finite support, for any particular amount of borrowing, it would always be possible to set the standard deviation ς to be sufficiently small to ensure that no default would occur.

With no issuance of government bonds, no bond purchases by the central bank (the supply of reserves is maintained at zero), and no international borrowing and lending, the bond-market clearing condition is

$$\frac{1}{3}B_{\mathrm{y},t} + \frac{1}{3}B_{\mathrm{m},t} = 0. \tag{2.11}$$

Equilibrium quantities in the bond market can be summarized by one variable: the *gross* amount of bonds issued.[9] Let B_t denote gross bond issuance per person, L_t the implied real value of the loans that are made per person, and D_t the real value of debt liabilities per person that fall due at time t. Assuming that (as will be confirmed later) the young will sell bonds and the middle-aged will buy them, these variables are given by:

$$B_t \equiv -\frac{B_{\mathrm{y},t}}{3}, \quad L_t \equiv \frac{Q_t B_t}{P_t}, \quad \text{and} \quad D_t \equiv \frac{B_{t-1}}{P_t}. \tag{2.12}$$

It is convenient to introduce variables for age-specific consumption, loans, and debt liabilities measured relative to GDP Y_t. These are denoted with lower-case letters. The real return (*ex post*) r_t between periods $t-1$ and t is defined as the percentage by which the real value of debt liabilities is greater than the real amount of the corresponding loans. These definitions are listed below:

$$c_{\mathrm{y},t} \equiv \frac{C_{\mathrm{y},t}}{Y_t}, \quad c_{\mathrm{m},t} \equiv \frac{C_{\mathrm{m},t}}{Y_t}, \quad c_{\mathrm{o},t} \equiv \frac{C_{\mathrm{o},t}}{Y_t}, \quad l_t \equiv \frac{L_t}{Y_t}, \quad d_t \equiv \frac{D_t}{Y_t}, \quad \text{and} \quad r_t \equiv \frac{D_t - L_{t-1}}{L_{t-1}}. \tag{2.13}$$

Using the definitions of the debt-to-GDP and loans-to-GDP ratios from [2.13] it follows that:

$$d_t = \left(\frac{1+r_t}{1+g_t}\right) l_{t-1}. \tag{2.14a}$$

The real interest rate ρ_t (*ex-ante* real return) between periods t and $t+1$ is defined as the conditional expectation of the real return between those periods:[10]

$$\rho_t = \mathbb{E}_t r_{t+1}. \tag{2.14b}$$

Using the age-specific incomes [2.3] and the definitions in [2.12] and [2.13], the budget identities in [2.9] for each generation can be written as:

$$c_{\mathrm{y},t} = 1 - \beta\gamma + 3l_t, \quad c_{\mathrm{m},t} = 1 + (1+\beta)\gamma - 3d_t - 3l_t, \quad \text{and} \quad c_{\mathrm{o},t} = 1 - \gamma + 3d_t. \tag{2.14c}$$

Similarly, after using the definitions in [2.12] and [2.13], the Euler equations in [2.10] become:

$$\delta\mathbb{E}_t\left[(1+r_{t+1})(1+g_{t+1})^{-\frac{1}{\sigma}}\left\{\frac{(1+g_{t+1})v_{\mathrm{m},t+1}}{\mathbb{E}_t\left[(1+g_{t+1})^{1-\alpha}v_{\mathrm{m},t+1}^{1-\alpha}\right]^{\frac{1}{1-\alpha}}}\right\}^{\frac{1}{\sigma}-\alpha}\left(\frac{c_{\mathrm{m},t+1}}{c_{\mathrm{y},t}}\right)^{-\frac{1}{\sigma}}\right] = 1$$
$$= \delta\mathbb{E}_t\left[(1+r_{t+1})(1+g_{t+1})^{-\frac{1}{\sigma}}\left\{\frac{(1+g_{t+1})v_{\mathrm{o},t+1}}{\mathbb{E}_t\left[(1+g_{t+1})^{1-\alpha}v_{\mathrm{o},t+1}^{1-\alpha}\right]^{\frac{1}{1-\alpha}}}\right\}^{\frac{1}{\sigma}-\alpha}\left(\frac{c_{\mathrm{o},t+1}}{c_{\mathrm{m},t}}\right)^{-\frac{1}{\sigma}}\right], \tag{2.14d}$$

[9] In equilibrium, the net bond positions of the household sector and the whole economy are of course both zero under the assumptions made.

[10] This real interest rate is important for saving and borrowing decisions, but there is no actual real risk-free asset to invest in.

where $v_{\text{m},t} \equiv V_{\text{m},t}/Y_t$ and $v_{\text{o},t} \equiv V_{\text{o},t}/Y_t$ denote the continuation values of middle-aged and old individuals relative to GDP. Using equation [2.1], these value functions satisfy:

$$v_{\text{m},t} = \left(c_{\text{m},t}^{1-\frac{1}{\sigma}} + \delta \left\{ \mathbb{E}_t \left[(1+g_{t+1})^{1-\alpha} v_{\text{o},t+1}^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \right\}^{1-\frac{1}{\sigma}} \right)^{\frac{1}{1-\frac{1}{\sigma}}}, \quad \text{and} \quad v_{\text{o},t} = c_{\text{o},t}. \qquad [2.14e]$$

The ex-post Fisher equation for the real return on nominal bonds is obtained from the no-arbitrage condition [2.8] and the definitions in [2.12]:

$$1 + r_t = \frac{1 + i_{t-1}}{1 + \pi_t}. \qquad [2.15]$$

Finally, goods-market clearing [2.7] with the definition of aggregate consumption [2.2] requires:

$$\frac{1}{3} c_{\text{y},t} + \frac{1}{3} c_{\text{m},t} + \frac{1}{3} c_{\text{o},t} = 1. \qquad [2.16]$$

Before examining the equilibrium of the economy under different monetary policies, it is helpful to study as a benchmark the hypothetical world of complete financial markets.

2.2 The complete financial markets benchmark

Suppose it were possible for individuals to take short and long positions in a range of Arrow-Debreu securities for each possible state of the world. Suppose markets are *sequentially* complete in that securities are traded period-by-period for states of the world that will be realized one period in the future, and that individuals only participate in financial markets during their actual lifetimes (instead of all trades taking place at the 'beginning of time').[11] Without loss of generality, assume the payoffs of these securities are specified in terms of real consumption, and their prices are quoted in real terms. Let K_{t+1} denote the kernel of prices for securities with payoffs of one unit of consumption at time $t+1$ in terms of consumption at time t. The prices are defined relative to the (conditional) probabilities of each state of the world.

Let $S_{\text{y},t+1}$ and $S_{\text{m},t+1}$ denote the per-person net positions in the Arrow-Debreu securities at the end of period t of the young and middle-aged respectively (with $S_{\text{o},t+1} = 0$ for the old, who hold no assets at the end of period t). These variables give the real payoffs individuals will receive (or make, if negative) at time $t+1$. The price of taking net position S_{t+1} at time t is $\mathbb{E}_t K_{t+1} S_{t+1}$ (if negative, this is the amount received from selling securities).

In what follows, the levels of consumption obtained with complete markets (and the corresponding value functions) are denoted with an asterisk to distinguish them from the outcomes with incomplete markets. The budget identities of the young, middle-aged, and old are:

$$C^*_{\text{y},t} + \mathbb{E}_t K_{t+1} S_{\text{y},t+1} = Y_{\text{y},t}, \quad C^*_{\text{m},t} + \mathbb{E}_t K_{t+1} S_{\text{m},t+1} = Y_{\text{m},t} + S_{\text{y},t}, \quad \text{and} \quad C^*_{\text{o},t} = Y_{\text{o},t} + S_{\text{m},t}. \qquad [2.17]$$

Maximizing utility [2.1] for each generation with respect to holdings of Arrow-Debreu securities,

[11] This distinction is relevant here. As will be seen, sequential completeness is the appropriate notion of complete markets for studying the issues that arise in this paper.

subject to the budget identities [2.17], implies the Euler equations:

$$\delta\left\{\frac{V^*_{\mathrm{m},t+1}}{\mathbb{E}_t\left[{V^*}^{1-\alpha}_{\mathrm{m},t+1}\right]^{\frac{1}{1-\alpha}}}\right\}^{\frac{1}{\sigma}-\alpha}\left(\frac{C^*_{\mathrm{m},t+1}}{C^*_{\mathrm{y},t}}\right)^{-\frac{1}{\sigma}} = K_{t+1} = \delta\left\{\frac{V^*_{\mathrm{o},t+1}}{\mathbb{E}_t\left[{V^*}^{1-\alpha}_{\mathrm{o},t+1}\right]^{\frac{1}{1-\alpha}}}\right\}^{\frac{1}{\sigma}-\alpha}\left(\frac{C^*_{\mathrm{o},t+1}}{C^*_{\mathrm{m},t}}\right)^{-\frac{1}{\sigma}}, \quad [2.18]$$

where these hold in all states of the world at time $t+1$. Market clearing for Arrow-Debreu securities requires:

$$\frac{1}{3}S_{\mathrm{y},t} + \frac{1}{3}S_{\mathrm{m},t} = 0. \quad [2.19]$$

Let S_{t+1} denote the gross quantities of Arrow-Debreu securities issued at the end of period t. By analogy with the definitions of L_t and D_t in the case of incomplete markets (from [2.12]), let L^*_t denote the value of all securities sold, which represents the amount lent to borrowers, and let D^*_t be the state-contingent quantity of securities liable for repayment, the equivalent of borrowers' debt liabilities. Supposing, as will be confirmed, that securities would be issued by the young and bought by the middle-aged, these variables are given by:

$$S_{t+1} \equiv -\frac{S_{\mathrm{y},t+1}}{3}, \quad L^*_t \equiv \mathbb{E}_t K_{t+1} S_{t+1}, \quad \text{and} \quad D^*_t \equiv S_t. \quad [2.20]$$

In what follows, let $c^*_{\mathrm{y},t}$, $c^*_{\mathrm{m},t}$, $c^*_{\mathrm{o},t}$, l^*_t, d^*_t, and r^*_t denote the complete-markets equivalents of the variables defined in [2.13].

Starting from the definitions in [2.13] and [2.20], it can be seen that equation [2.14a] also holds for the complete-markets variables d^*_t, r^*_t, and l^*_{t-1}. The real interest rate is defined as the expectation of the real return, so equation [2.14b] also holds for ρ^*_t and r^*_{t+1}. Using the age-specific income levels from [2.3] and the definitions from [2.13] and [2.20], the budget identities [2.17] can be written as in equation [2.14c] with l^*_t and d^*_t. The definition of the real return r^*_t together with [2.20] implies that $\mathbb{E}_t[(1+r^*_{t+1})K_{t+1}] = 1$. Using these definitions again, the Euler equations [2.18] imply that the equations in [2.14d] hold for $c^*_{\mathrm{y},t}$, $c^*_{\mathrm{m},t}$, $c^*_{\mathrm{o},t}$, $v^*_{\mathrm{m},t}$, $v^*_{\mathrm{o},t}$, and r^*_t, with the value functions satisfying the equivalent of [2.14e].

It is seen that all of equations [2.14a]–[2.14e] in the incomplete-markets model hold also under complete markets. The distinctive feature of complete markets is that the Euler equations in [2.18] also imply the following equation holds in all states of the world:

$$\left\{\frac{(1+g_{t+1})v^*_{\mathrm{m},t+1}}{\mathbb{E}_t\left[(1+g_{t+1})^{1-\alpha}{v^*}^{1-\alpha}_{\mathrm{m},t+1}\right]^{\frac{1}{1-\alpha}}}\right\}^{\frac{1}{\sigma}-\alpha}\left(\frac{c^*_{\mathrm{m},t+1}}{c^*_{\mathrm{y},t}}\right)^{-\frac{1}{\sigma}} = \left\{\frac{(1+g_{t+1})v^*_{\mathrm{o},t+1}}{\mathbb{E}_t\left[(1+g_{t+1})^{1-\alpha}{v^*}^{1-\alpha}_{\mathrm{o},t+1}\right]^{\frac{1}{1-\alpha}}}\right\}^{\frac{1}{\sigma}-\alpha}\left(\frac{c^*_{\mathrm{o},t+1}}{c^*_{\mathrm{m},t}}\right)^{-\frac{1}{\sigma}}. \quad [2.21]$$

This condition reflects the distribution of risk that is mutually agreeable among individuals who have access to a complete set of financial markets. The condition equates the growth rates of marginal utilities of those individuals whose lives overlap (and their consumption growth rates in the case of time-separable utility).

2.3 Equilibrium conditions

There are nine endogenous real variables: the age-specific consumption ratios $c_{\mathrm{y},t}$, $c_{\mathrm{m},t}$, and $c_{\mathrm{o},t}$; the value functions $v_{\mathrm{m},t}$ and $v_{\mathrm{o},t}$; the loans- and debt-to-GDP ratios l_t and d_t; and the real interest rate ρ_t and the ex-post real return r_t. Real GDP growth g_t is exogenous and given by [2.4]. Common to both incomplete and complete financial markets are the ten equations in [2.14a]–[2.14e] and [2.16]. By Walras' law, one of these equations is redundant, so the goods-market clearing condition [2.16] (seen to be implied by [2.14c]) is dropped in what follows.

With incomplete markets, the equilibrium conditions [2.14a]–[2.14e] are augmented by the ex-post Fisher equation [2.15], to which must be added a monetary policy rule since this equation refers to the nominal interest rate. Thus, two equations are added, corresponding to the two extra nominal variables, the inflation rate π_t and the nominal interest rate i_t. With complete markets, one extra equation [2.21] is appended to the system [2.14a]–[2.14e]. There are no extra endogenous variables, but condition [2.21] renders redundant one of the two equations in [2.14d]. Since none of the equilibrium conditions includes nominal variables, the complete-markets equilibrium is independent of monetary policy.

Finding the equilibrium with incomplete markets is complicated by the fact that the real payoff of the nominal bond is endogenous to monetary policy. However, owing to the substantial overlap between the equilibrium conditions under incomplete and complete markets, there is a sense in which there is only one degree of freedom for the equilibria in these two cases to differ, and thus only one degree of freedom for monetary policy to affect the equilibrium with incomplete markets.

To make this analysis precise, define Υ_t to be the realized debt-to-GDP ratio relative to its expected value:

$$\Upsilon_t \equiv \frac{d_t}{\mathbb{E}_{t-1} d_t}, \quad \text{with} \;\; \Upsilon_t = \left\{\frac{1+r_t}{1+g_t}\right\} \Big/ \mathbb{E}_{t-1}\left\{\frac{1+r_t}{1+g_t}\right\}. \qquad [2.22]$$

The second equation states that Υ_t is also the unexpected component of portfolio returns r_t relative to GDP growth g_t, where that equation follows from [2.14a]. With complete markets, equations [2.13] and [2.20] imply that $\Upsilon_t^* = (S_t/(1+g_t))/\mathbb{E}_{t-1}[S_t/(1+g_t)]$. Since the portfolio S_t is a variable determined by borrowers' and savers' choices, Υ_t^* is also determined. With incomplete markets, it can be seen from equation [2.15] that Υ_t will depend on monetary policy. But once Υ_t is determined, portfolio returns in all states of the world (relative to expectations) are known, which closes the model.

If Υ_t has been determined then equation [2.22] implies that the debt-to-GDP ratio d_t is a state variable. In the model, the debt-to-GDP ratio is a sufficient statistic for the wealth distribution at the beginning of period t. It would therefore be expected that there is a unique equilibrium conditional on having determined Υ_t. There are two caveats to this. First, since the model does not feature a representative agent, there is the possibility of multiple equilibria if substitution effects were too weak relative to income effects. Second, since the model features overlapping generations of individuals, there is the possibility of multiple equilibria due to rational bubbles. Suitable parameter restrictions will be imposed to rule out both of these types of multiplicity.

Given that d_t is a state variable, the uniqueness of the equilibrium will depend on the system of equations [2.14a]–[2.14e] having the saddlepath stability property together with a unique steady state. This issue is investigated by examining the perfect-foresight paths implied by equations [2.14a]–[2.14e]. Starting from time t_0 onwards, suppose there are no shocks to real GDP growth ($\varsigma = 0$ in [2.4]) so $g_t = \bar{g}$, and suppose there is no uncertainty about portfolio returns, hence $\Upsilon_t = 1$. With future expectations equal to the realized values of variables, equations [2.14b] and [2.14d] reduce to:

$$\rho_t = r_{t+1}, \quad \text{and} \quad \delta(1+r_{t+1})(1+g_{t+1})^{-\frac{1}{\sigma}}\left(\frac{c_{\text{m},t+1}}{c_{\text{y},t}}\right)^{-\frac{1}{\sigma}} = 1 = \delta(1+r_{t+1})(1+g_{t+1})^{-\frac{1}{\sigma}}\left(\frac{c_{\text{o},t+1}}{c_{\text{m},t}}\right)^{-\frac{1}{\sigma}}. \quad [2.23]$$

The perfect-foresight paths are determined by equations [2.14a], [2.14c], and [2.23] (with no uncertainty, [2.14e] is redundant). The analysis proceeds by reducing this system to two equations in two variables: one non-predetermined variable, the real interest rate ρ_t, and one state variable, the debt-to-GDP ratio d_t.

Proposition 1 *The system of equations* [2.14a], [2.14c], *and* [2.23] *has the following properties:*

(i) Any perfect foresight paths $\{\rho_{t_0}, \rho_{t_0+1}, \rho_{t_0+2}, \ldots\}$ *and* $\{d_{t_0}, d_{t_0+1}, d_{t_0+2}, \ldots\}$ *must satisfy a pair of first-order difference equations* $\mathscr{F}(\rho_t, d_t, \rho_{t+1}, d_{t+1}) = 0$.

(ii) The system of equations has a steady state:

$$\bar{d} = \frac{\gamma}{3}, \quad \bar{l} = \frac{\beta\gamma}{3}, \quad \bar{c}_{\text{y}} = \bar{c}_{\text{m}} = \bar{c}_{\text{o}} = 1, \quad \text{and} \ \bar{r} = \bar{\rho} = \frac{1+\bar{g}}{\beta} - 1 = \frac{(1+\bar{g})^{\frac{1}{\sigma}}}{\delta} - 1, \quad [2.24]$$

where [2.4] *and* [2.5] *imply* $\beta = \delta(1+\bar{g})^{1-\frac{1}{\sigma}}$ *when* $\varsigma = 0$. *The steady state is not dynamically inefficient* ($\bar{\rho} > \bar{g}$) *if* β *satisfies* $0 < \beta < 1$. *Given* $0 < \beta < 1$, *this steady state is unique if and only if:*

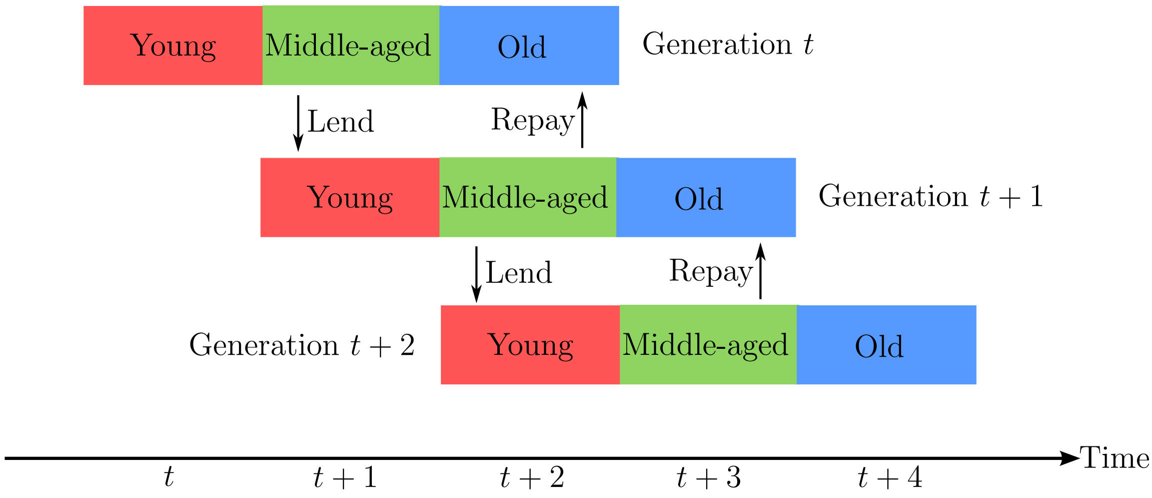
$$\sigma \geq \underline{\sigma}(\gamma, \beta), \quad \text{where} \ \frac{\beta\gamma}{1+\beta} < \underline{\sigma}(\gamma, \beta) < \frac{1}{2}, \quad \lim_{\gamma \to 0} \underline{\sigma}(\gamma, \beta) = 0, \quad \text{and} \ \frac{\partial \underline{\sigma}(\gamma, \beta)}{\partial \gamma} > 0. \quad [2.25]$$

(iii) If the parameter restrictions [2.5] *and* $\sigma \geq \underline{\sigma}(\gamma, \beta)$ *are satisfied then in the neighbourhood of the steady state there exists a stable manifold and an unstable manifold. The stable manifold is an upward-sloping line in* (d_t, ρ_t) *space, and the unstable manifold is either downward sloping or steeper than the stable manifold.*

Proof See appendix A.8. ■

Focusing first on the steady state, note that given the age profile of income in Figure 1 and a preference for consumption smoothing, the young would like to borrow and the middle-aged would like to save. In the absence of any fluctuations in real GDP, and with the parameterization of the age profile of income in [2.6], the model possesses a steady state where the age-profile of consumption is flat over the life-cycle. The parameterization [2.6] also has the convenient property that the value of debt obligations at maturity relative to GDP is solely determined by the income age-profile gradient parameter γ, while the formula for the equilibrium real interest rate is identical that found in an

Figure 2: *Borrowing and saving patterns*



economy with steady-state real GDP growth of $\bar{g}$ and a representative agent having discount factor δ and elasticity of intertemporal substitution σ. Greater changes in individuals' incomes over the life-cycle imply that there will be more borrowing in equilibrium, while faster GDP growth or greater impatience increase the real interest rate. With the parameter restriction $0 < \beta < 1$ from [2.5], the steady state is not dynamically inefficient (the real interest rate $\bar{\rho}$ exceeds the growth rate $\bar{g}$).

The equilibrium borrowing and saving patterns are depicted in Figure 2. The young borrow from the middle-aged and repay once they, the young, are middle-aged and the formerly middle-aged are old.[12] Lending to the young provides a way for the middle-aged to save. Note that all savings are held in the form of 'inside' financial assets (private IOUs) created by those who want to borrow. Under the model's simplifying assumptions, there are no 'outside' assets (for example, government bonds or fiat money).[13]

Given that $0 < \beta < 1$, Proposition 1 shows that the steady state [2.24] is unique if the elasticity of intertemporal substitution is sufficiently large relative to the gradient of the age-profile of income. These two conditions are sufficient to rule out multiple equilibria, and will be assumed in what follows. Out of steady state, the dynamics of the debt ratio and the real interest rate are determined by the first-order difference equation from Proposition 1, which in principle can be solved for (d_{t+1}, ρ_{t+1}) given (d_t, ρ_t). With a unique steady state, the model has the property of saddlepath

[12] The model is designed to represent a pure credit economy where the IOUs of private agents are exchanged for goods, and where IOUs can be created without the need for financial intermediation. The role of 'money' is confined to that of a unit of account and a standard of deferred payments (what borrowers are promising to deliver when their IOUs mature). The downplaying of money's role as a medium of exchange is in line with Woodford's (2003) 'cashless limit' analysis where the focus is on the use of money as a unit of account in setting prices of goods.

[13] The trade between the generations would not be feasible in an overlapping generations model with two-period lives. In that environment, saving is only possible by acquiring a physically storable asset or holding an 'outside' financial asset. In the three-period lives OLG model of Samuelson (1958), the age profile of income is monotonic, so there is little scope for trade between the generations. As a result, the equilibrium involving only 'inside' financial assets is dynamically inefficient. This inefficiency can be corrected by introducing an 'outside' financial asset. Here, under the parameter restrictions, the steady-state real interest rate is above the economy's growth rate, which is equivalent to the absence of dynamic inefficiency (Balasko and Shell, 1980). There are then no welfare gains from introducing an outside asset.

stability: starting from a particular debt ratio d_{t_0} at time t_0, there is only one real interest rate ρ_{t_0} consistent with convergence to the steady state.[14]

3 Monetary policy in a pure credit economy

This section analyses optimal monetary policy in an economy with incomplete markets subject to exogenous shocks to real GDP growth. A benchmark for monetary policy analysis is the equilibrium in the hypothetical case of complete financial markets.

3.1 The natural debt-to-GDP ratio

In monetary economics, it is conventional to use the prefix 'natural' to describe what the equilibrium would be in the absence of a particular friction, such as nominal rigidities or imperfect information. For instance, there are the concepts of the natural rate of unemployment, the natural rate of interest, and the natural level of output. Here, the friction is incomplete markets, not nominal rigidities, but it makes sense to refer to the equilibrium debt-to-GDP ratio in the absence of this friction as the 'natural debt-to-GDP ratio'. Just like any other 'natural' variable, the natural debt-to-GDP ratio is independent of monetary policy, while shocks will generally perturb the actual equilibrium debt-to-GDP ratio away from its natural level, to which it would otherwise converge. Furthermore, the natural debt-to-GDP ratio has efficiency properties that make it a desirable target for monetary policy.

The natural debt-to-GDP need not be constant when the economy is hit by shocks (just as the natural rate of unemployment may change over time), but there are two benchmark cases where it is in fact constant even though shocks occur. These cases require restrictions either on the utility function or on the stochastic process for GDP growth.

Proposition 2 *Consider the equilibrium of the economy with complete financial markets (the solution of equations [2.14a]–[2.14e] and [2.21]). If either of the following conditions is met:*

(i) the utility function is logarithmic ($\alpha = 1$ and $\sigma = 1$ in [2.1]);

(ii) real GDP follows a random walk (the random variable x_t in [2.4] is i.i.d.);

then the equilibrium is as follows, with a constant natural debt-to-GDP ratio:

$$d_t^* = \frac{\gamma}{3}, \quad l_t^* = \frac{\beta\gamma}{3}, \quad c_{\mathrm{y},t}^* = c_{\mathrm{m},t}^* = c_{\mathrm{o},t}^* = 1, \quad \rho_t^* = \frac{1 + \mathbb{E}_t g_{t+1}}{\beta} - 1, \quad \text{and } r_t^* = \frac{1 + g_t}{\beta} - 1. \quad [3.1]$$

The real interest rate is also constant ($\rho_t^* = (1 + \bar{g})/\beta - 1$) when GDP growth is i.i.d.

Proof See appendix A.9. ■

[14] Proposition 1 establishes the saddlepath stability property locally for parameters for which there is a unique steady state. Numerical analysis confirms the saddlepath stability property holds globally for these parameters. See appendix A.1 for further details, including a discussion of why non-convergent paths cannot be equilibria.

Intuitively, the case of real GDP following a random walk can be understood as follows. If a shock has the same effect on the level of GDP in the short run and the long run then it is feasible in all current and future time periods for each generation alive to receive the same consumption share of total output as before the shock. Since the utility function is homothetic, given relative prices for consumption in different time periods, individuals would choose future consumption plans proportional to their current consumption. If consumption shares are maintained then no change of relative prices is required. In this case, all individuals would have the same proportional exposure to consumption risk. Since each individual has a constant coefficient of relative risk aversion, and as this coefficient is the same across all individuals, constant consumption shares are equivalent to efficient risk sharing. For constant consumption shares to be consistent with individual budget constraints it is necessary that debt repayments move one-for-one with changes in GDP. Thus, the efficient financial contract between borrowers and savers resembles an equity share in GDP, which is equivalent to a constant natural debt-to-GDP ratio.

If the short-run and long-run effects of a shock to GDP differ then it is not feasible at all times for generations to maintain unchanged consumption shares because generations do not perfectly overlap. Relative prices of consumption at different times will have to change, which will generally change individuals' desired expenditure shares of lifetime income on consumption at different times. However, with a logarithmic utility function, current consumption will be an unchanging share of lifetime income, and so efficient risk sharing (given that all individuals have log utility) requires stabilization of individuals' consumption shares. This again requires debt repayments that move in line with GDP.

The 'debt gap' $\tilde{d}_t$ is defined as the actual debt-to-GDP ratio (d_t) relative to what the debt-to-GDP ratio would be with complete financial markets (d_t^*):

$$\tilde{d}_t \equiv \frac{d_t}{d_t^*}. \quad [3.2]$$

This concept is analogous to variables such as the output gap or interest-rate gap found in many monetary models. The next section justifies the claim that the goal of monetary policy should be close the debt gap, that is, to aim for $\tilde{d}_t = 1$.

3.2 Pareto efficient allocations

Before considering what can be achieved by a central bank setting monetary policy, first consider the economy from the perspective of a social planner who has the power to mandate allocations of consumption to specific individuals by directly making the appropriate transfers. The planner maximizes a weighted sum of individual utilities subject to the economy's resource constraint.

Starting at some time t_0, the welfare function maximized by the planner is

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0-2}\left[\frac{1}{3}\sum_{t=t_0-2}^{\infty} \beta^{t-t_0}\Omega_t\mathscr{U}_t\right], \quad [3.3]$$

which includes the utility functions [2.1] of all individuals alive at some point from time t_0 onwards.

The Pareto weight assigned to the generation born at time t is denoted by $\beta^{t-t_0}\Omega_t/3$, where the variable Ω_t is scaled for convenience by the term β^{t-t_0} (using β from [2.5] as a discount factor), and by the population share 1/3 of that generation when its members are alive. A Pareto-efficient allocation is a maximum of [3.3] subject to the economy's resource constraints for a particular sequence of Pareto weights $\{\Omega_{t_0-2}, \Omega_{t_0-1}, \Omega_{t_0}, \Omega_{t_0+1}, \ldots\}$, where the weight Ω_t for individuals born at time t may be a function of the state of the world at time t.[15]

The Lagrangian for maximizing the social welfare function subject to the economy's resource constraint $C_t = Y_t$ (with aggregate consumption C_t as defined in [2.2]) is:

$$\mathcal{L}_{t_0} = \mathbb{E}_{t_0-2}\left[\frac{1}{3}\sum_{t=t_0-2}^{\infty}\beta^{t-t_0}\Omega_t\mathscr{U}_t + \sum_{t=t_0}^{\infty}\beta^{t-t_0}\Lambda_t\left(Y_t - \frac{1}{3}C_{\mathrm{y},t} - \frac{1}{3}C_{\mathrm{m},t} - \frac{1}{3}C_{\mathrm{o},t}\right)\right], \qquad [3.4]$$

where the Lagrangian multiplier on the time-t resource constraint is $\beta^{t-t_0}\Lambda_t$ (the scaling by β^{t-t_0} is for convenience). Using the utility function [2.1], the first-order conditions for the consumption levels $C^\star_{\mathrm{y},t}$, $C^\star_{\mathrm{m},t}$ and $C^\star_{\mathrm{o},t}$ that maximize the welfare function [3.3] are:

$$\Omega_t {C^\star}_{\mathrm{y},t}^{-\frac{1}{\sigma}} = \Lambda^\star_t, \quad \Omega_{t-1}\left(\frac{\delta}{\beta}\right)\left\{\frac{V^\star_{\mathrm{m},t}}{\mathbb{E}_{t-1}[{V^\star}_{\mathrm{m},t}^{1-\alpha}]^{\frac{1}{1-\alpha}}}\right\}^{\frac{1}{\sigma}-\alpha} {C^\star}_{\mathrm{m},t}^{-\frac{1}{\sigma}} = \Lambda^\star_t, \quad \text{and}$$

$$\Omega_{t-2}\left(\frac{\delta}{\beta}\right)^2\left\{\frac{V^\star_{\mathrm{o},t}}{\mathbb{E}_{t-1}[{V^\star}_{\mathrm{o},t}^{1-\alpha}]^{\frac{1}{1-\alpha}}}\right\}^{\frac{1}{\sigma}-\alpha}\left\{\frac{V^\star_{\mathrm{m},t-1}}{\mathbb{E}_{t-2}[{V^\star}_{\mathrm{m},t-1}^{1-\alpha}]^{\frac{1}{1-\alpha}}}\right\}^{\frac{1}{\sigma}-\alpha} {C^\star}_{\mathrm{o},t}^{-\frac{1}{\sigma}} = \Lambda^\star_t \text{ for all } t \geq t_0. \qquad [3.5]$$

Since the first-order conditions are homogeneous of degree zero in the Pareto weights Ω_t and the Lagrangian multipliers Λ_t, one of the weights or one of the multipliers can be arbitrarily fixed. The normalization $\Lambda_{t_0} \equiv Y_{t_0}^{-1}$ is chosen, which has the convenient implication that a 0.01 change in the value of the welfare function is equivalent to an exogenous 1% change in real GDP in the initial period.[16] Since the normalization uses output Y_{t_0} at time t_0, the Pareto weights Ω_{t_0-2} and Ω_{t_0-1} may be functions of the state of the world at time t_0, but the ratio $\Omega_{t_0-1}/\Omega_{t_0-2}$ must depend only on variables known at time $t_0 - 1$. The welfare function [3.3] and first-order conditions [3.5] can be rewritten in terms of stationary variables as follows:

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0-2}\left[\frac{1}{3}\sum_{t=t_0-2}^{\infty}\beta^{t-t_0}\omega_t u_t\right], \quad \text{with } \omega_t \equiv \Omega_t Y_t^{1-\frac{1}{\sigma}}, \quad u_t \equiv \frac{\mathscr{U}_t}{Y_t^{1-\frac{1}{\sigma}}}, \quad \varphi_t \equiv \Lambda_t Y_t, \quad \text{and } \varphi_{t_0} \equiv 1. \qquad [3.6]$$

[15] This means that an 'individual' comprises not just a specific person but also a specific history of shocks up to the time of that person's birth. But the weight is not permitted to be a function of shocks realized after birth because this would result in an essentially vacuous notion of *ex-post* efficiency where every non-wasteful allocation of goods could be described as efficient for some sequence of weights that vary during individuals' lifetimes. See appendix A.3 for further discussion.

[16] Applying the envelope theorem to the Lagrangian [3.4] yields $\partial\mathscr{W}_{t_0}/\partial Y_{t_0} = \Lambda_{t_0}$, and hence by setting $\Lambda_{t_0} = Y_{t_0}^{-1}$ it follows that $\partial\mathscr{W}_{t_0}/\partial \log Y_{t_0} = 1$.

Manipulating the first-order conditions [3.5] and using the definitions in [2.13] and [3.6] leads to:

$$\omega_t = \frac{\varphi_t^\star}{{c^\star}_{\mathrm{y},t}^{-\frac{1}{\sigma}}}, \quad \text{and} \quad \frac{\varphi_{t+1}^\star}{\varphi_t^\star} = (1+g_{t+1})^{1-\frac{1}{\sigma}} \left\{ \frac{(1+g_{t+1})v_{\mathrm{m},t+1}^\star}{\mathbb{E}_t[(1+g_{t+1})^{1-\alpha}{v^\star}_{\mathrm{m},t+1}^{1-\alpha}]^{\frac{1}{1-\alpha}}} \right\}^{\frac{1}{\sigma}-\alpha} \left(\frac{c_{\mathrm{m},t+1}^\star}{c_{\mathrm{y},t}^\star} \right)^{-\frac{1}{\sigma}}$$

$$= (1+g_{t+1})^{1-\frac{1}{\sigma}} \left\{ \frac{(1+g_{t+1})v_{\mathrm{o},t+1}^\star}{\mathbb{E}_t[(1+g_{t+1})^{1-\alpha}{v^\star}_{\mathrm{o},t+1}^{1-\alpha}]^{\frac{1}{1-\alpha}}} \right\}^{\frac{1}{\sigma}-\alpha} \left(\frac{c_{\mathrm{o},t+1}^\star}{c_{\mathrm{m},t}^\star} \right)^{-\frac{1}{\sigma}} \quad \text{for all } t \geq t_0, \quad [3.7]$$

where these equations hold in all states of the world. There is a well-defined steady state for all of the transformed variables in [3.6]. Using Proposition 1 together with equations [2.1], [3.6], and [3.7], it follows that $\bar{\omega} = 1$ and $\bar{\varphi} = 1$. Given the parameter restriction [2.5], this shows the welfare function is finite-valued for any real GDP growth stochastic process consistent with [2.4].

The equations in [3.7] imply that the risk-sharing condition [2.21] is a necessary condition for any Pareto-efficient consumption allocation. This equation is an equilibrium condition with complete financial markets, so the complete-markets equilibrium will be Pareto efficient.[17] However, there are many other Pareto-efficient allocations satisfying the resource constraint [2.16] and the risk-sharing condition [2.21].

Now return to the analysis of monetary policy where the policymaker is a central bank with a single instrument, the nominal interest rate i_t. The central bank operates in an economy with incomplete markets where the equilibrium conditions are [2.14a]–[2.14e] and [2.15]. The central bank maximizes the welfare function [3.3] subject to the incomplete-markets equilibrium conditions as implementability constraints (including [2.14c], which implies the resource constraint [2.16]). The solution will depend on which Pareto weights Ω_t are used, which capture the distributional preferences of the policymaker.

Two questions regarding efficiency and distribution naturally arise when studying the central bank's constrained maximization problem. First, the extent to which the central bank will be able to achieve a Pareto-efficient consumption allocation. Second, the considerations that should guide the choice of the Pareto weights determining the policymaker's distributional preferences. The second question is less familiar in optimal monetary policy analysis because much existing work is based on models with a representative agent. The approach adopted here is to assume the central bank strives for Pareto efficiency and will always sacrifice distributional concerns to efficiency (that is, it has a 'lexicographic preference' for efficiency). The following result provides some guidance for such a central bank.

Proposition 3 *(i) A state-contingent consumption allocation $\{c_{\mathrm{y},t}^\star . c_{\mathrm{m},t}^\star, c_{\mathrm{o},t}^\star\}$ is Pareto efficient from $t \geq t_0$ onwards if and only if it satisfies the resource constraint [2.16] for all $t \geq t_0$, the risk-sharing condition [2.21] for all $t \geq t_0$, and is such that ${v^\star}_{\mathrm{o},t_0}^{\frac{1}{\sigma}-\alpha}{c^\star}_{\mathrm{o},t_0}^{-\frac{1}{\sigma}} / {v^\star}_{\mathrm{m},t_0}^{\frac{1}{\sigma}-\alpha}{c^\star}_{\mathrm{m},t_0}^{-\frac{1}{\sigma}}$ depends only on variables known at time $t_0 - 1$. The complete-markets equilibrium (with markets open from at least time $t_0 - 1$ onwards) is Pareto efficient from $t \geq t_0$.*

[17] There are two caveats to this claim specific to overlapping generations models: the question of whether the utility functions of the 'individuals' considered by the social planner should be evaluated as expectations over shocks realized *prior* to birth, and the possibility of dynamic inefficiency. As discussed in appendix A.3, while these issues are potentially important, neither of them is relevant in this paper.

(ii) If a Pareto-efficient consumption allocation can be implemented through monetary policy from time t_0 onwards then this allocation must be the complete-markets equilibrium (with markets open from time $t_0 - 1$ onwards).

PROOF See appendix A.10. ∎

The first part confirms that the complete-markets equilibrium is one of the many Pareto-efficient consumption allocations. More importantly, the second part states that the complete-markets equilibrium is the *only* Pareto-efficient allocation that can be implemented in an incomplete-markets economy by a central bank setting interest rates (rather than by a social planner who can make direct transfers). The intuition is that the risk-sharing condition [2.21] is necessary for Pareto efficiency, but this is also the only equation that differs between the equilibrium conditions of the incomplete- and complete-markets economies. This result is useful because it provides a unique answer to the question of the choice of Pareto weights for a central bank that always prioritizes efficiency over distributional concerns. This avoids the need to specify the political preferences of the central bank when analysing optimal monetary policy in a non-representative-agent economy. Therefore, in what follows, monetary policy is evaluated using the Pareto weights Ω_t^* consistent with the complete-markets equilibrium.

3.3 Optimal monetary policy

Optimal monetary policy is defined as the constrained maximum of the welfare function [3.3] subject to the equilibrium conditions [2.14a]–[2.14e] and [2.15] as constraints, and using Pareto weights Ω_t^* consistent with the complete-markets equilibrium. Monetary policy has a single instrument, and this can be used to generate any state-contingent path for one nominal variable, for example, the price level (accepting the equilibrium values of other nominal variables). For simplicity, monetary policy is modelled as directly choosing this nominal variable, while the question of what interest-rate policy would be needed to implement it is deferred for later analysis.

In characterizing the optimal policy it is helpful to introduce the definition of nominal GDP $M_t \equiv P_t Y_t$. Given the definitions of inflation π_t and real GDP growth g_t, the dynamics of nominal GDP can be written as $M_t = (1 + \pi_t)(1 + g_t)M_{t-1}$. Using this equation together with [2.14a] and [2.15], the following link between the unexpected components of the debt-to-GDP ratio d_t and nominal GDP is obtained:

$$\frac{d_t}{\mathbb{E}_{t-1} d_t} = \frac{M_t^{-1}}{\mathbb{E}_{t-1} M_t^{-1}}. \qquad [3.8]$$

This equation indicates that stabilizing the ratio of debt liabilities to income is related to stabilizing the nominal value of income. The intuition is that d_t can be written as a ratio of *nominal* debt liabilities to *nominal* income. Since nominal debt liabilities are not state contingent, any unpredictable change in the ratio is driven by unpredictable changes in nominal GDP. This leads to the main result of the paper.

Proposition 4 *The complete-markets equilibrium can be implemented by monetary policy in the incomplete-markets economy, closing the debt gap ($\tilde{d}_t = 1$) from [3.2]. This equilibrium is obtained if and only if monetary policy determines a level of nominal GDP M_t^* such that:*

$$M_t^* = d_t^{*-1} \mathcal{X}_{t-1}, \tag{3.9}$$

where d_t^* is the debt-to-GDP ratio in the complete-markets economy and $\mathcal{X}_{t-1}$ is any function of variables known at time $t-1$.

PROOF See appendix A.11. ■

To understand the intuition for this result, consider an economy where shocks to GDP are permanent or individuals have logarithmic utility functions. In those cases, Proposition 2 shows that efficient risk sharing requires debt repayments that rise and fall exactly in proportion to income. Decentralized implementation of this risk sharing entails individuals trading securities with state-contingent payoffs, or equivalently, writing contracts that spell out a complete schedule of varying repayments across different states of the world. Incomplete financial markets preclude this, and the assumption of the model is that individuals are restricted to the type of non-contingent nominal debt contracts commonly observed. In this environment, efficient risk sharing will break down when debtors are obliged to make fixed repayments from future incomes that are uncertain.

In an economy that is hit by aggregate shocks, irrespective of what monetary policy is followed, there will always be uncertainty about future real GDP. However, there is nothing in principle to prevent monetary policy stabilizing the nominal value of GDP. In the absence of idiosyncratic shocks, nominal GDP targeting would remove any uncertainty about nominal incomes, ensuring that even non-contingent nominal debt repayments maintain a stable ratio to income in all states of the world, and thus achieves efficient risk sharing.

3.4 Discussion

The importance of these arguments for nominal GDP targeting obviously depends on the plausibility of the incomplete-markets assumption in the context of household borrowing and saving. It seems reasonable to suppose that individuals will not find it easy to borrow by issuing Arrow-Debreu state-contingent bonds, but might there be other ways of reaching the same goal? Issuance of state-contingent bonds is equivalent to households agreeing loan contracts with financial intermediaries that specify a complete menu of state-contingent repayments. But such contracts would be much more time consuming to write, harder to understand, and more complicated to enforce than conventional non-contingent loan contracts, as well as making monitoring and assessment of default risk a more elaborate exercise.[18] Moreover, unlike firms, households cannot issue securities such as equity that feature state-contingent payments but do not require a complete description of the schedule of payments in advance.[19]

[18] For examples of theoretical work on endogenizing the incompleteness of markets through limited enforcement of contracts or asymmetric information, see Kehoe and Levine (1993) and Cole and Kocherlakota (2001).

[19] Consider an individual owner of a business that generates a stream of risky profits. If the firm's only external finance is non-contingent debt then the individual bears all the risk (except in the case of default). If the individual

Another possibility is that even if individuals are restricted to non-contingent borrowing, they can hedge their exposure to future income risk by purchasing an asset with returns that are negatively correlated with GDP. But there are several pitfalls to this. First, it may not be clear which asset reliably has a negative correlation with GDP (even if 'GDP securities' of the type proposed by Shiller (1993) were available, borrowers would need a short position in these). Second, the required gross positions for hedging may be very large. Third, an individual already intending to borrow will need to borrow even more to buy the asset for hedging purposes, and the amount of borrowing may be limited by an initial down-payment constraint and subsequent margin calls. In practice, a typical borrower does not have a significant portfolio of assets except for a house, and housing returns most likely lack the negative correlation with GDP required for hedging the relevant risks.

In spite of these difficulties, it might be argued the case for the incomplete markets assumption is overstated because the possibilities of renegotiation, default, and bankruptcy introduce some contingency into apparently non-contingent debt contracts. However, default and bankruptcy allow for only a crude form on contingency in extreme circumstances, and these options are not without their costs. Renegotiation is also not costless, and evidence from consumer mortgages in both the recent U.S. housing bust and the Great Depression suggests that the extent of renegotiation may be inefficiently low (White, 2009a, Piskorski, Seru and Vig, 2010, Ghent, 2011). Furthermore, even ex-post efficient renegotiation of a contract with no contingencies written in ex ante need not actually provide for efficient sharing of risk from an ex-ante perspective.

It is also possible to assess the completeness of markets indirectly through tests of the efficient risk-sharing condition, which is equivalent to correlation across consumption growth rates of individuals. These tests are the subject of a large literature (Cochrane, 1991, Nelson, 1994, Attanasio and Davis, 1996, Hayashi, Altonji and Kotlikoff, 1996), which has generally rejected the hypothesis of full risk sharing.

Finally, even if financial markets are incomplete, the assumption that contracts are written in terms of specifically nominal non-contingent payments is important for the analysis. The evidence presented in Doepke and Schneider (2006) indicates that household balance sheets contain significant quantities of nominal liabilities and assets (for assets, it is important to account for indirect exposure via households' ownership of firms and financial intermediaries). Furthermore, as pointed out by Shiller (1997), indexation of private debt contracts is extremely rare. This suggests the model's assumptions are not unrealistic.

The workings of nominal GDP targeting can also be seen from its implications for inflation and the real value of nominal liabilities. Indeed, nominal GDP targeting can be equivalently described as a policy of inducing a perfect negative correlation between the price level and real GDP, and ensuring these variables have the same volatility. When real GDP falls, inflation increases, which

wanted to share risk with other investors then one possibility would be to replace the non-contingent debt with state-contingent bonds where the payoffs on these bonds are positively related to the firm's profits. However, what is commonly observed is not issuance of state-contingent bonds but equity financing. Issuing equity also allows for risk sharing, but unlike state-contingent bonds does not need to spell out a schedule of payments in all states of the world. There is no right to any specific payment in any specific state at any specific time, only the right of being residual claimant. The lack of specific claims is balanced by control rights over the firm. However, there is no obvious way to be 'residual claimant' on or have 'control rights' over a household.

reduces the real value of fixed nominal liabilities in proportion to the fall in real income, and vice versa when real GDP rises. Thus the extent to which financial markets with non-contingent nominal assets are sufficiently complete to allow for efficient risk sharing is endogenous to the monetary policy regime: monetary policy can make the real value of fixed nominal repayments contingent on the realization of shocks. A strict policy of inflation targeting would be inefficient because it converts non-contingent nominal liabilities into non-contingent real liabilities. This points to an inherent tension between price stability and the efficient operation of financial markets.[20]

That optimal monetary policy in a non-representative-agent[21] model should feature inflation fluctuations is perhaps surprising given the long tradition of regarding inflation-induced unpredictability in the real values of contractual payments as one of the most important of all inflation's costs. As discussed in Clarida, Galí and Gertler (1999), there is a widely held view that the difficulties this induces in long-term financial planning ought to be regarded as the most significant cost of inflation, above the relative price distortions, menu costs, and deviations from the Friedman rule that have been stressed in representative-agent models. The view that unanticipated inflation leads to inefficient or inequitable redistributions between debtors and creditors clearly presupposes a world of incomplete markets, otherwise inflation would not have these effects. How then to reconcile this argument with the result that incompleteness of financial markets suggests nominal GDP targeting is desirable because it supports efficient risk sharing? (again, were markets complete, monetary policy would be irrelevant to risk sharing because all opportunities would already be exploited)

While nominal GDP targeting does imply unpredictable inflation fluctuations, the resulting real transfers between debtors and creditors are not an *arbitrary* redistribution — they are perfectly correlated with the relevant fundamental shock: unpredictable movements in aggregate real incomes. Since future consumption uncertainty is affected by income risk as well as risk from fluctuations in the real value of nominal contracts, it is not necessarily the case that long-term financial planning is compromised by inflation fluctuations that have known correlations with the economy's fundamentals. An efficient distribution of risk requires just such fluctuations because the provision of insurance is impossible without the possibility of ex-post transfers that cannot be predicted ex ante. Unpredictable movements in inflation orthogonal to the economy's fundamentals (such as would occur in the presence of monetary-policy shocks) are inefficient from a risk-sharing perspective, but there is no contradiction with nominal GDP targeting because such movements would only occur if policy failed to stabilize nominal GDP.[22]

It might be objected that if debtors and creditors really wanted such contingent transfers then they would write them into the contracts they agree, and it would be wrong for the central bank to try to second-guess their intentions. But the absence of such contingencies from observed contracts may simply reflect market incompleteness rather than what would be rationally chosen in a frictionless

[20] In a more general setting where the incompleteness of financial markets is endogenized, inflation fluctuations induced by nominal GDP targeting may play a role in minimizing the costs of contract renegotiation or default when the economy is hit by an aggregate shock.

[21] It is implicitly assumed different generations do not form the infinitely lived dynasties suggested by Barro (1974).

[22] The model could be applied to study the quantitative welfare costs of the arbitrary redistributions caused by inflation resulting from monetary-policy shocks. See section 5 for further details.

world. Reconciling the non-contingent nature of financial contracts with complete markets is not impossible, but it would require both substantial differences in risk tolerance across individuals and a high correlation of risk tolerance with whether an individual is a saver or a borrower. With assumptions on preferences that make borrowers risk neutral or savers extremely risk averse, it would not be efficient to share risk, even if no frictions prevented individuals writing contracts that implement it.

There are a number of problems with this alternative interpretation of the observed prevalence of non-contingent contracts. First, there is no compelling evidence to suggest that borrowers really are risk neutral or savers are extremely risk averse relative to borrowers. Second, while there is evidence suggesting considerable heterogeneity in individuals' risk tolerance (Barsky, Juster, Kimball and Shapiro, 1997, Cohen and Einav, 2007), most of this heterogeneity is not explained by observable characteristics such as age and net worth (even though many characteristics such as these have some correlation with risk tolerance). The dispersion in risk tolerance among individuals with similar observed characteristics suggests there should be a wide range of types of financial contract with different degrees of contingency. Risk neutral borrowers would agree non-contingent contracts with risk-averse savers, but contingent contracts would be offered to risk-averse borrowers.

Another problem with the complete markets but different risk preferences interpretation relates to the behaviour of the price level over time. While nominal GDP has never been an explicit target of monetary policy, nominal GDP targeting's implication of a countercyclical price level has been largely true in the U.S. during the post-war period (Cooley and Ohanian, 1991), albeit with a correlation coefficient much smaller than one in absolute value, and a lower volatility relative to real GDP. Whether by accident or design, U.S. monetary policy has had to a partial extent the features of nominal GDP targeting, resulting in the real values of fixed nominal payments positively co-moving with real GDP (but by less) on average. In a world of complete markets with extreme differences in risk tolerance between savers and borrowers, efficient contracts would undo the real contingency of payments brought about by the countercyclicality of the price level, for example, through indexation clauses. But as discussed in Shiller (1997), private nominal debt contracts have survived in this environment without any noticeable shift towards indexation. Furthermore, both the volatility of inflation and correlation of the price level with real GDP have changed significantly over time (the high volatility 1970s versus the 'Great Moderation', and the countercyclicality of the post-war price level versus its procyclicality during the inter-war period). The basic form of non-contingent nominal contracts has remained constant in spite of this change.[23]

Finally, while the policy recommendation of this paper goes against the long tradition of citing the avoidance of redistribution between debtors and creditors as an argument for price stability, it is worth noting that there is a similarly ancient tradition in monetary economics (which can be traced back at least to Bailey, 1837) of arguing that money prices should co-move inversely with productivity to promote 'fairness' between debtors and creditors. The idea is that if money prices fall when productivity rises, those savers who receive fixed nominal incomes are able to share in

[23] It could be argued that part of the reluctance to adopt indexation is a desire to avoid eliminating the risk-sharing offered by nominal contracts when the price level is countercyclical.

the gains, while the rise in prices at a time of falling productivity helps to ameliorate the burden of repayment for borrowers. This is equivalent to stabilizing the money value of incomes, in other words, nominal GDP targeting. The intellectual history of this idea (the 'productivity norm') is thoroughly surveyed in Selgin (1995). Like the older literature, this paper places distributional questions at the heart of monetary policy analysis, but studies policy through the lens of mitigating inefficiencies in incomplete financial markets, rather than with looser notions of fairness.

4 Equilibrium in a pure credit economy

In cases where Proposition 2 applies, [3.1] fully characterizes the equilibrium of the economy if the optimal monetary policy of nominal GDP targeting from Proposition 4 is followed. The equilibrium with optimal policy under conditions where Proposition 2 is not applicable, or where a non-optimal monetary policy is followed, cannot generally be found analytically. In what follows, log-linearization is used to find an approximate solution to the equilibrium in these cases.

4.1 Log-linear approximation of the equilibrium

The log-linearization is performed around the non-stochastic steady state of the model ($\varsigma = 0$ in [2.4]) as characterized in Proposition 1 (which is valid for sufficiently small values of the standard deviation ς of real GDP growth). Log deviations of variables from their steady-state values are denoted with sans serif letters,[24] for example, $\mathsf{d}_t \equiv \log d_t - \log \bar{d}$, while for variables that do not necessarily have a steady state,[25] the sans serif equivalent denotes simply the logarithm of the variable, for example, $\mathsf{Y}_t \equiv \log Y_t$. In the following, terms that are second-order or higher in deviations from the steady state are suppressed.

First consider the set of equations [2.14a]–[2.14e] common to the cases of complete and incomplete financial markets. The equation for debt dynamics [2.14a], the definition of the real interest rate [2.14b], the budget identities [2.14c], and the Euler equations [2.14d] for each generation have the following log-linear expressions:

$$\rho_t = \mathbb{E}_t \mathsf{r}_{t+1}, \quad \mathsf{d}_t = \mathsf{r}_t - \mathsf{g}_t + \mathsf{l}_{t-1}, \quad \mathsf{c}_{\mathrm{y},t} = \beta\gamma \mathsf{l}_t, \quad \mathsf{c}_{\mathrm{m},t} = -\gamma \mathsf{d}_t - \beta\gamma \mathsf{l}_t, \quad \mathsf{c}_{\mathrm{o},t} = \gamma \mathsf{d}_t, \tag{4.1a}$$

$$\mathsf{c}_{\mathrm{y},t} = \mathbb{E}_t \mathsf{c}_{\mathrm{m},t+1} - \sigma\rho_t + \mathbb{E}_t \mathsf{g}_{t+1}, \quad \text{and} \quad \mathsf{c}_{\mathrm{m},t} = \mathbb{E}_t \mathsf{c}_{\mathrm{o},t+1} - \sigma\rho_t + \mathbb{E}_t \mathsf{g}_{t+1}, \tag{4.1b}$$

observing that the value functions $\mathsf{v}_{\mathrm{m},t}$ and $\mathsf{v}_{\mathrm{o},t}$ and the coefficient of relative risk aversion α do not appear in these equations.

Proposition 5 *The log linear approximation of the solution of equations [4.1a]–[4.1b] is determined only up to a martingale difference stochastic process Υ_t ($\mathbb{E}_{t-1}\Upsilon_t = 0$) such that $\Upsilon_t = \mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t$ is the unexpected component of the debt-to-GDP ratio defined in [2.22]. Given Υ_t, the debt-to-GDP*

[24] For all variables that are either interest rates or growth rates, the log deviation is of the gross rate, for example, $\mathsf{g}_t \equiv \log(1 + g_t) - \log(1 + \bar{g})$.

[25] The level of GDP can be either stationary or non-stationary depending on the specification of the stochastic process for g_t.

ratio is given by

$$\mathsf{d}_t = \lambda \mathsf{d}_{t-1} + \chi\left(2\mathsf{f}_{t-1} + \mathbb{E}_{t-1}\mathsf{f}_t\right) + \Upsilon_t, \quad \textit{with } \mathsf{f}_t \equiv \beta\left(\frac{1-\sigma}{\sigma}\right)\sum_{\ell=1}^{\infty}\zeta^{\ell-1}\mathbb{E}_t\mathsf{g}_{t+\ell}. \qquad [4.2]$$

Given a debt ratio d_t satisfying [4.2], the other endogenous variables must satisfy:

$$\mathsf{l}_t = -\beta^{-1}\phi\mathsf{d}_t - \beta^{-1}\varkappa\mathsf{f}_t, \quad \rho_t = \frac{1}{\sigma}\mathbb{E}_t\mathsf{g}_{t+1} + \frac{\gamma}{\sigma}\left(\vartheta\mathsf{d}_t + \chi\left(\phi\mathsf{f}_t + \mathbb{E}_t\mathsf{f}_{t+1}\right)\right), \qquad [4.3a]$$

$$\mathsf{c}_{\mathrm{y},t} = -\gamma\left(\phi\mathsf{d}_t + \varkappa\mathsf{f}_t\right), \quad \mathsf{c}_{\mathrm{m},t} = -\gamma\left((1-\phi)\mathsf{d}_t - \varkappa\mathsf{f}_t\right), \quad \mathsf{c}_{\mathrm{o},t} = \gamma\mathsf{d}_t, \quad \textit{and} \qquad [4.3b]$$

$$\mathsf{r}_t = \mathsf{d}_t + \beta^{-1}\phi\mathsf{d}_{t-1} + \beta^{-1}\varkappa\mathsf{f}_{t-1} + \mathsf{g}_t. \qquad [4.3c]$$

All coefficients χ, $\varkappa$, ϕ, $\theta \equiv (\gamma/\sigma)\vartheta$, λ, and ζ are functions only of β and the ratio γ/σ, and all are increasing in the ratio γ/σ. Formulas for the coefficients are given in appendix A.4. The coefficients satisfy $0 < \chi < 1$, $0 < \phi < 1$, $|\lambda| < 1$, $|\zeta| < 1$, and both $\varkappa$ and θ are positive and bounded.

Proof See appendix A.12 ■

The variable f_t includes all that needs to be known about expectations of future real GDP growth to determine equilibrium saving and borrowing behaviour given individuals' desire for consumption smoothing over time. An increase in f_t leads to a reduction in lending l_t and a higher real interest rate ρ_t. However, whether expectations of future growth have a positive or negative effect on f_t depends on the relative strengths of income and substitution effects. With strong intertemporal substitution ($\sigma > 1$), expectations of future growth increase lending by the middle-aged to the young (the effects of f_t on the consumption of these two groups always have opposite signs because lending involves a transfer of resources), while the effect is the opposite if intertemporal substitution is weak ($\sigma < 1$).

Any unanticipated movements in the debt ratio d_t constitute transfers from the middle-aged to the old. These have the effect of pushing up real interest rates because aggregate desired saving falls following this transfer, and higher real interest rates reduce borrowing by the young. Consistent with this, consumption of the old is increasing in d_t, while consumption of both the young and the middle-aged is decreasing in d_t (the coefficient ϕ measures how the effects are spread between the young and middle-aged in equilibrium). Note that the size of these effects is increasing in the parameter γ, and as this parameter tends to zero, the economy behaves as if it contained a representative agent.[26]

With incomplete markets, the system of equations [4.1a]–[4.1b] is closed (that is, Υ_t is determined) by a description of monetary policy and equation [2.15], which has the following log-linear form:

$$\mathsf{r}_t = \mathsf{i}_{t-1} - \pi_t. \qquad [4.4]$$

With complete markets, the system [4.1a]–[4.1b] is closed by the risk-sharing equation [2.21]. This

[26]With $\gamma = 0$, equations [4.3a] and [4.3b] imply $\rho_t = (1/\sigma)\mathbb{E}_t\mathsf{g}_{t+1}$ and $\mathsf{c}_{\mathrm{y},t} = \mathsf{c}_{\mathrm{m},t} = \mathsf{c}_{\mathrm{o},t} = 0$. This means that $\mathsf{C}_{\mathrm{y},t} = \mathsf{C}_{\mathrm{m},t} = \mathsf{C}_{\mathrm{o},t} = \mathsf{C}_t = \mathsf{Y}_t$ and the representative-agent consumption Euler equation $\mathsf{Y}_t = \mathbb{E}_t\mathsf{Y}_{t+1} - \sigma\rho_t$ holds. Strictly speaking, this limiting case is not a representative-agent model, but because all individuals receive the same incomes, there is limited scope for trade, so to a first-order approximation, the economy behaves as if it contained a representative agent.

can be log-linearized as follows:

$$\frac{1}{\sigma}\left((c^*_{m,t+1} - c^*_{y,t}) - (c^*_{o,t+1} - c^*_{m,t})\right) + \left(\alpha - \frac{1}{\sigma}\right)\left((v^*_{m,t+1} - \mathbb{E}_t v^*_{m,t+1}) - (v^*_{o,t+1} - \mathbb{E}_t v^*_{o,t+1})\right) = 0,$$

$$\text{with}\quad v^*_{m,t} = \frac{1}{1+\beta}c^*_{m,t} + \frac{\beta}{1+\beta}\mathbb{E}_t[c^*_{o,t+1} + g_{t+1}], \quad \text{and} \quad v^*_{o,t} = c^*_{o,t}, \tag{4.5}$$

where the second line log linearizes the value functions appearing in [2.14e]. The following result first characterizes the complete-markets equilibrium, then states the equations for the 'gaps' between variables and their values in the hypothetical complete-markets equilibrium, and finally provides the link between these 'gaps' and the inflation rate.

Proposition 6 *The equilibrium with complete financial markets is given by equations* [4.2] *and* [4.3a]–[4.3c] *with* $\Upsilon^*_t = d^*_t - \mathbb{E}_{t-1}d^*_t$ *given by*

$$\Upsilon^*_t = \left(2 - \phi - \frac{\beta}{1+\beta}\frac{(\alpha\sigma - 1)}{\alpha\sigma}(1 - \phi + \lambda)\right)^{-1}\left\{\chi\left(\left(2 - \phi + \frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}\phi\right)(f_t - \mathbb{E}_{t-1}f_t)\right.\right.$$
$$\left.\left. + \frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}(\mathbb{E}_t f_{t+1} - \mathbb{E}_{t-1}f_{t+1})\right) + \frac{1}{\gamma}\frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}(\mathbb{E}_t g_{t+1} - \mathbb{E}_{t-1}g_{t+1})\right\}. \tag{4.6}$$

The debt gap $\tilde{d}_t \equiv d_t - d^*_t$ *(from* [3.2]*) in the incomplete-markets economy must satisfy:*

$$\mathbb{E}_t\tilde{d}_{t+1} = \lambda\tilde{d}_t. \tag{4.7a}$$

The debt gap is a sufficient statistic for describing all deviations of the economy from the hypothetical complete-markets equilibrium (for example, the real interest rate gap $\tilde{\rho}_t \equiv \rho_t - \rho^*_t$*):*

$$\tilde{l}_t = -\beta^{-1}\phi\tilde{d}_t, \quad \tilde{\rho}_t = \theta\tilde{d}_t, \quad \tilde{c}_{y,t} = -\gamma\phi\tilde{d}_t, \quad \tilde{c}_{m,t} = -\gamma(1-\phi)\tilde{d}_t, \quad \textit{and} \quad \tilde{c}_{o,t} = \gamma\tilde{d}_t. \tag{4.7b}$$

The inflation rate in the incomplete-markets economy satisfies:

$$\pi_t = i_{t-1} - \tilde{d}_t - \beta^{-1}\phi\tilde{d}_{t-1} - r^*_t. \tag{4.7c}$$

PROOF See appendix A.13 ■

The proposition shows how the complete-markets debt-to-GDP ratio d^*_t (the natural debt-to-GDP ratio) can be characterized for general utility function parameters and a general stochastic process for real GDP growth. In the absence of further shocks, the economy will approach d^*_t in the long run (the debt gap will shrink to zero according to equation [4.7a], noting that $|\lambda| < 1$). However, the debt gap is not automatically closed in the short run following shocks without a monetary policy intervention. The behaviour of the debt-to-GDP ratio following a shock depends on the behaviour of nominal GDP (see equation [3.8]):

$$d_t - \mathbb{E}_{t-1}d_t = -(M_t - \mathbb{E}_{t-1}M_t). \tag{4.8}$$

The class of policies that close the debt gap are characterized in Proposition 4. The simplest is a target for nominal GDP ($M_t = P_t + Y_t$) that moves inversely with the natural debt-to-GDP ratio

d_t^*, that is, $\mathsf{M}_t^* = -\mathsf{d}_t^*$. This policy achieves $\tilde{\mathsf{d}}_t = 0$, but requires fluctuations in inflation. The equilibrium inflation rate and nominal interest rate are:

$$\pi_t = -\mathsf{g}_t - (\mathsf{d}_t^* - \mathsf{d}_{t-1}^*), \quad \text{and} \quad \mathsf{i}_t = \rho_t^* - \mathbb{E}_t \mathsf{g}_{t+1} - (\mathbb{E}_t \mathsf{d}_{t+1}^* - \mathsf{d}_t^*). \qquad [4.9]$$

The optimal policy only allows nominal GDP to fluctuate if the natural debt-to-GDP ratio is time varying. With real GDP following a random walk or logarithmic utility, $\mathsf{d}_t^* = 0$, in which case the target reduces to $\mathsf{M}_t^* = 0$ and the required inflation fluctuations simply mirror the fluctuations in real GDP growth in the opposite direction. In general, it is a quantitative question how much optimal policy deviates from a completely stable level of nominal GDP.

4.2 Non-logarithmic utility and predictable variation in GDP growth

To study how much optimal policy deviates from a constant nominal GDP target when the utility function is not logarithmic and real GDP does not follow a random walk, consider the following stochastic process for real GDP growth:

$$\mathsf{g}_t = \epsilon_t + \xi \epsilon_{t-1}, \quad \text{with} \ \ \epsilon_t \sim \text{i.i.d.}(0, \varsigma_\epsilon). \qquad [4.10]$$

In this first-order moving-average process, the parameter ξ represents the difference between the long-run effect of a shock ϵ_t on the level of GDP minus its short-run effect ($\xi = 0$ corresponds to the case of a random walk where the long-run effect is identical to the short-run effect). If $\xi > 0$ then the long-run effect on GDP is greater than the effect in the short run, and vice versa for $\xi < 0$. Substituting this stochastic process into [4.6] yields an expression for the innovation $\Upsilon_t^* = \mathsf{d}_t^* - \mathbb{E}_{t-1}\mathsf{d}_t^*$ to the natural debt-to-GDP ratio:

$$\Upsilon_t^* = (\varpi^* - 1)(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t), \quad \text{where} \ \ \varpi^* = 1 + \frac{\xi\beta\left(\chi\frac{(1-\sigma)}{\sigma}\left(2 - \phi + \frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}\phi\right) + \frac{1}{\gamma}\frac{1}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}\right)}{2 - \phi - \frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}(1 - \phi + \lambda)}. \qquad [4.11]$$

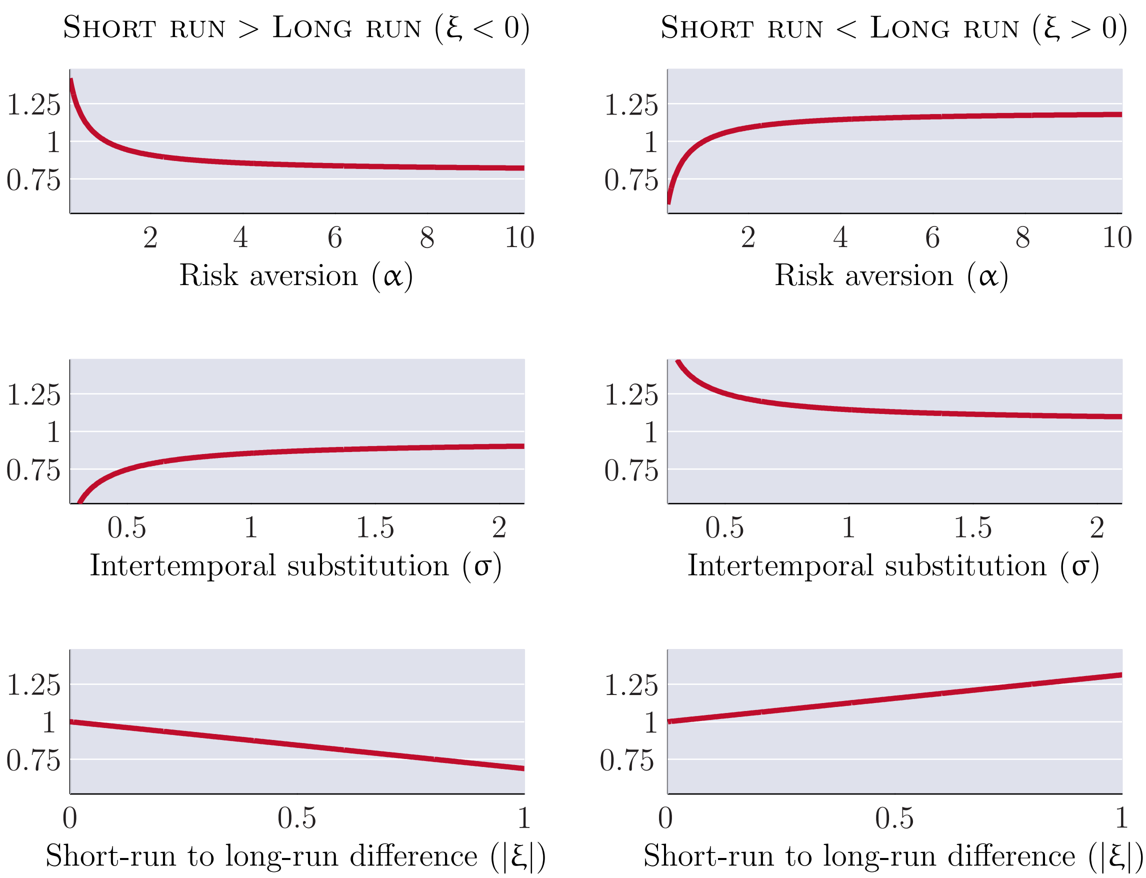
The coefficient ϖ^* determines how much the debt-to-GDP ratio should rise or fall following a shock to the level of GDP: the debt ratio should positively co-move with GDP if $\varpi^* > 1$, and negatively co-move if $\varpi^* < 1$. As can be seen from the expression for ϖ^*, determining which case prevails requires assumptions on the preference parameters and the GDP stochastic process. As an example, consider the plausible case where intertemporal substitution is relatively low ($\sigma < 1$) and risk aversion is at least what would be implied by a time-separable utility function ($\alpha \geq 1/\sigma$). In this case, following a negative shock to GDP, the debt-to-GDP ratio should rise if the shock's effects are smaller in the long run than the short run, while the ratio should fall if the long-run effects are larger. Intuitively, if the economy is expected to recover in the future, debt liabilities should fall by less than current income does, while if GDP is expected to deteriorate further, the real value of debt liabilities should fall by more than current income.

Proposition 7 *If real GDP growth is described by the stochastic process [4.10] then optimal monetary policy can be described as a constant target for weighted nominal GDP* $\mathsf{P}_t + \varpi^*\mathsf{Y}_t = 0$, *where* *the weight ϖ^* on real output is given in equation* [4.11].

PROOF See appendix A.14. ■

In the case where real GDP is described by stochastic process [4.10], Proposition 7 shows that optimal monetary policy can equivalently be expressed in terms of a target for a stable level of *weighted* nominal GDP, where ϖ is the weight on real GDP relative to the weight on the price level (standard nominal GDP targeting is $\varpi = 1$). The optimal policy implies $\mathsf{P}_t = -\varpi^* \mathsf{Y}_t$, so ϖ^* can also be interpreted as the optimal countercyclicality of the price level. As an example, consider the plausible case where the elasticity of intertemporal substitution is relative low ($\sigma < 1$) and risk aversion is relatively high ($\alpha \geq 1/\sigma$). It can be seen from [4.11] that when the long-run effect of a shock to GDP is smaller than its initial effect ($\xi < 0$) then $\varpi^* < 1$, so following a negative shock to real GDP, the price level should rise by less than if the shock were permanent. Given parameters α and σ, the size of the deviation of ϖ^* from 1 depends on the deviation of the parameter ξ from zero.

Figure 3: *Optimal monetary policy when GDP shocks have different short-run and long-run effects — weight ϖ^* assigned to real GDP relative to price level*



Notes: Monetary policy is $\mathsf{P}_t + \varpi^* \mathsf{Y}_t = 0$, where the formula for ϖ^* is given in [4.11]. The graphs show the effects of varying one parameter, holding other parameters constant at their baseline values, given in Table 1 (with the baseline value of ξ set to 0.5).

The quantitative deviation of the optimal monetary policy from pure nominal GDP targeting thus depends first on how much the stochastic process for real GDP differs from a random walk. There is an extensive literature that attempts to determine whether shocks to GDP have largely permanent or transitory effects, in other words, whether GDP is difference stationary or trend stationary (see, for example, Campbell and Mankiw, 1987, Durlauf, 1993, Murray and Nelson, 2000). This literature has not reached a consensus, but episodes such as the Great Depression and the recent 'Great Recession' point towards the existence of shocks where the economy has no strong tendency to return to the trend line that was expected prior to the shock. For the stochastic process [4.10], real GDP is described by a random walk when $\xi = 0$, while the level of GDP is stationary when $\xi = -1$, and when $-1 < \xi < 0$, a partial recovery is expected following a negative shock. The evidence then suggests a relative low value of ξ may be appropriate.

Even when the parameter ξ significantly differs from zero, how far optimal policy is from pure nominal GDP targeting depends on preference parameters. A range of plausible values for these are studied, as discussed later in section 5.6. For the coefficient of relative risk aversion, values between 0.25 and 10 are considered, with 5 as the baseline estimate. For the elasticity of intertemporal substitution, the range is 0.26 to 2, with 0.9 as the baseline. The values of β and γ are chosen to match the average real interest rate and debt-to-GDP ratio as described in section 5.6. The implied values of ϖ^* are shown in Figure 3. Apart from cases where risk aversion or intertemporal substitution are extremely low, the value of ϖ^* lies approximately between 0.75 and 1.25, even with almost complete trend reversion in real GDP. Therefore, the quantitative deviation of optimal policy from pure nominal GDP targeting due to trend reversion in real GDP appears to be small.

4.3 Implementation of optimal monetary policy

The analysis so far has assumed that the central bank can directly set the nominal price level or the nominal value of income. The optimal monetary policy results have thus been stated as targeting rules, rather than instrument rules. The following result shows how the nominal GDP target can be implemented by a rule for adjusting the nominal interest rate in response to deviations of nominal GDP from its target value. This is analogous to the Taylor rules that can be used to implement a policy of inflation targeting.[27]

Proposition 8 *Suppose the nominal interest rate is set according to the following rule:*

$$\mathsf{i}_t = \rho_t^* - (\mathbb{E}_t \mathsf{g}_{t+1} + \mathbb{E}_t \mathsf{d}_{t+1}^* - \mathsf{d}_t^*) + \psi(\mathsf{M}_t - \mathsf{M}_t^*), \qquad [4.12]$$

where $\mathsf{M}_t^* = -\mathsf{d}_t^*$ is the target for nominal GDP. If $\psi > 0$ then $\mathsf{M}_t = \mathsf{M}_t^*$ and $\tilde{\mathsf{d}}_t = 0$ is the unique equilibrium in which nominal variables remain bounded. If $\psi = 0$ then there are multiple equilibria for the debt gap $\tilde{\mathsf{d}}_t$, in all of which nominal variables remain bounded.

Proof See appendix A.15. ■

[27]The use of Taylor rules to determine inflation and the price level is studied by Woodford (2003). The determinacy properties of Taylor rules have been criticized by Cochrane (2011).

4.4 Consequences of directly targeting financial variables

Finally, given that the optimality of targeting nominal GDP derives from its effect on the ratio of debt liabilities to income, it might be argued that a more immediate way of implementing optimal policy would be to target the debt-to-GDP ratio directly. While a targeting rule of $\mathsf{d}_t = \mathsf{d}_t^*$ is feasible (there is one instrument and one target to hit), this policy has the serious drawback that it fails to provide a nominal anchor.

Proposition 9 *Suppose monetary policy is adjusted to meet the target $\mathsf{d}_t = \mathsf{d}_t^*$. The equilibria of the economy are:*

$$\tilde{\mathsf{d}}_t = 0, \quad \pi_t = \mathsf{e}_{t-1} + \rho_{t-1}^* - \mathsf{r}_t^*, \quad \textit{and} \ \ \mathsf{i}_t = \rho_t^* + \mathsf{e}_t, \tag{4.13}$$

where e_t is any arbitrary stochastic process observed at time t.

Proof See appendix A.16. ■

While this targeting rule achieves Pareto efficiency (because $\tilde{\mathsf{d}}_t = 0$), it does not uniquely determine inflation expectations because it specified solely in terms of a ratio. Nominal GDP targeting both achieves efficiency and provides a nominal anchor.

4.5 Consequences of inflation targeting

The choice of monetary policy in an economy with incomplete financial markets not only affects the distribution of risk, but also has implications for the quantity and price of credit. In particular, the model predicts that if the central bank reduces fluctuations in the price level below those consistent with an efficient distribution of risk then this increases the procyclicality of credit. The more the price level is stabilized, the more lending rises and interest rates fall during an expansion. In other words, more stable prices lead to larger fluctuations in financial variables.

Proposition 10 *Suppose monetary policy implements the targeting rule $\pi_t = 0$ (strict inflation targeting). The unique equilibrium of the economy is then:*

$$\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1} - (\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t) - (\mathsf{d}_t^* - \mathbb{E}_{t-1}\mathsf{d}_t^*), \tag{4.14}$$

with implied nominal interest rate $\mathsf{i}_t = \rho_t^* + \theta\tilde{\mathsf{d}}_t$. In the case where real GDP is described by the stochastic process [4.10], a monetary policy target of $\mathsf{P}_t + \varpi\mathsf{Y}_t = 0$ implies the equilibrium has the following features:

$$\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1} - (\varpi^* - \varpi)(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t), \quad \textit{and} \tag{4.15}$$
$$\tilde{\mathsf{l}}_t = \lambda\tilde{\mathsf{l}}_{t-1} + \beta^{-1}\varphi(\varpi^* - \varpi)(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t), \quad \textit{and} \ \ \tilde{\rho}_t = \lambda\tilde{\rho}_{t-1} - \theta(\varpi^* - \varpi)(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t),$$

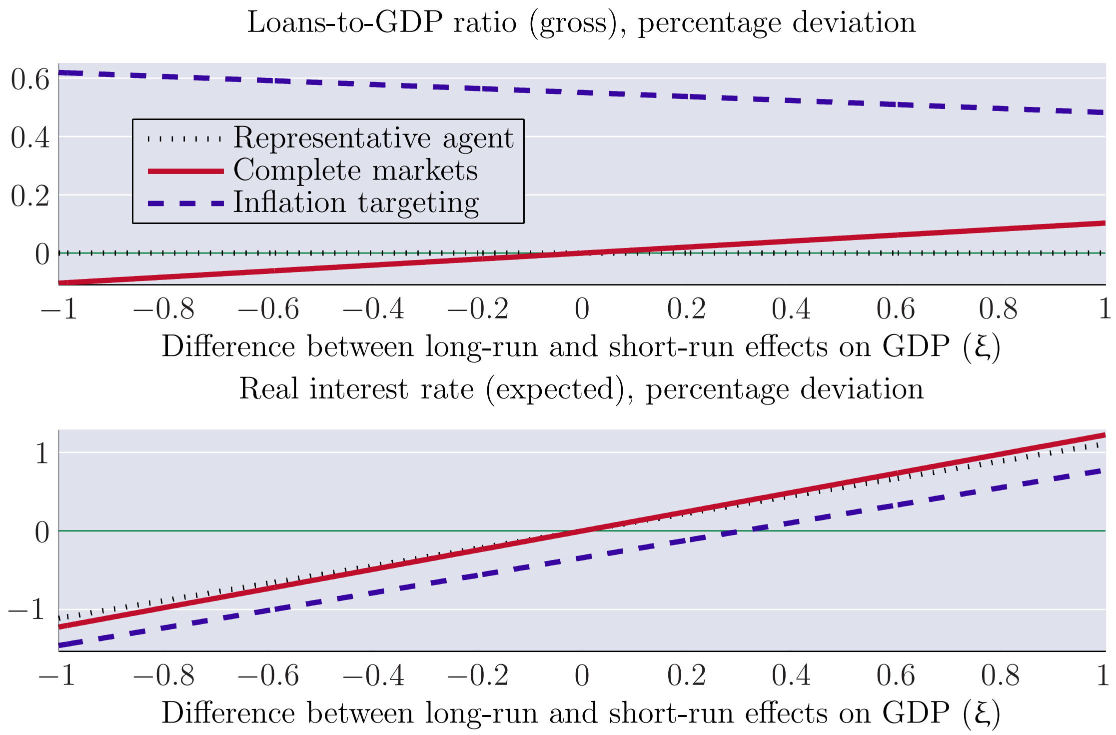
where ϖ^* is defined in [4.11].

Proof See appendix A.17. ■

The proposition reveals that too much price stability relative to that consistent with an efficient distribution of risk ($\varpi < \varpi^*$) implies the loan-to-GDP ratio rises by more than is efficient following a positive shock to real GDP, and the equilibrium real interest rate falls more than is efficient (that is, falls below the natural interest rate). Achieving greater stability in financial markets is seen to require some sacrifice of price stability in goods markets.

The size of these effects for plausible parameter values is depicted in Figure 4 (the coefficient of relative risk aversion α is set to 5, the elasticity of intertemporal substitution σ is set to 0.9, and the parameters β and γ are set to match the average real interest rate and debt-to-GDP ratio — see Table 1). With strict inflation targeting, lending increases by approximately 1.5% following a 1% rise in GDP, while for a temporary shock, the efficient outcome is for lending to rise by slightly less than GDP. The effects on the real interest rate are smaller, but strict inflation targeting leads to fall by about 0.3–0.4% more than is efficient (in the case of a permanent shock, the efficient outcome is for the real interest rate to remain unchanged).

Figure 4: *Response of financial variations to a positive shock to real GDP under different assumptions on the completeness of markets and monetary policy*



Notes: Percentage deviations from steady state on impact following an unexpected 1% increase in the level of real GDP. 'Representative agent' is the limiting case of $\gamma \to 0$. 'Complete markets' is also the outcome when the optimal monetary policy is followed under incomplete markets. 'Inflation targeting' is strict inflation targeting under incomplete markets. The long-run effect of the shock on GDP is larger than the short-run effect when $\xi > 0$, and vice versa for $\xi < 0$.

The intuition for these results can be understood by looking at the effects of a transfer between savers and borrowers. Given the pattern of earnings over the life-cycle, savers are older than borrowers. An unexpected change in inflation is thus economically equivalent to a redistribution between younger and older individuals. Overlapping generations models have been widely used to study the effects of such intergenerational transfers in the context of public debt and pensions (Samuelson, 1958, Diamond, 1965, Feldstein, 1974). A redistribution from younger to older individuals reduces desired saving and raises real interest rates (and reduces capital accumulation in a model with an investment technology). A policy of nominal GDP targeting that implies an unexpected decrease in inflation when real GDP unexpectedly rises thus generates a transfer from debtors (younger individuals) to creditors (older individuals). A policy of strict inflation targeting fails to generate this transfer following the shock to real GDP. Since the effects of the transfer are to reduce desired saving (and hence in equilibrium the amount of lending) and raise interest rates, strict inflation targeting is responsible for increasing lending too much in a boom and reducing the real interest rate too much.

These effects are also at work following a pure monetary policy shock, where an unexpected loosening of policy increases lending and reduces the real interest rate.

Proposition 11 *Suppose monetary policy is described by* $\mathsf{M}_t = \mathsf{M}_{t-1} + \epsilon_t$ *with an exogenous policy shock* $\epsilon_t \sim i.i.d.(0, \varsigma_\epsilon)$*. The equilibrium of the economy is then:*

$$\tilde{\mathsf{d}}_t = \lambda \tilde{\mathsf{d}}_{t-1} - \epsilon_t - (\mathsf{d}_t^* - \mathbb{E}_{t-1}\mathsf{d}_t^*), \quad \textit{and} \;\; \pi_t = \epsilon_t - \mathsf{g}_t, \tag{4.16}$$

with nominal interest rate $\mathsf{i}_t = \rho_t^* - \mathbb{E}_t \mathsf{g}_{t+1} + \theta \tilde{\mathsf{d}}_t$*. A positive shock* ϵ_t *reduces the real return* r_t*, the real interest rate* ρ_t*, and increases the loans-to-GDP ratio* l_t*.*

Proof See appendix A.18. ■

4.6 The maturity of debt

The analysis so far has assumed borrowers have one loan contract over their period of borrowing with a single monetary repayment at maturity. In this case, all inflation cumulated over the borrowing period that was not anticipated at the beginning of the contract reduces the real value of debt by the same percentage amount. In general, with repayments over the term of the loan, or with a sequence of loan contracts over the borrowing period, the effect of inflation is smaller (except for the case of a single jump in the price level before the first repayment, unanticipated when the initial loan contract was agreed).

The *duration* of a loan contract is defined as the average maturity of the repayments weighted by their contribution to the present discounted value of the loan. Duration is the elasticity of the value of the repayments with respect to a parallel shift in the term structure over the term of the loan. Now consider the case where any inflation that is unanticipated at the beginning of the loan period is spread evenly over the term of the loan. This inflation has a larger effect on the real value of repayments made later in the term of the loan. To introduce this into the model where borrowing takes place over one discrete time period, let μ denote the duration of debt relative to the period of borrowing ($0 < \mu \leq 1$), and let $i_{t+1}^\dagger$ denote the overall nominal interest rate between period t and

$t+1$. Assume this effective nominal rate is given by:

$$1 + i^{\dagger}_{t+1} = (1 + i_t)\left(\frac{1 + \pi_{t+1}}{1 + \mathbb{E}_t \pi_{t+1}}\right)^{1-\mu}, \qquad [4.17]$$

which implies an ex-post real return of $1 + r^{\dagger}_{t+1} = (1 + i^{\dagger}_{t+1})/(1 + \pi_{t+1})$. The standard case where the duration of debt is the same as the period of borrowing is obtained by setting $\mu = 1$.

Proposition 12 *If the effective nominal interest rate is given by [4.17] then all the results of Proposition 5 and Proposition 6 continue to hold with equation [4.7c] replaced by:*

$$\mu\pi_t + (1 - \mu)\mathbb{E}_{t-1}\pi_t = \mathsf{i}_{t-1} - \tilde{\mathsf{d}}_t - \beta^{-1}\phi\tilde{\mathsf{d}}_{t-1} - \mathsf{r}^*_t, \qquad [4.18]$$

where $\mathsf{i}_t = \mathbb{E}_t \mathsf{i}^{\dagger}_{t+1}$ is the expected nominal rate over the term of the loan. Unexpected changes in the debt-to-GDP ratio are associated with unexpected changes in weighted nominal GDP $\mathsf{P}_t + \varpi^{\dagger}\mathsf{Y}$:

$$\varpi^{\dagger}(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t) = -\left(\{\mathsf{P}_t + \varpi^{\dagger}\mathsf{Y}_t\} - \mathbb{E}_{t-1}\{\mathsf{P}_t + \varpi^{\dagger}\mathsf{Y}_t\}\right), \quad \textit{where} \;\; \varpi^{\dagger} = \mu^{-1}, \qquad [4.19]$$

which replaces equation [4.8]. Pareto efficiency is achieved using a monetary policy target of $\mathsf{P}_t + \varpi^{\dagger}\mathsf{Y}_t = -\varpi^{\dagger}\mathsf{d}^*_t$, and when the stochastic process for real GDP is [4.10], by using the target $\mathsf{P}_t + \varpi^*\varpi^{\dagger}\mathsf{Y}_t = 0$.

Proof See appendix A.19. ■

The effect of shorter maturity debt ($\mu < 1$) is to increase the amount of inflation required to achieve the efficient real state-contingency of debt obligations (assuming that inflation occurs uniformly over the term of borrowing). To implement this, the weight assigned to real GDP in the weighted nominal GDP target must be scaled by a factor of $\varpi^{\dagger} > 1$ (in addition to any scaling ϖ^* needed because of differences between the short-run and long-run effects of shocks).

5 Policy tradeoffs: Incomplete markets versus sticky prices

With fully flexible prices, the inflation fluctuations resulting from the optimal monetary policy of nominal GDP targeting are without cost, but the conventional argument for inflation targeting is that such inflation fluctuations lead to a misallocation of resources. This section adds sticky prices to the model to analyse optimal monetary policy subject to both incomplete financial markets and nominal rigidities in goods markets. To do this, it is necessary to introduce differentiated goods, imperfect competition, and a market for labour that can be hired by different firms.

5.1 Differentiated goods

Consumption in individuals' lifetime utility function [2.1] now denotes consumption of a composite good made up of a measure-one continuum of differentiated goods. Young, middle-aged, and old individuals share the same CES (Dixit-Stiglitz) consumption aggregator over these goods. The price level P_t is the minimum expenditure required per unit of the composite good:

$$P_t = \min \int_{[0,1]} P_t(\jmath) C_{i,t}(\jmath) \mathrm{d}\jmath \;\; \text{s.t.} \;\; C_{i,t} = 1, \quad \text{where} \;\; C_{i,t} \equiv \left(\int_{[0,1]} C_{i,t}(\jmath)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}\jmath \right)^{\frac{\varepsilon}{\varepsilon-1}} \;\; \text{for} \;\; i \in \{\mathrm{y}, \mathrm{m}, \mathrm{o}\}, \tag{5.1}$$

with $C_{i,t}(\jmath)$ denoting consumption of good $\jmath \in [0,1]$ per individual of generation i at time t and $P_t(\jmath)$ the nominal price of this good. The parameter ε ($\varepsilon > 1$) is the elasticity of substitution between differentiated goods. The price level and each individuals' expenditure-minimizing demand functions for the differentiated goods are given by:

$$P_t = \left(\int_{[0,1]} P_t(\jmath)^{1-\varepsilon} \mathrm{d}\jmath \right)^{\frac{1}{1-\varepsilon}}, \quad \text{and} \;\; C_{i,t}(\jmath) = \left(\frac{P_t(\jmath)}{P_t} \right)^{-\varepsilon} C_{i,t} \;\; \text{for all} \;\; \jmath \in [0,1] \;\; \text{and} \;\; i \in \{\mathrm{y}, \mathrm{m}, \mathrm{o}\}. \tag{5.2}$$

5.2 Firms

There is a measure-one continuum of firms in the economy, each of which has a monopoly on the production and sale of one of the differentiated goods. Each firm is operated by a team of owner-managers who each have an equal claim to the profits of the firm, but cannot trade their shares. Firms simply maximize the profits paid out to their owner-managers.[28]

Consider the firm that is the monopoly supplier of good $\jmath$. The firm's output $Y_t(\jmath)$ is subject to the linear production function

$$Y_t(\jmath) = A_t N_t(\jmath), \tag{5.3}$$

where $N_t(\jmath)$ is the number of hours of labour hired by the firm, and A_t is the exogenous level of TFP common to all firms. The firm is a wage taker in the perfectly competitive market for homogeneous labour, where the real wage in units of composite goods is w_t. The real profits of firm $\jmath$ are $J_t(\jmath) = P_t(\jmath)Y_t(\jmath)/P_t - w_t N_t(\jmath)$. Given the production function [5.3], the real marginal cost of production common to all firms irrespective of their levels of output is $k_t = w_t/A_t$.

Firm $\jmath$ faces a demand function derived from summing up consumption of good $\jmath$ over all generations (each of which has measure 1/3). Using each individual's demand function [5.2] for good $\jmath$ and the definition [2.2] of aggregate demand C_t for the composite good, the total demand function faced by firm $\jmath$ is $Y_t(\jmath) = (P_t(\jmath)/P_t)^{-\varepsilon} C_t$, and profits as a function of price $P_t(\jmath)$ are as follows (with the firm taking as given the general price level P_t, real aggregate demand C_t, and real marginal cost k_t):

$$J_t(\jmath) = \left\{ \left(\frac{P_t(\jmath)}{P_t} \right)^{1-\varepsilon} - k_t \left(\frac{P_t(\jmath)}{P_t} \right)^{-\varepsilon} \right\} C_t. \tag{5.4}$$

At the beginning of time period t, a group of firms is randomly selected to have access to all

[28] The participation of a specific team of managers is essential for production, and managers cannot commit to provide labour input to firms owned by outsiders. In this situation, managers will not be able to sell shares in firms, so the presence of firms does not affect the range of financial assets that can be bought and sold.

information available during period t when setting prices. For a firm $\jmath$ among this group, $P_t(\jmath)$ is chosen to maximize the expression for profits $J_t(\jmath)$ in [5.4]. Since the profit function [5.4] is the same across firms, all firms in this group will chose the same price, denoted by $\hat{P}_t$. The remaining group of firms must set a price in advance of period-t information being revealed, choosing $P_t(\jmath)$ to maximize expected profits $\mathbb{E}_{t-1}J_t(\jmath)$. All firms in this group will choose the same price $\check{P}_t$ that satisfies the first-order condition in expectation. The first-order conditions for $\hat{P}_t$ and $\check{P}_t$ are:

$$\frac{\hat{P}_t}{P_t} = \left(\frac{\varepsilon}{\varepsilon - 1}\right) k_t, \quad \text{and} \quad \mathbb{E}_{t-1}\left[\left(\frac{\check{P}_t}{P_t} - \left(\frac{\varepsilon}{\varepsilon-1}\right) k_t\right)\left(\frac{\check{P}_t}{P_t}\right)^{-\varepsilon} C_t\right] = 0, \qquad [5.5]$$

where the term $\varepsilon/(\varepsilon - 1)$ represents each firm's desired (gross) markup of price on marginal cost.[29] The proportion of firms setting a price using period $t-1$ information relative to those using period t information is denoted by the parameter κ ($0 < \kappa < \infty$), and firms are randomly assigned to these two groups.

5.3 Households

An individual born at time t has lifetime utility function [2.1], with the consumption levels $C_{\mathrm{y},t}$, $C_{\mathrm{m},t}$, and $C_{\mathrm{o},t}$ now referring to consumption of the composite good [5.1]. Labour is supplied inelastically, with the number of hours varying over the life cycle.[30] Young, middle-aged, and old individuals respectively supply Θ_{y}, Θ_{m}, and Θ_{o} hours of homogeneous labour. Individuals also derive income from their role as owner-managers of firms, and it is assumed that the amount of income from this source also varies over the life cycle in the same manner as labour income. Specifically, each young, middle-aged, and old individual belongs respectively to the managerial teams of Θ_{y}, Θ_{m}, and Θ_{o} firms. The non-financial real incomes of the generations alive at time t are:[31]

$$Y_{\mathrm{y},t} = \Theta_{\mathrm{y}} w_t + \Theta_{\mathrm{y}} J_t, \quad Y_{\mathrm{m},t} = \Theta_{\mathrm{m}} w_t + \Theta_{\mathrm{m}} J_t, \quad \text{and} \quad Y_{\mathrm{o},t} = \Theta_{\mathrm{o}} w_t + \Theta_{\mathrm{o}} J_t, \quad \text{with} \quad J_t \equiv \int_{[0,1]} J_t(\jmath)\mathrm{d}\jmath. \qquad [5.6]$$

The coefficients Θ_{y}, Θ_{m}, and Θ_{o} are parameterized in terms of γ and β as in [2.6].

The assumptions on financial markets are the same as those considered in section 2. In the benchmark case of incomplete markets with a one-period, risk-free, nominal bond, the budget identities are as given in [2.9]; in the hypothetical case of complete markets, the budget identities are as in [2.17], in both cases with consumption $C_{i,t}$ and income $Y_{i,t}$ reinterpreted according to equations [5.1] and [5.6].

[29] It is implicitly assumed that firms using the preset price will be willing to satisfy whatever level of demand is forthcoming. Technically, this requires that $\check{P}_t/P_t \geq k_t$ holds in all states of the world, which will be true for shocks within some bounds given the presence of a positive steady-state markup.

[30] The case of endogenous labour supply is taken up in appendix A.6, but it is possible to study the cost of relative price distortions in a model with an exogenous aggregate labour supply.

[31] Individuals receive fixed fractions of total profits J_t because all variation in profits between different firms is owing to the random selection of which firms receive access to full information when setting their prices.

5.4 Equilibrium

The young, middle-aged, and old have per-person labour supplies $H_{\mathrm{y},t} = \Theta_{\mathrm{y}}$, $H_{\mathrm{m},t} = \Theta_{\mathrm{m}}$, and $H_{\mathrm{o},t} = \Theta_{\mathrm{o}}$. The aggregate supply of homogeneous labour is therefore $H_t = (1/3)H_{\mathrm{y},t} + (1/3)H_{\mathrm{m},t} + (1/3)H_{\mathrm{o},t}$, which is fixed at $H_t = 1$ given [2.3]. Given aggregate demand C_t, market clearing $(1/3)C_{\mathrm{y},t}(\jmath) + (1/3)C_{\mathrm{m},t}(\jmath) + (1/3)C_{\mathrm{o},t}(\jmath) = Y_t(\jmath)$ for differentiated good $\jmath$ holds because firm $\jmath$ meets all forthcoming demand. The aggregate goods- and labour-market clearing conditions are:

$$C_t = Y_t, \quad \text{where } Y_t \equiv \int_{[0,1]} \frac{P_t(\jmath)}{P_t} Y_t(\jmath)\mathrm{d}\jmath, \quad \text{and} \quad \int_{[0,1]} N_t(\jmath)\mathrm{d}\jmath = 1, \tag{5.7}$$

with Y_t now being the real value of output summed over all firms, which must equal C_t given [5.1]. Using the definition of profits $J_t(\jmath)$ and equations [5.6] and [5.7], it follows that $J_t = Y_t - w_t$, and hence $Y_{\mathrm{y},t} = \Theta_{\mathrm{y}} Y_t$, $Y_{\mathrm{m},t} = \Theta_{\mathrm{m}}$, and $Y_{\mathrm{o},t} = \Theta_{\mathrm{o}} Y_t$, as in equation [2.3].

Given the aggregate goods-market clearing condition from [5.7], and the individual demand and production functions in [5.2] and [5.3], satisfaction of the labour-market clearing equation in [5.7] is equivalent to real GDP given by the aggregate production function:

$$Y_t = \frac{A_t}{\Psi_t}, \quad \text{with } \Psi_t \equiv \left(\int_{[0,1]} \left(\frac{P_t(\jmath)}{P_t} \right)^{-\varepsilon} \mathrm{d}\jmath \right)^{-1}, \tag{5.8}$$

where the term Ψ_t represents the effects of relative-price distortions on aggregate productivity.

Let $\hat{p}_t \equiv \hat{P}_t/P_t$ denote the relative price of goods sold by the fraction $1/(1+\kappa)$ of firms that set a price using period t information, and $\check{p}_t \equiv \check{P}_t/P_t$ the relative price for the fraction $\kappa/(1+\kappa)$ of firms using period $t-1$ information. The formula for the price index P_t in [5.2] implies $\hat{p}_t = (1 - \kappa(\check{p}_t^{1-\varepsilon} - 1))^{\frac{1}{1-\varepsilon}}$, while equation [5.5] is equivalent to $\hat{p}_t = (\varepsilon/(\varepsilon-1))k_t$. Using these equations, the first-order condition [5.5], the aggregate goods-market clearing condition [5.7], the definitions of real GDP growth g_t and inflation π_t, and $\mathbb{E}_{t-1}\check{P}_t = \check{P}_t$, it follows that:

$$\frac{1+\pi_t}{1+\mathbb{E}_{t-1}\pi_t} = \frac{\check{p}_t^{-1}}{\mathbb{E}_{t-1}\check{p}_t^{-1}}, \quad \text{and } \mathbb{E}_{t-1}\left[\left(\check{p}_t - \left(1 - \kappa\left(\check{p}_t^{1-\varepsilon} - 1\right)\right)^{\frac{1}{1-\varepsilon}} \right) \check{p}_t^{-\varepsilon}(1+g_t) \right] = 0. \tag{5.9a}$$

Using equation [5.8], real GDP growth g_t and relative-price distortions Ψ_t are given by:

$$1 + g_t = (1+a_t)\frac{\Psi_{t-1}}{\Psi_t}, \quad \text{and } \Psi_t = \left(\frac{\kappa \check{p}_t^{-\varepsilon} + \left(1 - \kappa\left(\check{p}_t^{1-\varepsilon} - 1\right)\right)^{-\frac{\varepsilon}{1-\varepsilon}}}{1+\kappa} \right)^{-1}, \tag{5.9b}$$

where $a_t \equiv (A_t - A_{t-1})/A_{t-1}$ is TFP growth. The equilibrium of the model with incomplete markets (given exogenous TFP A_t) is then the solution of equations [2.15]–[2.14e] and [5.9a]–[5.9b], augmented with a monetary policy equation.

Consider first the hypothetical case where all prices are flexible and set using full information ($\kappa = 0$), with the resulting equilibrium values being denoted with a $\hat{\cdot}$. In this case, [5.9b] implies $\hat{\Psi}_t = 1$, so equilibrium real GDP growth with flexible prices is $\hat{g}_t = (A_t - A_{t-1})/A_{t-1}$, which is simply equal to growth in exogenous TFP. This corresponds to the Pareto-efficient level of aggregate output $\hat{Y}_t = A_t$.

Returning to the analysis for a general value of κ, in a non-stochastic steady state, the unique solution of equations [5.9a] and [5.9b] is $\bar{\check{p}} = 1$ and $\bar{\Psi} = 1$. Assuming the steady-state growth rate of A_t is zero, the steady state of the model is then as described in Proposition 1. Log-linearizing equations [5.9a]–[5.9b] around the unique steady state yields:

$$\mathsf{g}_t = \mathsf{A}_t - \mathsf{A}_{t-1}, \quad \Psi_t = 0, \quad \text{and} \quad \pi_t - \mathbb{E}_{t-1}\pi_t = -\check{\mathsf{p}}_t. \qquad [5.10]$$

This means that real GDP growth is equal to the exogenous growth rate of TFP up to a first-order approximation.

5.5 Optimal monetary policy

Optimal monetary policy maximizes social welfare [3.3] using the Pareto weights derived from the equilibrium with complete financial markets and flexible prices:

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0-2}\left[\frac{1}{3}\sum_{t=t_0-2}^{\infty}\beta^{t-t_0}\hat{\Omega}_t^*\mathscr{U}_t\right], \qquad [5.11]$$

where $\hat{\Omega}_t^*$ is constructed using $\hat{Y}_t$ as real GDP and $\hat{g}_t$ as real GDP growth.[32] With both incomplete financial markets and sticky goods prices, monetary policy has competing objectives to meet with the nominal interest rate as the single policy instrument.

Proposition 13 *The welfare function $\mathscr{W}_{t_0}$ in [5.11] can be written as* $\mathscr{W}_{t_0} = -\mathbb{E}_{t_0-2}\mathscr{L}_{t_0}$ + terms independent of monetary policy + third- and higher-order terms, *where $\mathscr{L}_{t_0}$ is the quadratic loss function:*

$$\mathscr{L}_{t_0} = \frac{1}{2}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\aleph\tilde{\mathsf{d}}_t^2 + \varepsilon\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t)^2\right], \quad \textit{where} \qquad [5.12a]$$

$$\aleph = \frac{\gamma^2}{3}\left(\frac{2}{\sigma}\left(1-\phi+\phi^2\right) + \left(\alpha - \frac{1}{\sigma}\right)\left(1-\beta\lambda^2\right)\left(1 + \frac{(1-\phi-\beta\lambda)^2}{1+\beta}\right)\right). \qquad [5.12b]$$

The coefficient $\aleph$ on the squared debt-to-GDP gap $\tilde{\mathsf{d}}_t = \mathsf{d}_t - \mathsf{d}_t^*$ is strictly positive.

PROOF See appendix A.20 ■

The quadratic loss function [5.12a] shows that just two variables capture all that needs to be known about the economy's deviation from Pareto efficiency. First, the loss from imperfect risk-sharing in incomplete financial markets is proportional to the square of the gap $\tilde{\mathsf{d}}_t = \mathsf{d}_t - \mathsf{d}_t^*$ between the debt-to-GDP ratio and its value with complete markets. Second, the loss from misallocation of resources owing to sticky prices is proportional to the square of the inflation surprise $\pi_t - \mathbb{E}_{t-1}\pi_t$.

Optimal monetary policy minimizes the quadratic loss function using the nominal interest rate i_t as the instrument, and subject to first-order approximations of the constraints involving the endogenous variables, the debt-to-GDP gap $\tilde{\mathsf{d}}_t$, and inflation π_t. The debt-to-GDP gap must satisfy

[32] As discussed in section 3.2, the complete-markets weights are the only ones for which monetary policy can achieve efficient risk-sharing. The use of flexible-price output ensures the weights are independent of monetary policy, unlike in general those derived using actual GDP.

equation [4.7a], while in the general case where debt has average maturity μ, inflation must satisfy equation [4.18]. The two constraints are:

$$\lambda\tilde{\mathsf{d}}_t = \mathbb{E}_t\tilde{\mathsf{d}}_{t+1}, \quad \text{and} \quad \mu\pi_t + (1-\mu)\mathbb{E}_{t-1}\pi_t = \mathsf{i}_{t-1} - \tilde{\mathsf{d}}_t - \beta^{-1}\phi\tilde{\mathsf{d}}_{t-1} - \mathsf{r}^*_t, \tag{5.13}$$

where r^*_t an exogenous variable determined using [4.3c] with the real GDP growth rate from [5.10].

Proposition 14 *The first-order condition for minimizing the loss function* [5.12a] *subject to the constraints in* [5.13] *is*

$$\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = \frac{\varepsilon\kappa(1-\beta\lambda^2)}{\mu\aleph}(\pi_t - \mathbb{E}_{t-1}\pi_t). \tag{5.14}$$

The first-order condition is satisfied if monetary policy achieves the following target:

$$\mathsf{P}_t + \hat{\varpi}\varpi^\dagger\mathsf{Y}_t = -\hat{\varpi}\varpi^\dagger\mathsf{d}^*_t, \quad \textit{with} \;\; \hat{\varpi} = \left(1 + \frac{\varepsilon\kappa(1-\beta\lambda^2)}{\mu^2\aleph}\right)^{-1}, \tag{5.15}$$

and with $\varpi^\dagger$ *is as defined in* [4.19], *or if the stochastic process for productivity growth is given by* [4.10], *the target is* $\mathsf{P}_t + \hat{\varpi}\varpi^\dagger\varpi^*\mathsf{Y}_t = 0$, *with* ϖ^* *is as defined in* [4.11].

Proof See appendix A.21. ■

The optimal monetary policy can be expressed as target for weighted nominal GDP (the weight on real GDP is scaled by ϖ^* and $\varpi^\dagger$ even with fully flexible prices). Compared to the case of flexible prices, the weight on real GDP relative to the price level must be scaled down by $\hat{\varpi} < 1$. This pushes monetary policy in the direction of strict inflation targeting, which corresponds to the case where $\hat{\varpi} = 0$. The optimal monetary policy is essentially a comprise between the nominal GDP target that would achieve efficient risk sharing, and the strict inflation target that would avoid relative-price distortions.

The value of $\hat{\varpi}$ is larger when risk aversion α is higher or when the life-cycle income gradient γ is higher (both of which increase the term $\aleph$ in [5.15]). Intuitively, these parameters increase the importance of risk sharing. The value of $\hat{\varpi}$ is lower when the price elasticity ε is larger, or κ is higher so prices are stickiness. These parameters increase the importance of avoiding relative-price distortions. A quantitative assessment of whether optimal monetary policy is closer to nominal GDP targeting or strict inflation targeting requires calibrating these parameters.

5.6 Calibration

Let T denote the length in years of one discrete time period. In the model, the length of an individual's lifetime is $3T$, while shocks to GDP occur every T years. In choosing T there is a trade-off between a realistic representation of the length of an individual's lifetime (suggesting T between 15 and 20, excluding childhood) and allowing for the relevant shocks to occur at a realistic frequency. Given that the model is more likely relevant for permanent shocks to GDP rather than for transient business-cycle episodes, T is set to 10 years, which still allows for a realistic horizon over which individuals borrow and save (the term of borrowing and saving is for T years). Values of T between 5 and 15 years are considered in the sensitivity analysis. The parameters of the model α, σ, β, γ, μ, ε, and κ are then set to match features of U.S. data. The calibration is summarized in Table 1 and justified below.

Table 1: *Calibration of parameters*

Parameter	Value	Target
Directly calibrated		
Relative risk aversion (α)	5	Values well within range of estimates obtained in
Intertemporal substitution (σ)	0.9	the literature — see discussion in text
Price elasticity of demand (ε)	3	″
Borrowing/saving period (T)	10	
Indirectly calibrated		
Discount factor (β)	0.59	Real interest rate of 7%; real GDP growth of 1.7%*
Life-cycle income gradient (γ)	0.66	Household gross debt-to-income ratio of 130%*
Debt maturity (μ)	0.5	Average duration (T_{f}) of debt of 5 years†
Price stickiness (κ)	0.0044	Median duration (T_{p}) of a price spell of 8 months§

* *Source*: Author's calculations using series from Federal Reserve Economic Data (http://research.stlouisfed.org/fred2)

† *Source*: Doepke and Schneider (2006)

§ *Source*: Nakamura and Steinsson (2008)

The parameter β is related to the steady-state real interest rate and real GDP growth rate (see Proposition 1). Let $\mathcal{R}$ and $\mathcal{G}$ denote the annual rates of interest and GDP growth, so that $1+\bar{\rho} = e^{\mathcal{R}T}$ and $1+\bar{g} = e^{\mathcal{G}T}$. Equation [2.24] implies that $\beta = e^{-(\mathcal{R}-\mathcal{G})T}$. Given the focus on household debt, it is natural to consider interest rates on the types of loans offered to households in choosing $\mathcal{R}$.

From 1972 through to 2011, there was an average annual nominal interest rate of 8.8% on 30-year mortgages, 10% on 4-year auto loans, and 13.7% on two-year personal loans, while the average annual change in the personal consumption expenditure (PCE) price index over the same time period was 3.8%. The average credit-card interest rate between 1995 and 2011 was 14%. For comparison, 30-year Treasury bonds had an average yield of 7.7% over the periods 1977–2001 and 2006–2011. The implied real interest rates are 4.2% on Treasury bonds, 5% on mortgages, 6.2% on auto loans, 9.9% on personal loans, and 12% on credit cards.[33] Given this wide range of interest rates, the sensitivity analysis considers values of $\mathcal{R}$ from 4% up to 10%. The baseline real interest rate is set to 7% as the midpoint of this range.[34]

Over the period 1972–2011 used to calibrate the interest rate, the average annual growth rate of real GDP per capita was 1.7%. Together with the baseline real interest rate of 7%, this implies

[33] Average PCE inflation over the periods 1977–2001 and 2006–2011 was 3.5%, and 2% over the period 1995–2011. The real interest rate on government bonds is close to the conventional calibration of a 4% annual real interest rate used in many real business cycle models.

[34] This would imply a spread of 2.8% between the interest rates on loans to households and Treasury bonds. Cúrdia and Woodford (2009) consider a spread of 2% between borrowing and saving rates.

that $\beta \approx 0.59$ using $\beta = e^{-(\mathcal{R}-\mathcal{G})T}$.

In the model, the parameter γ sets the gradient of the age-profile of income (see Figure 1), but also determines the steady-state debt-to-GDP ratio (see Proposition 1). Given the focus on debt rather than on the specific reasons for household borrowing, γ is chosen to match observed levels of household debt. Let $\mathcal{D}$ denote the measured ratio of gross household debt to annual household income. This corresponds to what is defined as the loans-to-GDP ratio in the model (the empirical debt ratio being based on the amount borrowed rather than the subsequent value of loans at maturity), with an adjustment made for the fact that the level of GDP in the model is total income over T years.

According to equation [2.24], the steady-state loans-to-GDP ratio is $\bar{l} = \beta\gamma/3$, and thus $\mathcal{D} = \beta\gamma T/3$, from which it follows that $\gamma = 3\mathcal{D}/\beta T$. Note that in the model, all GDP is consumed, so for consistency between the data and the model's prediction for the debt-to-GDP ratio, either the numerator of the ratio should be total gross debt (not only household debt), or the denominator should be disposable personal income or private consumption. Since the model is designed to represent household borrowing, and because the implications of corporate and government debt may be different, the latter approach is taken.

In the U.S., like a number of other countries, the ratio of household debt to income has grown significantly in recent decades. To focus on the implications of levels of debt recently experienced, the model is calibrated to match average debt ratios during the five years from 2006 to 2010. The sensitivity analysis considers the full range of possible debt ratios from 0% to the model's theoretical maximum (approximately 196%, corresponding to $\gamma = 1$ with $\beta \approx 0.59$). Over 2006–2010, the average ratio of gross household debt to disposable personal income was approximately 124%, while the ratio of debt to consumption was approximately 135%. Taking the average of these numbers, the target chosen is a model-consistent debt-to-income ratio of 130%, which implies $\gamma \approx 0.66$.[35]

There is an extensive literature estimating the elasticity of intertemporal substitution σ. Taking the balance of evidence as pointing towards an elasticity less than one, but not substantially so, the baseline value of σ is set to 0.9.[36] The sensitivity analysis explores a range of values between 0.26 (the lower bound $\underline{\sigma}(\gamma, \beta)$ consistent with the model having a unique steady state according to Proposition 1 with $\gamma \approx 0.66$ and $\beta \approx 0.59$) and 2.[37]

[35] This calibration implies the log difference between the peak and initial income levels over the life-cycle is approximately 1.2 (see equation [2.6]). Empirical age-earnings profiles are less steep than this, see for example Murphy and Welch (1990), where the peak-initial log difference of income is approximately 0.8. In the model, that would be consistent with $\gamma \approx 0.42$ and a debt-to-GDP ratio of approximately 83%, which is considered in the sensitivity analysis. The model does not however capture all the reasons for household borrowing so it is to be expected that observed debt levels are higher than can be explained by the age-profile of income.

[36] Since $\bar{g} \approx 0.19$ with $\mathcal{G}$ equal to 1.7%, and given that $\beta = \delta(1+\bar{g})^{1-\frac{1}{\sigma}}$ in steady state, the baseline value of σ implies $\delta \approx 0.6$.

[37] There is limited consensus among the various studies in the literature. Early estimates suggested large elasticities, such as those from the instrumental variables method applied by Hansen and Singleton (1982). That work suggested an elasticity somewhere between 1 and 2 (this early literature has one parameter to capture both intertemporal substitution and risk aversion). Those high estimates have been criticized for bias due to time aggregation by Hall (1988), who finds elasticities as low as 0.1 and often insignificantly different from zero. Using cohort data, Attanasio and Weber (1993) obtain values for the elasticity of intertemporal substitution in the range 0.7–0.8, while Beaudry and van Wincoop (1996) find an elasticity close to one using a panel of data from U.S. states. A recent study

In estimating the coefficient of relative risk aversion α, one possibility would be to choose values consistent with household portfolios of risky and safe assets. But since Mehra and Prescott (1985) it has been known that matching the equity risk premium may require a risk aversion coefficient above 30, while values in excess of 10 are considered by many to be highly implausible. Subsequent analysis of the 'equity risk premium puzzle' has attempted to build models consistent with the large risk premium but with much more modest degrees of risk aversion.[38]

Alternative approaches to estimating risk aversion have made use of laboratory experiments, observed behaviour on game shows, and in a recent study, the choice of deductible for car insurance policies (Cohen and Einav, 2007).[39] The survey evidence presented by Barsky, Juster, Kimball and Shapiro (1997) potentially provides a way to measure risk aversion over stakes that are large as a fraction of lifetime income and wealth.[40] The results suggests considerable risk aversion, but most likely not in the high double-digit range for the majority of individuals. Overall, the weight of evidence from the studies suggests a coefficient of relative risk aversion above one, but not significantly more than 10. A conservative baseline value of 5 is adopted, and the sensitivity analysis considers values from as low as 0.25 up to 10.

In the model, the parameter μ represents the elasticity of the real value of debt liabilities with respect to the total amount of inflation occurring over loan period that was not initially anticipated. This follows from equation [4.17], which implies an ex-post real return of $r^{\dagger}_{t+1} = \rho_t - \mu(\pi_{t+1} - \mathbb{E}_t \pi_{t+1})$. To calibrate μ, the strategy is to use data on the *duration* of household debt liabilities. The duration T_f of a sequence of loan repayments is defined as the average maturity of those payments weighted by their contribution to the present discounted value of all repayments.

Doepke and Schneider (2006) present evidence on the duration of household nominal debt liabilities. For the most recent year in their data (2004), the duration lies between 5 and 6 years, while the duration has not been less than 4 years over the entire period covered by the study (1952–2004). This suggests a baseline duration of $T_f \approx 5$ years. The sensitivity analysis considers the effects of having durations as short as one quarter, and longer durations up to the theoretical maximum of 10 years (given $T = 10$).

by Gruber (2006) makes use of variation in capital income tax rates across individuals and obtains an elasticity of approximately 2. Following Weil (1989), it has also been argued that low values of the intertemporal elasticity lead to a 'risk-free rate puzzle', and many papers in the finance literature assume elasticities larger than one (for example, Bansal and Yaron, 2004, use 1.5). Finally, contrary to these larger estimates, the survey evidence of Barsky, Juster, Kimball and Shapiro (1997) produces an estimate of 0.18.

[38]For example, Bansal and Yaron (2004) assume a risk aversion coefficient of 10, while Barro (2006) chooses a more conservative value of 4.

[39]Converting the estimates of absolute risk aversion into coefficients of relative risk aversion (using average annual after-tax income as a proxy for the relevant level of wealth) leads to a mean of 82 and a median of 0.4. The stakes are relatively small and many individuals are not far from being risk neutral, though a minority are extremely risk averse. As discussed in Cohen and Einav (2007), the estimated level of mean risk aversion is above that found in other studies, which are generally consistent with single-digit coefficients of relative risk aversion.

[40]Respondents to the U.S. Health and Retirement Study survey are asked a series of questions about whether they would be willing to leave a job bringing in a secure income for another job with a chance of either a 50% increase in income or a 50% fall. By asking a series of questions that vary the probabilities of these outcomes, the answers can in principle be used to elicit risk preferences. One finding is that approximately 65% of individuals' answers fall in a category for which the theoretically consistent coefficient of relative risk aversion is at least 3.8. The arithmetic mean coefficient is approximately 12, while the harmonic mean is 4.

The definition of duration (in years) implies that it is equal to the percentage change in the real value of a sequence of repayments following a parallel upward shift by 1% (at an annual rate) of the nominal term structure. To relate this to the model, suppose that any inflation occurring between period t and $t+1$ is uniformly spread over that time period. Inflation $\pi_{t+1} - \mathbb{E}_t\pi_{t+1}$ that is unexpected when contracts covering the period were written would therefore shift up the nominal term structure by $(\pi_{t+1} - \mathbb{E}_t\pi_{t+1})/T$ (at an annual rate) once the shock triggering it becomes known. Given that μ is the elasticity of the real value of debt liabilities with respect to total unexpected inflation over T years, this suggests setting $\mu = T_\mathrm{f}/T$, and hence $\mu \approx 0.5$.

In the model, the extent of nominal rigidity is captured by the parameter κ. As was seen in section 5.5, the only role of this parameter in determining optimal monetary policy is as part of the coefficient of the squared unexpected inflation term in the loss function [5.12a]. The form of nominal rigidity in the model is that some fraction of prices are predetermined before shocks to GDP are realized. However, it is desirable to evaluate the welfare costs of inflation using the more conventional Calvo (1983) pricing model with staggered price adjustment taking place at a higher frequency.

Woodford (2003) demonstrates that Calvo pricing implies that the welfare costs of inflation appear in the utility-based loss function as squared inflation terms (additively separable from other terms, as in [5.12a]). Supposing that individual price adjustment occurs at a constant rate within each discrete time period, and with inflation uniformly spread over each period (to be consistent with the analysis of inflation's effects on the real value of debt liabilities), appendix A.5 shows that the formula for the welfare costs of inflation with Calvo pricing are bounded by:

$$\mathscr{L}_{\pi,t_0} \leq \frac{\varepsilon}{2}\left(\frac{T_\mathrm{p}}{T}\right)^2 \sum_{t=t_0}^{\infty} \beta^{t-t_0}\mathbb{E}_{t_0}\pi_t^2, \qquad [5.16]$$

where T_p is the expected duration of a price spell (in years). The term $\mathscr{L}_{\pi,t}$ denotes the welfare costs of inflation as a fraction of the initial T years' steady-state real GDP, which is in same units as the loss function [5.12a] given the normalization of the Pareto weights adopted in section 3.2, hence $\mathscr{L}_{t_0}$ and $\mathscr{L}_{\pi,t_0}$ are comparable.

The calibration strategy for the parameter κ is to set it so that the coefficient of the inflation term in the loss function is the same as would be found in the Calvo model for parameters consistent with the measured average duration of a price spell.[41] Comparison of [5.12a] and [5.16] suggests setting $\kappa = (T_\mathrm{p}/T)^2$ to capture the welfare costs of inflation.[42] There is now an extensive literature

[41]This strategy is much simpler than the alternative of actually building Calvo price adjustment into the model, which would entail working with a quarterly or monthly time period to capture high-frequency price adjustment. The debt contracts in this alternative model would span many discrete time periods, vastly increasing the dimensionality of the model's state space. The simplification adopted does ignore the possibility of interactions between staggered price adjustment and nominal debt contracts, though arguably there is no obvious reason to suggest such interactions might be quantitatively important.

[42]The only difference between the utility-based loss functions of the two forms of nominal rigidity is that the predetermining pricing assumption implies the term in inflation is unanticipated inflation squared, rather than all inflation squared. In the model, anticipated inflation $\mathbb{E}_{t-1}\pi_t$ is inflation that is anticipated before financial contracts over the period between $t-1$ and t are written. Such inflation has no bearing on the real value of debt liabilities arising from these contracts. As can be seen from equation [5.14], optimal monetary policy is therefore completely characterized by the behaviour of unanticipated inflation $\pi_t - \mathbb{E}_{t-1}\pi_t$, and so can be implemented by a target that

measuring the frequency of price adjustment across a representative sample of goods. Using the dataset underlying the U.S. CPI index, Nakamura and Steinsson (2008) find the median duration of a price spell is 7–9 months, excluding sales but including product substitutions. Klenow and Malin (2010) survey a wide range of studies reporting median durations in a range from 3–4 months to one year. The baseline duration is taken to be 8 months ($T_{\mathrm{p}} \approx 2/3$), implying $\kappa \approx 0.0044$. The sensitivity analysis considers average durations from 3 to 12 months.

There are two main strategies for calibrating the price elasticity of demand ε. The direct approach draws on studies estimating consumer responses to price differences within narrow consumption categories. A price elasticity of approximately three is typical of estimates at the retail level (see, for example, Nevo, 2001), while estimates of consumer substitution across broad consumption categories suggest much lower price elasticities, typically lower than one (Blundell, Pashardes and Weber, 1993). Indirect approaches estimate the price elasticity based on the implied markup $1/(\varepsilon-1)$, or as part of the estimation of a DSGE model. Rotemberg and Woodford (1997) estimate an elasticity of approximately 7.9 and point out this is consistent with the markups in the range of 10%–20%. Since it is the price elasticity of demand that directly matters for the welfare consequences of inflation rather than its implications for markups as such, the direct approach is preferred here and the baseline value of ε is set to 3. A range of values from the theoretical minimum elasticity of 1 up to 36 is considered in the sensitivity analysis, with the extremely large range chosen to allow for possible real rigidities that raise the welfare cost of inflation in exactly the same way as a higher price elasticity.[43]

The mapping between calibration targets and parameters is summarized below:

$$\beta = e^{-(\mathcal{R}-\mathcal{G})T}, \quad \gamma = \frac{3\mathcal{D}}{\beta T}, \quad \mu = \frac{T_{\mathrm{f}}}{T}, \quad \text{and} \quad \kappa = \left(\frac{T_{\mathrm{p}}}{T}\right)^2. \qquad [5.17]$$

5.7 Results

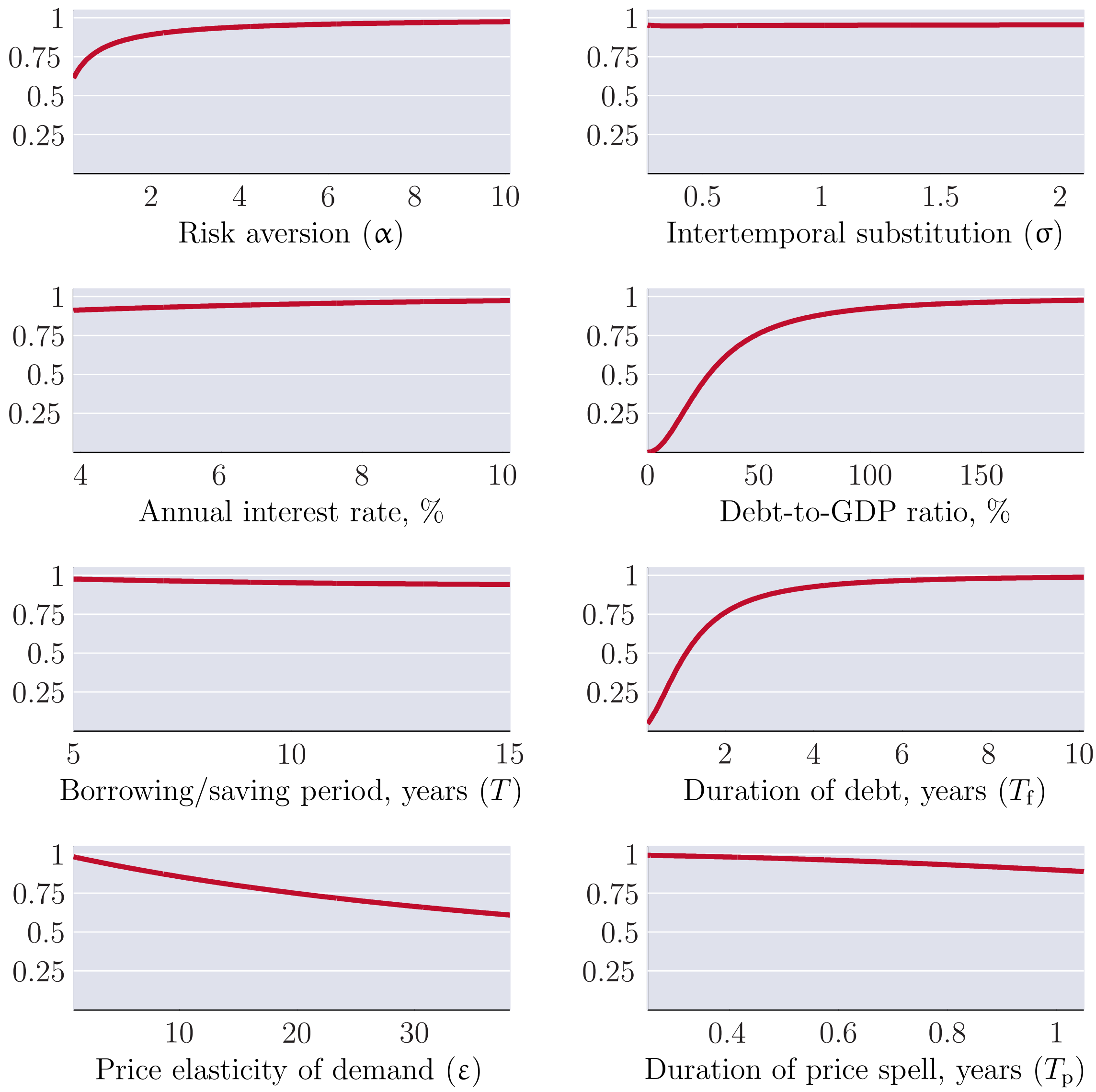
The consequences of sticky prices for optimal monetary policy can be seen from the $\hat{\varpi}$ coefficient in equation [5.15], which represents the weight on the monetary policy optimal with fully flexible prices relative to the weight on strict inflation targeting (as would be optimal were financial markets complete). The value of $\hat{\varpi}$ under the baseline calibration is 0.95, indicating that the quantitatively dominant concern is to allow inflation fluctuations to help complete financial markets, rather than avoid these to minimize relative-price distortions.

The extent to which this conclusion is sensitive to the calibration targets and the resulting parameter values can be seen in Figure 5. The panels plot the value of $\hat{\varpi}$ as each target is varied

is consistent with zero expected inflation at the beginning of the time period.

[43] The model does not include real rigidities, but these would increase the welfare cost of inflation. For example, if marginal cost is increasing in firm-level output then the ε multipying squared inflation in the loss function needs to be replaced by $\varepsilon \times (1 + \varepsilon \times$ *elasticity of marginal cost w.r.t. firm-level output*). Assuming a Cobb-Douglas production function with a conventional labour elasticity of 2/3, the elasticity of marginal cost with respect to output is 1/2. Taking the value of $\varepsilon = 7.8$ from Rotemberg and Woodford (1997), the term ε in the loss function should be set to 36 rather than 7.8 to capture this effect. The sensitivity analysis allows for this by considering a wider range of ε values to mimic the effects of real rigidities of this size. Assuming large real rigidities is controversial: Bils, Klenow and Malin (2012) present some critical evidence.

Figure 5: *Optimal monetary policy with sticky prices — weight ($\hat{\varpi}$) assigned to flexible-price optimal monetary policy target relative to strict inflation targeting*



Notes: The formula for the weight $\hat{\varpi}$ is given in equation [5.15]. Strict inflation targeting corresponds to $\hat{\varpi} = 0$, while the optimal monetary policy with flexible prices corresponds to $\hat{\varpi} = 1$. Each panel varies one parameter or calibration target holding constant all others at the baseline values given in Table 1.

over the plausible ranges identified earlier. It can be seen immediately that the calibration targets for σ, the real interest rate (and hence β), T, and T_p make little difference to the results. The results are most sensitive to the steady-state debt-to-GDP ratio, the coefficient of relative risk aversion, the duration of debt contracts, and the price elasticity of demand. However, within a very wide range of plausible values of these calibration targets, the weight on nominal GDP targeting is never reduced significantly below 0.5.

6 Conclusions

This paper has shown how a monetary policy of nominal GDP targeting facilitates efficient risk sharing in incomplete financial markets where contracts are denominated in terms of money. In an environment where risk derives from uncertainty about future real GDP, strict inflation targeting would lead to a very uneven distribution of risk, with leveraged borrowers' consumption highly exposed to any unexpected change in their incomes when monetary policy prevents any adjustment of the real value of their liabilities. This concentration of risk implies that volumes of credit, long-term real interest rates, and asset prices would be excessively volatile. Strict inflation targeting does provide savers with a risk-free real return, but fundamentally, the economy lacks any technology that delivers risk-free real returns, so the safety of savers' portfolios is simply the flip-side of borrowers' leverage and high levels of risk. Absent any changes in the physical investment technology available to the economy, aggregate risk cannot be annihilated, only redistributed.

That leaves the question of whether the distribution of risk is efficient. The combination of incomplete markets and strict inflation targeting implies a particularly inefficient distribution of risk when individuals are risk averse. If complete financial markets were available, borrowers would issue state-contingent debt where the contractual repayment is lower in a recession and higher in a boom. These securities would resemble equity shares in GDP, and they would have the effect of reducing the leverage of borrowers and hence distributing risk more evenly. In the absence of such financial markets, in particular because of the inability of households to sell such securities, a monetary policy of nominal GDP targeting can effectively complete the market even when only non-contingent nominal debt is available. Nominal GDP targeting operates by stabilizing the debt-to-GDP ratio. With financial contracts specifying liabilities fixed in terms of money, a policy that stabilizes the monetary value of real incomes ensures that borrowers are not forced to bear too much of the aggregate risk, converting nominal debt into real equity.

While the model is far too simple to apply to the recent financial crises and deep recessions experienced by a number of economies, one policy implication does resonate with the predicament of several economies faced with high levels of debt combined with stagnant or falling GDPs. Nominal GDP targeting is equivalent to a countercyclical price level, so the model suggests that higher inflation can be optimal in recessions. In other words, while each of the 'stagnation' and 'inflation' that make up the word 'stagflation' is bad in itself, if stagnation cannot immediately be remedied, some inflation might be a good idea to compensate for the inefficiency of incomplete financial markets. And even if policymakers were reluctant to abandon inflation targeting, the model does suggest that they have the strongest incentives to avoid deflation during recessions (a procyclical price level). Deflation would raise the real value of debt, which combined with falling real incomes would be the very opposite of the risk sharing stressed in this paper, and even worse than an unchanging inflation rate.

It is important to stress that the policy implications of the model in recessions are matched by equal and opposite prescriptions during an expansion. Thus, it is not just that optimal monetary policy tolerates higher inflation in a recession — it also requires lower inflation or even deflation during a period of high growth. Pursuing higher inflation in recessions without following a symmetric policy during an expansion is both inefficient and jeopardizes an environment of low inflation on average. Therefore the model also argues that more should be done by central banks to 'take away the punch bowl' during a boom even were inflation to be stable.

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A Appendices

A.1 Uniqueness of the equilibrium

As shown by Proposition 1, there are two possible reasons why the steady state may fail to be unique. The first is that the elasticity of intertemporal substitution might be too low. The second is that individuals may not be sufficiently impatient relative to the economy's average rate of growth (taking into account individuals' willingness to substitute over time). The first possibility is because the model does not feature a representative agent, while the second is due to the overlapping generations structure of the model.

It is well known that a problem of non-uniqueness of equilibria can arise in non-representative-agent general equilibrium models where substitution effects are too weak relative to income effects (irrespective of whether there are overlapping generations of finitely lived individuals, or multiple infinitely lived individuals). In the model, the only substitution effect is intertemporal substitution in response to changes in the real interest rate. All individuals have an incentive to save less when the real interest rate falls. The strength of this substitution effect is determined by the parameter σ.

Income effects of interest-rate changes affect borrowers and savers differently. Lower interest rates make borrowers better off, causing them to consume more now, which entails saving less. This income effect goes in the same direction as the substitution effect. On the other hand, savers are worse off with lower interest rates, which induces them to consume less, and hence save even more now, going against the substitution effect. If the substitution effect is too weak then savers may increase saving following a fall in the interest rate. Since borrowers borrow more when interest rates fall, this opens up the possibility that the bond market might clear at multiple interest rates. Therefore the elasticity of substitution σ must be above a threshold to ensure this does not happen. This threshold $\underline{\sigma}(\gamma, \beta)$ is increasing in γ because the size of income effects from interest-rate changes depend on the amounts borrowed and saved, which are increasing in the value of γ. The reason is that higher values of γ mean steeper age-profiles of income, and thus a greater need for borrowing and saving to smooth consumption.

The second potential non-uniqueness problem is related to the issue of dynamic inefficiency in overlapping generations models. If the real interest rate is below the economy's growth rate then 'bubble' assets might emerge that offer no return other than capital gains. If these assets can be sold at a positive price then this changes the consumption possibilities of individuals. However, if the real interest rate is above the economy's growth rate then the price of these assets would grow faster than the economy, so eventually it would be impossible for them to sold on to the next generation of individuals, causing the bubble to burst. Anticipating this, the bubble would not form to begin with. In the model, the growth rate of real GDP is given, while the equilibrium real interest rate is increasing in individuals' impatience (given by the discount factor δ) and the GDP growth rate $\bar{g}$ (the size of the second effect depends on the elasticity of intertemporal substitution σ). The equilibrium real interest rate is below the growth rate when the parameter restriction $0 < \beta < 1$ from [2.5] is satisfied.

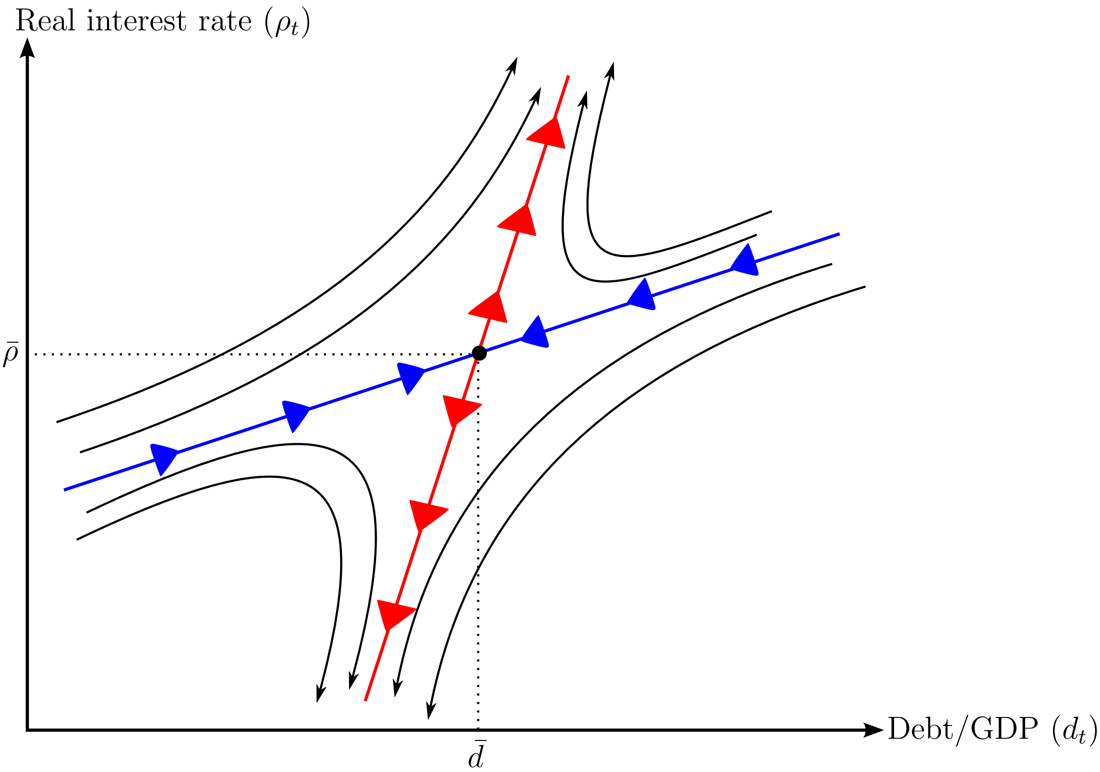
Conditional on having a unique steady state, the economy has a unique equilibrium if the difference equations of the model have the saddlepath stability property. Saddlepath stability can be determined by studying the perfect-foresight paths of the model, which Proposition 1 shows must satisfy the equations in [A.8.1]. These equations have been reduced to two variables: the debt-to-GDP ratio d_t and the real interest rate ρ_t. Proposition 1 demonstrates that when there is a unique steady state, the difference equations [A.8.1] have the saddlepath stability property in the neighbourhood of the steady state. Locally, there is an upward-sloping stable saddlepath and a steeper or downward-sloping unstable saddlepath approaching the steady state as depicted in Figure 6. To guarantee a unique equilibrium, saddlepath stability should be a global property of the system of equations. This is extremely difficult to verify analytically. Numerical analysis is used in appendix A.2 to confirm the model has this property for parameters where the steady state is unique. It is assumed throughout the parameters are such that global saddlepath stability holds.

The saddlepath stability property ensures a unique equilibrium because the debt-to-GDP ratio d_t is a state variable, while the real interest rate ρ_t is not predetermined. As discussed in section 2.3, with incomplete markets the choice of monetary policy determines the variable Υ_t from [2.22]. Conditional on Υ_t being known, d_t behaves as a state variable. Then starting from a particular value of d_{t_0} at time t_0, the real interest rate ρ_{t_0} adjusts to allow the economy to jump to the stable saddlepath.

This analysis is based on the idea that perfect foresight paths not on the stable saddlepath cannot be equilibria. The argument is not the usual one that explosive paths would violate a transversality condition. With finitely lived individuals, there are no transversality conditions among the equilibrium conditions. Technically, any perfect-foresight path that can be continued indefinitely could be an equilibrium even if it diverges from or cycles around the steady state.[44] What the numerical analysis in appendix A.2 shows is that perfect-foresight paths not on the stable saddlepath lead into a region of (d_t, ρ_t) space for which it

[44] All equilibrium conditions would be satisfied by construction of the perfect-foresight path. Assuming that the initial debt level d_{t_0} does not give creditors a claim to more than the maximum debtors could feasibly pay, all non-negativity constraints will remain satisfied on the path because it is always feasible for individuals to choose strictly positive consumption at all times and because marginal utility tends to infinity as consumption tends to zero.

Figure 6: *Perfect foresight paths and saddlepaths*



is impossible to find any economically meaningful solution to the equations in [A.8.1] to continue the path further. As the inability to continue the path can be foreseen, the corresponding starting point cannot be an equilibrium at time t_0. For this reason, all equilibria must lie on the stable saddlepath.

A.2 Computing perfect-foresight paths of the non-linear equations

It is shown in Lemma 1 that any perfect foresight path of the equations [2.14a], [2.14c], and [2.23] must satisfy the difference equations:

$$\wp_t^\sigma \left(1 - \beta\gamma + \beta\gamma\frac{\Delta_{t+1}}{\wp_t}\right)(\wp_{t+1} + \beta\wp_{t+1}^\sigma) - (1 + (1+\beta)\gamma - \gamma\Delta_{t+1})\wp_{t+1} - \beta(1-\gamma) = 0; \qquad \text{[A.2.1a]}$$

$$\gamma(1 + \beta\wp_t^{\sigma-1})\Delta_{t+1} - (1 + (1+\beta)\gamma - \gamma\Delta_t)\wp_t^\sigma + (1-\gamma) = 0. \qquad \text{[A.2.1b]}$$

where $\wp_t$ and Δ_t are the variables defined in [A.7.1] in terms of the real interest rate ρ_t and the debt-to-GDP ratio d_t. The implied values of ρ_t and d_t can be recovered using:

$$\rho_t = \frac{(1+\bar{g})^{\frac{1}{\sigma}}\wp_t}{\delta} - 1, \quad \text{and} \quad d_t = \frac{\gamma\Delta_t}{3}. \qquad \text{[A.2.1c]}$$

The objective of numerical analysis of the difference equations [A.2.1a]–[A.2.1b] is to confirm that the properties established analytically in the neighbourhood of the unique steady state by Proposition 1 also hold globally. In particular, in $(\wp_t, \Delta_t)$ space, all perfect foresight paths must lie on a one-dimensional manifold (the stable saddlepath). This means that starting from a specific value of Δ_{t_0}, there is one and only one starting point $\wp_{t_0}$ such that the subsequent perfect-foresight path is well defined.

This requires a procedure for calculating the vector $\Xi_{t+1} = \begin{pmatrix}\wp_{t+1} & \Delta_{t+1}\end{pmatrix}'$ given the vector Ξ_t. First, if $\wp_t$ and Δ_t are known, Δ_{t+1} can be obtained directly by solving equation [A.2.1b]:

$$\Delta_{t+1} = \frac{(1 + (1+\beta)\gamma - \gamma\Delta_t)\,\wp_t^\sigma - (1-\gamma)}{\gamma\left(1 + \beta\wp_t^{\sigma-1}\right)}. \qquad \text{[A.2.2a]}$$

To find $\wp_{t+1}$, equation [A.2.1a] can be rearranged as follows (making use of [A.2.1b]):

$$\beta\wp_t^{\sigma}\left(1-\beta\gamma+\beta\gamma\frac{\Delta_{t+1}}{\wp_t}\right)\wp_{t+1}^{\sigma}+\left((2+\gamma-\gamma\Delta_t)\wp_t^{\sigma}-(2+\beta\gamma)\right)\wp_{t+1}-\beta(1-\gamma)=0. \quad \text{[A.2.2b]}$$

With $\wp_t$, Δ_t, and Δ_{t+1} known, equation [A.2.2b] can be solved numerically for a value of $\wp_{t+1}$. A perfect-foresight path can be constructed by applying this procedure recursively, starting from $\Xi_{t_0}=\begin{pmatrix}\wp_{t_0} & \Delta_{t_0}\end{pmatrix}'$. A path is only valid if it can be continued indefinitely. If there is no economically meaningful solution ($0<\wp_{t+1}<\infty$) to equation [A.2.2b] then the starting value of $\wp_{t_0}$ must be rejected.

The saddlepath stability property can be confirmed numerically using a version of the 'shooting' algorithm. For a fixed Δ_{t_0}, different starting points for $\wp_{t_0}$ are considered, with an attempt to construct a perfect-foresight path for each. It is then recorded how many iterations occur before each path reaches a point where there is no economically meaningful value for the next iteration. A valid path can be iterated infinitely many times. Plotting the number of possible iterations against the starting point $\wp_{t_0}$ (conditional on a given value of Δ_{t_0}) reveals an asymptote at one value of $\wp_{t_0}$, indicating the presence of a manifold on which all perfect-foresight paths must lie.

That this manifold represents a stable saddlepath can be confirmed using the 'reverse shooting' algorithm. To apply this algorithm, it is necessary to have a procedure to calculate Ξ_t given Ξ_{t+1}, that is, to trace out perfect-foresight paths in reverse. Note that equation [A.2.1a] can be written as:

$$(1-\beta\gamma)(\wp_{t+1}+\beta\wp_{t+1}^{\sigma})\wp_t^{\sigma}+\beta\gamma\Delta_{t+1}(\wp_{t+1}+\beta\wp_{t+1}^{\sigma})\wp_t^{\sigma-1}-(1+(1+\beta)\gamma-\gamma\Delta_{t+1})\wp_{t+1}-\beta(1-\gamma)=0. \quad \text{[A.2.3a]}$$

Given $\wp_{t+1}$ and Δ_{t+1}, this equation can be solved numerically for a value of $\wp_t$. Equation [A.2.1b] can then be arranged to obtain an expression for Δ_t in terms of Δ_{t+1} and this value of $\wp_t$:

$$\Delta_t=\frac{(1+(1+\beta)\gamma)\wp_t^{\sigma}-\gamma(1+\beta\wp_t^{\sigma-1})\Delta_{t+1}-(1-\gamma)}{\gamma\wp_t}. \quad \text{[A.2.3b]}$$

Using the local characterization of the stable saddlepath provided by Proposition 1, a starting point Ξ_{t_0} can be chosen that lies on the stable saddlepath in the neighbourhood of the steady state. The reverse shooting algorithm can then be used to trace out the path along which this point would be approached.

A.3 Pareto efficiency

Proposition 3 shows that the set of Pareto-efficient consumption allocations is characterized by those satisfying equations [2.16] and [2.21] (and an initial condition at time t_0), with value functions as given in [2.14e]. This claim is made subject to the parameter restriction $0<\beta<1$ from [2.5] and allowing the Pareto weight Ω_t (in the social planner's problem [3.3]) for the generation born at time t to depend on the state of the world at time t. These assumptions are needed to rule out two issues specific to overlapping generations models which can mean that even the (sequential) complete-markets equilibrium is not Pareto efficient. But as described below, dropping these assumptions leads to potential inefficiencies that cannot be corrected by monetary policy (interpreted as setting the nominal interest rate), and are hence beyond the scope of this paper.

The first issue relates to the possibility of dynamic inefficiency, which occurs when the equilibrium real interest rate is permanently below the economy's growth rate of real GDP. As shown in Proposition 1, ruling out dynamic inefficiency requires the parameter restriction $0<\beta<1$ from [2.5]. With $\beta>1$, the real interest rate would be below the growth rate. In that case, with the (scaled) Pareto weights ω_t^* that support the complete-markets equilibrium (which are bounded given that the shocks to the growth rate are bounded), the social welfare function $\mathscr{W}_{t_0}$ in [3.6] would have an infinite value. When the objective function is unbounded over the feasible set, the constrained maximum is not well defined, so the first-order conditions do not correctly characterize a Pareto-efficient allocation.

With dynamic inefficiency, a Pareto improvement can be obtained by a permanent sequence of transfers to old individuals from younger individuals. Sufficiently large transfers would correct the over-saving problem underlying dynamic inefficiency, and in equilibrium, the real interest rate would rise above the growth rate. This calls for a policy intervention directly specifying such transfers, or a monetary policy that creates a 'bubble' which that can implement the same sequence of transfers in equilibrium. However, a conventional monetary policy of setting the nominal interest will fail to correct dynamic inefficiency. The reason is that Proposition 1 shows the steady-state real interest rate is invariant to all policies of this type, and unless monetary policy can permanently raise the real interest rate, it cannot succeed in eliminating dynamic inefficiency.

The second issue is that of risk sharing prior to birth. Even though there are no idiosyncratic shocks in the model, individuals' lifetime consumption possibilities are affected by the realizations of aggregate shocks that become known before and up to their births. It is then possible to argue that individuals' welfare should be assessed by taking an expectation over the outcomes for lifetime utility that will be obtained for different shocks realized prior to birth, even though the individual does not actually experience the uncertainty of not knowing his lifetime consumption path prior to birth (unlike when he is alive but is subject to uncertainty about future consumption). If this argument is accepted then it entails treating individuals as autonomous prior to their births, in other words, an individual born during an economic boom is considered to be one and the same as the individual he would be had he been born during a deep recession, even though he would never face uncertainty of this type while he is actually alive.

The Pareto efficiency criterion requires that autonomous individuals receive a fixed Pareto weight when solving the social planner's problem [3.3], that is, the Pareto weight Ω_t for individuals born at time t can be a function only of the state of the world when the planner's problem is first solved (at some time t_0). This is more demanding than the requirement adopted in section 3.2 that the weight Ω_t may be a function of the state of the world at time t (but not a function of the realization of shocks during individuals' lifetimes). This weaker requirement effectively denies the autonomy of individuals prior to birth because it is as if the social planner treats an individual born in good times as a different person from the individual he would have been had he been born in bad times (though individuals are autonomous once born, represented by a Pareto weight that depends only on the state of the world at their time of birth — which ensures that efficiency is not reduced to a vacuous notion of any non-wasteful consumption allocation). The combination of Pareto weights fixed at time t_0 and evaluating pre-birth expected utility given a concave utility function implies that additional risk-sharing conditions beyond [2.21] are necessary for Pareto efficiency. There is a literature that analyses these risk-sharing conditions and their policy implications (see for example, Gordon and Varian, 1988).

The sequential complete-markets equilibrium need not belong to the set of pre-birth Pareto-efficient allocations in general. To restore the efficiency of complete financial markets with the more demanding notion of pre-birth efficiency it would be necessary for these markets to be open at the beginning of time (period t_0) to all individuals, even those not yet born. Trading in these 'beginning-of-time' complete financial markets would lead to an equilibrium that is one of many pre-birth Pareto-efficient allocations.

More importantly, there is the question of whether this 'beginning-of-time' complete-markets equilibrium (or some other pre-birth Pareto-efficient allocation) could be implemented via a suitable monetary policy in the incomplete-markets economy. Since the resource constraint [2.16] and the risk-sharing condition [2.21] remain necessary conditions for any such allocation, the set of pre-birth Pareto-efficient consumption allocations is contained within the set of regular Pareto-efficient allocations. But Proposition 3 demonstrates that the only consumption allocation in this set that could be implemented through monetary policy in an incomplete markets economy is the sequential complete-markets equilibrium, which need not in general meet the more stringent requirement of pre-birth Pareto efficiency.

Whether the sequential complete-markets equilibrium satisfies the requirements for pre-birth Pareto efficiency can be tested by considering the weights Ω_t^* that support this consumption allocation. If these weights depend only on the state of the world at time t_0 (and not up to t, as allowed in section 3.2), the equilibrium will be pre-birth efficient. Since the consumption of the young at time t in the sequential complete-markets equilibrium will generally depend on the state of the world at time t, equations [3.6] and [3.7] indicate that Ω_t^* will generally depend on the time t state of the world as well, so this equilibrium will not generally be pre-birth efficient.

There is at least one special case in which the sequential complete markets is pre-birth efficient. With a logarithmic utility function ($\alpha = 1$ and $\sigma = 1$), Proposition 2 shows that the equilibrium features $c_{y,t}^* = 1$. Using equations [3.6] and [3.7], it follows that $\Omega_t^* = 1$, which is independent of the realization of any shocks. With Pareto weights equal to constants, the consumption allocation is pre-birth efficient.

A.4 Coefficients in log-linearized equations

Formulas for the coefficients χ, $\varkappa$, ϕ, θ, λ, and ζ of the equations introduced in Proposition 5 are given below in terms of the parameters β, γ, and σ:

$$\chi = \frac{2}{(1+2\beta) + \left(1 - \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}; \quad \text{[A.4.1a]}$$

$$\varkappa = \frac{3}{(1+2\beta) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}; \quad \text{[A.4.1b]}$$

$$\phi = \frac{2\beta\left(1 + \frac{\gamma}{\sigma}\right)}{(1+2\beta) + \left(1 + \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}; \quad \text{[A.4.1c]}$$

$$\vartheta = \frac{2\left(1+\beta+\beta^2\right)}{2\left(1+\beta+\beta^2\right) + (1-\beta)\frac{\beta\gamma}{\sigma} + (1+\beta)\sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}, \quad \text{and } \theta \equiv \frac{\gamma}{\sigma}\vartheta; \quad \text{[A.4.1d]}$$

$$\lambda = \frac{2\left(\frac{\beta\gamma}{\sigma} - 1\right)}{(1+2\beta) + \left(1 - \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}; \quad \text{[A.4.1e]}$$

$$\zeta = \frac{2\beta\left(\frac{\gamma}{\sigma} - 1\right)}{(1+2\beta) + \left(1 - \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}. \quad \text{[A.4.1f]}$$

A.5 Cost of inflation with Calvo price setting

Suppose price adjustment can occur continuously between time $t-1$ and t, with $\mathsf{P}(\tau)$ denoting the log price level at time τ (where $\mathsf{P}(t-1) = \mathsf{P}_{t-1}$ and $\mathsf{P}(t) = \mathsf{P}_t$ in terms of the existing discrete-time notation). Inflation $\pi_t = \mathsf{P}_t - \mathsf{P}_{t-1}$ between time $t-1$ and t is the integral of the continuous-time inflation rate $\dot{P}(\tau) = \mathrm{d}\mathsf{P}(\tau)/\mathrm{d}\tau$ over this period:

$$\pi_t = \int_{\tau\in[t-1,t]} \dot{\mathsf{P}}(\tau)\mathrm{d}\tau. \quad \text{[A.5.1]}$$

The time interval $[t-1,t]$ is assumed to represent T years. The steady-state real interest rate is denoted by $\mathcal{R}$, and the steady-state real GDP growth rate by $\mathcal{G}$ (both are given as continuous annual rates). In terms of the existing discrete-time notation, $1+\bar{\rho} = e^{\mathcal{R}T}$ and $1+\bar{g} = e^{\mathcal{G}T}$. Let $\varrho = \mathcal{R} - \mathcal{G}$ be the difference between the real interest rate and the economy's growth rate. Proposition 1 demonstrates that $\beta = (1+\bar{g})/(1+\bar{\rho})$, and hence $\beta = e^{-\varrho T}$. The expected duration of an individual price spell is T_{p} years. It is assumed that firms receive opportunities to change prices at a constant rate as in the Calvo (1983) pricing model, the annual arrival rate being $1/T_{\mathrm{p}}$.

For the purposes of deriving the welfare cost of inflation, suppose that each interval of T years is divided into n discrete subperiods, each being an equal fraction $h = 1/n$ of the T years, that is, each is hT years long. The assumption of Calvo pricing means there is a probability $e^{-hT/T_{\mathrm{p}}}$ that an individual price will remain unchanged during a subperiod. The discount factor over a subperiod is $e^{-hT\mathrm{R}}$, while growth scales up real payoffs by $e^{hT\mathrm{G}}$ over the same period of time. The effective discount factor over a subperiod of hT years is therefore $e^{-\varrho hT}$.

The welfare costs of inflation can be derived by taking the continuous-time limit as $h \to 0$ ($n \to \infty$) of the formula for the discrete-time case reported in Woodford (2003).[45] Let $\mathscr{L}_{\pi,t}$ denote the expected welfare cost of all inflation over the time interval $[t-1,\infty)$, reported as a fraction of the first T years' steady-state GDP. The Calvo probability of no price adjustment in the formula is set to $e^{-hT/T_{\mathrm{p}}}$ and the discount factor is set to $e^{-\varrho hT}$. The formula gives the welfare cost in terms of one discrete period's steady-state GDP (hT years), so the expression must be multiplied by h to obtain the cost in terms of T years' GDP. The welfare cost is therefore:[46]

$$\mathscr{L}_{\pi,t} = \frac{\varepsilon}{2}\lim_{h\to 0} h\frac{e^{-hT/T_{\mathrm{p}}}}{\left(1-e^{-hT/T_{\mathrm{p}}}\right)\left(1-e^{-\varrho hT}e^{-hT/T_{\mathrm{p}}}\right)}\sum_{\ell=0}^{\infty}\left(e^{-\varrho hT}\right)^{\ell}\mathbb{E}_t\left[\left(\mathsf{P}(t-1+h(\ell+1))-\mathsf{P}(t-1+h\ell)\right)^2\right].$$

Noting that $e^{-\varrho hnT}=\beta$ since $hn=1$, this equation can be written recursively as $\mathscr{L}_{\pi,t}=\mathfrak{L}_{\pi,t}+\beta\mathbb{E}_t\mathscr{L}_{\pi,t+1}$, where $\mathfrak{L}_{\pi,t}$ is defined by:

$$\mathfrak{L}_{\pi,t} \equiv \frac{\varepsilon}{2}\lim_{h\to 0}\frac{e^{-hT/T_{\mathrm{p}}}}{(1-e^{-hT/T_{\mathrm{p}}})(1-e^{-(\varrho+T_{\mathrm{p}}^{-1})hT})}\sum_{\ell=0}^{n-1}e^{-\varrho hT\ell}\left(\mathsf{P}(t-1+h(\ell+1))-\mathsf{P}(t-1+h\ell)\right)^2 h.$$

This expression for $\mathfrak{L}_{\pi,t}$ can be rearranged as follows:

$$\mathfrak{L}_{\pi,t} = \frac{\varepsilon}{2}\frac{T_{\mathrm{p}}^2}{(1+\varrho T_{\mathrm{p}})T^2}\lim_{h\to 0}\frac{e^{-hT/T_{\mathrm{p}}}}{\frac{1-e^{-hT/T_{\mathrm{p}}}}{hT/T_{\mathrm{p}}}\cdot\frac{1-e^{-h(\varrho+T_{\mathrm{p}}^{-1})T}}{h(\varrho+T_{\mathrm{p}}^{-1})T}}\sum_{\ell=0}^{n-1}e^{-\varrho hT\ell}\left(\frac{\mathsf{P}(t-1+h(\ell+1))-\mathsf{P}(t-1+h\ell)}{h}\right)^2 h. \quad \text{[A.5.2]}$$

Noting the following limits as $h\to 0$:

$$\lim_{h\to 0}e^{-hT/T_{\mathrm{p}}}=1,\quad \lim_{h\to 0}\frac{1-e^{-hT/T_{\mathrm{p}}}}{hT/T_{\mathrm{p}}}=1,\quad \lim_{h\to 0}\frac{1-e^{-h(\varrho+T_{\mathrm{p}}^{-1})T}}{h(\varrho+T_{\mathrm{p}}^{-1})T}=1,\quad \text{and}\quad \lim_{t\to 0}\frac{\mathsf{P}(t+h)-\mathsf{P}(t)}{h}=\dot{\mathsf{P}}(t),$$

the limit in [A.5.2] is given by:

$$\mathfrak{L}_{\pi,t} = \frac{\varepsilon}{2}\frac{T_{\mathrm{p}}^2}{(1+\varrho T_{\mathrm{p}})T^2}\int_{\tau\in[t-1,t]}e^{-\varrho T(\tau-(t-1))}\dot{\mathsf{P}}(\tau)^2\mathrm{d}\tau. \quad \text{[A.5.3]}$$

Suppose that total inflation π_t between $t-1$ and t is evenly spread over that period of time.[47] It follows from [A.5.1] that $\dot{\mathsf{P}}(\tau)=\pi_t$ for all $\tau\in[t-1,t]$. Substituting this into [A.5.3] and evaluating the integral leads to:

$$\mathfrak{L}_{\pi,t} = \frac{\varepsilon}{2}\left(\frac{T_{\mathrm{p}}}{T}\right)^2\left(\frac{1}{1+\varrho T_{\mathrm{p}}}\right)\left(\frac{1-e^{-\varrho T}}{\varrho T}\right)\pi_t^2. \quad \text{[A.5.4]}$$

Since $\varrho>0$, it must be the case that $1/(1+\varrho T_{\mathrm{p}})<1$. If $\varrho T\geq 1$ then $(1-e^{-\varrho T})/\varrho T<1$. When $\varrho T<1$, since $(1-e^{-\varrho T})/\varrho T<1$ is equivalent to $(1-\varrho T)^{-1}>e^{\varrho T}$, which holds given the series expansions of these functions, it is shown that $(1-e^{-\varrho T})/\varrho T<1$ in all cases. Given that $\mathscr{L}_{\pi,t_0}=\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\mathfrak{L}_{\pi,t}$, it follows from [A.5.4] that:

$$\mathscr{L}_{\pi,t} \leq \frac{\varepsilon}{2}\left(\frac{T_{\mathrm{p}}}{T}\right)^2\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\pi_t^2. \quad \text{[A.5.5]}$$

This provides an upper bound for the welfare costs of inflation in the Calvo model (expressed as a fraction of the initial T years' steady-state GDP).

[45] It can be verified numerically that assuming a finite h (for example, $h=1/4T$ for subperiods equal to one quarter) leads to smaller welfare costs of inflation than the continuous-time limit. The limiting case thus provides an upper bound for the cost of inflation.

[46] The model here includes no real rigidity, but as discussed in section 5.6, the presence of real rigidity is isomorphic to an increase in the price elasticity of demand ε.

[47] Given π_t and equation [A.5.1], evenly spread inflation would minimize the loss [A.5.3] in the case where $\varrho=0$. When ϱ is relatively small, the loss-minimizing inflation time-path would not deviate too much from that constant path. If inflation followed the optimal path, the loss would be smaller, so the assumption of evenly spread inflation provides an upper bound for the welfare cost.

A.6 Endogenous labour supply

This section adds an endogenous labour supply decision, which will imply that both the incompleteness of financial markets and monetary policy have first-order consequences for aggregate output.

A.6.1 Households

The population and age structure of households is the same as that described in section 2, but the lifetime utility function of individuals born at time t is now

$$\mathscr{U}_t = \left\{\log C_{y,t} - \frac{H_{y,t}^{\eta}}{\eta\Theta_y^{\eta-1}}\right\} + \delta\mathbb{E}_t\left\{\log C_{m,t+1} - \frac{H_{m,t+1}^{\eta}}{\eta\Theta_m^{\eta-1}}\right\} + \delta^2\mathbb{E}_t\left\{\log C_{o,t+2} - \frac{H_{o,t+2}^{\eta}}{\eta\Theta_o^{\eta-1}}\right\}, \qquad \text{[A.6.1]}$$

where $H_{y,t}$, $H_{m,t}$, and $H_{o,t}$ are respectively the per-person hours of labour supplied by young, middle-aged, and old individuals at time t. The utility function is additively separable between consumption and hours, and utility is logarithmic in consumption (the parameter restrictions $\alpha = 1$ and $\sigma = 1$ are imposed), with the composite consumption good being [5.1], as in section 5. The parameter η ($1 < \eta < \infty$) is related to the Frisch elasticity of labour supply, the Frisch elasticity being $(\eta - 1)^{-1}$. The parameters Θ_y, Θ_m, and Θ_o, which in section 2 specified the shares of the exogenous income endowment received by each generation, are now interpreted as age-specific differences in the disutility of working. A higher value of Θ_j indicates that generation $j \in \{y, m, o\}$ has a relatively low disutility of labour.

Hours of labour supplied by individuals of different ages are not perfect substitutes, so wages are age specific. Let $w_{y,t}$, $w_{m,t}$, and $w_{o,t}$ denote the hourly (real) wages of the young, middle-aged, and old, respectively. As in section 5, individuals earn remuneration as owner-managers of firms. Managerial labour is assumed to have no disutility and is supplied inelastically, with Θ_y, Θ_m, and Θ_o denoting the per-person proportions of total profits J_t received by individuals of each generation. Individuals are also subject to age-specific lump-sum taxes $T_{y,t}$, $T_{m,t}$, and $T_{o,t}$. The per-person real non-financial incomes of individuals from different generations are:

$$Y_{y,t} = w_{y,t}H_{y,t} + \Theta_y J_t - T_{y,t}, \quad Y_{m,t} = w_{m,t}H_{m,t} + \Theta_m J_t - T_{m,t}, \quad \text{and } Y_{o,t} = w_{y,t}H_{y,t} + \Theta_y J_t - T_{y,t}, \quad \text{[A.6.2]}$$

where total profits J_t are as defined in [5.6].

Given additive separability of the utility function between consumption and hours, the consumption Euler equations for each generation are the same (after setting $\alpha = 1$ and $\sigma = 1$) as those in [2.10] in the case of incomplete markets, and [2.18] in the hypothetical case of complete markets. Irrespective of the assumptions on financial markets, the optimality conditions for maximizing utility [A.6.1] with respect to labour supply $H_{j,t}$ subject to [A.6.2] and the appropriate budget constraint are:

$$C_{y,t}\left(\frac{H_{y,t}}{\Theta_y}\right)^{\eta-1} = w_{y,t}, \quad C_{m,t}\left(\frac{H_{m,t}}{\Theta_m}\right)^{\eta-1} = w_{m,t}, \quad \text{and } C_{o,t}\left(\frac{H_{o,t}}{\Theta_o}\right)^{\eta-1} = w_{o,t}. \qquad \text{[A.6.3]}$$

A.6.2 Firms

There is a range of differentiated goods and monopolistically competitive firms as in the model of section 5. The production function is [5.3], but now the labour $N_t(\jmath)$ used by firm $\jmath$ is reinterpreted as a composite labour input drawing on hours of labour from young, middle-aged, and old workers. The labour aggregator is assumed to have the Cobb-Douglas form:

$$N_t(\jmath) \equiv \frac{N_{y,t}(\jmath)^{\frac{\Theta_y}{3}} N_{m,t}(\jmath)^{\frac{\Theta_m}{3}} N_{o,t}(\jmath)^{\frac{\Theta_o}{3}}}{3\Theta_y^{\frac{\Theta_y}{3}}\Theta_m^{\frac{\Theta_m}{3}}\Theta_o^{\frac{\Theta_o}{3}}}, \qquad \text{[A.6.4]}$$

where $N_{j,t}(\jmath)$ is the firm's employment of hours of labour by individuals of age $j \in \{y, m, o\}$, and Θ_y, Θ_m, and Θ_o are the same parameters that appear in [A.6.1]. The Cobb-Douglas form of [A.6.4] implies an elasticity of substitution between different labour inputs of one.

The firm takes the real wages $w_{y,t}$, $w_{m,t}$, and $w_{o,t}$ in each age-specific labour market as given. Firms are assumed to receive a proportional wage-bill subsidy of s from the government. Irrespective of how firms set prices, labour inputs $N_{i,t}(\jmath)$ are chosen to minimize the total cost of obtaining the number of units $N_t(\jmath)$

of composite labour that enter the production function [5.3] subject to the aggregator [A.6.4]. Conditional on $N_t(\jmath)$, the cost-minimizing labour demand functions are:

$$N_{\mathrm{y},t}(\jmath) = \frac{\Theta_{\mathrm{y}} w_t}{3(1-s)w_{\mathrm{y},t}} N_t(\jmath), \quad N_{\mathrm{m},t}(\jmath) = \frac{\Theta_{\mathrm{m}} w_t}{3(1-s)w_{\mathrm{m},t}} N_t(\jmath), \quad \text{and} \;\; N_{\mathrm{o},t}(\jmath) = \frac{\Theta_{\mathrm{o}} w_t}{3(1-s)w_{\mathrm{o},t}} N_t(\jmath), \quad \text{[A.6.5]}$$

where w_t is the minimized value of (post-subsidy) wage costs per unit of composite labour:

$$w_t = (1-s) w_{\mathrm{y},t}^{\frac{\Theta_{\mathrm{y}}}{3}} w_{\mathrm{m},t}^{\frac{\Theta_{\mathrm{m}}}{3}} w_{\mathrm{o},t}^{\frac{\Theta_{\mathrm{o}}}{3}}, \quad \text{with} \;\; (1-s)\left(w_{\mathrm{y},t}N_{\mathrm{y},t}(\jmath) + w_{\mathrm{m},t}N_{\mathrm{m},t}(\jmath) + w_{\mathrm{o},t}N_{\mathrm{o},t}(\jmath)\right) = w_t N_t(\jmath). \quad \text{[A.6.6]}$$

Given the cost-minimizing choice of labour inputs, firms can be analysed as if they were directly hiring units of composite labour at real wage w_t for use in the production function [5.3].

A.6.3 Fiscal policy

The only role of fiscal policy in the model is to raise lump-sum taxes to fund the wage-bill subsidy paid to firms. The government does not spend, borrow, or save. The total cost of the wage-bill subsidy is:

$$T_t = s \int_{[0,1]} \left(w_{\mathrm{y},t}N_{\mathrm{y},t}(\jmath) + w_{\mathrm{m},t}N_{\mathrm{m},t}(\jmath) + w_{\mathrm{o},t}N_{\mathrm{o},t}(\jmath)\right) \mathrm{d}\jmath. \quad \text{[A.6.7]}$$

The subsidy is set so that $s = 1/\varepsilon$, where ε is the elasticity of substitution between different goods in [5.1] (this implies a well-defined subsidy because $\varepsilon > 1$). The government distributes the tax burden so that the per-person age-specific tax levels are:

$$T_{\mathrm{y},t} = \Theta_{\mathrm{y}} T_t, \quad T_{\mathrm{m},t} = \Theta_{\mathrm{m}} T_t, \quad \text{and} \;\; T_{\mathrm{o},t} = \Theta_{\mathrm{o}} T_t, \quad \text{[A.6.8]}$$

where Θ_{y}, Θ_{m}, and Θ_{o} are the same parameters as in [A.6.1] and [A.6.4].

A.6.4 Equilibrium

There is a separate market-clearing condition for each age-specific labour market:

$$\int_{[0,1]} N_{j,t}(\jmath)\mathrm{d}\jmath = \frac{1}{3} H_{j,t} \;\; \text{for each} \;\; j \in \{\mathrm{y}, \mathrm{m}, \mathrm{o}\}. \quad \text{[A.6.9]}$$

Following the same steps as in [5.8] leads to the aggregate production function

$$Y_t = \frac{A_t N_t}{\Psi_t}, \quad \text{with} \;\; N_t \equiv \int_{[0,1]} N_t(\jmath)\mathrm{d}\jmath, \quad \text{[A.6.10]}$$

where N_t denotes the aggregate usage of units of composite labour, and Ψ_t is the measure of relative-price distortions defined in [5.8]. The goods-market clearing condition $C_t = Y_t$ from [5.7].

Using the cost-minimizing labour demand functions [A.6.5], the market clearing condition [A.6.9], and the definition of aggregate demand N_t for units of composite labour from [A.6.10]:

$$w_{j,t} H_{j,t} = \Theta_j \left(\frac{w_t N_t}{1-s}\right) \;\; \text{for each} \;\; j \in \{\mathrm{y}, \mathrm{m}, \mathrm{o}\}. \quad \text{[A.6.11]}$$

From [5.6], [5.7], [A.6.6], and [A.6.9], total profits are given by $J_t = Y_t - w_t N_t$. Using [A.6.6], [A.6.7], and [A.6.10], total taxes are $T_t = (s/(1-s))w_t N_t$. With [A.6.8], age-specific non-financial incomes from [A.6.2] are $Y_{i,t} = w_{i,t}H_{i,t} + \Theta_i(J_t - T_t)$ for each i. Together with the expressions for J_t and T_t, equation [A.6.11] therefore implies that $Y_{i,t} = \Theta_i Y_t$, and hence age-shares of total GDP are constant and equal to those assumed in section 2.

Using the aggregate production function [A.6.10] to note that $N_t = \Psi_t Y_t / A_t$ and substituting this into [A.6.11], and then substituting this into the labour supply first-order condition [A.6.3] implies that the age-specific real wages $w_{i,t}$ satisfy:

$$w_{j,t} = c_{j,t} \frac{\Psi_t^{\eta-1} Y_t^{\eta}}{A_t^{\eta-1}} \left(\frac{w_t}{(1-s)w_{j,t}}\right)^{\eta-1} \;\; \text{for each} \;\; j \in \{\mathrm{y}, \mathrm{m}, \mathrm{o}\}. \quad \text{[A.6.12]}$$

With the formula [A.6.6] for the minimum cost w_t of a unit of composite labour and the size of the wage-bill subsidy $s = \varepsilon^{-1}$, the equation above implies that:

$$w_t = (1 - \varepsilon^{-1}) \left(c_{\mathrm{y},t}^{\frac{\Theta_\mathrm{y}}{3}} c_{\mathrm{m},t}^{\frac{\Theta_\mathrm{m}}{3}} c_{\mathrm{o},t}^{\frac{\Theta_\mathrm{o}}{3}} \right) \frac{\Psi_t^{\eta-1} Y_t^{\eta}}{A_t^{\eta-1}}.$$

Assuming the parameterization [2.6] for Θ_j and substituting the expression for the real wage into the equation $k_t = w_t/A_t$ for real marginal cost:

$$k_t = (1 - \varepsilon^{-1}) \left(c_{\mathrm{y},t}^{\frac{1-\beta\gamma}{3}} c_{\mathrm{m},t}^{\frac{1+(1+\beta)\gamma}{3}} c_{\mathrm{o},t}^{\frac{1-\gamma}{3}} \right) \Psi_t^{\eta-1} \left(\frac{Y_t}{A_t} \right)^{\eta}. \qquad \text{[A.6.13]}$$

Output Y_t is endogenously determined by the equation above given an assumption on how prices are set. All other equilibrium conditions from section 2 continue to hold.

A.6.5 Flexible prices

If all firms have fully flexible prices then all set the same price ($P_t(\jmath) = P_t$) because all face the same marginal cost of production. Equation [5.8] then implies $\Psi_t = 1$, meaning that there are no relative-price distortions and that $N_t = Y_t/A_t$. From [5.5], profit maximization implies that $k_t = 1 - \varepsilon^{-1}$, and by substituting this into [A.6.13] and solving for Y_t/A_t:

$$\frac{Y_t}{A_t} = N_t = \left(c_{\mathrm{y},t}^{\frac{1-\beta\gamma}{3}} c_{\mathrm{m},t}^{\frac{1+(1+\beta)\gamma}{3}} c_{\mathrm{o},t}^{\frac{1-\gamma}{3}} \right)^{\frac{1}{\eta}}. \qquad \text{[A.6.14]}$$

In a non-stochastic steady state with TFP growth $a_t = (A_t - A_{t-1})/A_{t-1}$ equal to a constant $\bar{a}$, Proposition 1 implies the consumption ratios have steady-state values $\bar{c}_\mathrm{y} = \bar{c}_\mathrm{m} = \bar{c}_\mathrm{o} = 1$. Equation [A.6.14] then implies steady-state employment is $\bar{N} = 1$, and thus steady-state real GDP growth is $\bar{g} = \bar{a}$.

With complete financial markets, Proposition 2 characterizes the equilibrium since $\alpha = 1$ and $\sigma = 1$. The natural debt-to-GDP ratio is $d_t^* = \gamma/3$ and the complete-markets consumption ratios are $c_{\mathrm{y},t}^* = c_{\mathrm{m},t}^* = c_{\mathrm{o},t}^* = 1$. Using equation [A.6.14], complete-markets employment with flexible prices is $N_t^* = 1$, and the corresponding level and growth rate of real GDP are $Y_t^* = A_t$ and $g_t^* = a_t$.

With incomplete markets, a log-linear approximation of the equilibrium can be found by noting

$$\frac{1}{3}\left((1-\beta\gamma)\mathsf{c}_{\mathrm{y},t} + (1+(1+\beta)\gamma)\mathsf{c}_{\mathrm{m},t} + (1-\gamma)\mathsf{c}_{\mathrm{o},t}\right) = -\nu \mathsf{d}_t, \quad \text{where } \nu = \frac{\gamma^2}{3}\left((2+\beta) - (1+2\beta)\phi\right), \qquad \text{[A.6.15]}$$

where the expression for ν is derived using the results of Proposition 6, with $\tilde{\mathsf{d}}_t = \mathsf{d}_t$ because $\mathsf{d}_t^* = 0$ when $\alpha = 1$ and $\sigma = 1$. The output gap $\tilde{Y}_t = Y_t/Y_t^*$ and the employment gap $\tilde{N}_t \equiv N_t/N_t^*$ can then be found using equation [A.6.14]:

$$\tilde{\mathsf{Y}}_t = \tilde{\mathsf{N}}_t = \mathsf{N}_t = -\frac{\nu}{\eta}\mathsf{d}_t. \qquad \text{[A.6.16]}$$

Pareto efficiency requires a constant debt-to-GDP ratio, which can be achieved through stabilizing the level of nominal GDP.

A.6.6 Sticky prices

Now suppose as in section 5.2 there are firms using a predetermined price $\check{P}_t$ and firms using a flexible price $\hat{P}_t$, with κ denoting the fraction of the former relative to the latter. In what follows, the benchmark is the hypothetical economy with complete financial markets and flexible prices with $\hat{Y}_t^* = A_t$ and $\hat{N}_t^* = 1$. The output gap relative to this benchmark is defined as $\tilde{Y}_t \equiv Y_t/\hat{Y}_t^*$. Since $\alpha = 1$ and $\sigma = 1$, the results of Proposition 2 apply, and hence $d_t^* = \gamma/3$ and $(1 + r_t^*) = (1 + g_t)/\beta$.

The formula for the price index in [5.2] implies $\hat{p}_t = (1 - \kappa(\check{p}_t^{1-\varepsilon} - 1))^{\frac{1}{1-\varepsilon}}$, while the first-order condition in [5.5] implies $\hat{p}_t = (\varepsilon/(\varepsilon-1))k_t$. Using the expression for real marginal cost k_t in [A.6.13] and the definition of the output gap $\tilde{Y}_t$:

$$\left(1 - \kappa\left(\check{p}_t^{1-\varepsilon} - 1\right)\right)^{\frac{1}{1-\varepsilon}} = \left(c_{\mathrm{y},t}^{\frac{1-\beta\gamma}{3}} c_{\mathrm{m},t}^{\frac{1+(1+\beta)\gamma}{3}} c_{\mathrm{o},t}^{\frac{1-\gamma}{3}} \right) \Psi_t^{\eta-1} \tilde{Y}_t^{\eta}. \qquad \text{[A.6.17]}$$

The non-stochastic steady state features $\bar{\bar{p}} = 1$, $\bar{\Psi} = 1$, and $\bar{c}_y = \bar{c}_m = \bar{c}_o = 1$, hence it follows that $\bar{\bar{Y}} = 1$.

Log linearizing equation [A.6.17] around the steady state implies the Phillips curve:

$$\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t) = \eta\tilde{\mathsf{Y}}_t - \nu\mathsf{d}_t, \qquad \text{[A.6.18a]}$$

where equations [5.10] and [A.6.15] have been used, together with $\Psi_t = 0$.

With complete markets and flexible prices, the ex-post real return satisfies $1 + \hat{r}^*_t = (1 + \hat{g}_t)/\beta$. Using the definition of the output gap, it follows that $1 + r^*_t = (\tilde{Y}_t/\tilde{Y}_{t-1})(1 + \hat{r}^*_t)$. Equation [4.7c] implies that the inflation rate must satisfy

$$\pi_t = \mathsf{i}_{t-1} - \mathsf{d}_t - \beta^{-1}\phi\mathsf{d}_{t-1} - \tilde{\mathsf{Y}}_t + \tilde{\mathsf{Y}}_{t-1} - \hat{\mathsf{r}}^*_t, \qquad \text{[A.6.18b]}$$

noting that $\mathsf{d}_t = \tilde{\mathsf{d}}_t$ since $\mathsf{d}^*_t = 0$. The final equation is [4.7a] with $\mathsf{d}_t = \tilde{\mathsf{d}}_t$:

$$\lambda\mathsf{d}_t = \mathbb{E}_t\mathsf{d}_{t+1}. \qquad \text{[A.6.18c]}$$

Therefore, the constraints faced by the policymaker are the Phillips curve [A.6.18a], the link between inflation, the output gap and debt in [A.6.18b], and the debt-dynamics equation [A.6.18c]. There are four endogenous variables: inflation π_t, the debt-to-GDP ratio d_t, the output gap $\tilde{\mathsf{Y}}_t$, and the nominal interest rate i_t; and one exogenous variable $\hat{\mathsf{r}}^*_t = \hat{\mathsf{g}}_t = \mathsf{a}_t$, which depends only on the growth rate of exogenous TFP A_t. With $\alpha = 1$ and $\sigma = 1$, the Pareto weights supporting the complete-markets equilibrium are $\Omega^*_t = 1$, and these are used to calculate the social welfare function [3.3]. The following result presents the second-order approximation of the welfare function and optimal monetary policy.

Proposition 15 *The welfare function can be written* $\mathscr{W}_{t_0} = -\mathbb{E}_{t_0-2}\mathscr{L}_{t_0}$ + terms independent of monetary policy + third- and higher-order terms, *where the loss function* $\mathscr{L}_{t_0}$ *is:*

$$\mathscr{L}_{t_0} = \frac{1}{2}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\aleph\mathsf{d}_t^2 + \varepsilon\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t)^2 + \eta\tilde{\mathsf{Y}}_t^2\right], \qquad \text{[A.6.19]}$$

and where the weight $\aleph$ *on the debt-to-GDP ratio is:*

$$\aleph = \frac{\gamma^2}{3}\left(1 + \frac{1}{\eta}\right)\left(1 + \phi^2 + (1-\phi)^2\right) + \frac{\gamma^3}{3}\frac{1}{\eta}\left((1+\beta)(1-\phi)^2 - \beta\phi^2 - 1\right) - \frac{\nu^2}{\eta}. \qquad \text{[A.6.20]}$$

The first-order condition for minimizing the loss function subject to the constraints [A.6.18a]–[A.6.18c] *is:*

$$\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t = \left\{\frac{\kappa\left(\varepsilon + \frac{\kappa}{\eta} + (\varepsilon-1)\frac{\nu}{\eta}\right)}{\left(\frac{\aleph + \frac{\nu^2}{\eta}}{1-\beta\lambda^2}\right)\left(1 + \beta\lambda^2\frac{\kappa}{\eta}\right) + (\aleph - \nu)\frac{\kappa}{\eta}}\right\}(\pi_t - \mathbb{E}_{t-1}\pi_t), \qquad \text{[A.6.21]}$$

and optimal monetary policy is a weighted nominal GDP target:

$$\mathsf{P}_t + \hat{\varpi}\mathsf{Y}_t = 0, \quad \textit{where } \hat{\varpi} = \left(1 + \frac{\kappa\left(\varepsilon + \frac{\kappa}{\eta} + (\varepsilon-1)\frac{\nu}{\eta}\right)}{\left(\frac{\aleph + \frac{\nu^2}{\eta}}{1-\beta\lambda^2}\right)\left(1 + \beta\lambda^2\frac{\kappa}{\eta}\right) + (\aleph - \nu)\frac{\kappa}{\eta}}\right)^{-1}. \qquad \text{[A.6.22]}$$

PROOF See appendix A.22 ■

There are three terms in the loss function: the debt-to-GDP ratio, the inflation surprise, and the output gap (all terms are squared percentage deviations from their steady-state values). The coefficient of the inflation surprise is the same as that in the model with a fixed labour supply, and the coefficient of the debt-to-GDP ratio converges to that in the fixed labour supply model as $\eta \to \infty$, that is, as the Frisch elasticity approaches zero (noting that the assumption of log utility requires $\alpha = 1$ and $\sigma = 1$). The coefficient of the output gap is increasing in η (and thus decreasing in the Frisch elasticity).

With flexible prices, the optimal monetary policy given the parameter restrictions $\alpha = 1$, $\sigma = 1$, and $\mu = 1$ is exactly a target for the level of (unweighted) nominal GDP. With sticky prices, optimal monetary policy is a weighted nominal GDP target, where $\hat{\varpi} < 1$ is the weight assigned to output. There are two additional effects present relative to the case of a fixed labour supply. First, the Phillips curve [A.6.18a] implies that unexpected inflation leads to fluctuations in the output gap. Holding the debt-to-GDP ratio constant, stabilizing inflation is equivalent to stabilizing the output gap. This effect of an endogenous labour supply thus increases the importance of stabilizing inflation. The second effect is that fluctuations in the debt-to-GDP ratio lead to shifts of the Phillips curve that must increase the volatility of inflation or the output gap, or both. This effect increases the importance of stabilizing the debt-to-GDP ratio, which requires a policy of targeting nominal GDP. It is thus a quantitative question whether the addition of an endogenous labour supply decision to the model increases or decreases the weight on nominal GDP targeting.

Imposing $\alpha = 1$, $\sigma = 1$, and $\mu = 1$ but calibrating other parameters as in section 5.6 (see Table 1) leads to $\beta \approx 0.59$ and $\gamma \approx 0.66$. With these parameters, the expression for $\hat{\varpi}$ in [A.6.22] is decreasing in η, and hence increasing in the Frisch elasticity. The effect of fluctuations in the debt-to-GDP ratio on the position of the Phillips curve is thus the quantitatively dominant effect. The size of the overall effect is small, though. In the limiting case of inelastic labour supply ($\eta \to \infty$) the optimal weight is $\hat{\varpi} \approx 0.95$, while with the Frisch elasticity of 2/3 ($\eta \approx 2.5$) from Hall (2009), the optimal weight is $\hat{\varpi} \approx 0.96$.

A.7 Preliminary results

Lemma 1 *Define the variables $\wp_t$ and Δ_t as follows in terms of ρ_t and d_t and let $\boldsymbol{\Xi}_t$ denote the vector of these variables:*

$$\wp_t \equiv \frac{\delta(1+\rho_t)}{(1+\bar{g})^{\frac{1}{\sigma}}}, \quad \Delta_t \equiv \frac{3d_t}{\gamma}, \quad \text{and } \boldsymbol{\Xi}_t \equiv \begin{pmatrix} \wp_t \\ \Delta_t \end{pmatrix}. \qquad \text{[A.7.1]}$$

Any perfect-foresight path that is a solution of the system of equations [2.14a], [2.14c], *and* [2.23], *with $g_t = \bar{g}$ and $\Upsilon_t = 1$, must imply values of $\wp_t$ and Δ_t that satisfy the following equations:*

$$\mathfrak{F}(\boldsymbol{\Xi}_{t+1}, \boldsymbol{\Xi}_t) = \mathbf{0}, \quad \text{where } \mathfrak{F}(\boldsymbol{\Xi}', \boldsymbol{\Xi}) \equiv \begin{pmatrix} \mathfrak{f}_\rho(\wp', \Delta', \wp) \\ \mathfrak{f}_d(\Delta', \wp, \Delta) \end{pmatrix}, \qquad \text{[A.7.2]}$$

with functions $\mathfrak{f}_\rho(\wp', \Delta', \wp)$ and $\mathfrak{f}_d(\Delta', \wp, \Delta)$ defined by:

$$\mathfrak{f}_\rho(\wp', \Delta', \wp) \equiv \wp^\sigma \left(1 - \beta\gamma + \beta\gamma\frac{\Delta'}{\wp}\right)\left(\wp' + \beta\wp'^\sigma\right) - \left(1 + (1+\beta)\gamma - \gamma\Delta'\right)\wp' - \beta(1-\gamma); \qquad \text{[A.7.3a]}$$

$$\mathfrak{f}_d(\Delta', \wp, \Delta) \equiv \gamma(1 + \beta\wp^{\sigma-1})\Delta' - (1 + (1+\beta)\gamma - \gamma\Delta)\wp^\sigma + (1-\gamma). \qquad \text{[A.7.3b]}$$

Proof Any solution of [2.23] must feature a future ex post real return r_{t+1} equal to the real interest rate ρ_t, and hence from [2.14a] it follows (with $g_t = \bar{g}$) that the current loan ratio is given by:

$$l_t = \frac{(1+\bar{g})d_{t+1}}{1+\rho_t}. \qquad \text{[A.7.4]}$$

Substituting this equation into the age-specific budget constraints from [2.14c]:

$$c_{\mathrm{y},t} = 1 - \beta\gamma + \frac{3(1+\bar{g})d_{t+1}}{1+\rho_t}, \quad c_{\mathrm{m},t} = 1 + (1+\beta)\gamma - 3d_t - \frac{3(1+\bar{g})d_{t+1}}{1+\rho_t}, \quad \text{and } c_{\mathrm{o},t} = 1 - \gamma + d_t.$$

Written in terms of the new variables $\wp_t$ and Δ_t from [A.7.1], and making use of the definition of β from [2.5] which implies $\beta = \delta(1+\bar{g})^{1-\frac{1}{\sigma}}$ when $\varsigma = 0$, the budget constraints above are:

$$c_{\mathrm{y},t} = 1 - \beta\gamma + \beta\gamma\frac{\Delta_{t+1}}{\wp_t}, \quad c_{\mathrm{m},t} = 1 + (1+\beta)\gamma - \gamma\Delta_t - \beta\gamma\frac{\Delta_{t+1}}{\wp_t}, \quad \text{and } c_{\mathrm{o},t} = 1 - \gamma + \gamma\Delta_t. \qquad \text{[A.7.5a]}$$

Using $\rho_t = r_{t+1}$ and $g_{t+1} = \bar{g}$, the Euler equations from [2.23] become

$$\frac{\delta(1+\rho_t)}{(1+\bar{g})^{\frac{1}{\sigma}}}\left(\frac{c_{\mathrm{m},t+1}}{c_{\mathrm{y},t}}\right)^{-\frac{1}{\sigma}} = 1 = \frac{\delta(1+\rho_t)}{(1+\bar{g})^{\frac{1}{\sigma}}}\left(\frac{c_{\mathrm{o},t+1}}{c_{\mathrm{m},t}}\right)^{-\frac{1}{\sigma}},$$

and hence by noting the definition of $\wp_t$ in [A.7.1], these can be written as:

$$c_{\mathrm{m},t+1} = \wp_t^\sigma c_{\mathrm{y},t}, \quad \text{and } c_{\mathrm{o},t+1} = \wp_t^\sigma c_{\mathrm{m},t}. \qquad \text{[A.7.5b]}$$

Substituting the budget constraints from [A.7.5a] into the second Euler equation from [A.7.5b]:

$$1-\gamma+\gamma\Delta_{t+1}=\wp_t^{\sigma}\left(1+(1+\beta)\gamma-\gamma\Delta_t-\beta\gamma\frac{\Delta_{t+1}}{\wp_t}\right),$$

and by rearranging this equation it can be seen that:

$$\gamma\left(1+\beta\wp_t^{\sigma-1}\right)\Delta_{t+1}-\left(1+(1+\beta)\gamma-\gamma\Delta_t\right)\wp_t^{\sigma}+(1-\gamma)=0. \quad \text{[A.7.6]}$$

Comparison with the definition [A.7.3b] shows that this equation is $\mathfrak{f}_d(\Delta_{t+1},\wp_t,\Delta_t)=0$.

The next step is to substitute the budget constraints from [A.7.5a] into the first Euler equation from [A.7.5b]:

$$\left(1+(1+\beta)\gamma-\gamma\Delta_{t+1}\right)-\beta\gamma\frac{\Delta_{t+2}}{\wp_{t+1}}=\wp_t^{\sigma}\left(1-\beta\gamma+\beta\gamma\frac{\Delta_{t+1}}{\wp_t}\right). \quad \text{[A.7.7]}$$

Note that equation [A.7.6] implies:

$$\beta\gamma\frac{\Delta_{t+2}}{\wp_{t+1}}=\frac{\beta\left(1+(1+\beta)\gamma-\gamma\Delta_{t+1}\right)\wp_{t+1}^{\sigma}-\beta(1-\gamma)}{\wp_{t+1}+\beta\wp_{t+1}^{\sigma}},$$

which can be rearranged to obtain:

$$\left(1+(1+\beta)\gamma-\gamma\Delta_{t+1}\right)-\beta\gamma\frac{\Delta_{t+2}}{\wp_{t+1}}=\frac{\left(1+(1+\beta)\gamma-\gamma\Delta_{t+1}\right)\wp_{t+1}+\beta(1-\gamma)}{\wp_{t+1}+\beta\wp_{t+1}^{\sigma}}.$$

Substituting this into equation [A.7.7] yields:

$$\wp_t^{\sigma}\left(1-\beta\gamma+\beta\gamma\frac{\Delta_{t+1}}{\wp_t}\right)\left(\wp_{t+1}+\beta\wp_{t+1}^{\sigma}\right)-\left(1+(1+\beta)\gamma-\gamma\Delta_{t+1}\right)\wp_{t+1}-\beta(1-\gamma)=0, \quad \text{[A.7.8]}$$

which given the definition in [A.7.3a] is the equation $\mathfrak{f}_\rho(\wp_{t+1},\Delta_{t+1},\wp_t)=0$. This demonstrates that $\mathfrak{F}(\mathbf{\Xi}_{t+1},\mathbf{\Xi}_t)=\mathbf{0}$ as defined in [A.7.2] must hold, completing the proof. ■

Lemma 2 *Let $\mathscr{G}(z)$ be the following quadratic equation:*

$$\mathscr{G}(z)\equiv\beta\left(1-\frac{\gamma}{\sigma}\right)z^2+\left(2(1+\beta)-\frac{\beta\gamma}{\sigma}\right)z+\left(1-\frac{\beta\gamma}{\sigma}\right). \quad \text{[A.7.9]}$$

Assume the parameters are such that $0<\beta<1$, $0<\gamma<1$, $\sigma>0$. If the following condition is satisfied:

$$\frac{\gamma}{\sigma}<\frac{1+\beta}{\beta}, \quad \text{[A.7.10]}$$

then $\mathscr{G}(z)$ can be factorized uniquely as

$$\mathscr{G}(z)=\frac{z(1-\lambda z^{-1})(1-\zeta z)}{\chi}, \quad \text{[A.7.11]}$$

in terms of real roots λ and ζ^{-1} of $\mathscr{G}(z)=0$ and a non-zero coefficient χ. These are given by the following *formulas:*

$$\chi = \frac{2}{(1+2\beta) + \left(1 - \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}; \qquad \text{[A.7.12a]}$$

$$\begin{aligned}\lambda &= \frac{\sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)} - (1+2\beta) - \left(1 - \frac{\beta\gamma}{\sigma}\right)}{2\beta\left(1 - \frac{\gamma}{\sigma}\right)} \\ &= \frac{2\left(\frac{\beta\gamma}{\sigma} - 1\right)}{(1+2\beta) + \left(1 - \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}};\end{aligned} \qquad \text{[A.7.12b]}$$

$$\begin{aligned}\zeta &= \frac{\sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)} - (1+2\beta) - \left(1 - \frac{\beta\gamma}{\sigma}\right)}{2\left(1 - \frac{\beta\gamma}{\sigma}\right)} \\ &= \frac{2\beta\left(\frac{\gamma}{\sigma} - 1\right)}{(1+2\beta) + \left(1 - \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}.\end{aligned} \qquad \text{[A.7.12c]}$$

In the range of parameters consistent with [A.7.10]*, all of* χ*,* λ *and* ζ *are strictly increasing in the ratio* γ/σ *and lie between the following bounds:*

$$\frac{1}{2(1+\beta)} < \chi < 1, \quad -\frac{1}{2} < \lambda < \beta, \quad \textit{and} \quad -\frac{1}{2} < \zeta < 1, \qquad \text{[A.7.13]}$$

and hence $|\lambda| < 1$ *and* $|\zeta| < 1$.

Proof Evaluate the quadratic $\mathscr{G}(z)$ in [A.7.9] at $z = -1$ and $z = 1$:

$$\mathscr{G}(-1) = -\left(1 + \beta\left(1 + \frac{\gamma}{\sigma}\right)\right) < 0, \quad \text{and } \mathscr{G}(1) = 3\left((1+\beta) - \frac{\beta\gamma}{\sigma}\right).$$

Given that condition [A.7.10] is assumed to hold, it follows that $\mathscr{G}(1) > 0$, and hence that $\mathscr{G}(z)$ changes sign over the interval $[-1, 1]$. Thus, by continuity, $\mathscr{G}(z) = 0$ always has a root in the interval $(-1, 1)$. Let this root be denoted by λ, which must satisfy $|\lambda| < 1$.

Since [A.7.9] holds, it must be the case that

$$2(1+\beta) > \frac{\beta\gamma}{\sigma},$$

and thus that the coefficient of z in [A.7.9] is never zero. The coefficient of z^2 can be zero, though, so $\mathscr{G}(z)$ is either quadratic or purely linear. This means that either $\mathscr{G}(z)$ has only one root or has two distinct roots. As one root is known to be real, complex roots are not possible. Given that $\mathscr{G}(z)$ is at most quadratic and has a sign change on $[-1, 1]$, there can be no more than one root in this interval. A second root, if it exists, lies in either $(-\infty, -1)$ or $(1, \infty)$. If there is a second root, let ζ denote the reciprocal of this root. If there is no second root, let $\zeta = 0$. In either case, ζ is a real number satisfying $|\zeta| < 1$.

When $\zeta = 0$, the function given in [A.7.11] is linear with single root at $z = \lambda$. When $\zeta \neq 0$, [A.7.11] is a quadratic function with roots $z = \lambda$ and $z = \zeta^{-1}$. Therefore, the factorization [A.7.11] must hold for some non-zero coefficient χ.

Take the case of $\gamma < \sigma$ first. This means the coefficient of z^2 in [A.7.9] is positive, so the quadratic is u-shaped. Given that $\mathscr{G}(-1) < 0$, it follows that the second root ζ^{-1} is found in $(-\infty, -1)$, that is, to the left of λ. Now consider the case of $\gamma > \sigma$, where the coefficient of z^2 in $\mathscr{G}(z)$ is negative, and $\mathscr{G}(z)$ is n-shaped. With $\mathscr{G}(1) > 0$ this means that the second root ζ^{-1} is found in $(1, \infty)$, lying to the right of λ. In applying the quadratic root formula to find λ, observe that the denominator of the formula is positive in the case where $\gamma < \sigma$ (with $\zeta^{-1} < \lambda$) and negative when $\gamma > \sigma$ (with $\lambda < \zeta^{-1}$). Therefore, the root λ is always associated with the upper branch of the quadratic root function:

$$\lambda = \frac{-\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right) + \sqrt{\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right)^2 - 4\beta\left(1-\frac{\gamma}{\sigma}\right)\left(1-\frac{\beta\gamma}{\sigma}\right)}}{2\beta\left(1-\frac{\gamma}{\sigma}\right)}. \qquad \text{[A.7.14]}$$

Since λ is known to be a real number, the term inside the square root must be non-negative.

When a second root exists, ζ^{-1} is given by the lower branch of the quadratic root function, and hence an expression for ζ is:

$$\zeta = \frac{-2\beta\left(1-\frac{\gamma}{\sigma}\right)}{\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right) + \sqrt{\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right)^2 - 4\beta\left(1-\frac{\gamma}{\sigma}\right)\left(1-\frac{\beta\gamma}{\sigma}\right)}}. \qquad \text{[A.7.15]}$$

Using [A.7.9], the formula for the product $\lambda\zeta^{-1}$ of the roots of $\mathscr{G}(z) = 0$ implies:

$$\lambda = \frac{\left(1-\frac{\beta\gamma}{\sigma}\right)}{\beta\left(1-\frac{\gamma}{\sigma}\right)}\zeta. \qquad \text{[A.7.16]}$$

Substituting for ζ from [A.7.15] provides an alternative expression for λ:

$$\lambda = \frac{-2\left(1-\frac{\beta\gamma}{\sigma}\right)}{\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right) + \sqrt{\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right)^2 - 4\beta\left(1-\frac{\gamma}{\sigma}\right)\left(1-\frac{\beta\gamma}{\sigma}\right)}}. \qquad \text{[A.7.17]}$$

Given that condition [A.7.10] holds and that the term in the square root is positive, [A.7.17] provides a well-defined formula for λ in all cases, including $\gamma = \sigma$. Similarly, it can be seen from [A.7.15] that $\zeta = 0$ if and only if $\gamma = \sigma$, which given the definition [A.7.9] is equivalent to $\mathscr{G}(z)$ being purely linear. Therefore, formulas [A.7.15] and [A.7.17] are well defined for all configurations of γ and σ consistent with [A.7.10]. An alternative expression for ζ can be obtained by rearranging [A.7.16] to deduce $\zeta = (\beta(1-\gamma/\sigma)/(1-\beta\gamma/\sigma))\lambda$ and by substituting the expression for λ from [A.7.14]:

$$\zeta = \frac{\sqrt{\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right)^2 - 4\beta\left(1-\frac{\gamma}{\sigma}\right)\left(1-\frac{\beta\gamma}{\sigma}\right)} - (1+2\beta) - \left(1-\frac{\beta\gamma}{\sigma}\right)}{2\left(1-\frac{\beta\gamma}{\sigma}\right)}. \qquad \text{[A.7.18]}$$

Multiplying out the terms in the factorization [A.7.11] yields:

$$\mathscr{G}(z) = \frac{-\zeta z^2 + (1+\lambda\zeta)z - \lambda}{\chi}.$$

Equating the constant term with that in [A.7.9] implies $-\lambda/\chi = (1-\beta\gamma/\sigma)$, which leads to the following expression for χ:

$$\chi = \frac{\lambda}{\left(\frac{\beta\gamma}{\sigma} - 1\right)}.$$

Substituting for λ from [A.7.17] shows that χ is given by:

$$\chi = \frac{2}{\left(\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right) + \sqrt{\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right)^2 - 4\beta\left(1-\frac{\gamma}{\sigma}\right)\left(1-\frac{\beta\gamma}{\sigma}\right)}\right)}. \qquad \text{[A.7.19]}$$

Given [A.7.10] holds and the term in the square root is positive, it follows that χ is strictly positive. Observe that the term in the square root can be simplified as follows:

$$\left(2(1+\beta) - \beta\frac{\gamma}{\sigma}\right)^2 - 4\beta\left(1-\frac{\gamma}{\sigma}\right)\left(1-\frac{\beta\gamma}{\sigma}\right) = (1+2\beta)^2 + 3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right). \qquad \text{[A.7.20]}$$

Substituting [A.7.20] into [A.7.19], [A.7.14], [A.7.17], [A.7.15], and [A.7.18] yields the formulas in [A.7.12a]–[A.7.12c].

It follows immediately from the expression in [A.7.12a] that χ is strictly increasing in the ratio γ/σ. To consider its effects on λ and ζ, denote the ratio by $\Phi \equiv \gamma/\sigma$. In terms of Φ, $\mathscr{G}(z)$ can be written as:

$$\mathscr{G}(z) = \beta(1-\Phi)z^2 + (2(1+\beta) - \beta\Phi)z + (1-\beta\Phi). \qquad \text{[A.7.21]}$$

The coefficient λ is the only solution in $(-1, 1)$ of the equation $\mathscr{G}(z) = 0$. Since $\mathscr{G}(-1) < 0$ and $\mathscr{G}(1) > 0$ when condition [A.7.10] holds, it follows that $\mathscr{G}'(\lambda) > 0$. Similarly, since ζ is the only root lying in either $(-\infty, 1)$ or $(1, \infty)$, given that $\mathscr{G}(-1) < 0$ and $\mathscr{G}(1) > 0$ it must be the case that $\mathscr{G}'(\zeta^{-1}) < 0$. From [A.7.21] it follows that:

$$\frac{\partial\mathscr{G}(\lambda)}{\partial\Phi} = -\beta(1+\lambda+\lambda^2) < 0, \quad \text{and} \quad \frac{\partial\mathscr{G}(\zeta^{-1})}{\partial\Phi} = -\beta\left(1+\zeta^{-1}+\zeta^{-2}\right) < 0, \qquad \text{[A.7.22]}$$

because $\lambda \in (-1, 1)$ and $\zeta^{-1} \in (-1, 1)$. Given $\mathscr{G}(\lambda) = 0$ and $\mathscr{G}(\zeta^{-1}) = 0$, the effects of Φ on λ and ζ are determined by:

$$\frac{\partial\lambda}{\partial\Phi} = -\frac{1}{\mathscr{G}'(\lambda)}\frac{\partial\mathscr{G}(\lambda)}{\partial\Phi}, \quad \text{and} \quad \frac{\partial\zeta}{\partial\Phi} = \frac{\zeta^2}{\mathscr{G}'(\zeta^{-1})}\frac{\partial\mathscr{G}(\zeta^{-1})}{\partial\Phi},$$

both of which are strictly positive. Therefore, λ and ζ are both increasing in the ratio γ/σ.

Note that the restriction [A.7.10] means that Φ is restricted to the range $0 < \Phi < (1+\beta)/\beta$. If $\Phi = 0$ then $(1+2\beta) + (1-\Phi) + \sqrt{(1+2\beta)^2 + 3(1-\beta^2\Phi^2)} = 2(1+\beta+\sqrt{1+\beta+\beta^2}) < 2(1+\beta)$. If $\Phi = (1+\beta)/\beta$ then $(1+2\beta) + (1-\Phi) + \sqrt{(1+2\beta)^2 + 3(1-\beta^2\Phi^2)} = 2$. The bounds on χ given in [A.7.13] follow immediately from [A.7.12a] given that χ is increasing in Φ. Since [A.7.12a], [A.7.12b], and [A.7.12c] imply that $\lambda = (\beta\Phi - 1)\chi$ and $\zeta = \beta(\Phi - 1)\chi$, the bounds on λ and ζ in [A.7.13] follow from those established for χ given that Φ is restricted to the range $0 < \Phi < (1+\beta)/\beta$. This completes the proof. ■

A.8 Proof of Proposition 1

(i) The pair of difference equations can be written explicitly as:

$$d_{t+1} = \frac{(1+(1+\beta)\gamma - 3d_t)\beta^\sigma(1+\rho_t)^\sigma - (1-\gamma)}{3\left(1+\beta^\sigma(1+\rho_t)^{\sigma-1}\right)}; \text{ and } \beta^{2\sigma}\left(1-\beta\gamma+\frac{3d_{t+1}}{1+\rho_t}\right)(1+\rho_t)^\sigma(1+\rho_{t+1})^\sigma$$
$$+ \left(\beta^\sigma(1+\rho_t)^\sigma(2+\gamma-3d_t) - (2+\beta\gamma)\right)(1+\rho_{t+1}) = (1-\gamma). \qquad \text{[A.8.1]}$$

Let $\wp_t \equiv \delta(1+\rho_t)/(1+\bar{g})^{\frac{1}{\sigma}}$ and $\Delta_t \equiv 3d_t/\gamma$ denote linear functions of the real interest rate ρ_t and the debt ratio d_t, and let $\Xi_t \equiv \begin{pmatrix}\wp_t & \Delta_t\end{pmatrix}'$ denote the vector containing these two variables, as defined in equation [A.7.1] of Lemma 1. It is shown in Lemma 1 that any solution of the system of equations [2.14a], [2.14c], and [2.23], with $g_t = \bar{g}$ and $\Upsilon_t = 1$, must imply values of Ξ_t satisfying the vector of equations $\mathfrak{F}(\Xi_{t+1}, \Xi_t) = \mathbf{0}$ defined in [A.7.3]. Writing out the equations $\mathfrak{f}_d(\Delta_{t+1}, \wp_t, \Delta_t) = 0$ and $\mathfrak{f}_\rho(\wp_{t+1}, \Delta_{t+1}, \wp_t) = 0$ explicitly:

$$\gamma(1+\beta\wp_t^{\sigma-1})\Delta_{t+1} - (1+(1+\beta)\gamma - \gamma\Delta_t)\,\wp_t^\sigma + (1-\gamma) = 0; \qquad \text{[A.8.2a]}$$

$$\wp_t^\sigma\left(1-\beta\gamma+\beta\gamma\frac{\Delta_{t+1}}{\wp_t}\right)\left(\wp_{t+1}+\beta\wp_{t+1}^\sigma\right) - (1+(1+\beta)\gamma - \gamma\Delta_{t+1})\,\wp_{t+1} - \beta(1-\gamma) = 0. \qquad \text{[A.8.2b]}$$

The definitions of $\wp_t$ and Δ_t imply that the values of ρ_t and d_t can be recovered using:

$$1+\rho_t = \frac{(1+\bar{g})^{\frac{1}{\sigma}}\wp_t}{\delta}, \quad \text{and} \quad d_t = \frac{\gamma\Delta_t}{3}. \qquad \text{[A.8.3]}$$

Substituting these definitions into [A.8.2a] leads immediately to the first equation in [A.8.1]. Next, note that [A.8.2b] can be rearranged as follows:

$$\beta\left(1-\beta\gamma+\beta\gamma\frac{\Delta_{t+1}}{\wp_t}\right)\wp_t^\sigma\wp_{t+1}^\sigma + \left(\gamma\left(1+\beta\wp_t^{\sigma-1}\right)\Delta_{t+1} + \wp_t^\sigma(1-\beta\gamma) - (1+(1+\beta)\gamma)\right)\wp_{t+1} - \beta(1-\gamma) = 0.$$

Using [A.8.2a], this equation can be written as:

$$\beta\left(1-\beta\gamma+\beta\gamma\frac{\Delta_{t+1}}{\wp_t}\right)\wp_t^\sigma\wp_{t+1}^\sigma + \left((2+\gamma-\gamma\Delta_t)\wp_t^\sigma - (2+\beta\gamma)\right)\wp_{t+1} - \beta(1-\gamma) = 0,$$

and with the definitions in [A.8.3], it is seen to be equivalent to the second equation in [A.8.1].

(ii) Now consider steady states (with $g_t = \bar{g}$) of the system of equations [2.14a], [2.14c], and [2.23]. Given Lemma 1, such a steady state is equivalent to a constant perfect-foresight path $\boldsymbol{\Xi}_t = \bar{\boldsymbol{\Xi}}$ of the state vector $\boldsymbol{\Xi}_t = \begin{pmatrix}\wp_t & \Delta_t\end{pmatrix}'$. The vector $\boldsymbol{\Xi}$ must satisfy $\mathfrak{F}(\bar{\boldsymbol{\Xi}}, \bar{\boldsymbol{\Xi}}) = \mathbf{0}$. The definition [A.7.2] of the vector-valued function $\mathfrak{F}(\boldsymbol{\Xi}', \boldsymbol{\Xi})$ implies that the steady-state values $\bar{\wp}$ and $\bar{\Delta}$ are the solutions of the pair of equations $\mathfrak{f}_\rho(\bar{\wp}, \bar{\Delta}, \bar{\wp}) = 0$ and $\mathfrak{f}_d(\bar{\Delta}, \bar{\wp}, \bar{\Delta}) = 0$. Explicit expressions for these functions are given in [A.7.3]:

$$\bar{\wp}^\sigma \left(1 - \beta\gamma + \beta\gamma\frac{\bar{\Delta}}{\bar{\wp}}\right)(\bar{\wp} + \beta\bar{\wp}^\sigma) - (1 + (1+\beta)\gamma - \gamma\bar{\Delta})\bar{\wp} - \beta(1-\gamma) = 0; \quad \text{[A.8.4a]}$$

$$\gamma\left(1 + \beta\bar{\wp}^{\sigma-1}\right)\bar{\Delta} - (1 + (1+\beta)\gamma - \gamma\bar{\Delta})\bar{\wp}^\sigma + (1-\gamma) = 0. \quad \text{[A.8.4b]}$$

The first equation [A.8.4a] can be rearranged as follows:

$$(1 - \beta\gamma + \beta\gamma\frac{\bar{\Delta}}{\bar{\wp}})\bar{\wp}^\sigma - \left(\frac{(1 + \gamma(1+\beta) - \gamma\bar{\Delta})\bar{\wp} + \beta(1-\gamma)}{\bar{\wp} + \beta\bar{\wp}^\sigma}\right) = 0. \quad \text{[A.8.5]}$$

The second equation [A.8.4b] implies:

$$\beta\gamma\frac{\bar{\Delta}}{\bar{\wp}} = \frac{\beta(1 + (1+\beta)\gamma - \gamma\bar{\Delta})\bar{\wp}^\sigma - \beta(1-\gamma)}{\bar{\wp} + \beta\bar{\wp}^\sigma},$$

from which it follows that:

$$(1 + (1+\beta)\gamma - \gamma\bar{\Delta}) - \beta\gamma\frac{\bar{\Delta}}{\bar{\wp}} = \frac{(1 + (1+\beta)\gamma - \gamma\bar{\Delta})\bar{\wp} + \beta(1-\gamma)}{\bar{\wp} + \beta\bar{\wp}^\sigma}.$$

Substituting the above equation into [A.8.5] implies:

$$\bar{\wp}^\sigma\left(1 - \beta\gamma + \beta\gamma\frac{\bar{\Delta}}{\bar{\wp}}\right) - \left(1 + (1+\beta)\gamma - \gamma\bar{\Delta} - \beta\gamma\frac{\bar{\Delta}}{\bar{\wp}}\right) = 0,$$

which can be rearranged to deduce that:

$$\gamma\frac{\bar{\Delta}}{\bar{\wp}}(\beta + \beta\bar{\wp}^\sigma + \bar{\wp}) = (1 + (1+\beta)\gamma) - (1 - \beta\gamma)\bar{\wp}^\sigma. \quad \text{[A.8.6]}$$

It also follows directly from equation [A.8.4b] that:

$$\gamma\frac{\bar{\Delta}}{\bar{\wp}}\left(\bar{\wp} + \beta\bar{\wp}^\sigma + \bar{\wp}^{\sigma+1}\right) = (1 + (1+\beta)\gamma)\bar{\wp}^\sigma - (1-\gamma). \quad \text{[A.8.7]}$$

Combining equations [A.8.6] and [A.8.7] allows the variable $\bar{\Delta}$ to be eliminated, leaving the following equation that involves only $\bar{\wp}$:

$$(\beta + \beta\bar{\wp}^\sigma + \bar{\wp})\left((1 + (1+\beta)\gamma)\bar{\wp}^\sigma - (1-\gamma)\right) = \left(\bar{\wp} + \beta\bar{\wp}^\sigma + \bar{\wp}^{\sigma+1}\right)\left((1 + (1+\beta)\gamma) - (1-\beta\gamma)\bar{\wp}^\sigma\right). \quad \text{[A.8.8]}$$

In finding the set of solutions of [A.8.8] for $\bar{\wp}$, it is convenient to make use of the change of variable $z \equiv \bar{\wp}^\sigma$. Using $\bar{\wp} = z^{\frac{1}{\sigma}}$, equation [A.8.8] is equivalent to:

$$\left(\beta(1+z) + z^{\frac{1}{\sigma}}\right)\left((1 + (1+\beta)\gamma)z - (1-\gamma)\right) - \left(\beta z + (1+z)z^{\frac{1}{\sigma}}\right)\left((1 + (1+\beta)\gamma) - (1-\beta\gamma)z\right) = 0.$$

By multiplying out the brackets of the above equation and grouping terms in $z^{\frac{1}{\sigma}}$:

$$\beta\left((1 + (1+\beta)\gamma)(1+z)z - (1-\gamma)(1+z) - (1 + (1+\beta)\gamma)z + (1-\beta\gamma)z^2\right)$$
$$+ \left((1 + (1+\beta)\gamma)z - (1-\gamma) - (1 + (1+\beta)\gamma)(1+z) + (1-\beta\gamma)(1+z)z\right)z^{\frac{1}{\sigma}} = 0.$$

Simplifying this equation shows that it is equivalent to:

$$\mathscr{F}(z) = 0, \quad \text{where } \mathscr{F}(z) \equiv \left((1-\beta\gamma)(1+z+z^2) - 3\right)z^{\frac{1}{\sigma}} + \beta\left(3z^2 - (1-\gamma)(1+z+z^2)\right). \quad \text{[A.8.9]}$$

With economically meaningful real interest rates and real GDP growth rates in the range $-1 < \bar{\rho} < \infty$ and $-1 < \bar{g} < \infty$, the variable $\bar{\wp} = \delta(1+\bar{\rho})/(1+\bar{g})^{\frac{1}{\sigma}}$ must be restricted to be a positive real number. Given that $z = \bar{\wp}^\sigma$, a solution of $\mathscr{F}(z) = 0$ is an economically meaningful steady state if and only if $z \in (0, \infty)$.

In solving the equation $\mathscr{F}(z) = 0$, first note that the function $\mathscr{F}(z)$ in [A.8.9] is continuous and differentiable for all $z \in (0, \infty)$. It is helpful to write $\mathscr{F}(z)$ as

$$\mathscr{F}(z) = \beta\mathcal{P}(z) + z^{\frac{1}{\sigma}}\mathcal{Q}(z), \quad \text{[A.8.10]}$$

in terms of two quadratic functions $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ defined as follows:

$$\mathcal{P}(z) \equiv (2+\gamma)z^2 - (1-\gamma)z - (1-\gamma), \quad \text{and} \quad \mathcal{Q}(z) \equiv (1-\beta\gamma)z^2 + (1-\beta\gamma)z - (2+\beta\gamma). \quad \text{[A.8.11]}$$

Since $\mathcal{P}(0) = -(1-\gamma) < 0$ and $\mathcal{P}(1) = 3\gamma > 0$ (with $0 < \gamma < 1$), and as the coefficient of z^2 in $\mathcal{P}(z)$ is positive, it follows that the equation $\mathcal{P}(z) = 0$ has a unique positive root $\underline{z}(\gamma, \beta)$ satisfying $0 < \underline{z}(\gamma, \beta) < 1$. The value of the root depends only on the parameters γ and β. If $z \in [0, \underline{z}(\gamma, \beta))$ then $\mathcal{P}(z) < 0$, and if $z \in (\underline{z}(\gamma, \beta), \infty)$ then $\mathcal{P}(z) > 0$.

Now consider similar arguments for the quadratic function $\mathcal{Q}(z)$. Since $\mathcal{Q}(0) = -(2+\beta\gamma) < 0$ and $\mathcal{Q}(1) = -3\beta\gamma < 0$, and as the coefficient of z^2 in $\mathcal{Q}(z)$ is positive (given $0 < \beta < 1$), the equation $\mathcal{Q}(z) = 0$ has a unique positive root $\overline{z}(\gamma, \beta)$ satisfying $\overline{z}(\gamma, \beta) > 1$, and thus $\underline{z}(\gamma, \beta) < \overline{z}(\gamma, \beta)$. If $z \in [0, \overline{z}(\gamma, \beta))$ then $\mathcal{Q}(z) < 0$, and if $z \in (\overline{z}(\gamma, \beta), \infty)$ then $\mathcal{Q}(z) > 0$.

Given these findings, for any $z \in [0, \underline{z}(\gamma, \beta))$, since $\mathcal{P}(z) < 0$ and $\mathcal{Q}(z) < 0$, it follows from [A.8.10] that $\mathscr{F}(z) < 0$. Similarly, for any $z \in (\overline{z}(\gamma, \beta), \infty)$, as $\mathcal{P}(z) > 0$ and $\mathcal{Q}(z) > 0$, it follows that $\mathscr{F}(z) > 0$. Therefore, the search for solutions of the equation $\mathscr{F}(z) = 0$ can be confined to the finite interval $[\underline{z}(\gamma, \beta), \overline{z}(\gamma, \beta)]$.

Existence of a steady state

Using equation [A.8.9] to evaluate $\mathscr{F}(z)$ at $z = 1$:

$$\mathscr{F}(1) = 3((1-\beta\gamma) - 1) + 3\beta(1 - (1-\gamma)) = 0,$$

hence $z = 1$ is a solution of $\mathscr{F}(z) = 0$ for all possible parameter values. Given that $\bar{\wp} = z^{\frac{1}{\sigma}}$, $z = 1$ implies $\bar{\wp} = 1$. Furthermore, equation [A.8.3] shows that the steady-state real interest rate $\bar{\rho} = (1+\bar{g})^{\frac{1}{\sigma}}/\delta - 1$, which is as given in [2.24]. Similarly, given [2.23], the steady-state real return has the same value $\bar{r} = \bar{\rho}$. The following is required for $\bar{\rho} > \bar{g}$ to be satisfied:

$$\frac{1}{\delta(1+\bar{g})^{1-\frac{1}{\sigma}}} > 1,$$

which is equivalent to the parameter restriction on β in [2.5] given that $\beta = \delta(1+\bar{g})^{1-\frac{1}{\sigma}}$ when $\varsigma = 0$.

Given the value of $\bar{\wp}$, the steady-state value $\bar{\Delta}$ can be obtained directly from equation [A.8.6]:

$$\bar{\Delta} = \frac{\bar{\wp}\left((1+(1+\beta)\gamma) - (1-\beta\gamma)\bar{\wp}^{\sigma}\right)}{\gamma\left(\beta + \beta\bar{\wp}^{\sigma} + \bar{\wp}\right)}.$$

Substituting $\bar{\wp} = 1$ into this equation yields $\bar{\Delta} = 1$, and from [A.8.3] it follows that $\bar{d} = \gamma/3$. In steady state, [2.14a] demonstrates that $\bar{l} = (1+\bar{g})\bar{d}/(1+\bar{r})$, and since it has been shown that $1+\bar{r} = (1+\bar{g})^{\frac{1}{\sigma}}/\delta$, it follows that $\bar{l} = \delta\gamma(1+\bar{g})^{1-\frac{1}{\sigma}}/3$. With $\beta = \delta(1+\bar{g})^{1-\frac{1}{\sigma}}$ from [2.5] when $\varsigma = 0$, this is equivalent to $\bar{l} = \beta\gamma/3$. Finally, the budget identities from [2.14c] in steady state imply $\bar{c}_{\mathrm{y}} = 1 - \beta\gamma + 3\bar{l}$, $\bar{c}_{\mathrm{m}} = 1 + (1+\beta)\gamma - 3\bar{d} - 3\bar{l}$, and $\bar{c}_{\mathrm{o}} = 1 - \gamma + 3\bar{d}$. Given the values of $\bar{d}$ and $\bar{l}$, it is seen that $\bar{c}_{\mathrm{y}} = \bar{c}_{\mathrm{m}} = \bar{c}_{\mathrm{o}} = 1$.

Existence of parameters for which the steady state is unique

Let $\mathcal{F}(z; \sigma)$ denote the function $\mathscr{F}(z)$ from [A.8.9] with the dependence on the parameter σ made explicit. This can be written as:

$$\mathcal{F}(z;\sigma) \equiv (1-\beta\gamma)(1+z+z^2)z^{\frac{1}{\sigma}} - 3z^{\frac{1}{\sigma}} + 3\beta z^2 - \beta(1-\gamma)(1+z+z^2). \quad \text{[A.8.12]}$$

First, consider the limiting case of $\sigma \to \infty$. For any $z \geq \underline{z}(\gamma, \beta) > 0$, [A.8.12] reduces to:

$$\mathcal{F}(z;\infty) = (1-\beta\gamma)(1+z+z^2) - 3 + 3\beta z^2 - \beta(1-\gamma)(1+z+z^2) = (1+2\beta)z^2 + (1-\beta)z - (2+\beta).$$

This is a quadratic equation in z. Given that $0 < \beta < 1$, the coefficients of powers of z in the polynomial change sign exactly once. Hence, by Leibniz's rule of signs, the equation has at most one positive root. This shows that $z = 1$ is the unique positive solution of the equation $\mathcal{F}(z; \infty) = 0$.

Next, consider the special case of $\sigma = 1/2$, in which case [A.8.12] reduces to:

$$\begin{aligned}\mathcal{F}(z;1/2) &= (1-\beta\gamma)(1+z+z^2)z^2 - 3z^2 + 3\beta z^2 - \beta(1-\gamma)(1+z+z^2) \\ &= (1-\beta\gamma)z^4 + (1-\beta\gamma)z^3 - 2(1-\beta)z^2 - \beta(1-\gamma)z - \beta(1-\gamma).\end{aligned}$$

This is a quartic equation in z. Given that $0 < \beta < 1$ and $0 < \gamma < 1$, the coefficients of powers of z change sign exactly once. Hence, by Leibniz's rule of signs, the equation has at most one positive root, proving that $z = 1$ is the unique positive solution.

To analyse a general value of σ, make use of the expression for $\mathscr{F}(z)$ in [A.8.10] to take the derivative of $\mathcal{F}(z;\sigma)$ with respect to σ, holding z constant:

$$\frac{\partial \mathcal{F}(z;\sigma)}{\partial \sigma} = -\frac{1}{\sigma^2}(\log z) z^{\frac{1}{\sigma}} \mathcal{Q}(z). \qquad \text{[A.8.13]}$$

Now take any $z \in (\underline{z}(\gamma,\beta), \overline{z}(\gamma,\beta))$, for which $\mathcal{Q}(z) < 0$ given the definition of $\overline{z}(\gamma,\beta)$. For z values in this range, if $z < 1$ then it can be seen from [A.8.13] that $\mathcal{F}(z;\sigma)$ is decreasing in σ, while for $z > 1$, $\mathcal{F}(z;\sigma)$ is increasing in σ. It follows that for any parameter σ in the range $1/2 \leq \sigma < \infty$ and any $z \in (\underline{z}(\gamma,\beta), \overline{z}(\gamma,\beta))$ that $\mathcal{F}(z;\sigma)$ lies somewhere between the values of $\mathcal{F}(z;1/2)$ and $\mathcal{F}(z;\infty)$. Formally:

$$\min\{\mathcal{F}(z;1/2), \mathcal{F}(z;\infty)\} \leq \mathcal{F}(z;\sigma) \leq \max\{\mathcal{F}(z;1/2), \mathcal{F}(z;\infty)\}. \qquad \text{[A.8.14]}$$

Now suppose the equation $\mathcal{F}(z;\sigma) = 0$ were to have a root $z \neq 1$ for some parameter σ satisfying $1/2 \leq \sigma < \infty$. This root would need to lie in the interval $(\underline{z}(\gamma,\beta), \overline{z}(\gamma,\beta))$. It has already been established that the equations $\mathcal{F}(z;1/2) = 0$ and $\mathcal{F}(z;\infty) = 0$ have only one root $z = 1$ on the interval $z \in (\underline{z}(\gamma,\beta), \overline{z}(\gamma,\beta))$. Since $\mathcal{F}(z;\sigma)$ is a continuous function of z for all σ, and as $\mathcal{F}(\underline{z}(\gamma,\beta);\sigma) < 0$ and $\mathcal{F}(\overline{z}(\gamma,\beta);\sigma) > 0$, it follows that $\mathcal{F}(z;1/2)$ and $\mathcal{F}(z;\infty)$ are both negative for all $z \in (\underline{z}(\gamma,\beta), 1)$. Similarly, $\mathcal{F}(z;1/2)$ and $\mathcal{F}(z;\infty)$ are both positive for all $z \in (1, \overline{z}(\gamma,\beta))$. Given the bounds in [A.8.14], it follows that $\mathcal{F}(z;\sigma)$ is negative for all $z \in (\underline{z}(\gamma,\beta), 1)$ and positive for all $z \in (1, \overline{z}(\gamma,\beta))$ for any $1/2 \leq \sigma < \infty$. It is therefore shown that $z = 1$ is the only positive root of $\mathscr{F}(z) = 0$ for any σ satisfying $1/2 \leq \sigma < \infty$.

Existence of parameters for which the steady state is not unique

Given the definitions of $\underline{z}(\gamma,\beta)$ and $\overline{z}(\gamma,\beta)$ it follows from equation [A.8.10] that:

$$\mathscr{F}(\underline{z}(\gamma,\beta)) < 0, \quad \text{and } \mathscr{F}(\overline{z}(\gamma,\beta)) > 0, \quad \text{where } \underline{z}(\gamma,\beta) < 1 < \overline{z}(\gamma,\beta). \qquad \text{[A.8.15]}$$

Now note that the derivative of $\mathscr{F}(z)$ from [A.8.9] is

$$\mathscr{F}'(z) = (1-\beta\gamma)(1+2z)z^{\frac{1}{\sigma}} + \frac{1}{\sigma}\left((1-\beta\gamma)(1+z+z^2) - 3\right) z^{\frac{1}{\sigma}-1} + \beta\left(6z - (1-\gamma)(1+2z)\right). \qquad \text{[A.8.16]}$$

Evaluating this derivative at $z = 1$:

$$\mathscr{F}'(1) = 3(1-\beta\gamma) + \frac{3}{\sigma}\left((1-\beta\gamma) - 1\right) + 3\beta\left(2 - (1-\gamma)\right) = 3\left(1 - \beta\gamma - \frac{\beta\gamma}{\sigma} + \beta + \beta\gamma\right)$$
$$= 3(1+\beta)\left(1 - \frac{\beta}{1+\beta}\frac{\gamma}{\sigma}\right). \qquad \text{[A.8.17]}$$

If $\gamma/\sigma > (1+\beta)/\beta$ then [A.8.17] implies that $\mathscr{F}'(1) < 0$. Since $\mathscr{F}(1) = 0$, this means that $\mathscr{F}(z)$ is strictly positive in a neighbourhood below $z = 1$, and strictly negative in a neighbourhood above $z = 1$. Given the statements in [A.8.15] and the continuity of $\mathscr{F}(z)$, it follows that $\mathscr{F}(z) = 0$ has solutions in the ranges $(\underline{z}(\gamma,\beta), 1)$ and $(1, \overline{z}(\gamma,\beta))$. The steady state $z = 1$ would not then be unique.

To analyse the case where $\gamma/\sigma = (1+\beta)/\beta$, use [A.8.16] to obtain the second derivative of $\mathscr{F}(z)$:

$$\mathscr{F}''(z) = 2(1-\beta\gamma)z^{\frac{1}{\sigma}} + \frac{1}{\sigma}(1-\beta\gamma)(1+2z)z^{\frac{1}{\sigma}-1} + \frac{1}{\sigma}(1-\beta\gamma)(1+2z)z^{\frac{1}{\sigma}-1}$$
$$+ \frac{1}{\sigma}\left(\frac{1}{\sigma} - 1\right)\left((1-\beta\gamma)(1+z+z^2) - 3\right) z^{\frac{1}{\sigma}-2} + \beta\left(6 - 2(1-\gamma)\right). \qquad \text{[A.8.18]}$$

Evaluating this derivative at $z = 1$:

$$\mathscr{F}''(1) = 2(1-\beta\gamma) + \frac{3}{\sigma}(1-\beta\gamma) + \frac{3}{\sigma}(1-\beta\gamma) - \frac{3\beta\gamma}{\sigma}\left(\frac{1}{\sigma} - 1\right) + 2\beta(2+\gamma)$$
$$= 2(1+2\beta) + \frac{6}{\sigma}(1-\beta\gamma) - \frac{3\beta\gamma}{\sigma}\left(\frac{1}{\sigma} - 1\right). \qquad \text{[A.8.19]}$$

Note that when $\gamma/\sigma = (1+\beta)/\beta$:

$$\frac{1}{\sigma} = \frac{1+\beta}{\beta\gamma},$$

which can be substituted into [A.8.19] to obtain:

$$\mathscr{F}''(1) = \frac{1}{\beta\gamma}\left(2\beta\gamma(1+2\beta) + 6(1+\beta)(1-\beta\gamma) - 3(1+\beta)((1+\beta)-\beta\gamma)\right)$$

$$= \frac{1}{\beta\gamma}\left(2\beta\gamma(1+2\beta) + 3(1+\beta)(1-\beta-\beta\gamma)\right) = \frac{1-\beta}{\beta\gamma}\left(3(1+\beta)-\beta\gamma\right). \quad \text{[A.8.20]}$$

Therefore, from [A.8.17] and [A.8.20], when $\gamma/\sigma = (1+\beta)/\beta$ it follows that $\mathscr{F}'(1) = 0$ and $\mathscr{F}''(1) > 0$, the latter by noting $0 < \beta < 1$ and $0 < \gamma < 1$. Since $\mathscr{F}(1) = 0$, it must be the case that $\mathscr{F}(z)$ is positive in a neighbourhood below $z = 1$. Combined with [A.8.15], this means that there exists a solution of $\mathscr{F}(z) = 0$ in the range $(\underline{z}(\gamma,\beta), 1)$, demonstrating that the steady state $z = 1$ is not unique.

Hence, it is established that there exist multiple steady states if the following condition holds:

$$\frac{\gamma}{\sigma} \geq \frac{1+\beta}{\beta}. \quad \text{[A.8.21]}$$

Existence of a threshold intertemporal elasticity of substitution for uniqueness of the steady state

Consider given values of β and γ satisfying $0 < \beta < 1$ and $0 < \gamma < 1$. It has been shown so far that the steady state $z = 1$ is unique for any $\sigma \geq 1/2$, but from [A.8.21] it is known not to be unique if $\sigma \leq (\beta/(1+\beta))\gamma < 1/2$. Any solutions to $\mathscr{F}(z) = 0$ must lie in the interval $[\underline{z}(\gamma,\beta), \overline{z}(\gamma,\beta)]$, and given the definitions of $\underline{z}(\gamma,\beta)$ and $\overline{z}(\gamma,\beta)$, it follows from [A.8.10] that $\mathscr{F}(\underline{z}(\gamma,\beta)) < 0$ and $\mathscr{F}(\overline{z}(\gamma,\beta)) > 0$ for any parameter values.

Now suppose there is a value of σ for which $z = 1$ is the unique solution of $\mathscr{F}(z) = 0$. By continuity of $\mathscr{F}(z)$ given the signs of this function at $z = \underline{z}(\gamma,\beta)$ and $z = \overline{z}(\gamma,\beta)$ and since $\mathscr{F}(1) = 0$, this is equivalent to $\mathscr{F}(z) < 0$ for all $z \in (\underline{z}(\gamma,\beta), 1)$ and $\mathscr{F}(z) > 0$ for all $z \in (1, \overline{z}(\gamma,\beta))$. Now consider the derivative of $\mathscr{F}(z)$ with respect to σ as given in [A.8.13]. Since $\mathcal{Q}(z) < 0$ for $z \in (\underline{z}(\gamma,\beta), \overline{z}(\gamma,\beta))$, the derivative is negative for $z \in (\underline{z}(\gamma,\beta), 1)$ and positive for $z \in (1, \overline{z}(\gamma,\beta))$. This implies that following any increase in σ, it remains the case that $\mathscr{F}(z) < 0$ for all $z \in (\underline{z}(\gamma,\beta), 1)$ and $\mathscr{F}(z) > 0$ for all $z \in (1, \overline{z}(\gamma,\beta))$, so $z = 1$ remains the unique solution of $\mathscr{F}(z) = 0$. Therefore it is demonstrated that there exists a threshold for σ (depending on γ and β), denoted by $\underline{\sigma}(\gamma,\beta)$, such that $z = 1$ is the unique solution if and only if $\sigma \geq \underline{\sigma}(\gamma,\beta)$.

To find the dependence of the threshold $\underline{\sigma}(\gamma,\beta)$ on γ, differentiate $\mathscr{F}(z)$ with respect to γ (holding other parameters and z constant):

$$\frac{\partial\mathscr{F}(z)}{\partial\gamma} = \beta(1+z+z^2)\left(1 - z^{\frac{1}{\sigma}}\right),$$

where this expression uses [A.8.10] and [A.8.11]. For any $0 < \sigma < \infty$, this shows that $\mathscr{F}(z)$ is increasing in γ for $z < 1$ and decreasing for $z > 1$. Since $\mathscr{F}(\underline{z}(\gamma,\beta)) < 0$, $\mathscr{F}(1) = 0$, $\mathscr{F}(\overline{z}(\gamma,\beta)) > 0$, and solutions of $\mathscr{F}(z) = 0$ must lie in the interval $(\underline{z}(\gamma,\beta), \overline{z}(\gamma,\beta))$, it follows that if $z = 1$ is not the unique solution for some γ, any higher γ value will also imply multiple solutions. This means that the required threshold for σ to ensure a unique solution is increasing in γ. Finally, note that when $\gamma \to 0$, $\mathcal{P}(1) \to 0$ and $\mathcal{Q}(1) \to 0$, so $\underline{z}(\gamma,\beta) \to 1$ and $\overline{z}(\gamma,\beta) \to 1$. In this limiting case, $z = 1$ is the only possible solution of $\mathscr{F}(z) = 0$, hence it follows that $\lim_{\gamma\to 0}\underline{\sigma}(\gamma,\beta) = 0$.

(iv) Lemma 1 shows that any perfect foresight path consistent with the system of equations [2.14a], [2.14c], and [2.23] must be such that the sequence $\{\Xi_{t_0}, \Xi_{t_0+1}, \Xi_{t_0+2}, \ldots\}$ satisfies $\mathfrak{F}(\Xi_{t+1}, \Xi_t) = \mathbf{0}$ for all $t \geq t_0$. These equations implicitly define a time-invariant transition equation:

$$\Xi_{t+1} = \mathfrak{T}(\Xi_t), \quad \text{where } \mathfrak{F}(\mathfrak{T}(\Xi), \Xi) = \mathbf{0} \text{ for all } \Xi. \quad \text{[A.8.22]}$$

Now suppose that $\sigma \geq \underline{\sigma}(\gamma,\beta)$, which has been shown to be the condition for there to be a unique steady state. In terms of the components of the vector Ξ_t defined in [A.7.1], this steady state is $\bar{\wp} = 1$

and $\bar{\Delta} = 1$, or equivalently, $\bar{\Xi} = \boldsymbol{\iota} \equiv \begin{pmatrix} 1 & 1 \end{pmatrix}'$. The local dynamics of the system of equations [A.8.1] are determined by the Jacobian matrix of the transition equations $\mathfrak{T}(\cdot)$ from [A.8.22] evaluated at the steady state. This matrix is denoted by $\nabla_{\Xi}\mathfrak{T}(\boldsymbol{\iota})$, and by applying the implicit function theorem to [A.8.22], the Jacobian must satisfy:

$$\nabla_{\Xi'}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota})\nabla_{\Xi}\mathfrak{T}(\boldsymbol{\iota}) + \nabla_{\Xi}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota}) = \mathbf{0}.$$

Suppose z is an eigenvalue of $\nabla_{\Xi}\mathfrak{T}(\boldsymbol{\iota})$ and $\mathbf{v}$ is the corresponding eigenvector, that is, $\nabla_{\Xi}\mathfrak{T}(\boldsymbol{\iota})\mathbf{v} = z\mathbf{v}$. From the equation above, it follows that

$$\nabla_{\Xi}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota})\mathbf{v} = z(-\nabla_{\Xi'}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota}))\mathbf{v}, \quad \text{[A.8.23]}$$

thus, z is a generalized eigenvalue of the pair of matrices $\{\nabla_{\Xi}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota}), -\nabla_{\Xi'}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota})\}$.

To find the matrices in [A.8.23], the partial derivatives of the functions in [A.7.3] that make up the elements of $\mathfrak{F}(\cdot,\cdot)$ in [A.7.2] are evaluated at the steady state $\bar{\wp} = 1$ and $\bar{\Delta} = 1$:

$$\frac{\partial \mathfrak{f}_\rho(1,1,1)}{\partial \wp'} = \beta(\sigma-\gamma), \quad \frac{\partial \mathfrak{f}_\rho(1,1,1)}{\partial \Delta'} = (1+\beta+\beta^2)\gamma, \quad \frac{\partial \mathfrak{f}_\rho(1,1,1)}{\partial \wp} = (1+\beta)(\sigma-\beta\gamma),$$
$$\frac{\partial \mathfrak{f}_d(1,1,1)}{\partial \Delta'} = (1+\beta)\gamma, \quad \frac{\partial \mathfrak{f}_d(1,1,1)}{\partial \wp} = -(\sigma+\beta\gamma), \quad \text{and} \quad \frac{\partial \mathfrak{f}_d}{\partial \Delta} = -\gamma,$$

and these are inserted into the Jacobian matrices $\nabla_{\Xi'}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota})$ and $\nabla_{\Xi}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota})$ to obtain:

$$\nabla_{\Xi'}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota}) = \begin{pmatrix} \beta(\sigma-\gamma) & (1+\beta+\beta^2)\gamma \\ 0 & (1+\beta)\gamma \end{pmatrix}, \quad \text{and} \quad \nabla_{\Xi}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota}) = \begin{pmatrix} (1+\beta)(\sigma-\beta\gamma) & 0 \\ -(\sigma+\beta\gamma) & \gamma \end{pmatrix}.$$

Equation [A.8.23] shows that the generalized eigenvalues can be found from the following matrix:

$$\nabla_{\Xi}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota}) + z\nabla_{\Xi'}\mathfrak{F}(\bar{\Xi},\bar{\Xi}) = \begin{pmatrix} (1+\beta)(\sigma-\beta\gamma)+\beta(\sigma-\gamma)z & (1+\beta+\beta^2)\gamma z \\ -(\sigma+\beta\gamma) & \gamma+(1+\beta)\gamma z \end{pmatrix}, \quad \text{[A.8.24]}$$

which has determinant:

$$\det\left(\nabla_{\Xi}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota}) + z\nabla_{\Xi'}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota})\right) = \gamma\sigma(1+\beta)\left(\beta\left(1-\frac{\gamma}{\sigma}\right)z^2 + \left(2(1+\beta)-\frac{\beta\gamma}{\sigma}\right)z + \left(1-\frac{\beta\gamma}{\sigma}\right)\right).$$

Comparing this expression to the quadratic function $\mathscr{G}(z)$ defined in equation [A.7.9] of Lemma 2, the determinant can be written as:

$$\det\left(\nabla_{\Xi}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota}) + z\nabla_{\Xi'}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota})\right) = \gamma\sigma(1+\beta)\mathscr{G}(z). \quad \text{[A.8.25]}$$

The results of Lemma 2 and equation [A.8.25] reveal that $\det\left(\nabla_{\Xi}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota}) + z\nabla_{\Xi'}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota})\right) = 0$ if and only if $z = \lambda$ or $z = \zeta^{-1}$, where λ and ζ are as defined in [A.7.12b] and [A.7.12c]. Lemma 2 establishes that $|\lambda| < 1$ and $|\zeta| < 1$, so $z = \lambda$ is a stable eigenvalue and $z = \zeta^{-1}$ is an unstable eigenvalue. To find the associated eigenvectors, let $\mathbf{v} = \begin{pmatrix} \mathrm{v} & 1 \end{pmatrix}'$, where the second element of the eigenvector is normalized to unity without loss of generality. For a given eigenvalue z, equation [A.8.23] shows that $(\nabla_{\Xi}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota})\mathbf{v} + z\nabla_{\Xi'}\mathfrak{F}(\boldsymbol{\iota},\boldsymbol{\iota}))\mathbf{v} = \mathbf{0}$, and hence by using [A.8.24], the second row of this equation is:

$$-(\sigma+\beta\gamma)\mathrm{v} + (\gamma+(1+\beta)\gamma z) = 0.$$

Dividing both sides by σ and solving for v_λ (when $z = \lambda$) and v_ζ (when $z = \zeta^{-1}$) yields:

$$\mathrm{v}_\lambda = \frac{\frac{\gamma}{\sigma}(1+(1+\beta)\lambda)}{1+\frac{\beta\gamma}{\sigma}}, \quad \text{and} \quad \mathrm{v}_\zeta = \frac{\frac{\gamma}{\sigma}\left(1+(1+\beta)\zeta^{-1}\right)}{1+\frac{\beta\gamma}{\sigma}}. \quad \text{[A.8.26]}$$

These results demonstrate the existence of both a stable manifold and an unstable manifold in the neighbourhood of the unique steady state. If a phase diagram were drawn with Δ_t and $\wp_t$ on the horizontal and vertical axes respectively, since $\mathbf{v} = \begin{pmatrix} \mathrm{v} & 1 \end{pmatrix}'$, it follows that v_λ is the slope of the stable manifold and v_ζ is the slope of the unstable manifold. Lemma 2 proves that $\lambda > -1/2$, and since $0 < \beta < 1$, it must be the case that $1+(1+\beta)\lambda > 0$, so $\mathrm{v}_\lambda > 0$. If $\zeta > 0$, then since Lemma 2 shows that $|\lambda| < 1$ and $|\zeta| < 1$, it would follow that $1+(1+\beta)\lambda < 1+(1+\beta)\zeta^{-1}$, hence $\mathrm{v}_\lambda < \mathrm{v}_\zeta$. Alternatively, if $\zeta < 0$ then $1+(1+\beta)\zeta^{-1} < 0$, and hence $\mathrm{v}_\zeta < 0$. This completes the proof.

A.9 Proof of Proposition 2

Consider the possibility of an equilibrium in which the following are time-invariant: the debt-to-GDP ratio ($d_t^* = d^*$), the loans-to-GDP ratio ($l_t^* = l^*$), and the age-specific consumption ratios ($c_{\mathrm{y},t}^* = c_{\mathrm{y}}^*$, $c_{\mathrm{m},t}^* = c_{\mathrm{m}}^*$, and $c_{\mathrm{o},t}^* = c_{\mathrm{o}}^*$). With time-invariant debt, loans, and consumption ratios, the budget identities in [2.14c] require:

$$c_{\mathrm{y}}^* = 1 - \beta\gamma + 3l^*, \quad c_{\mathrm{m}}^* = 1 + (1+\beta)\gamma - 3d^* - 3l^*, \quad \text{and } \; c_{\mathrm{o}}^* = 1 - \gamma + 3d^*. \tag{A.9.1}$$

Satisfying equation [2.14a] requires $d^*(1+g_t) = l^*(1+r_t^*)$. Since the stochastic process for growth g_t specified in [2.4] is stationary, the equilibrium real return (ex post) r_t^* must also be a stationary stochastic process with a time-invariant mean $\mathbb{E}r_t^*$. Taking unconditional expectations of $d^*(1+g_t) = l^*(1+r_t^*)$ and of the equation in [2.14b] implies that

$$d^* = \left(\frac{1+\mathbb{E}r_t^*}{1+\bar{g}}\right) l^*, \quad \text{and } \; \mathbb{E}\rho_t^* = \mathbb{E}r_t^*, \tag{A.9.2}$$

where $\bar{g} = \mathbb{E}g_t$ is the time-invariant mean growth rate from [2.4]. Hence from [2.14a] and [2.14b] it follows that the equilibrium real return (ex post) r_t^* and real interest rate ρ_t^* must satisfy the equations:

$$\frac{1+r_t^*}{1+g_t} = \frac{1+\mathbb{E}r_t^*}{1+\bar{g}}, \quad \text{and } \; \frac{1+\rho_t^*}{1+\mathbb{E}_t g_{t+1}} = \frac{1+\mathbb{E}r_t^*}{1+\bar{g}}. \tag{A.9.3}$$

Therefore, if equations [A.9.2] and [A.9.3] hold then equations [2.14a] and [2.14b] are satisfied by construction.

(i) Consider the case of log utility ($\alpha = 1$ and $\sigma = 1$). Given that $\alpha = 1/\sigma$ in this special case, the Euler equations [2.14d] and the risk-sharing condition [2.21] reduce to:

$$\delta\mathbb{E}_t\left[\left(\frac{1+r_{t+1}^*}{1+g_{t+1}}\right)\left(\frac{c_{\mathrm{m}}^*}{c_{\mathrm{y}}^*}\right)^{-1}\right] = 1 = \delta\mathbb{E}_t\left[\left(\frac{1+r_{t+1}^*}{1+g_{t+1}}\right)\left(\frac{c_{\mathrm{o}}^*}{c_{\mathrm{m}}^*}\right)^{-1}\right], \quad \text{and } \left(\frac{c_{\mathrm{m}}^*}{c_{\mathrm{y}}^*}\right)^{-1} = \left(\frac{c_{\mathrm{o}}^*}{c_{\mathrm{m}}^*}\right)^{-1}. \tag{A.9.4}$$

The value function equations [2.14e] are redundant in this case.

(ii) Consider the case where the stochastic process x_t in [2.4] is independent and identically distributed over time. Using the i.i.d. property together with the definition of β in [2.5], note that:

$$\beta = \delta\mathbb{E}_t\left[(1+g_{t+1})^{1-\alpha}\right]^{\frac{1-1/\sigma}{1-\alpha}}. \tag{A.9.5}$$

With this expression for β and equation [2.14e], the time-invariant consumption ratios imply time-invariant value function ratios:

$$v_{\mathrm{m},t}^* = v_{\mathrm{m}}^* = \left({c^*_{\mathrm{m}}}^{1-\frac{1}{\sigma}} + \beta {c^*_{\mathrm{o}}}^{1-\frac{1}{\sigma}}\right)^{\frac{1}{1-\frac{1}{\sigma}}}, \quad \text{and } \; v_{\mathrm{o},t}^* = v_{\mathrm{o}}^* = c_{\mathrm{o}}^*.$$

With the time-invariant values v_{m}^* and v_{o}^*, the Euler equations in [2.14d] become:

$$\frac{\delta\mathbb{E}_t\left[\left(\frac{1+r_{t+1}^*}{1+g_{t+1}}\right)\left(\frac{c_{\mathrm{m}}^*}{c_{\mathrm{y}}^*}\right)^{-\frac{1}{\sigma}}(1+g_{t+1})^{1-\alpha}\right]}{\left(\mathbb{E}_t\left[(1+g_{t+1})^{1-\alpha}\right]^{\frac{1}{1-\alpha}}\right)^{\frac{1}{\sigma}-\alpha}} = 1 = \frac{\delta\mathbb{E}_t\left[\left(\frac{1+r_{t+1}^*}{1+g_{t+1}}\right)\left(\frac{c_{\mathrm{o}}^*}{c_{\mathrm{m}}^*}\right)^{-\frac{1}{\sigma}}(1+g_{t+1})^{1-\alpha}\right]}{\left(\mathbb{E}_t\left[(1+g_{t+1})^{1-\alpha}\right]^{\frac{1}{1-\alpha}}\right)^{\frac{1}{\sigma}-\alpha}}, \tag{A.9.6a}$$

and the risk-sharing condition [2.21] reduces to:

$$\left(\frac{c_{\mathrm{m}}^*}{c_{\mathrm{y}}^*}\right)^{-\frac{1}{\sigma}} = \left(\frac{c_{\mathrm{o}}^*}{c_{\mathrm{m}}^*}\right)^{-\frac{1}{\sigma}}. \tag{A.9.6b}$$

Now consider the following equations:

$$\beta\left(\frac{1+\mathbb{E}r_t^*}{1+\bar{g}}\right)\left(\frac{c_{\mathrm{m}}^*}{c_{\mathrm{o}}^*}\right)^{-\frac{1}{\sigma}} = 1 = \beta\left(\frac{1+\mathbb{E}r_t^*}{1+\bar{g}}\right)\left(\frac{c_{\mathrm{m}}^*}{c_{\mathrm{o}}^*}\right)^{-\frac{1}{\sigma}}. \tag{A.9.7}$$

Using [A.9.3] and [A.9.5], satisfaction of these equations implies [A.9.4] in special case (i) and [A.9.6a] and [A.9.6b] in special case (ii).

Next, consider the system of equations [2.14a], [2.14c], and [2.23] (with $\varsigma = 0$, so $g_t = \bar{g}$) that determines the non-stochastic steady-state equilibrium values $\bar{d}$, $\bar{l}$, $\bar{c}_{\mathrm{y}}$, $\bar{c}_{\mathrm{m}}$, $\bar{c}_{\mathrm{o}}$, $\bar{r}$, and $\bar{\rho}$. Note that these equations are of an identical form to those in [A.9.1], [A.9.2], and [A.9.7] in variables d^*, l^*, c^*_{y}, c^*_{m}, c^*_{o}, $\mathbb{E}r^*_t$, and $\mathbb{E}\rho^*_t$ (the only difference is that β may depend on the standard deviation ς, and is not generally the same as in a non-stochastic steady state). Using the argument of Proposition 1, the unique solution of these equations is:

$$d^* = \frac{\gamma}{3}, \quad l^* = \frac{\beta\gamma}{3}, \quad c^*_{\mathrm{y}} = c^*_{\mathrm{m}} = c^*_{\mathrm{o}} = 1, \quad \text{and } \mathbb{E}r^*_t = \mathbb{E}\rho^*_t = \frac{1+\bar{g}}{\beta} - 1. \qquad \text{[A.9.8]}$$

Combining this result with equation [A.9.3] yields the equilibrium given in [3.1]. This completes the proof.

A.10 Proof of Proposition 3

Using the transformations in [3.6], the first-order conditions [3.5] of the social planner's problem can be rewritten as:

$$\varphi^\star_t = \omega_t c^{\star\,-\frac{1}{\sigma}}_{\mathrm{y},t}, \quad \varphi^\star_t = \frac{\delta}{\beta}\omega_{t-1}(1+g_t)^{1-\frac{1}{\sigma}}\left\{\frac{(1+g_t)v^\star_{\mathrm{m},t}}{\mathbb{E}_{t-1}[(1+g_t)^{1-\alpha}v^{\star\,1-\alpha}_{\mathrm{m},t}]^{\frac{1}{1-\alpha}}}\right\}^{\frac{1}{\sigma}-\alpha} c^{\star\,-\frac{1}{\sigma}}_{\mathrm{m},t}, \quad \varphi^\star_t = \left(\frac{\delta}{\beta}\right)^2\omega_{t-2}$$
$$\times\left((1+g_{t-1})(1+g_t)\right)^{1-\frac{1}{\sigma}}\left\{\frac{(1+g_{t-1})v^\star_{\mathrm{m},t-1}}{\mathbb{E}_{t-2}[(1+g_{t-1})^{1-\alpha}v^{\star\,1-\alpha}_{\mathrm{m},t-1}]^{\frac{1}{1-\alpha}}}\frac{(1+g_t)v^\star_{\mathrm{o},t}}{\mathbb{E}_{t-1}[(1+g_t)^{1-\alpha}v^{\star\,1-\alpha}_{\mathrm{o},t}]^{\frac{1}{1-\alpha}}}\right\}^{\frac{1}{\sigma}-\alpha} c^{\star\,-\frac{1}{\sigma}}_{\mathrm{o},t}, \qquad \text{[A.10.1]}$$

for all $t \geq t_0$. With the normalization in [3.6], $\varphi^\star_{t_0} = 1$, ω_t must depend only on the state of the world at time t (for all $t \geq t_0$), while $\omega_{t_0-1}/\omega_{t_0-2}$ must depend only on the state of the world at time $t_0 - 1$.

(i) For a specific set of Pareto weights ω_t for $t \geq t_0 - 2$, let $\{c^\star_{\mathrm{y},t}, c^\star_{\mathrm{m},t}, c^\star_{\mathrm{o},t}\}$ denote the consumption allocation (relative to real GDP) that is the solution to the planner's problem. This must satisfy the resource constraint [2.16] for all $t \geq t_0$. Next, the equations in [3.7] (which follow from [A.10.1]) imply that the risk-sharing condition [2.21] must hold for all $t \geq t_0$. Finally, considering the equations in [A.10.1] at $t = t_0$ and taking the ratio of the equations involving ω_{t_0-1} and ω_{t_0-2}, the following expression can be deduced:

$$\frac{v^{\star\,\frac{1}{\sigma}-\alpha}_{\mathrm{o},t_0} c^{\star\,-\frac{1}{\sigma}}_{\mathrm{o},t_0}}{v^{\star\,\frac{1}{\sigma}-\alpha}_{\mathrm{m},t_0} c^{\star\,-\frac{1}{\sigma}}_{\mathrm{m},t_0}} = \left(\frac{\omega_{t_0-1}}{\omega_{t_0-2}}\right)\left(\frac{\beta}{\delta(1+g_{t_0-1})^{1-\frac{1}{\sigma}}}\right)\left\{\frac{(1+g_{t_0-1})v^\star_{\mathrm{m},t_0-1}}{\mathbb{E}_{t_0-2}[(1+g_{t_0-1})^{1-\alpha}v^{\star\,1-\alpha}_{\mathrm{m},t_0-1}]^{\frac{1}{1-\alpha}}}\right\}^{\alpha-\frac{1}{\sigma}}$$
$$\times\left\{\frac{\mathbb{E}_{t_0-1}[(1+g_{t_0})^{1-\alpha}v^{\star\,1-\alpha}_{\mathrm{m},t_0}]^{\frac{1}{1-\alpha}}}{\mathbb{E}_{t_0-1}[(1+g_{t_0})^{1-\alpha}v^{\star\,1-\alpha}_{\mathrm{o},t_0}]^{\frac{1}{1-\alpha}}}\right\}^{\alpha-\frac{1}{\sigma}}. \qquad \text{[A.10.2]}$$

Since all terms on the right-hand side are functions of variables known at time $t_0 - 1$ or earlier, it follows that

$$\frac{v^{\star\,\frac{1}{\sigma}-\alpha}_{\mathrm{o},t_0} c^{\star\,-\frac{1}{\sigma}}_{\mathrm{o},t_0}}{v^{\star\,\frac{1}{\sigma}-\alpha}_{\mathrm{m},t_0} c^{\star\,-\frac{1}{\sigma}}_{\mathrm{m},t_0}} = \mathcal{X}^\star_{t_0-1}, \qquad \text{[A.10.3]}$$

where $\mathcal{X}^\star_{t_0-1}$ is a function of variables known at time $t_0 - 1$.

Now consider the converse. Take any state-contingent consumption allocation $\{c^\star_{\mathrm{y},t}, c^\star_{\mathrm{m},t}, c^\star_{\mathrm{o},t}\}$, with value functions $\{v^\star_{\mathrm{m},t}, v^\star_{\mathrm{o},t}\}$ obtained from [2.14e]. Suppose the consumption allocation satisfies the resource constraint [2.16], the risk-sharing condition [2.21] for $t \geq t_0$, and is such that equation [A.10.3] holds for some $\mathcal{X}^\star_{t_0-1}$ which depends only on the state of the world at time $t_0 - 1$. The following steps explicitly construct a sequence of Pareto weights $\omega^\star_t$ for $t \geq t_0 - 2$ and Lagrangian multipliers $\varphi^\star_t$ for $t \geq t_0$ that support the consumption allocation. In what follows, the normalization $\varphi^\star_{t_0} = 1$ (from [3.6]) is adopted.

Starting from a point where $\varphi^\star_t$ has been determined in all states of the world at some time $t \geq t_0$, the value of the Pareto weight $\omega^\star_t$ is set as follows:

$$\omega^\star_t = \frac{\varphi^\star_t}{{c^\star}_{\mathrm{y},t}^{-\frac{1}{\sigma}}}, \qquad \text{[A.10.4]}$$

which is in accordance with the first equation in [A.10.1]. Next, the state-contingent value of $\varphi^\star_{t+1}$ is set to

$$\varphi^\star_{t+1} = (1+g_{t+1})^{1-\frac{1}{\sigma}} \left\{ \frac{(1+g_{t+1})v^\star_{\mathrm{m},t+1}}{\mathbb{E}_t[(1+g_{t+1})^{1-\alpha}{v^\star}^{1-\alpha}_{\mathrm{m},t+1}]^{\frac{1}{1-\alpha}}} \right\}^{\frac{1}{\sigma}-\alpha} \left(\frac{c^\star_{\mathrm{m},t+1}}{c^\star_{\mathrm{y},t}} \right)^{-\frac{1}{\sigma}} \varphi^\star_t. \qquad \text{[A.10.5]}$$

By construction, this is consistent with the first expression for $\varphi^\star_{t+1}/\varphi^\star_t$ in [3.7]. Since the consumption allocation is assumed to satisfy the risk-sharing condition [2.21] for all $t \geq t_0$, the value of $\varphi^\star_{t+1}/\varphi^\star_t$ implied by [A.10.5] also agrees with the second expression in [3.7]. Hence, given $\varphi^\star_t$, equations [A.10.4] and [A.10.5] can be used to calculate $\omega^\star_t$ and $\varphi^\star_{t+1}$ in all states of the world. Thus, proceeding recursively from the starting point of $\varphi^\star_{t_0}$ it is possible to construct sequences $\{\omega^\star_t\}$ and $\{\varphi^\star_t\}$ that satisfy [3.7] in all states of the world and from $t \geq t_0$.

The remaining Pareto weights $\omega^\star_{t_0-1}$ and $\omega^\star_{t_0-2}$ are constructed to satisfy the second and third equations in [A.10.1] at time $t = t_0$ (given the normalization in [3.6], these weights may be functions of the state of the world at time t_0). Together with [3.7], it then follows that all the first-order conditions in [A.10.1] hold for all $t \geq t_0$. The construction of the weights $\omega^\star_{t_0-1}$ and $\omega^\star_{t_0-2}$ implies equation [A.10.2]. The consumption allocation is assumed to be such that the left-hand side of [A.10.2] depends only on the state of the world at time $t_0 - 1$, thus it follows that $\omega^\star_{t_0-1}/\omega^\star_{t_0-2}$ also depends only on the state of the world at time $t_0 - 1$. The consumption allocation is therefore Pareto efficient because it is the solution of the planner's problem for a well-defined set of Pareto weights.

Now consider the equilibrium of the economy with complete financial markets open from $t \geq t_0 - 1$. The budget identities [2.14c] imply the resource constraint [2.16] holds for all $t \geq t_0$. The risk-sharing condition [2.21] holds for all $t \geq t_0$ as an equilibrium condition. Given that the risk-sharing condition [2.21] also holds at $t = t_0 - 1$, it can be rearranged to deduce that:

$$\frac{{v^*}_{\mathrm{o},t_0}^{\frac{1}{\sigma}-\alpha}{c^*}_{\mathrm{o},t_0}^{-\frac{1}{\sigma}}}{{v^*}_{\mathrm{m},t_0}^{\frac{1}{\sigma}-\alpha}{c^*}_{\mathrm{m},t_0}^{-\frac{1}{\sigma}}} = \left\{ \frac{\mathbb{E}_{t_0-1}[(1+g_{t_0})^{1-\alpha}{v^*}^{1-\alpha}_{\mathrm{o},t_0}]^{\frac{1}{1-\alpha}}}{\mathbb{E}_{t_0-1}[(1+g_{t_0})^{1-\alpha}{v^*}^{1-\alpha}_{\mathrm{m},t_0}]^{\frac{1}{1-\alpha}}} \right\}^{\frac{1}{\sigma}-\alpha} \left(\frac{c^*_{\mathrm{m},t_0-1}}{c^*_{\mathrm{y},t_0-1}} \right)^{-\frac{1}{\sigma}}, \qquad \text{[A.10.6]}$$

and hence [A.10.3] holds for some $\mathcal{X}^*_{t_0-1}$ that depends only on the state of the world at time $t_0 - 1$. The complete-markets equilibrium (from $t \geq t_0 - 1$) is therefore Pareto efficient from $t \geq t_0$ onwards.

(ii) Suppose that with one instrument of monetary policy (the nominal interest rate), the central bank is able to implement a Pareto-efficient consumption allocation. Since this allocation must be an equilibrium with incomplete markets for a particular monetary policy, it must satisfy the incomplete-markets equilibrium conditions [2.14a]–[2.14e] and [2.15], as well as satisfying the necessary conditions for Pareto efficiency. These necessary conditions imply that the risk-sharing condition [2.21] must hold for all $t \geq t_0$. Another requirement for Pareto efficiency is that equation [A.10.3] holds for some $\mathcal{X}^\star_{t_0-1}$ that is a function only of variables known as of time $t_0 - 1$. Multiplying numerator and denominator of [A.10.3] by $(1+r_{t_0})(1+g_{t_0})^{-\alpha}$ and taking expectations conditional on information available at time $t_0 - 1$ yields:

$$\mathcal{X}^\star_{t_0-1} = \frac{\mathbb{E}_{t_0-1}\left[(1+r_{t_0})(1+g_{t_0})^{-\frac{1}{\sigma}}\{(1+g_{t_0})v^\star_{\mathrm{o},t_0}\}^{\frac{1}{\sigma}-\alpha}{c^\star}_{\mathrm{o},t_0}^{-\frac{1}{\sigma}}\right]}{\mathbb{E}_{t_0-1}\left[(1+r_{t_0})(1+g_{t_0})^{-\frac{1}{\sigma}}\{(1+g_{t_0})v^\star_{\mathrm{m},t_0}\}^{\frac{1}{\sigma}-\alpha}{c^\star}_{\mathrm{m},t_0}^{-\frac{1}{\sigma}}\right]}. \qquad \text{[A.10.7]}$$

Next, note that the Euler equations in [2.14d] imply that:

$$\frac{\mathbb{E}_{t_0-1}\left[(1+r_{t_0})(1+g_{t_0})^{-\frac{1}{\sigma}}\{(1+g_{t_0})v^\star_{\mathrm{o},t_0}\}^{\frac{1}{\sigma}-\alpha}{c^\star}_{\mathrm{o},t_0}^{-\frac{1}{\sigma}}\right]}{\mathbb{E}_{t_0-1}\left[(1+r_{t_0})(1+g_{t_0})^{-\frac{1}{\sigma}}\{(1+g_{t_0})v^\star_{\mathrm{m},t_0}\}^{\frac{1}{\sigma}-\alpha}{c^\star}_{\mathrm{m},t_0}^{-\frac{1}{\sigma}}\right]} = \left\{ \frac{\mathbb{E}_{t_0-1}[(1+g_{t_0})^{1-\alpha}{v^\star}^{1-\alpha}_{\mathrm{o},t_0}]^{\frac{1}{1-\alpha}}}{\mathbb{E}_{t_0-1}[(1+g_{t_0})^{1-\alpha}{v^\star}^{1-\alpha}_{\mathrm{m},t_0}]^{\frac{1}{1-\alpha}}} \right\}^{\frac{1}{\sigma}-\alpha} \frac{{c^\star}_{\mathrm{m},t_0-1}^{-\frac{1}{\sigma}}}{{c^\star}_{\mathrm{y},t_0-1}^{-\frac{1}{\sigma}}}.$$

[A.10.8]

Putting together equations [A.10.3], [A.10.7], and [A.10.8] implies that equation [A.10.6] must hold, which is equivalent to the risk-sharing condition [2.21] at time $t = t_0 - 1$. Therefore, the consumption allocation must satisfy all the equilibrium conditions for the economy with complete markets from $t = t_0 - 1$ onwards, and hence coincide with the complete-markets equilibrium. This completes the proof.

A.11 Proof of Proposition 4

Consider the conditions under which $\Upsilon_t = \Upsilon_t^*$, where $\Upsilon_t \equiv d_t / \mathbb{E}_{t-1} d_t$ is the value of the variable defined in [2.22] with incomplete markets, while $\Upsilon_t^* \equiv d_t^* / \mathbb{E}_{t-1} d_t^*$ is the value of this variable with complete markets. Using equation [3.8], $\Upsilon_t = \Upsilon_t^*$ is equivalent to:

$$\frac{M_t^{-1}}{\mathbb{E}_{t-1} M_t^{-1}} = \frac{d_t^*}{\mathbb{E}_{t-1} d_t^*}, \quad \text{[A.11.1]}$$

where $M_t \equiv P_t Y_t$ is nominal GDP. If nominal GDP is at its target value $M_t^* = {d_t^*}^{-1} \mathcal{X}_{t-1}$ from [3.9] then this clearly satisfies equation [A.11.1] because $\mathcal{X}_{t-1}$ depends only on variables known at time $t-1$. Conversely, take any M_t satisfying equation [A.11.1]. The equation implies:

$$M_t = {d_t^*}^{-1} \left(\frac{\mathbb{E}_{t-1} d_t^*}{\mathbb{E}_{t-1} M_t^{-1}} \right),$$

and hence [3.9] holds with $\mathcal{X}_{t-1} = \mathbb{E}_{t-1} d_t^* / \mathbb{E}_{t-1} {M_t^*}^{-1}$, which depends only on variables known at time $t-1$. Thus, the nominal GDP target [3.9] is necessary and sufficient for $\Upsilon_t = \Upsilon_t^*$.

With a particular monetary policy, if the incomplete-markets equilibrium were to coincide with the complete-markets equilibrium then the debt gap from [3.2] would be closed ($\tilde{d}_t = 1$), and hence $\Upsilon_t = \Upsilon_t^*$ given the definition of Υ_t in [2.22]. Conversely, suppose that $\Upsilon_t = \Upsilon_t^*$. From equation [2.22] it follows that the state-contingent unexpected component of the incomplete-markets real return r_t is the same as the real return r_t^* on the equilibrium portfolio with complete markets (the state-contingent realization of real GDP growth g_t is the same in both cases). The equilibrium conditions for incomplete markets and complete markets share all of equations [2.14a]–[2.14e], and only differ in that [2.15] (determining r_t) is used with incomplete markets, while [2.21] (determining the complete-markets portfolio, and hence r_t^*) is used with complete markets. Conditional on the behaviour of the real return, the budget identities and Euler equations determining borrowing and saving behaviour are identical in both cases. The equilibrium under incomplete markets is then the same as the complete-markets equilibrium, so $d_t = d_t^*$, closing the debt gap.

Given that monetary policy can always generate any state-contingent path for one nominal variable (accepting the equilibrium nominal values of other variables), it is feasible for the central bank to choose the nominal GDP target [3.9].

With the Pareto weights Ω_t^* supporting the complete-markets equilibrium, the complete-markets consumption allocation is the maximum of the welfare function [3.3] subject to the resource constraint [2.16]. It has been shown that a central bank can achieve the same consumption allocation subject to the incomplete-markets equilibrium conditions [2.14a]–[2.14e] and [2.15] as implementability constraints (which imply the resource constraint [2.16]). The nominal GDP target in [3.9] is therefore the constrained maximum of the welfare function.

A.12 Proof of Proposition 5

The debt liabilities definition [4.1a] implies the ex-post real return r_t must satisfy the identity $\mathsf{r}_t = \mathsf{d}_t - \mathsf{l}_{t-1} - \mathsf{g}_t$. Since the real interest rate is $\rho_t = \mathbb{E}_t \mathsf{r}_{t+1}$, this means that:

$$\rho_t = \mathbb{E}_t \mathsf{d}_{t+1} - \mathsf{l}_t + \mathbb{E}_t \mathsf{g}_{t+1}. \quad \text{[A.12.1]}$$

Substituting this expression for ρ_t and the budget identities [4.1a] into the Euler equations [4.1b] of the young and the middle-aged yields:

$$\beta\gamma \mathsf{l}_t = -\gamma\mathbb{E}_t\mathsf{d}_{t+1} - \beta\gamma\mathbb{E}_t\mathsf{l}_{t+1} - \sigma(\mathbb{E}_t\mathsf{d}_{t+1} - \mathsf{l}_t + \mathbb{E}_t\mathsf{g}_{t+1}) + \mathbb{E}_t\mathsf{g}_{t+1};$$
$$-\gamma\mathsf{d}_t - \beta\gamma\mathsf{l}_t = \gamma\mathbb{E}_t\mathsf{d}_{t+1} - \sigma(\mathbb{E}_t\mathsf{d}_{t+1} - \mathsf{l}_t + \mathbb{E}_t\mathsf{g}_{t+1}) + \mathbb{E}_t\mathsf{g}_{t+1}.$$

After collecting terms, simplifying, and dividing both sides of the equations by the positive coefficient σ, a pair of equations in debt d_t and loans l_t is obtained:

$$\left(1+\frac{\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{d}_{t+1} = \left(1-\frac{\beta\gamma}{\sigma}\right)\mathsf{l}_t - \frac{\beta\gamma}{\sigma}\mathbb{E}_t\mathsf{l}_{t+1} + \left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\mathsf{g}_{t+1}; \quad \text{[A.12.2a]}$$

$$\left(1+\frac{\beta\gamma}{\sigma}\right)\mathsf{l}_t = \left(1-\frac{\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{d}_{t+1} - \frac{\gamma}{\sigma}\mathsf{d}_t - \left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\mathsf{g}_{t+1}. \quad \text{[A.12.2b]}$$

The class of solutions for the debt ratio d_t is found by solving the system of simultaneous equations [A.12.2]. This is done by eliminating the variable l_t. Equation [A.12.2a] is multiplied by the positive coefficient $(1+\beta\gamma/\sigma)$ to obtain:

$$\left(1+\frac{\gamma}{\sigma}\right)\left(1+\frac{\beta\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{d}_{t+1} = \left(1-\frac{\beta\gamma}{\sigma}\right)\left(1+\frac{\beta\gamma}{\sigma}\right)\mathsf{l}_t - \frac{\beta\gamma}{\sigma}\left(1+\frac{\beta\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{l}_{t+1} + \left(\frac{1-\sigma}{\sigma}\right)\left(1+\frac{\beta\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{g}_{t+1}.$$

Equation [A.12.2b] can now be substituted into the above to eliminate terms in l_t:

$$\left(1+\frac{\gamma}{\sigma}\right)\left(1+\frac{\beta\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{d}_{t+1} = \left(1-\frac{\beta\gamma}{\sigma}\right)\left\{\left(1-\frac{\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{d}_{t+1} - \frac{\gamma}{\sigma}\mathsf{d}_t - \left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\mathsf{g}_{t+1}\right\}$$
$$- \frac{\beta\gamma}{\sigma}\mathbb{E}_t\left[\left(1-\frac{\gamma}{\sigma}\right)\mathbb{E}_{t+1}\mathsf{d}_{t+2} - \frac{\gamma}{\sigma}\mathsf{d}_{t+1} - \left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_{t+1}\mathsf{g}_{t+2}\right] + \left(\frac{1-\sigma}{\sigma}\right)\left(1+\frac{\beta\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{g}_{t+1},$$

which after applying the law of iterated expectations, collecting terms, and simplifying the coefficients becomes:

$$\frac{\gamma}{\sigma}\left(1-\frac{\beta\gamma}{\sigma}\right)\mathsf{d}_t + \frac{\gamma}{\sigma}\left(2(1+\beta)-\frac{\beta\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{d}_{t+1} + \frac{\beta\gamma}{\sigma}\left(1-\frac{\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{d}_{t+2} = \frac{\beta\gamma}{\sigma}\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\left[2\mathsf{g}_{t+1}+\mathsf{g}_{t+2}\right].$$

Finally, cancelling the non-zero coefficient γ/σ from both sides yields an equation for d_t:

$$\left(1-\frac{\beta\gamma}{\sigma}\right)\mathsf{d}_t + \left(2(1+\beta)-\frac{\beta\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{d}_{t+1} + \beta\left(1-\frac{\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{d}_{t+2} = \beta\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\left[2\mathsf{g}_{t+1}+\mathsf{g}_{t+2}\right]. \quad \text{[A.12.3]}$$

Using the quadratic function $\mathscr{G}(z)$ defined in equation [A.7.9] in Lemma 2, the expectational difference equation [A.12.3] for d_t can be written in terms of $\mathscr{G}(z)$ and the forward $\mathbb{F}$ and identity operators $\mathbb{I}$:

$$\mathbb{E}_t\left[\mathscr{G}(\mathbb{F})\mathsf{d}_t\right] = \beta\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\left[(2\mathbb{I}+\mathbb{F})\mathbb{F}\mathsf{g}_t\right]. \quad \text{[A.12.4]}$$

Proposition 1 shows that uniqueness of the steady state is equivalent to the restriction $\sigma \geq \underline{\sigma}(\gamma,\beta)$, and also demonstrates that $\underline{\sigma}(\gamma,\beta) > \beta\gamma/(1+\beta)$. Therefore, as parameters consistent with a unique steady state are used, it must be the case that $\gamma/\sigma < (1+\beta)/\beta$. Lemma 2 implies that this restriction justifies the factorization of the quadratic $\mathscr{G}(z)$ given in [A.7.11] with real values of λ, ζ, and χ satisfying $|\lambda| < 1$, $|\zeta| < 1$, and $0 < \chi < 1$. The factorization[A.7.11] is used to rewrite equation [A.12.4] as follows:

$$\mathbb{E}_t\left[\frac{1}{\chi}(\mathbb{I}-\zeta\mathbb{F})(\mathbb{F}-\lambda\mathbb{I})\mathsf{d}_t\right] = \beta\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\left[(2\mathbb{I}+\mathbb{F})\mathbb{F}\mathsf{g}_t\right],$$

where the coefficients λ, ζ, and χ are the terms from equations [A.7.12a]–[A.7.12c]. These agree with the formulas given in [A.4.1a], [A.4.1e], and [A.4.1f]. Writing the above equation out explicitly and using the law of iterated expectations leads to:

$$(\mathbb{E}_t\mathsf{d}_{t+1} - \lambda\mathsf{d}_t) - \zeta\mathbb{E}_t[\mathbb{E}_{t+1}\mathsf{d}_{t+2} - \lambda\mathsf{d}_{t+1}] = \chi\beta\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t[2\mathsf{g}_{t+1}+\mathsf{g}_{t+2}]. \quad \text{[A.12.5]}$$

To characterize the general class of solutions consistent with the economy always remaining on the stable manifold, first consider the term f_t defined in [4.2]. It is assumed that GDP growth g_t is a bounded stochastic process, and given that Lemma 2 implies $|\zeta| < 1$ for parameters consistent with a unique steady state, it must also be the case that f_t is a bounded stochastic process. Taking this definition of f_t, define the stochastic process $\mathcal{F}_t$ as follows:

$$\mathcal{F}_t \equiv \mathbb{E}_t \mathsf{d}_{t+1} - \lambda \mathsf{d}_t - \chi \left(2\mathsf{f}_t + \mathbb{E}_t \mathsf{f}_{t+1}\right). \quad [A.12.6]$$

This definition refers to the debt ratio d_t, so each potential solution for d_t corresponds to a particular stochastic process $\mathcal{F}_t$. Rearranging the definition implies that expectations of the future debt ratio are given by:

$$\mathbb{E}_t \mathsf{d}_{t+1} = \lambda \mathsf{d}_t + \chi \left(2\mathsf{f}_t + \mathbb{E}_t \mathsf{f}_{t+1}\right) + \mathcal{F}_t, \quad [A.12.7]$$

and then substituting this equation into [A.12.5] yields:

$$\mathcal{F}_t - \zeta \mathbb{E}_t \mathcal{F}_{t+1} + \chi \mathbb{E}_t \left[(\mathsf{f}_{t+1} - \zeta \mathbb{E}_{t+1} \mathsf{f}_{t+2}) + 2(\mathsf{f}_t - \zeta \mathbb{E}_t \mathsf{f}_{t+1})\right] = \chi\beta \left(\frac{1-\sigma}{\sigma}\right) \mathbb{E}_t \left[2\mathsf{g}_{t+1} + \mathsf{g}_{t+2}\right]. \quad [A.12.8]$$

Observe that the definition of f_t in [4.2] implies that it satisfies the following recursive equation:

$$\mathsf{f}_t - \zeta \mathbb{E}_t \mathsf{f}_{t+1} = \beta \left(\frac{1-\sigma}{\sigma}\right) \mathbb{E}_t \mathsf{g}_{t+1}, \quad [A.12.9]$$

which can be substituted into [A.12.8] to obtain:

$$\mathcal{F}_t = \zeta \mathbb{E}_t \mathcal{F}_{t+1}. \quad [A.12.10]$$

This equation characterizes the whole class of solutions for $\mathcal{F}_t$.

First, if $\zeta = 0$, equation [A.12.10] implies $\mathcal{F}_t = 0$. Now consider the general case $\zeta \neq 0$. Make the definition $\upsilon_t \equiv \mathcal{F}_t - \mathbb{E}_{t-1}\mathcal{F}_t$ (so that υ_t is the unpredictable component of $\mathcal{F}_t$) and divide both sides of equation [A.12.10] by ζ to deduce:

$$\mathcal{F}_t = \zeta^{-1} \mathcal{F}_{t-1} + \upsilon_t. \quad [A.12.11]$$

Make one further definition $\Upsilon_t \equiv \mathsf{d}_t - \mathbb{E}_{t-1}\Upsilon_t$ (so that Υ_t is the unpredictable component of the debt ratio d_t). Substituting this definition of Υ_t into [A.12.7] and rearranging to obtain an expression for d_t:

$$\mathsf{d}_t = \lambda \mathsf{d}_{t-1} + \chi(2\mathsf{f}_{t-1} + \mathbb{E}_{t-1}\mathsf{f}_t) + \mathcal{F}_{t-1} + \Upsilon_t. \quad [A.12.12]$$

Equation [A.12.12] characterizes the general solution for the debt ratio d_t up to two degrees of freedom: any stochastic process $\mathcal{F}_t$ satisfying equation [A.12.10], and any stochastic process Υ_t satisfying $\mathbb{E}_{t-1}\Upsilon_t = 0$ (in other words, any martingale difference sequence). For $\mathcal{F}_t$, it has been seen that either $\mathcal{F}_t = 0$ in the case $\zeta = 0$, or that $\mathcal{F}_t$ satisfies [A.12.11] for $\zeta \neq 0$. In the latter case, a solution for $\mathcal{F}_t$ can be generated by any stochastic process υ_t satisfying $\mathbb{E}_{t-1}\upsilon_t = 0$ (any martingale difference sequence). However, since $|\zeta| < 1$ and υ_t must be uncorrelated with $\mathcal{F}_{t-1}$, equation [A.12.11] implies that $|\mathcal{F}_t| \to \infty$ as $t \to \infty$ for any υ_t process, with the exception of $\upsilon_t = 0$. As Υ_t must be uncorrelated with d_{t-1} and $\mathcal{F}_{t-1}$, if $|\mathcal{F}_t| \to \infty$ then $|\mathsf{d}_t| \to \infty$ as well. Therefore, the only set of solutions for d_t consistent with the economy remaining on the stable manifold are those with $\upsilon_t = 0$, and hence $\mathcal{F}_t = 0$. From equation [A.12.12], the class of solutions is then reduced to that given in equation [4.2].

Since the stochastic process f_t is bounded and as $|\lambda| < 1$, it can be seen from equation [4.2] that any bounded martingale difference sequence Υ_t is consistent with the economy remaining on the stable manifold. The model places no further restrictions on Υ_t. This confirms the general class of solutions is that given in equation [4.2]. Since any choice of Υ_t must satisfy $\mathbb{E}_{t-1}\Upsilon_t = 0$, by taking expectations of [4.2], the requirement that the economy remain on the stable saddlepath uniquely determines the expected future debt ratio:

$$\mathbb{E}_t \mathsf{d}_{t+1} = \lambda \mathsf{d}_t + \chi \left(2\mathsf{f}_t + \mathbb{E}_t \mathsf{f}_{t+1}\right). \quad [A.12.13]$$

Finally, note that the formulas for χ, λ, and ζ given in [A.4.1a], [A.4.1e], and [A.4.1f], the bounds on these coefficients, and the fact that all are increasing in the ratio γ/σ, are confirmed by the results of Lemma 2.

Starting with the equilibrium loans to GDP ratio, substitute the expression in [A.12.13] for the expected future debt ratio $\mathbb{E}_t \mathsf{d}_{t+1}$ into equation [A.12.2b]:

$$\left(1 + \frac{\beta\gamma}{\sigma}\right) \mathsf{l}_t = \left(1 - \frac{\gamma}{\sigma}\right) \left(\lambda \mathsf{d}_t + \chi \left(2\mathsf{f}_t + \mathbb{E}_t \mathsf{f}_{t+1}\right)\right) - \frac{\gamma}{\sigma} \mathsf{d}_t - \left(\frac{1-\sigma}{\sigma}\right) \mathbb{E}_t \mathsf{g}_{t+1}.$$

Use equation [A.12.9] to eliminate the term involving $\mathbb{E}_t \mathsf{g}_{t+1}$ in the above, and then collect similar terms to obtain:

$$\left(1+\frac{\beta\gamma}{\sigma}\right)\mathsf{l}_t = -\left(\frac{\gamma}{\sigma}-\left(1-\frac{\gamma}{\sigma}\right)\lambda\right)\mathsf{d}_t - \beta^{-1}\mathbb{E}_t\left[\left(1+2\beta\left(\frac{\gamma}{\sigma}-1\right)\chi\right)\mathsf{f}_t - \left(\zeta-\beta\left(\frac{\gamma}{\sigma}-1\right)\chi\right)\mathsf{f}_{t+1}\right]. \tag{A.12.14}$$

Lemma 2 gives expressions in [A.7.12a] and [A.7.12c] for χ and ζ, from which it can be seen that $\zeta = \beta(\gamma/\sigma - 1)\chi$. Substituting for ζ in [A.12.14] yields the equation:

$$\left(1+\frac{\beta\gamma}{\sigma}\right)\mathsf{l}_t = -\beta^{-1}\left(\beta\left(\frac{\gamma}{\sigma}-\left(1-\frac{\gamma}{\sigma}\right)\lambda\right)\right)\mathsf{d}_t - \beta^{-1}(1+2\zeta)\mathsf{f}_t. \tag{A.12.15}$$

By making the following definitions:

$$\phi \equiv \frac{\beta\left(\frac{\gamma}{\sigma}-\left(1-\frac{\gamma}{\sigma}\right)\lambda\right)}{1+\frac{\beta\gamma}{\sigma}}, \quad \text{and} \quad \varkappa \equiv \frac{1+2\zeta}{1+\frac{\beta\gamma}{\sigma}}, \tag{A.12.16}$$

it can be seen that [A.12.15] implies the equation for the loan ratio l_t given in [4.3a].

Explicit expressions for ϕ and $\varkappa$ are now derived to find the properties of these coefficients. Using the first expression for λ given in [A.7.12b], it follows that:

$$\frac{\beta\gamma}{\sigma} - \beta\left(1-\frac{\gamma}{\sigma}\right)\lambda = \frac{1}{2}\left((1+2\beta)+\left(1+\frac{\beta\gamma}{\sigma}\right)-\sqrt{(1+2\beta)^2+3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right)}\right). \tag{A.12.17}$$

Note that this expression is equivalent to:

$$\beta\left(\frac{\gamma}{\sigma}-\left(1-\frac{\gamma}{\sigma}\right)\lambda\right) = \frac{\frac{1}{2}\left(\left((1+2\beta)+\left(1+\frac{\beta\gamma}{\sigma}\right)\right)^2-\left((1+2\beta)^2+3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right)\right)\right)}{(1+2\beta)+\left(1+\frac{\beta\gamma}{\sigma}\right)+\sqrt{(1+2\beta)^2+3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right)}},$$

where this formula has been obtained by multiplying and dividing [A.12.18] by the term appearing the denominator. By expanding the brackets in the numerator, the formula can be simplified to:

$$\beta\left(\frac{\gamma}{\sigma}-\left(1-\frac{\gamma}{\sigma}\right)\lambda\right) = \frac{2\beta\left(1+\frac{\gamma}{\sigma}\right)\left(1+\frac{\beta\gamma}{\sigma}\right)}{(1+2\beta)+\left(1+\frac{\beta\gamma}{\sigma}\right)+\sqrt{(1+2\beta)^2+3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}.$$

Substituting this into the definition of ϕ from [A.12.16] yields the expression for ϕ in [A.4.1c].

Using the formulas for χ and ϕ from [A.4.1a] and [A.4.1c] it can be seen these are related as follows:

$$\phi^{-1} = \frac{\chi^{-1}+\frac{\beta\gamma}{\sigma}}{\beta\left(1+\frac{\gamma}{\sigma}\right)},$$

from which it follows that:

$$\phi = \frac{\beta\chi+\frac{\beta\gamma\chi}{\sigma}}{1+\frac{\beta\gamma\chi}{\sigma}}, \quad \text{and} \quad 1-\phi = \frac{1-\beta\chi}{1+\frac{\beta\gamma\chi}{\sigma}}.$$

Since $\chi > 0$, the first formula demonstrates that ϕ is positive. It has been shown that χ is increasing in the ratio γ/σ, and so the second formula shows that ϕ is also increasing in γ/σ, but always satisfies $\phi < 1$ given that $\chi < 1$.

Turning to the coefficient $\varkappa$, the first expression for ζ in [A.7.12c] can be used to deduce that

$$1+2\zeta = \frac{\sqrt{(1+2\beta)^2+3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right)}-(1+2\beta)}{1-\frac{\beta\gamma}{\sigma}}. \tag{A.12.18}$$

This expression is equivalent to

$$1+2\zeta = \frac{\left((1+2\beta)^2 + 3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right)\right) - (1+2\beta)^2}{\left(1-\frac{\beta\gamma}{\sigma}\right)\left((1+2\beta)+\sqrt{(1+2\beta)^2+3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right)}\right)},$$

which has been obtained by multiplying and dividing [A.12.18] by the second term in parentheses in the denominator above. Expanding the brackets and simplifying leads to:

$$1+2\zeta = \frac{3\left(1+\frac{\beta\gamma}{\sigma}\right)}{(1+2\beta)+\sqrt{(1+2\beta)^2+3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right)}},$$

noting that the term $(1-\beta\gamma/\sigma)$ in the denominator cancels out. Substituting into the definition of $\varkappa$ from [A.12.16] yields the expression for $\varkappa$ given in equation [A.4.1b]. From this formula it follows immediately that $\varkappa$ is strictly positive and increasing in the ratio γ/σ.

The next variable to consider is the real interest rate ρ_t. Substituting the solution [A.12.13] for $\mathbb{E}_t\mathsf{d}_{t+1}$ and the solution [4.3a] for l_t into equation [A.12.1] yields:

$$\rho_t = (\lambda\mathsf{d}_t + \chi(2\mathsf{f}_t + \mathbb{E}_t\mathsf{f}_{t+1})) + \left(\beta^{-1}\phi\mathsf{d}_t + \beta^{-1}\varkappa\mathsf{f}_t\right) + \mathbb{E}_t\mathsf{g}_{t+1}.$$

Grouping similar terms in the above equation leads to:

$$\rho_t = \frac{1}{\sigma}\mathbb{E}_t\mathsf{g}_{t+1} + \left(\lambda+\beta^{-1}\phi\right)\mathsf{d}_t + \beta^{-1}\left((2\beta\chi+\varkappa)\mathsf{f}_t + \beta\chi\mathbb{E}_t\mathsf{f}_{t+1} - \beta\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\mathsf{g}_{t+1}\right). \qquad \text{[A.12.19]}$$

Substituting the original definition of ϕ from [A.12.16] implies:

$$\lambda+\beta^{-1}\phi = \frac{\gamma}{\sigma}\vartheta, \quad \text{where } \vartheta \equiv \frac{1+(1+\beta)\lambda}{1+\frac{\beta\gamma}{\sigma}}. \qquad \text{[A.12.20]}$$

Substituting the equation above into [A.12.19] and using [A.12.9] to write $\beta(1-\sigma)/\sigma\mathbb{E}_t\mathsf{g}_{t+1}$ in terms of f_t and $\mathbb{E}_t\mathsf{f}_{t+1}$:

$$\rho_t = \frac{1}{\sigma}\mathbb{E}_t\mathsf{g}_{t+1} + \frac{\gamma}{\sigma}\vartheta\mathsf{d}_t + \beta^{-1}\left((2\beta\chi+\varkappa-1)\mathsf{f}_t + (\beta\chi+\zeta)\mathbb{E}_t\mathsf{f}_{t+1}\right).$$

Now noting that [A.4.1a] and [A.4.1f] imply that $\zeta = \beta(\gamma/\sigma-1)\chi$, the equation above can be written as:

$$\rho_t = \frac{1}{\sigma}\mathbb{E}_t\mathsf{g}_{t+1} + \frac{\gamma}{\sigma}\vartheta\mathsf{d}_t + \left(\beta^{-1}(2\beta\chi+\varkappa-1)\mathsf{f}_t + \frac{\gamma}{\sigma}\chi\mathbb{E}_t\mathsf{f}_{t+1}\right). \qquad \text{[A.12.21]}$$

Using the original definition of $\varkappa$ from [A.12.16]:

$$2\beta\chi+\varkappa-1 = \chi\left(2\beta + \frac{\left(2\zeta-\beta\frac{\gamma}{\sigma}\right)\chi^{-1}}{1+\frac{\beta\gamma}{\sigma}}\right) = \beta\frac{\gamma}{\sigma}\chi\left(\frac{2(1+\beta)-\chi^{-1}}{1+\frac{\beta\gamma}{\sigma}}\right), \qquad \text{[A.12.22]}$$

where the second equality follows by substituting $\zeta = \beta(\gamma/\sigma-1)\chi$. The expression for χ from [A.4.1a] implies that

$$2(1+\beta)-\chi^{-1} = (1+\beta) + \frac{1}{2}\frac{\beta\gamma}{\sigma} - \frac{1}{2}\sqrt{(1+2\beta)^2+3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right)},$$

and by making use of the first formula for λ in [A.7.12b] it follows that $2(1+\beta)-\chi^{-1} = \beta\gamma/\sigma - \beta(1-\gamma/\sigma)\lambda$. Substituting this result into [A.12.22] implies:

$$\beta^{-1}(2\beta\chi+\varkappa-1) = \frac{\gamma}{\sigma}\chi\left(\frac{\beta\left(\frac{\gamma}{\sigma}-\left(1-\frac{\gamma}{\sigma}\right)\lambda\right)}{1+\frac{\beta\gamma}{\sigma}}\right),$$

and comparison with the original definition of ϕ in [A.12.16] shows that this allows [A.12.21] to be written as:

$$\rho_t = \frac{1}{\sigma}\mathbb{E}_t g_{t+1} + \frac{\gamma}{\sigma}\vartheta d_t + \left(\frac{\gamma}{\sigma}\phi\chi f_t + \frac{\gamma}{\sigma}\chi\mathbb{E}_t f_{t+1}\right).$$

Factoring out common terms yields the equation for ρ_t in [4.3a].

Now consider the coefficient ϑ defined in [A.12.20]. The formula for λ in [A.4.1e] implies that the numerator of ϑ can be written as:

$$1 + (1+\beta)\lambda = \frac{(1+2\beta)\frac{\beta\gamma}{\sigma} + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}{2(1+\beta) - \frac{\beta\gamma}{\sigma} + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}. \quad \text{[A.12.23]}$$

By expanding the brackets and simplifying it follows that:

$$\left((1+2\beta)\frac{\beta\gamma}{\sigma} - \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}\right)\left((1+2\beta)\frac{\beta\gamma}{\sigma} + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}\right)$$
$$= \left((1+2\beta)\frac{\beta\gamma}{\sigma}\right)^2 - \left((1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)\right) = 4(1+\beta+\beta^2)\left(\frac{\beta\gamma}{\sigma} - 1\right)\left(1 + \frac{\beta\gamma}{\sigma}\right),$$

and:

$$\left((1+2\beta)\frac{\beta\gamma}{\sigma} - \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}\right)\left(2(1+\beta) - \frac{\beta\gamma}{\sigma} + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}\right)$$
$$= 2\left(\frac{\beta\gamma}{\sigma} - 1\right)\left(2(1+\beta+\beta^2) + (1-\beta)\frac{\beta\gamma}{\sigma} + (1+\beta)\sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}\right).$$

Hence by multiplying numerator and denominator of [A.12.23] by the first bracket appearing in the equations above, it follows that:

$$1 + (1+\beta)\lambda = \frac{2(1+\beta+\beta^2)\left(1 + \frac{\beta\gamma}{\sigma}\right)}{2(1+\beta+\beta^2) + (1-\beta)\frac{\beta\gamma}{\sigma} + (1+\beta)\sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}},$$

where the term $(\beta\gamma/\sigma - 1)$ cancels out. Substituting this expression into [A.12.20] yields the formula for ϑ in [A.4.1d]. Defining $\theta = (\gamma/\sigma)\vartheta$ and comparing [A.8.26] with [A.12.20] shows that $\theta = v_\lambda$, where v_λ is the slope of the stable saddlepath. It can be seen from [A.4.1d] that ϑ is strictly positive, and since $\gamma/\sigma > 0$, so it θ. Given the definition [A.12.20], it follows that $\theta = \lambda + \beta^{-1}\phi$. Since both λ and ϕ are increasing in the ratio γ/σ, this property is also possessed by θ.

Substituting the solution for the loans ratio l_t from [4.3a] into the budget identities [4.1a] yields the expressions for consumption in [4.3b]. Finally, substituting the Fisher equation [4.4] into the definition of debt liabilities from [4.1a] and using [4.3a] yields the expression for the real return r_t in [4.3c]. This completes the proof.

A.13 Proof of Proposition 6

The system of equations describing the economy with sequentially complete markets is [4.1a]–[4.1b] and [4.5]. The risk-sharing equation [4.5] implies the following restrictions on the innovations to consumption and the value functions at time t:

$$\frac{1}{\sigma}\left((c^*_{o,t} - \mathbb{E}_{t-1}c^*_{o,t}) - (c^*_{m,t} - \mathbb{E}_{t-1}c^*_{m,t})\right) + \left(\alpha - \frac{1}{\sigma}\right)\left((v^*_{o,t} - \mathbb{E}_{t-1}v^*_{o,t}) - (v^*_{m,t} - \mathbb{E}_{t-1}v^*_{m,t})\right) = 0. \quad \text{[A.13.1]}$$

Using [4.5], the innovations to the value functions are:

$$\mathsf{v}^*_{\mathrm{m},t} - \mathbb{E}_{t-1}\mathsf{v}^*_{\mathrm{m},t} = \frac{1}{1+\beta}\left(\mathsf{c}^*_{\mathrm{m},t} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{m},t}\right) + \frac{\beta}{1+\beta}\left(\mathbb{E}_t[\mathsf{c}^*_{\mathrm{o},t+1} + \mathsf{g}_{t+1}] - \mathbb{E}_{t-1}[\mathsf{c}^*_{\mathrm{o},t+1} + \mathsf{g}_{t+1}]\right), \tag{A.13.2a}$$

$$\mathsf{v}^*_{\mathrm{o},t} - \mathbb{E}_{t-1}\mathsf{v}^*_{\mathrm{o},t} = \mathsf{c}^*_{\mathrm{o},t} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{o},t}. \tag{A.13.2b}$$

Subtracting equation [A.13.2a] from [A.13.2b]:

$$(\mathsf{v}^*_{\mathrm{o},t} - \mathbb{E}_{t-1}\mathsf{v}^*_{\mathrm{o},t}) - (\mathsf{v}^*_{\mathrm{m},t} - \mathbb{E}_{t-1}\mathsf{v}^*_{\mathrm{m},t}) = (\mathsf{c}^*_{\mathrm{o},t} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{o},t}) - (\mathsf{c}^*_{\mathrm{m},t} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{m},t})$$
$$- \frac{\beta}{1+\beta}\left((\mathbb{E}_t\mathsf{c}^*_{\mathrm{o},t+1} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{o},t+1}) - (\mathsf{c}^*_{\mathrm{m},t} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{m},t}) + (\mathbb{E}_t\mathsf{g}_{t+1} - \mathbb{E}_{t-1}\mathsf{g}_{t+1})\right),$$

then substituting this into [A.13.1] and dividing both sides by α:

$$\frac{\beta}{1+\beta}\left(1 - \frac{1}{\alpha\sigma}\right)\left((\mathbb{E}_t\mathsf{c}^*_{\mathrm{o},t+1} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{o},t+1}) - (\mathsf{c}^*_{\mathrm{m},t} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{m},t}) + (\mathbb{E}_t\mathsf{g}_{t+1} - \mathbb{E}_{t-1}\mathsf{g}_{t+1})\right)$$
$$= (\mathsf{c}^*_{\mathrm{o},t} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{o},t}) - (\mathsf{c}^*_{\mathrm{m},t} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{m},t}). \tag{A.13.3}$$

Since equation [4.3b] holds for $\mathsf{c}^*_{\mathrm{m},t}$ and $\mathsf{c}^*_{\mathrm{o},t}$ with a debt ratio d^*_t, the innovations to these consumption levels are given by:

$$\mathsf{c}^*_{\mathrm{m},t} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{m},t} = -\gamma\left((1-\phi)\Upsilon^*_t - \varkappa(\mathsf{f}_t - \mathbb{E}_{t-1}\mathsf{f}_t)\right), \quad \text{and} \quad \mathsf{c}^*_{\mathrm{o},t} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{o},t} = \gamma\Upsilon^*_t, \tag{A.13.4}$$

where $\Upsilon^*_t = \mathsf{d}^*_t - \mathbb{E}_{t-1}\mathsf{d}^*_t$. It also follows from [4.3b] that

$$\mathbb{E}_t\mathsf{c}^*_{\mathrm{o},t+1} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{o},t+1} = \gamma(\mathbb{E}_t\mathsf{d}^*_{t+1} - \mathbb{E}_{t-1}\mathsf{d}^*_{t+1}),$$

and since d^*_t must satisfy [4.2] for $\Upsilon^*_t = \mathsf{d}^*_t - \mathbb{E}_{t-1}\mathsf{d}^*_t$:

$$\mathbb{E}_t\mathsf{c}^*_{\mathrm{o},t+1} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{o},t+1} = \gamma\left(\lambda\Upsilon^*_t + 2\chi(\mathsf{f}_t - \mathbb{E}_{t-1}\mathsf{f}_t) + \chi(\mathbb{E}_t\mathsf{f}_{t+1} - \mathbb{E}_{t-1}\mathsf{f}_{t+1})\right). \tag{A.13.5}$$

Now substituting [A.13.4] and [A.13.5] into [A.13.3] and dividing both sides by γ yields:

$$\frac{\beta}{1+\beta}\left(\frac{\alpha\sigma-1}{\alpha\sigma}\right)\left((1-\phi+\lambda)\Upsilon^*_t + (2\chi-\varkappa)(\mathsf{f}_t - \mathbb{E}_{t-1}\mathsf{f}_t) + \chi(\mathbb{E}_t\mathsf{f}_{t+1} - \mathbb{E}_{t-1}\mathsf{f}_{t+1}) + \frac{1}{\gamma}(\mathbb{E}_t\mathsf{g}_{t+1} - \mathbb{E}_{t-1}\mathsf{g}_{t+1})\right)$$
$$= (2-\phi)\Upsilon^*_t - \varkappa(\mathsf{f}_t - \mathbb{E}_{t-1}\mathsf{f}_t).$$

By rearranging this equation, collecting terms in Υ^*_t on one side:

$$\left(2 - \phi - \frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}(1-\phi+\lambda)\right)\Upsilon^*_t = \left(\varkappa + \frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}(2\chi-\varkappa)\right)(\mathsf{f}_t - \mathbb{E}_{t-1}\mathsf{f}_t)$$
$$+ \frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}\chi(\mathbb{E}_t\mathsf{f}_{t+1} - \mathbb{E}_{t-1}\mathsf{f}_{t+1}) + \frac{1}{\gamma}\frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}(\mathbb{E}_t\mathsf{g}_{t+1} - \mathbb{E}_{t-1}\mathsf{g}_{t+1}). \tag{A.13.6}$$

The term $\varkappa$ appearing in the coefficient of $\mathsf{f}_t - \mathbb{E}_{t-1}\mathsf{f}_t$ in [A.13.6] can be eliminated as follows. The proof of Proposition 5 defines ϕ in equation [A.12.16], which implies that:

$$2 - \phi = \frac{2 + \frac{\beta\gamma}{\sigma} + \beta\left(1 - \frac{\gamma}{\sigma}\right)\lambda}{1 + \frac{\beta\gamma}{\sigma}},$$

and hence:

$$(2-\phi)\chi = \frac{2\beta\left(\frac{\gamma}{\sigma} - 1\right)\chi + \left(2(1+\beta) - \frac{\beta\gamma}{\sigma} + \beta\left(1 - \frac{\gamma}{\sigma}\right)\lambda\right)\chi}{1 + \frac{\beta\gamma}{\sigma}}. \tag{A.13.7}$$

By using the formulas for χ and λ given in [A.7.12a] and [A.7.12b] of Lemma 2 it follows that:

$$2(1+\beta) - \frac{\beta\gamma}{\sigma} + \beta\left(1 - \frac{\gamma}{\sigma}\right)\lambda = \frac{(1+2\beta) + \left(1 - \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}{2} = \chi^{-1},$$

and comparison of the expressions for χ and ζ in [A.7.12a] and [A.7.12c] demonstrates that $\zeta = \beta(\gamma/\sigma - 1)\chi$.

Substituting this and the equation above into [A.13.7] leads to:

$$(2-\phi)\chi = \frac{1+2\zeta}{1+\frac{\beta\gamma}{\sigma}}.$$

The coefficient $\varkappa$ is defined in [A.12.16] in the proof of Proposition 5. Comparison with the equation above shows that $\varkappa = (2-\phi)\chi$, which can be substituted into [A.13.6] to obtain:

$$\left(2-\phi-\frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}(1-\phi+\lambda)\right)\Upsilon_t^* = \chi\left(2-\phi+\frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}\phi\right)(f_t-\mathbb{E}_{t-1}f_t)$$
$$+\chi\frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}(\mathbb{E}_t f_{t+1}-\mathbb{E}_{t-1}f_{t+1})+\frac{1}{\gamma}\frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}(\mathbb{E}_t g_{t+1}-\mathbb{E}_{t-1}g_{t+1}). \quad \text{[A.13.8]}$$

Before using [A.13.8] to solve for Υ_t^*, it must be verified that the coefficient of Υ_t^* is non-zero. To do this, use the definition of ϕ from [A.12.16] to note that:

$$2\phi-1 = \frac{\left(\frac{\beta\gamma}{\sigma}-1\right)-2\beta\left(1-\frac{\gamma}{\sigma}\right)\lambda}{1+\frac{\beta\gamma}{\sigma}}.$$

Comparison of [A.7.12a] and [A.7.12b] shows that $\lambda = (\beta\gamma/\sigma-1)\chi$, and hence:

$$2\phi-1 = \left(\frac{\beta\gamma}{\sigma}-1\right)\left(\frac{1+2\beta\left(\frac{\gamma}{\sigma}-1\right)\chi}{1+\frac{\beta\gamma}{\sigma}}\right) = \left(\frac{\beta\gamma}{\sigma}-1\right)\left(\frac{1+2\zeta}{1+\frac{\beta\gamma}{\sigma}}\right),$$

where the second equality follows from a comparison of the expressions for χ and ζ in [A.7.12a] and [A.7.12c]. The definition of $\varkappa$ in [A.12.16] then establishes that $2\phi-1 = (\beta\gamma/\sigma-1)\varkappa$. Together with $\lambda = (\beta\gamma/\sigma-1)\chi$, this means that:

$$1-\phi+\lambda = \frac{1}{2}\left(1+(2\chi-\varkappa)\left(\frac{\beta\gamma}{\sigma}-1\right)\right) = \frac{1}{2}\left(1+\chi\phi\left(\frac{\beta\gamma}{\sigma}-1\right)\right),$$

with the latter using $(2-\phi)\chi = \varkappa$. Since $0<\chi<1$ and $0<\phi<1$ according to Proposition 5, this proves that $1-\phi+\lambda>0$. Given that $\alpha>0$, $\sigma>0$, and $0<\beta<1$:

$$-\infty < \frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma} < 1,$$

so when combined with $0<\phi<1$, $1-\phi+\lambda>0$ and $\lambda<1$, it follows that for all parameters:

$$2-\phi-\frac{\beta}{1+\beta}\frac{(\alpha\sigma-1)}{\alpha\sigma}(1-\phi+\lambda) > 0.$$

Dividing both sides of [A.13.8] by the coefficient of Υ_t^* then leads to [4.6].

Since $\mathbb{E}_t\Upsilon_{t+1}=0$ and $\mathbb{E}_t\Upsilon_{t+1}^*=0$, taking conditional expectations of equation [4.2] with d_t and with d_t^* implies:

$$\mathbb{E}_t d_{t+1} = \lambda d_t + \chi(2f_t+\mathbb{E}_t f_{t+1}), \quad \text{and} \quad \mathbb{E}_t d_{t+1}^* = \lambda d_t^* + \chi(2f_t+\mathbb{E}_t f_{t+1}).$$

Subtracting these two equations and using the definition of $\tilde{d}_t \equiv d_t - d_t^*$ leads to equation [4.7a]. Similarly subtracting the equations in [4.3a] and [4.3b] in the complete-markets case from their incomplete-markets equivalents implies those in [4.7b]. Finally, the expression for inflation in [4.7c] is obtained by using the Fisher equation [4.4] in conjunction with equation [4.3c] for the real return in both the incomplete-markets and complete-markets cases. This completes the proof.

A.14 Proof of Proposition 7

Now suppose the stochastic process for real GDP is that given in [4.10]. If monetary policy achieves the target $P_t + \varpi^* Y_t = 0$ then:

$$\pi_t + g_t = (1-\varpi^*)g_t.$$

Since $M_t - M_{t-1} = \pi_t + g_t$ and $g_t - \mathbb{E}_{t-1}g_t = Y_t - \mathbb{E}_{t-1}Y_t$, it follows that:

$$M_t - \mathbb{E}_{t-1}M_t = (1-\varpi^*)(Y_t - \mathbb{E}_{t-1}Y_t).$$

Given [4.8], this means $\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t = (\varpi^* - 1)(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t)$, and therefore $\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = 0$ using [4.11]. That implies $\tilde{\mathsf{d}}_t = 0$ given [4.7a]. This completes the proof.

A.15 Proof of Proposition 8

Suppose the nominal interest rate is set so that [4.12] holds for some $\psi \geq 0$. Any equilibrium must feature an inflation rate satisfying equation [4.7c]. Taking time-t conditional expectations of this equation at time $t+1$ implies:

$$\mathbb{E}_t\pi_{t+1} = \mathsf{i}_t - (\lambda + \beta^{-1}\phi)\tilde{\mathsf{d}}_t - \rho_t^*, \qquad \text{[A.15.1]}$$

where $\rho_t^* = \mathbb{E}_t\mathsf{r}_t^*$ according to the definition in [4.1a], and where equation [4.7a] has been used to replace the term in $\mathbb{E}_t\tilde{\mathsf{d}}_{t+1}$. The proof of Proposition 5 demonstrates that the coefficient of $\tilde{\mathsf{d}}_t$ in the equation above is $\theta = \lambda + \beta^{-1}\phi$, which is a strictly positive number. Substituting the interest-rate rule [4.12] into [A.15.1] yields:

$$\mathbb{E}_t\pi_{t+1} = \psi(\mathsf{M}_t - \mathsf{M}_t^*) - \mathbb{E}_t\mathsf{g}_{t+1} - \mathbb{E}_t\mathsf{d}_{t+1}^* + \mathsf{d}_t^* - \theta\tilde{\mathsf{d}}_t,$$

where the target for nominal GDP is $\mathsf{M}_t^* = -\mathsf{d}_t^*$. Moving the terms in GDP growth and the natural debt-to-GDP ratio to the left-hand side and using the definition of target nominal GDP to replace terms in d_t^* with $-\mathsf{M}_t^*$:

$$\mathbb{E}_t[\pi_{t+1} + \mathsf{g}_{t+1}] - (\mathsf{M}_{t+1}^* - \mathsf{M}_t^*) = \psi(\mathsf{M}_t - \mathsf{M}_t^*) - \theta\tilde{\mathsf{d}}_t.$$

The definition of nominal GDP $\mathsf{M}_t = \mathsf{P}_t + \mathsf{Y}_t$ implies $\pi_{t+1} + \mathsf{g}_{t+1} = \mathsf{M}_{t+1} - \mathsf{M}_t$. Now define $\tilde{\mathsf{M}}_t \equiv \mathsf{M}_t - \mathsf{M}_t^*$ to be the gap between actual nominal GDP and the central bank's target for nominal GDP. Substituting these definitions into the equation above yields:

$$\mathbb{E}_t\tilde{\mathsf{M}}_{t+1} = (1+\psi)\tilde{\mathsf{M}}_t - \theta\tilde{\mathsf{d}}_t. \qquad \text{[A.15.2]}$$

Since this equation must hold for all t, by applying the law of iterated expectations it follows that $\mathbb{E}_t\tilde{\mathsf{M}}_{t+2} = (1+\psi)\mathbb{E}_t\tilde{\mathsf{M}}_{t+1} - \theta\mathbb{E}_t\tilde{\mathsf{d}}_{t+1}$. Substituting equation [A.15.2] and iterating for ℓ periods leads to:

$$\mathbb{E}_t\tilde{\mathsf{M}}_{t+\ell} = (1+\psi)^\ell\tilde{\mathsf{M}}_t - \theta\left((1+\psi)^{\ell-1}\tilde{\mathsf{d}}_t + (1+\psi)^{\ell-2}\mathbb{E}_t\tilde{\mathsf{d}}_{t+1} + \cdots + (1+\psi)\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell-2} + \mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell-1}\right).$$

Using [4.7a] to note that $\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell} = \lambda^\ell\tilde{\mathsf{d}}_t$, this equation can be written as:

$$\mathbb{E}_t\tilde{\mathsf{M}}_{t+\ell} = (1+\psi)^\ell\tilde{\mathsf{M}}_t - \theta(1+\psi)^{\ell-1}\left(1 + \frac{\lambda}{1+\psi} + \cdots + \left(\frac{\lambda}{1+\psi}\right)^{\ell-2} + \left(\frac{\lambda}{1+\psi}\right)^{\ell-1}\right)\tilde{\mathsf{d}}_t,$$

and by summing the geometric series and simplifying this becomes:

$$\mathbb{E}_t\tilde{\mathsf{M}}_{t+\ell} = (1+\psi)^\ell\tilde{\mathsf{M}}_t - \left(\frac{\theta}{1-\lambda+\psi}\right)\left((1+\psi)^\ell - \lambda^\ell\right)\tilde{\mathsf{d}}_t,$$

where this formula is well defined since $-1 < \lambda < 1$ and $\psi \geq 0$. After collecting terms, the expression for $\mathbb{E}_t\tilde{\mathsf{M}}_{t+\ell}$ can be written as follows:

$$\mathbb{E}_t\tilde{\mathsf{M}}_{t+\ell} = \lambda^\ell\left(\frac{\theta}{1-\lambda+\psi}\tilde{\mathsf{d}}_t\right) + (1+\psi)^\ell\left(\tilde{\mathsf{M}}_t - \frac{\theta}{1-\lambda+\psi}\tilde{\mathsf{d}}_t\right). \qquad \text{[A.15.3]}$$

Next, note that the choice of nominal GDP target $\mathsf{M}_t^* = -\mathsf{d}_t^*$ implies that $\mathsf{d}_t^* - \mathbb{E}_{t-1}\mathsf{d}_t^* = -(\mathsf{M}_t^* - \mathbb{E}_{t-1}\mathsf{M}_t^*)$, which can be combined with equation [4.8] and the definitions of $\tilde{\mathsf{M}}_t$ and $\tilde{\mathsf{d}}_t$ to deduce:

$$\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = -(\tilde{\mathsf{M}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{M}}_t). \qquad \text{[A.15.4]}$$

First consider the case where $\psi > 0$. Since $|\lambda| < 1$, for any $\tilde{\mathsf{d}}_t$:

$$\lim_{\ell\to\infty}\lambda^\ell\left(\frac{\theta}{1-\lambda+\psi}\tilde{\mathsf{d}}_t\right) = 0.$$

Therefore, using equation [A.15.3], $|\mathbb{E}_t\tilde{\mathsf{M}}_{t+\ell}| \to \infty$ as $\ell \to \infty$ unless the following condition holds:

$$\tilde{\mathsf{M}}_t - \frac{\theta}{1-\lambda+\psi}\tilde{\mathsf{d}}_t = 0. \qquad \text{[A.15.5]}$$

From this equation it follows that:

$$\tilde{\mathsf{M}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{M}} = \frac{\theta}{1-\lambda+\psi}(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t), \quad \text{and hence} \quad \left(1 + \frac{\theta}{1-\lambda+\psi}\right)(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t) = 0,$$

where the second equation is derived by combining the first with [A.15.4]. Since the coefficient of $\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t$ in the second equation is positive, it follows that $\tilde{\mathsf{d}}_t = \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t$. Together with equation [4.7a], the only solution must be $\tilde{\mathsf{d}}_t = 0$ for all t. Equation [A.15.5] then implies $\tilde{\mathsf{M}}_t = 0$ for all t is the unique solution.

Now consider the case of $\psi = 0$. Conjecture that there is a solution with $\tilde{\mathsf{M}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{M}}_t = \upsilon_t$, for some martingale difference sequence υ_t ($\mathbb{E}_{t-1}\upsilon_t = 0$). Since υ_t is the forecast error for $\tilde{\mathsf{M}}_t$, it follows that $\mathbb{E}_t\tilde{\mathsf{M}}_{t+1} = \tilde{\mathsf{M}}_{t+1} - \upsilon_{t+1}$. Substituting the definition of υ_t into [A.15.4] also shows that $\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = -\upsilon_t$. Finally, substituting these results into equations [4.7a] and [A.15.5] shows that a solution is:

$$\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1} - \upsilon_t, \quad \text{and} \quad \tilde{\mathsf{M}}_t = \tilde{\mathsf{M}}_{t-1} - \theta\tilde{\mathsf{d}}_{t-1} + \upsilon_t. \tag{A.15.6}$$

There exist non-zero martingale difference sequences $\{\upsilon_t\}$ where $\tilde{\mathsf{d}}_t \neq 0$ such that $\tilde{\mathsf{M}}_t$ remains bounded. Therefore, there are multiple equilibria. This completes the proof.

A.16 Proof of Proposition 9

The target criterion is $\tilde{\mathsf{d}}_t = 0$, which it is feasible to achieve with one instrument of monetary policy. The required state-contingent inflation rate is given in [4.7c]:

$$\pi_t = \mathsf{i}_{t-1} - \mathsf{r}^*_t. \tag{A.16.1}$$

Taking a conditional expectation of this equation implies $\mathsf{i}_t = \rho^*_t + \mathbb{E}_t\pi_{t+1}$, where $\rho^*_t = \mathbb{E}_t\mathsf{r}^*_{t+1}$. Let $\mathsf{e}_t \equiv \mathbb{E}_t\pi_{t+1}$ denote current expectations of inflation one period ahead. This leads to the nominal interest rate equation $\mathsf{i}_t = \rho^*_t + \mathsf{e}_t$ from [4.13]. Substituting this into [A.16.1] yields the inflation equation in [4.13]. Thus, all the equations of the model are satisfied with no restrictions placed on the stochastic process for e_t other that it is known at time t. This completes the proof.

A.17 Proof of Proposition 10

With inflation $\pi_t = 0$, the definition of nominal GDP implies $\mathsf{M}_t - \mathsf{M}_{t-1} = \mathsf{g}_t$. The unexpected change in nominal GDP is therefore:

$$\mathsf{M}_t - \mathbb{E}_{t-1}\mathsf{M}_t = \mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t,$$

and hence using [4.8] it follows that $\Upsilon_t = \mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t = -(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t)$. Since $\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1} + (\Upsilon_t - \Upsilon^*_t)$ and $\Upsilon^*_t = \mathsf{d}^*_t - \mathbb{E}_{t-1}\mathsf{d}^*_t$, the law of motion for $\tilde{\mathsf{d}}_t$ in [4.14] is obtained. The definition of the real interest rate ρ_t in [2.14b] and the Fisher equation [4.4] imply that $\mathsf{i}_t = \rho^*_t + \tilde{\rho}_t + \mathbb{E}_t\pi_{t+1}$. Since $\mathbb{E}_t\pi_{t+1} = 0$ and [4.7b] implies $\tilde{\rho}_t = \theta\tilde{\mathsf{d}}_t$, the expression for the nominal interest rate is obtained.

Now suppose [4.10] is the stochastic process for real GDP, but the policymaker pursues target $\mathsf{P}_t + \varpi\mathsf{Y}_t = 0$ with $\varpi \neq \varpi^*$. This implies that $\pi_t = -\varpi\mathsf{g}_t$ and hence:

$$\mathsf{M}_t - \mathbb{E}_{t-1}\mathsf{M}_t = (1-\varpi)(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t).$$

Using [4.8] and [4.11]:

$$\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t = (\varpi^* - 1)(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t) - (\varpi^* - \varpi)(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t) = \Upsilon^*_t - (\varpi^* - \varpi)(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t).$$

Hence, $\Upsilon_t - \Upsilon^*_t = -(\varpi^*_t - \varpi)(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t)$, and when combined with $\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1} + (\Upsilon_t - \Upsilon^*_t)$ this leads to [4.15]. Substituting equation [4.15] into the expressions for $\tilde{\mathsf{l}}_t$ and $\tilde{\rho}_t$ from [4.7b] implies the other equations in [4.15], completing the proof.

A.18 Proof of Proposition 11

With monetary policy $\mathsf{M}_t = \mathsf{M}_{t-1} + \epsilon_t$, the innovation to nominal GDP is $\mathsf{M}_t - \mathbb{E}_{t-1}\mathsf{M}_t = \epsilon_t$, and hence using [4.8]:

$$\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t = -\epsilon_t.$$

Since $\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1} + (\Upsilon_t - \Upsilon^*_t)$, this leads to the expression for $\tilde{\mathsf{d}}_t$ in [4.16]. The definition of nominal GDP directly implies the expression for inflation in [4.16]. With $\mathbb{E}_t\pi_{t+1} = -\mathbb{E}_t\mathsf{g}_{t+1}$, and $\mathsf{i}_t = \rho^* + \tilde{\rho}_t + \mathbb{E}_t\pi_{t+1}$, the expression for i_t is obtained using [4.7b]. The effects on the real return, real interest rate, and loans-to-GDP ratio follow from Proposition 6. This completes the proof.

A.19 Proof of Proposition 12

The ex-post real return $r^\dagger_t$ is such that $1 + r^\dagger_t = (1 + i^\dagger_t)/(1 + \pi_t)$. Using equation [4.17]:

$$1 + r^\dagger_t = \frac{1 + i_{t-1}}{(1 + \pi_t)^\mu (1 + \mathbb{E}_{t-1}\pi_t)^{1-\mu}}.$$

Log-linearizing this equation leads to $\mathsf{r}^\dagger_t = \mathsf{i}_{t-1} - \mu\pi_t - (1 - \mu)\mathbb{E}_{t-1}\pi_t$. Substituting into [4.3c] implies:

$$\mu\pi_t + (1 - \mu)\mathbb{E}_{t-1}\pi_t = \mathsf{i}_{t-1} - \mathsf{d}_t - \beta^{-1}\phi\mathsf{d}_{t-1} - \beta^{-1}\varkappa\mathsf{f}_{t-1} - \mathsf{g}_t. \quad \text{[A.19.1]}$$

Combining this with the equivalent of equation [4.3c] in the case of complete markets implies equation [4.18], which replaces [4.7c]. Equating the unexpected components of both sides of [A.19.1] and using $\pi_t - \mathbb{E}_{t-1}\pi_t = \mathsf{P}_t - \mathbb{E}_{t-1}\mathsf{P}_t$ and $\mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t = \mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t$ implies:

$$\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t = -\mu(\mathsf{P}_t - \mathbb{E}_{t-1}\mathsf{P}_t) - (\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t).$$

Dividing both sides by μ then leads to [4.19], noting the definition of $\varpi^\dagger = 1/\mu$. This completes the proof.

A.20 Proof of Proposition 13

The social welfare function is [3.3] using Pareto weights $\hat{\Omega}^*_t$ for the sequentially complete markets equilibrium with output equal to its flexible-price level $\hat{Y}_t$ at all times. The scaled Pareto weights $\hat{\omega}^*_t$ and corresponding scaled Lagrangian multipliers $\hat{\varphi}^*_t$ are constructed as in [3.6]. All these variables are independent of monetary policy because $\hat{Y}_t$ is independent of monetary policy, and conditional on real GDP, the complete-markets equilibrium is independent of policy.

Using the definitions in [3.6], the first-order conditions [3.5] of the social planner's problem can be written in terms of $\hat{\omega}^*_t$ and $\hat{\varphi}^*_t$ as follows:

$$\hat{\omega}^*_t = \hat{\varphi}^*_t \hat{c}^{*\frac{1}{\sigma}}_{\mathrm{y},t}, \quad \hat{\omega}^*_t = \frac{(\beta/\delta)\hat{\varphi}^*_{t+1}\hat{c}^{*\frac{1}{\sigma}}_{\mathrm{m},t+1}}{(1+\hat{g}_{t+1})^{1-\frac{1}{\sigma}}}\left\{\frac{(1+\hat{g}_{t+1})\hat{v}^*_{\mathrm{m},t+1}}{\mathbb{E}_t[(1+\hat{g}_{t+1})^{1-\alpha}\hat{v}^{*1-\alpha}_{\mathrm{m},t+1}]^{\frac{1}{1-\alpha}}}\right\}^{\alpha-\frac{1}{\sigma}}, \quad \text{and}$$

$$\hat{\omega}^*_t = \frac{(\beta/\delta)^2\hat{\varphi}^*_{t+2}\hat{c}^{*\frac{1}{\sigma}}_{\mathrm{o},t+2}}{((1+\hat{g}_{t+1})(1+\hat{g}_{t+2}))^{1-\frac{1}{\sigma}}}\left\{\frac{(1+\hat{g}_{t+2})\hat{v}^*_{\mathrm{o},t+2}}{\mathbb{E}_{t+1}[(1+\hat{g}_{t+2})^{1-\alpha}\hat{v}^{*1-\alpha}_{\mathrm{o},t+2}]^{\frac{1}{1-\alpha}}}\frac{(1+\hat{g}_{t+1})\hat{v}^*_{\mathrm{m},t+1}}{\mathbb{E}_t[(1+\hat{g}_{t+1})^{1-\alpha}\hat{v}^{*1-\alpha}_{\mathrm{m},t+1}]^{\frac{1}{1-\alpha}}}\right\}^{\alpha-\frac{1}{\sigma}}. \quad \text{[A.20.1]}$$

Using [3.7], it can be seen that $\hat{\varphi}^*_t$ has steady-state value $\bar{\varphi} = 1$ and $\hat{\omega}^*_t$ has steady-state value $\bar{\omega} = 1$.

Now let $u_t \equiv \mathscr{U}_t/Y_t^{1-\frac{1}{\sigma}}$ denote the scaled utility level defined in [3.6]. The utility function [2.1] implies that u_t can be written in terms of $v_{\mathrm{y},t} \equiv V_{\mathrm{y},t}/Y_t$:

$$u_t = \frac{v_{\mathrm{y},t}^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \quad \text{where} \quad v_{\mathrm{y},t} = \left(c_{\mathrm{y},t}^{1-\frac{1}{\sigma}} + \delta\left\{\mathbb{E}_t\left[(1+g_{t+1})^{1-\alpha}v^{1-\alpha}_{\mathrm{m},t+1}\right]^{\frac{1}{1-\alpha}}\right\}^{1-\frac{1}{\sigma}}\right)^{\frac{1}{1-\frac{1}{\sigma}}}, \quad \text{[A.20.2]}$$

and where $v_{\mathrm{m},t}$ and $v_{\mathrm{o},t}$ are determined by the equations in [2.14e]. Given that the steady state is such that $\bar{c}_\mathrm{y} = \bar{c}_\mathrm{m} = \bar{c}_\mathrm{o} = 1$, equations [2.14e] and [A.20.2] imply that $\bar{v}_\mathrm{y}^{1-1/\sigma} = 1 + \beta + \beta^2$, $\bar{v}_\mathrm{m}^{1-1/\sigma} = 1 + \beta$, and $\bar{v}_\mathrm{o}^{1-1/\sigma} = 1$, and hence the scaled utility value has a well-defined steady state $\bar{u} = (1 + \beta + \beta^2)/(1 - 1/\sigma)$.

Flexible-price output is $\hat{Y}_t = A_t$, and hence the relative-price distortions term Ψ_t from [5.8] is such that $\hat{Y}_t/Y_t = \Psi_t$ since $\hat{\Psi}_t = 1$. Using the definitions of $\hat{\omega}^*_t$ and u_t from [3.6], the social welfare function from [3.3] can be written as:

$$\mathscr{W}_{t_0} = \frac{1}{3}\sum_{t=t_0-2}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0-2}\left[\hat{\omega}^*_t\Psi_t^{-(1-\frac{1}{\sigma})}u_t\right]. \quad \text{[A.20.3]}$$

Taking a second-order accurate approximation of the relative-price distortions term $\mathit{\Psi}_t$ from [5.9b] yields:

$$\Psi_t = \frac{\varepsilon\kappa}{2}\check{\mathsf{p}}_t^2 + \mathscr{O}_3, \qquad \text{[A.20.4]}$$

where $\mathscr{O}_3$ denotes third- and higher-order terms in deviations from the steady state. Since [5.10] implies that $\check{\mathsf{p}}_t = -(\pi_t - \mathbb{E}_{t-1}\pi_t) + \mathscr{O}_2$, substituting this into [A.20.4] yields:

$$\Psi_t = \frac{\varepsilon\kappa}{2}(\pi_t - \mathbb{E}_{t-1}\pi_t)^2 + \mathscr{O}_3. \qquad \text{[A.20.5a]}$$

The absence of first-order terms from this equation demonstrates that:

$$\Psi_t = \mathscr{O}_2. \qquad \text{[A.20.5b]}$$

Taking a second-order accurate approximation of the time-t terms in [A.20.3]:

$$\hat{\omega}_t^* \mathit{\Psi}_t^{-\left(1-\frac{1}{\sigma}\right)} u_t = \bar{u} + (u_t - \bar{u}) + \bar{u}\left(\hat{\omega}_t^* + \frac{1}{2}\hat{\omega}_t^{*2} - \left(1-\frac{1}{\sigma}\right)\left(\Psi_t - \frac{1}{2}\left(1-\frac{1}{\sigma}\right)\Psi_t^2 + \hat{\omega}_t^*\Psi_t\right)\right)$$
$$+\left(\hat{\omega}_t^* - \left(1-\frac{1}{\sigma}\right)\Psi_t\right)(u_t - \bar{u}) + \mathscr{O}_3.$$

Since the weights $\hat{\omega}_t^*$ are independent of monetary policy, and since [A.20.5b] shows that Ψ_t is a second-order term, it follows that these terms can be reduced to:

$$\hat{\omega}_t^* \mathit{\Psi}_t^{-\left(1-\frac{1}{\sigma}\right)} u_t = (u_t - \bar{u}) - (1+\beta+\beta^2)\Psi_t + \hat{\omega}_t^*(u_t - \bar{u}) + \mathscr{I} + \mathscr{O}_3, \qquad \text{[A.20.6]}$$

using the formula for $\bar{u}$ and noting that this constant is independent of policy (such terms being denoted by $\mathscr{I}$).

To find an approximation for the utility deviation $u_t - \bar{u}$, write the equations [2.14e] and [A.20.2] for the continuation values $v_{\text{y},t}$ and $v_{\text{m},t}$ as:

$$v_{\text{y},t}^{1-\frac{1}{\sigma}} = c_{\text{y},t}^{1-\frac{1}{\sigma}} + \delta z_{\text{m},t}^{1-\frac{1}{\sigma}}, \quad \text{where } z_{\text{m},t}^{1-\alpha} = \mathbb{E}_t\left[(1+g_{t+1})^{1-\alpha} v_{\text{m},t+1}^{1-\alpha}\right], \quad \text{and} \qquad \text{[A.20.7a]}$$

$$v_{\text{m},t}^{1-\frac{1}{\sigma}} = c_{\text{m},t}^{1-\frac{1}{\sigma}} + \delta z_{\text{o},t}^{1-\frac{1}{\sigma}}, \quad \text{where } z_{\text{o},t}^{1-\alpha} = \mathbb{E}_t\left[(1+g_{t+1})^{1-\alpha} v_{\text{o},t+1}^{1-\alpha}\right]. \qquad \text{[A.20.7b]}$$

From equations [A.20.2] and [A.20.7a] it follows that $u_t = (c_{\text{y},t}^{1-\frac{1}{\sigma}} + \delta z_{\text{m},t}^{1-\frac{1}{\sigma}})/(1-1/\sigma)$, which can be approximated as follows:

$$u_t - \bar{u} = \mathsf{c}_{\text{y},t} + \frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathsf{c}_{\text{y},t}^2 + \beta(1+\beta)\left(\mathsf{z}_{\text{m},t} + \frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathsf{z}_{\text{m},t}^2\right) + \mathscr{O}_3. \qquad \text{[A.20.8]}$$

First- and second-order accurate approximations of the equation for $z_{\text{m},t}$ in [A.20.7a] are:

$$\mathsf{z}_{\text{m},t} = \mathbb{E}_t[\mathsf{v}_{\text{m},t+1} + \mathsf{g}_{t+1}] + \mathscr{O}_2, \quad \mathsf{z}_{\text{m},t} + \frac{1}{2}(1-\alpha)\mathsf{z}_{\text{m},t}^2 = \mathbb{E}_t[\mathsf{v}_{\text{m},t+1} + \mathsf{g}_{t+1}] + \frac{1}{2}(1-\alpha)\mathbb{E}_t\left[(\mathsf{v}_{\text{m},t+1} + \mathsf{g}_{t+1})^2\right] + \mathscr{O}_3.$$

The second-order approximation can be rearranged as follows:

$$\mathsf{z}_{\text{m},t} + \frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathsf{z}_{\text{m},t}^2 = \mathbb{E}_t[\mathsf{v}_{\text{m},t+1} + \mathsf{g}_{t+1}] + \frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathbb{E}_t\left[(\mathsf{v}_{\text{m},t+1} + \mathsf{g}_{t+1})^2\right]$$
$$+\frac{1}{2}\left(\frac{1}{\sigma} - \alpha\right)\left(\mathbb{E}_t\left[(\mathsf{v}_{\text{m},t+1} + \mathsf{g}_{t+1})^2\right] - \mathsf{z}_{\text{m},t}^2\right) + \mathscr{O}_3,$$

and hence by combining it with the first-order approximation equation:

$$\mathsf{z}_{\text{m},t} + \frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathsf{z}_{\text{m},t}^2 = \mathbb{E}_t[\mathsf{v}_{\text{m},t+1} + \mathsf{g}_{t+1}] + \frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathbb{E}_t\left[(\mathsf{v}_{\text{m},t+1} + \mathsf{g}_{t+1})^2\right]$$
$$+\frac{1}{2}\left(\frac{1}{\sigma} - \alpha\right)\mathbb{E}_t\left[((\mathsf{v}_{\text{m},t+1} + \mathsf{g}_{t+1}) - \mathbb{E}_t[\mathsf{v}_{\text{m},t+1} + \mathsf{g}_{t+1}])^2\right] + \mathscr{O}_3. \qquad \text{[A.20.9]}$$

Next, considering the value function equation for $v_{\text{m},t}$ from [A.20.7b], a first-order accurate approximation is:

$$\mathsf{v}_{\mathrm{m},t} = \frac{1}{1+\beta}\mathsf{c}_{\mathrm{m},t} + \frac{\beta}{1+\beta}\mathsf{z}_{\mathrm{o},t} + \mathscr{O}_2, \qquad \text{[A.20.10a]}$$

and a second-order accurate approximation is:

$$\mathsf{v}_{\mathrm{m},t}+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathsf{v}_{\mathrm{m},t}^2 = \frac{1}{1+\beta}\left(\mathsf{c}_{\mathrm{m},t}+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathsf{c}_{\mathrm{m},t}^2\right)+\frac{\beta}{1+\beta}\left(\mathsf{z}_{\mathrm{o},t}+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathsf{z}_{\mathrm{o},t}^2\right)+\mathscr{O}_3. \qquad \text{[A.20.10b]}$$

Using equations [A.20.10a] and [A.20.10b] it follows that:

$$(\mathsf{v}_{\mathrm{m},t+1}+\mathsf{g}_{t+1})+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)(\mathsf{v}_{\mathrm{m},t+1}+\mathsf{g}_{t+1})^2 = \frac{1}{1+\beta}\left((\mathsf{c}_{\mathrm{m},t+1}+\mathsf{g}_{t+1})+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)(\mathsf{c}_{\mathrm{m},t+1}+\mathsf{g}_{t+1})^2\right)$$
$$+\frac{\beta}{1+\beta}\left((\mathsf{z}_{\mathrm{o},t+1}+\mathsf{g}_{t+1})+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)(\mathsf{z}_{\mathrm{o},t+1}+\mathsf{g}_{t+1})^2\right)+\mathscr{O}_3. \qquad \text{[A.20.11]}$$

Using the same method that led to the approximation [A.20.9] for $z_{\mathrm{m},t}$, the equation in [A.20.7b] for $z_{\mathrm{o},t}$ can be approximated as follows:

$$\mathsf{z}_{\mathrm{o},t} = \mathbb{E}_t[\mathsf{c}_{\mathrm{o},t+1}+\mathsf{g}_{t+1}]+\mathscr{O}_2, \quad \mathsf{z}_{\mathrm{o},t}+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathsf{z}_{\mathrm{o},t}^2 = \mathbb{E}_t[\mathsf{c}_{\mathrm{o},t+1}+\mathsf{g}_{t+1}]+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathbb{E}_t\left[(\mathsf{c}_{\mathrm{o},t+1}+\mathsf{g}_{t+1})^2\right]$$
$$+\frac{1}{2}\left(\frac{1}{\sigma}-\alpha\right)\mathbb{E}_t\left[\left((\mathsf{c}_{\mathrm{o},t+1}+\mathsf{g}_{t+1})-\mathbb{E}_t[\mathsf{c}_{\mathrm{o},t+1}+\mathsf{g}_{t+1}]\right)^2\right]+\mathscr{O}_3,$$

noting that $\mathsf{v}_{\mathrm{o},t} = \mathsf{c}_{\mathrm{o},t}$ according to [2.14e]. These approximations can be used to deduce that:

$$(\mathsf{z}_{\mathrm{o},t+1}+\mathsf{g}_{t+1})+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)(\mathsf{z}_{\mathrm{o},t+1}+\mathsf{g}_{t+1})^2 = \frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathbb{E}_{t+1}\left[(\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+1}+\mathsf{g}_{t+2})^2\right]$$
$$+\mathbb{E}_{t+1}[\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+1}+\mathsf{g}_{t+2}]+\frac{1}{2}\left(\frac{1}{\sigma}-\alpha\right)\mathbb{E}_{t+1}\left[\left((\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+2})-\mathbb{E}_{t+1}[\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+2}]\right)^2\right]+\mathscr{O}_3. \qquad \text{[A.20.12]}$$

Substituting equation [A.20.12] into [A.20.11] yields:

$$(1+\beta)\mathbb{E}_t[\mathsf{v}_{\mathrm{m},t+1}+\mathsf{g}_{t+1}] = \mathbb{E}_t[\mathsf{c}_{\mathrm{m},t+1}+\mathsf{g}_{t+1}]+\beta\mathbb{E}_t[\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+1}+\mathsf{g}_{t+2}]+\mathscr{O}_2, \quad \text{and} \qquad \text{[A.20.13a]}$$

$$(1+\beta)\left(\mathbb{E}_t[\mathsf{v}_{\mathrm{m},t+1}+\mathsf{g}_{t+1}]+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathbb{E}_t\left[(\mathsf{v}_{\mathrm{m},t+1}+\mathsf{g}_{t+1})^2\right]\right) = \mathbb{E}_t[\mathsf{c}_{\mathrm{m},t+1}+\mathsf{g}_{t+1}]$$
$$+\beta\mathbb{E}_t[\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+1}+\mathsf{g}_{t+2}]+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)\left(\mathbb{E}_t\left[(\mathsf{c}_{\mathrm{m},t+1}+\mathsf{g}_{t+1})^2\right]+\mathbb{E}_t\left[(\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+1}+\mathsf{g}_{t+2})^2\right]\right)$$
$$+\frac{1}{2}\left(\frac{1}{\sigma}-\alpha\right)\mathbb{E}_{t+1}\left[\left((\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+2})-\mathbb{E}_{t+1}[\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+2}]\right)^2\right]+\mathscr{O}_3. \qquad \text{[A.20.13b]}$$

To complete the approximation of the utility deviation $u_t-\bar{u}$, substitute equations [A.20.13a] and [A.20.13b] into [A.20.9] and then into [A.20.8] to obtain a second-order accurate approximation:

$$u_t-\bar{u} = \mathsf{c}_{\mathrm{y},t}+\beta\mathbb{E}_t[\mathsf{c}_{\mathrm{m},t+1}+\mathsf{g}_{t+1}]+\beta^2\mathbb{E}_t[\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+1}+\mathsf{g}_{t+2}]$$
$$+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)\left(\mathsf{c}_{\mathrm{y},t}^2+\beta\mathbb{E}_t[(\mathsf{c}_{\mathrm{m},t+1}+\mathsf{g}_{t+1})^2]+\beta^2\mathbb{E}_t[(\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+1}+\mathsf{g}_{t+2})^2]\right)$$
$$+\frac{1}{2}\left(\frac{1}{\sigma}-\alpha\right)\frac{\beta}{1+\beta}\mathbb{E}_t\Bigg[\Bigg(\left((\mathsf{c}_{\mathrm{m},t+1}+\mathsf{g}_{t+1})-\mathbb{E}_t[\mathsf{c}_{\mathrm{m},t+1}+\mathsf{g}_{t+1}]\right)$$
$$+\beta\left(\mathbb{E}_{t+1}[\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+1}+\mathsf{g}_{t+2}]-\mathbb{E}_t[\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+1}+\mathsf{g}_{t+2}]\right)\Bigg)^2\Bigg]$$
$$+\frac{1}{2}\left(\frac{1}{\sigma}-\alpha\right)\beta^2\mathbb{E}_t\left[\left((\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+2})-\mathbb{E}_{t+1}[\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+2}]\right)^2\right]+\mathscr{O}_3, \qquad \text{[A.20.14]}$$

from which it can be seen that a first-order accurate approximation of the utility deviation is:

$$u_t-\bar{u} = \mathsf{c}_{\mathrm{y},t}+\beta\mathbb{E}_t[\mathsf{c}_{\mathrm{m},t+1}+\mathsf{g}_{t+1}]+\beta^2\mathbb{E}_t[\mathsf{c}_{\mathrm{o},t+2}+\mathsf{g}_{t+1}+\mathsf{g}_{t+2}]+\mathscr{O}_2. \qquad \text{[A.20.15]}$$

Given the definitions of $c_{\mathrm{y},t}$, $c_{\mathrm{m},t}$, $c_{\mathrm{o},t}$, g_t, and Ψ_t and the steady-state values $\bar{c}_{\mathrm{y}} = \bar{c}_{\mathrm{m}} = \bar{c}_{\mathrm{o}} = 1$, $\bar{g} = 0$, and $\bar{\Psi} = 1$, the log deviations of these variables can be written in terms of log differences of consumption and output levels:

$$\mathsf{c}_{\mathrm{y},t} = \mathsf{C}_{\mathrm{y},t} - \mathsf{Y}_t, \quad \mathsf{c}_{\mathrm{m},t} = \mathsf{C}_{\mathrm{m},t} - \mathsf{Y}_t, \quad \mathsf{c}_{\mathrm{o},t} = \mathsf{C}_{\mathrm{o},t} - \mathsf{Y}_t, \quad \mathsf{g}_t = \mathsf{Y}_t - \mathsf{Y}_{t-1}, \quad \text{and} \;\; \Psi_t = \hat{\mathsf{Y}}_t - \mathsf{Y}_t. \quad \text{[A.20.16]}$$

Thus, the terms in the first-order approximation [A.20.15] of the utility function are exactly equal to:

$$\mathsf{c}_{\mathrm{y},t} + \beta\mathbb{E}_t[\mathsf{c}_{\mathrm{m},t+1} + \mathsf{g}_{t+1}] + \beta^2\mathbb{E}_t[\mathsf{c}_{\mathrm{o},t+2} + \mathsf{g}_{t+1} + \mathsf{g}_{t+2}] = \mathsf{C}_{\mathrm{y},t} + \beta\mathbb{E}_t\mathsf{C}_{\mathrm{m},t+1} + \beta^2\mathbb{E}_t\mathsf{C}_{\mathrm{o},t+2} - (1+\beta+\beta^2)\mathsf{Y}_t. \quad \text{[A.20.17]}$$

Since $\Psi_t = \mathscr{O}_2$ according to [A.20.5b], it follows that $\hat{\mathsf{g}}_{t+1} - \mathsf{g}_{t+1} = \Psi_{t+1} - \Psi_t = \mathscr{O}_2$ and $\hat{\mathsf{g}}_{t+2} - \mathsf{g}_{t+2} = \Psi_{t+2} - \Psi_{t+1} = \mathscr{O}_2$. Thus, the terms on the second line of equation [A.20.14] can be written as:

$$\begin{aligned}
&\mathsf{c}^2_{\mathrm{y},t} + \beta\mathbb{E}_t[(\mathsf{c}_{\mathrm{m},t+1} + \mathsf{g}_{t+1})^2] + \beta^2\mathbb{E}_t[(\mathsf{c}_{\mathrm{o},t+2} + \mathsf{g}_{t+1} + \mathsf{g}_{t+2})^2] \\
&= (\mathsf{c}_{\mathrm{y},t} - \Psi_t)^2 + \beta\mathbb{E}_t[(\mathsf{c}_{\mathrm{m},t+1} + \hat{\mathsf{g}}_{t+1} - \Psi_{t+1})^2] + \beta^2\mathbb{E}_t[(\mathsf{c}_{\mathrm{o},t+2} + \hat{\mathsf{g}}_{t+1} + \hat{\mathsf{g}}_{t+2} - \Psi_{t+2})^2] + \mathscr{O}_3 \\
&= \mathsf{C}^2_{\mathrm{y},t} + \beta\mathbb{E}_t\mathsf{C}^2_{\mathrm{m},t+1} + \beta^2\mathbb{E}_t\mathsf{C}^2_{\mathrm{o},t+2} - 2\hat{\mathsf{Y}}_t\left(\mathsf{C}_{\mathrm{y},t} + \beta\mathbb{E}_t\mathsf{C}_{\mathrm{m},t+1} + \beta^2\mathbb{E}_t\mathsf{C}_{\mathrm{o},t+2}\right) + (1+\beta+\beta^2)\hat{\mathsf{Y}}^2_t + \mathscr{O}_3 \\
&= \mathsf{C}^2_{\mathrm{y},t} + \beta\mathbb{E}_t\mathsf{C}^2_{\mathrm{m},t+1} + \beta^2\mathbb{E}_t\mathsf{C}^2_{\mathrm{o},t+2} - 2\hat{\mathsf{Y}}_t\left(\mathsf{C}_{\mathrm{y},t} + \beta\mathbb{E}_t\mathsf{C}_{\mathrm{m},t+1} + \beta^2\mathbb{E}_t\mathsf{C}_{\mathrm{o},t+2}\right) + \mathscr{I} + \mathscr{O}_3, \quad \text{[A.20.18]}
\end{aligned}$$

where the third line makes use of the identities in [A.20.16] and expands the brackets and the final line notes that $\hat{Y}_t$ is independent of monetary policy. The identities in [A.20.16] can also be used to rewrite the final two terms appearing in [A.20.14] as follows:

$$\begin{aligned}
&\mathbb{E}_t\left[(((\mathsf{c}_{\mathrm{m},t+1} + \mathsf{g}_{t+1}) - \mathbb{E}_t[\mathsf{c}_{\mathrm{m},t+1} + \mathsf{g}_{t+1}]) + \beta\left(\mathbb{E}_{t+1}[\mathsf{c}_{\mathrm{o},t+2} + \mathsf{g}_{t+1} + \mathsf{g}_{t+2}] - \mathbb{E}_t[\mathsf{c}_{\mathrm{o},t+2} + \mathsf{g}_{t+1} + \mathsf{g}_{t+2}]\right))^2\right] \\
&= \mathbb{E}_t\left[((\mathsf{C}_{\mathrm{m},t+1} - \mathbb{E}_t\mathsf{C}_{\mathrm{m},t+1}) + \beta(\mathbb{E}_{t+1}\mathsf{C}_{\mathrm{o},t+2} - \mathbb{E}_t\mathsf{C}_{\mathrm{o},t+2}))^2\right], \quad \text{and} \quad \text{[A.20.19a]} \\
&\mathbb{E}_t\left[((\mathsf{c}_{\mathrm{o},t+2} + \mathsf{g}_{t+2}) - \mathbb{E}_{t+1}[\mathsf{c}_{\mathrm{o},t+2} + \mathsf{g}_{t+2}])^2\right] = \mathbb{E}_t\left[(\mathsf{C}_{\mathrm{o},t+2} - \mathbb{E}_{t+1}\mathsf{C}_{\mathrm{o},t+2})^2\right]. \quad \text{[A.20.19b]}
\end{aligned}$$

The next step is to consider the cross-product between the Pareto weight $\hat{\omega}^*_t$ and the utility deviation $u_t - \bar{u}$ that appears in [A.20.6]. By using equation [A.20.17] and $\Psi_t = \mathscr{O}_2$ and $\mathsf{g}_t - \hat{\mathsf{g}}_t = \mathscr{O}_2$:

$$\begin{aligned}
&\hat{\omega}^*_t(u_t - \bar{u}) = \hat{\omega}^*_t\left(\mathsf{c}_{\mathrm{y},t} + \beta\mathbb{E}_t[\mathsf{c}_{\mathrm{m},t+1} + \mathsf{g}_{t+1}] + \beta^2\mathbb{E}_t[\mathsf{c}_{\mathrm{o},t+2} + \mathsf{g}_{t+1} + \mathsf{g}_{t+2}]\right) + \mathscr{O}_3 \\
&= \hat{\omega}^*_t\left((\mathsf{c}_{\mathrm{y},t} - \Psi_t) + \beta\mathbb{E}_t[\mathsf{c}_{\mathrm{m},t+1} + \hat{\mathsf{g}}_{t+1} - \Psi_{t+1}] + \beta^2\mathbb{E}_t[\mathsf{c}_{\mathrm{o},t+2} + \hat{\mathsf{g}}_{t+1} + \hat{\mathsf{g}}_{t+2} - \Psi_{t+2}]\right) + \mathscr{O}_3 \\
&= \hat{\omega}^*_t\left(\mathsf{C}_{\mathrm{y},t} + \beta\mathbb{E}_t\mathsf{C}_{\mathrm{m},t+1} + \beta^2\mathbb{E}_t\mathsf{C}_{\mathrm{o},t+2} - (1+\beta+\beta^2)\hat{\mathsf{Y}}_t\right) + \mathscr{O}_3 \\
&= \hat{\omega}^*_t\left(\mathsf{C}_{\mathrm{y},t} + \beta\mathbb{E}_t\mathsf{C}_{\mathrm{m},t+1} + \beta^2\mathbb{E}_t\mathsf{C}_{\mathrm{o},t+2}\right) + \mathscr{I} + \mathscr{O}_3, \quad \text{[A.20.20]}
\end{aligned}$$

where the final line uses that $\hat{Y}_t$ is independent of monetary policy. Substituting equations [A.20.17], [A.20.18] and [A.20.19] into [A.20.14], and then substituting this together with [A.20.20] into equation [A.20.6] yields:

$$\begin{aligned}
&\hat{\omega}^*_t\Psi_t^{-\left(1-\frac{1}{\sigma}\right)}u_t = \left(\mathsf{C}_{\mathrm{y},t} + \beta\mathbb{E}_t\mathsf{C}_{\mathrm{m},t+1} + \beta^2\mathbb{E}_t\mathsf{C}_{\mathrm{o},t+2}\right) + \frac{1}{2}\left(1 - \frac{1}{\sigma}\right)\left(\mathsf{C}^2_{\mathrm{y},t} + \beta\mathbb{E}_t\mathsf{C}^2_{\mathrm{m},t+1} + \beta^2\mathbb{E}_t\mathsf{C}^2_{\mathrm{o},t+2}\right) \\
&- \left(1 - \frac{1}{\sigma}\right)\hat{\mathsf{Y}}_t\left(\mathsf{C}_{\mathrm{y},t} + \beta\mathbb{E}_t\mathsf{C}_{\mathrm{m},t+1} + \beta^2\mathbb{E}_t\mathsf{C}_{\mathrm{o},t+2}\right) + \hat{\omega}^*_t\left(\mathsf{C}_{\mathrm{y},t} + \beta\mathbb{E}_t\mathsf{C}_{\mathrm{m},t+1} + \beta^2\mathbb{E}_t\mathsf{C}_{\mathrm{o},t+2}\right) \\
&+ \frac{1}{2}\left(\frac{1}{\sigma} - \alpha\right)\left(\frac{\beta}{1+\beta}\right)\mathbb{E}_t\left[((\mathsf{C}_{\mathrm{m},t+1} - \mathbb{E}_t\mathsf{C}_{\mathrm{m},t+1}) + \beta(\mathbb{E}_{t+1}\mathsf{C}_{\mathrm{o},t+2} - \mathbb{E}_t\mathsf{C}_{\mathrm{o},t+2}))^2\right] \\
&+ \frac{1}{2}\left(\frac{1}{\sigma} - \alpha\right)\beta^2\mathbb{E}_t\left[(\mathsf{C}_{\mathrm{o},t+2} - \mathbb{E}_{t+1}\mathsf{C}_{\mathrm{o},t+2})^2\right] + \mathscr{I} + \mathscr{O}_3. \quad \text{[A.20.21]}
\end{aligned}$$

To analyse the cross-products between the Pareto-weights and consumption, second-order accurate approximations of the equations in [A.20.1] are derived:

$$\hat{\omega}^*_t = \hat{\varphi}^*_t + \frac{1}{\sigma}\hat{c}^*_{y,t},$$

$$\hat{\omega}^*_t = \hat{\varphi}^*_{t+1} - \left(1 - \frac{1}{\sigma}\right)\hat{g}_{t+1} + \frac{1}{\sigma}\hat{c}^*_{m,t+1} + \left(\alpha - \frac{1}{\sigma}\right)\left((\hat{v}^*_{m,t+1} + \hat{g}_{t+1}) - \mathbb{E}_t[\hat{v}^*_{m,t+1} + \hat{g}_{t+1}]\right) + \mathscr{O}_2, \quad \text{and}$$

$$\hat{\omega}^*_t = \hat{\varphi}^*_{t+2} - \left(1 - \frac{1}{\sigma}\right)(\hat{g}_{t+1} + \hat{g}_{t+2}) + \frac{1}{\sigma}\hat{c}^*_{o,t+2} + \left(\alpha - \frac{1}{\sigma}\right)\left((\hat{v}^*_{m,t+1} + \hat{g}_{t+1}) - \mathbb{E}_t[\hat{v}^*_{m,t+1} + \hat{g}_{t+1}]\right)$$
$$+ \left(\alpha - \frac{1}{\sigma}\right)\left((\hat{v}^*_{o,t+2} + \hat{g}_{t+2}) - \mathbb{E}_{t+1}[\hat{v}^*_{o,t+2} + \hat{g}_{t+2}]\right) + \mathscr{O}_2.$$

By using equation [A.20.13a] and noting that [2.14e] implies $v_{o,t} = c_{o,t}$, the terms involving the value functions can be replaced by terms in consumption and growth rates:

$$\hat{\omega}^*_t = \hat{\varphi}^*_{t+1} - \left(1 - \frac{1}{\sigma}\right)\hat{g}_{t+1} + \frac{1}{\sigma}\hat{c}^*_{m,t+1} + \left(\alpha - \frac{1}{\sigma}\right)\left(\frac{1}{1+\beta}\left((\hat{c}^*_{m,t+1} + \hat{g}_{t+1}) - \mathbb{E}_t[\hat{c}^*_{m,t+1} + \hat{g}_{t+1}]\right)\right.$$
$$\left. + \frac{\beta}{1+\beta}\left(\mathbb{E}_{t+1}[\hat{c}^*_{o,t+2} + \hat{g}_{t+1} + \hat{g}_{t+2}] - \mathbb{E}_t[\hat{c}^*_{o,t+2} + \hat{g}_{t+1} + \hat{g}_{t+2}]\right)\right) + \mathscr{O}_2,$$

and also in the equation:

$$\hat{\omega}^*_t = \hat{\varphi}^*_{t+2} - \left(1 - \frac{1}{\sigma}\right)(\hat{g}_{t+1} + \hat{g}_{t+2}) + \frac{1}{\sigma}\hat{c}^*_{o,t+2} + \left(\alpha - \frac{1}{\sigma}\right)\left(\frac{1}{1+\beta}\left((\hat{c}^*_{m,t+1} + \hat{g}_{t+1}) - \mathbb{E}_t[\hat{c}^*_{m,t+1} + \hat{g}_{t+1}]\right)\right.$$
$$\left. + \frac{\beta}{1+\beta}\left(\mathbb{E}_{t+1}[\hat{c}^*_{o,t+2} + \hat{g}_{t+1} + \hat{g}_{t+2}] - \mathbb{E}_t[\hat{c}^*_{o,t+2} + \hat{g}_{t+1} + \hat{g}_{t+2}]\right)\right)$$
$$+ \left(\alpha - \frac{1}{\sigma}\right)\left((\hat{c}^*_{o,t+2} + \hat{g}_{t+2}) - \mathbb{E}_{t+1}[\hat{c}^*_{o,t+2} + \hat{g}_{t+2}]\right) + \mathscr{O}_2.$$

Using the identities in [A.20.16] these equations can be written as:

$$\hat{\omega}^*_t = \hat{\varphi}^*_t + \frac{1}{\sigma}\hat{c}^*_{y,t} = \hat{\varphi}^*_{t+1} - \left(1 - \frac{1}{\sigma}\right)\hat{g}_{t+1} + \frac{1}{\sigma}\hat{c}^*_{m,t+1} + \left(\alpha - \frac{1}{\sigma}\right)\left(\frac{1}{1+\beta}\right)\left((\hat{C}^*_{m,t+1} - \mathbb{E}_t\hat{C}^*_{m,t+1})\right.$$
$$\left. + \beta(\mathbb{E}_{t+1}\hat{C}^*_{o,t+2} - \mathbb{E}_t\hat{C}^*_{o,t+2})\right) + \mathscr{O}_2 = \hat{\varphi}^*_{t+2} - \left(1 - \frac{1}{\sigma}\right)(\hat{g}_{t+1} + \hat{g}_{t+2}) + \frac{1}{\sigma}\hat{c}^*_{o,t+2}$$
$$+ \left(\alpha - \frac{1}{\sigma}\right)\left(\frac{1}{1+\beta}\right)\left((\hat{C}^*_{m,t+1} - \mathbb{E}_t\hat{C}^*_{m,t+1}) + \beta(\mathbb{E}_{t+1}\hat{C}^*_{o,t+2} - \mathbb{E}_t\hat{C}^*_{o,t+2})\right)$$
$$+ \left(\alpha - \frac{1}{\sigma}\right)(\hat{C}^*_{o,t+2} - \mathbb{E}_{t+1}\hat{C}^*_{o,t+2}) + \mathscr{O}_2. \quad \text{[A.20.22]}$$

The identities in [A.20.16] also imply that

$$\hat{Y}_t\left(C_{y,t} + \beta\mathbb{E}_t C_{m,t+1} + \beta^2\mathbb{E}_t C_{o,t+2}\right) + \beta\mathbb{E}_t[\hat{g}_{t+1}C_{m,t+1}] + \beta^2\mathbb{E}_t[(\hat{g}_{t+1} + \hat{g}_{t+2})C_{o,t+2}]$$
$$= \hat{Y}_t C_{y,t} + \beta\mathbb{E}_t\hat{Y}_{t+1}C_{m,t+1} + \beta^2\mathbb{E}_t\hat{Y}_{t+2}C_{o,t+2},$$

and hence by using the equations in [A.20.22] it follows that:

$$\hat{\omega}^*_t\left(C_{y,t} + \beta\mathbb{E}_t C_{m,t+1} + \beta^2\mathbb{E}_t C_{o,t+2}\right) - \left(1 - \frac{1}{\sigma}\right)\hat{Y}_t\left(C_{y,t} + \beta\mathbb{E}_t C_{m,t+1} + \beta^2\mathbb{E}_t C_{o,t+2}\right)$$
$$= \left(\hat{\varphi}^*_t C_{y,t} + \beta\mathbb{E}_t\hat{\varphi}^*_{t+1}C_{m,t+1} + \beta^2\mathbb{E}_t\hat{\varphi}^*_{t+2}C_{o,t+2}\right) + \frac{1}{\sigma}\left(\hat{c}^*_{y,t}C_{y,t} + \beta\mathbb{E}_t\hat{c}^*_{m,t+1}C_{m,t+1} + \beta^2\mathbb{E}_t\hat{c}^*_{o,t+2}C_{o,t+2}\right)$$
$$-\left(1 - \frac{1}{\sigma}\right)\left(\hat{Y}_t C_{y,t} + \beta\mathbb{E}_t\hat{Y}_{t+1}C_{m,t+1} + \beta^2\mathbb{E}_t\hat{Y}_{t+2}C_{o,t+2}\right) - \left(\frac{1}{\sigma} - \alpha\right)\beta^2\mathbb{E}_t\left[(\hat{C}^*_{o,t+2} - \mathbb{E}_{t+1}\hat{C}^*_{o,t+2})C_{o,t+2}\right]$$
$$-\left(\frac{1}{\sigma} - \alpha\right)\left(\frac{\beta}{1+\beta}\right)\mathbb{E}_t\left[\left(\left(\hat{C}^*_{m,t+1} - \mathbb{E}_t\hat{C}^*_{m,t+1}\right) + \beta\left(\mathbb{E}_{t+1}\hat{C}^*_{o,t+2} - \mathbb{E}_t\hat{C}^*_{o,t+2}\right)\right)(C_{m,t+1} + \beta C_{o,t+2})\right].$$
[A.20.23]

Next, note the expression below can be rearranged as follows:

$$\frac{1}{2}\mathbb{E}_t\left[(\mathsf{C}_{\mathrm{o},t+2}-\mathbb{E}_{t+1}\mathsf{C}_{\mathrm{o},t+2})^2\right]-\mathbb{E}_t\left[(\hat{\mathsf{C}}^*_{\mathrm{o},t+2}-\mathbb{E}_{t+1}\hat{\mathsf{C}}^*_{\mathrm{o},t+2})\mathsf{C}_{\mathrm{o},t+2}\right]$$
$$=\frac{1}{2}\mathbb{E}_t\left[(\mathsf{C}_{\mathrm{o},t+2}-\mathbb{E}_{t+1}\mathsf{C}_{\mathrm{o},t+2})^2\right]-\mathbb{E}_t\left[(\hat{\mathsf{C}}^*_{\mathrm{o},t+2}-\mathbb{E}_{t+1}\hat{\mathsf{C}}^*_{\mathrm{o},t+2})\left(\mathsf{C}_{\mathrm{o},t+2}-\mathbb{E}_{t+1}\mathsf{C}_{\mathrm{o},t+2}\right)\right]$$
$$=\frac{1}{2}\mathbb{E}_t\left[\left((\mathsf{C}_{\mathrm{o},t+2}-\mathbb{E}_{t+1}\mathsf{C}_{\mathrm{o},t+2})-(\hat{\mathsf{C}}^*_{\mathrm{o},t+2}-\mathbb{E}_{t+1}\hat{\mathsf{C}}^*_{\mathrm{o},t+2})\right)^2\right]+\mathscr{I}$$
$$=\frac{1}{2}\mathbb{E}_t\left[\left((\tilde{\mathsf{c}}_{\mathrm{o},t+2}-(\hat{\mathsf{c}}^*_{\mathrm{o},t+2}-\mathsf{c}^*_{\mathrm{o},t+2})-\Psi_{t+2})-\mathbb{E}_{t+1}[\tilde{\mathsf{c}}_{\mathrm{o},t+2}-(\hat{\mathsf{c}}^*_{\mathrm{o},t+2}-\mathsf{c}^*_{\mathrm{o},t+2})-\Psi_{t+2}]\right)^2\right]+\mathscr{I}, \quad \text{[A.20.24]}$$

where the final line uses the definition $\tilde{\mathsf{c}}_{\mathrm{o},t+2}\equiv\mathsf{c}_{\mathrm{o},t+2}-\mathsf{c}^*_{\mathrm{o},t+2}$ and the identities in [A.20.16] to observe that:

$$\mathsf{C}_{\mathrm{o},t+2}-\hat{\mathsf{C}}^*_{\mathrm{o},t+2}=(\mathsf{c}_{\mathrm{o},t+2}+\mathsf{Y}_{t+2})-(\hat{\mathsf{c}}^*_{\mathrm{o},t+2}+\hat{\mathsf{Y}}_{t+2})=(\mathsf{c}_{\mathrm{o},t+2}-\mathsf{c}^*_{\mathrm{o},t+2})-(\hat{\mathsf{c}}^*_{\mathrm{o},t+2}-\mathsf{c}^*_{\mathrm{o},t+2})-\Psi_{t+2}.$$

The terms $\hat{\mathsf{c}}^*_{\mathrm{o},t+2}$ and $\mathsf{c}^*_{\mathrm{o},t+2}$ denote consumption with sequentially complete markets in the cases where output is $\hat{Y}_t$ and Y_t respectively. A first-order accurate approximation for these variables is derived in Proposition 6 and the solution is a linear function of current, past, and expected future values of growth rates g_t (directly, and through f_t, which itself depends only on terms in g_t in accordance with [4.2]). Thus, the difference between $\hat{\mathsf{c}}^*_{\mathrm{o},t+2}$ and $\mathsf{c}^*_{\mathrm{o},t+2}$ is a function of terms in the difference between $\hat{\mathsf{g}}_t$ and g_t up to second- and higher-order terms. However, since $\hat{\mathsf{g}}_t-\mathsf{g}_t=\Psi_t-\Psi_{t-1}=\mathscr{O}_2$, it follows that:

$$\hat{\mathsf{c}}^*_{\mathrm{y},t}-\mathsf{c}^*_{\mathrm{y},t}=\mathscr{O}_2,\quad \hat{\mathsf{c}}^*_{\mathrm{m},t}-\mathsf{c}^*_{\mathrm{m},t}=\mathscr{O}_2,\quad\text{and}\quad \hat{\mathsf{c}}^*_{\mathrm{o},t}-\mathsf{c}^*_{\mathrm{o},t}=\mathscr{O}_2. \quad \text{[A.20.25]}$$

Using these findings and expanding the bracket in the final line of [A.20.24] implies:

$$\frac{1}{2}\mathbb{E}_t\left[(\mathsf{C}_{\mathrm{o},t+2}-\mathbb{E}_{t+1}\mathsf{C}_{\mathrm{o},t+2})^2\right]-\mathbb{E}_t\left[(\hat{\mathsf{C}}^*_{\mathrm{o},t+2}-\mathbb{E}_{t+1}\hat{\mathsf{C}}^*_{\mathrm{o},t+2})\mathsf{C}_{\mathrm{o},t+2}\right]=\frac{1}{2}\mathbb{E}_t\left[(\tilde{\mathsf{c}}_{\mathrm{o},t+2}-\mathbb{E}_{t+1}\tilde{\mathsf{c}}_{\mathrm{o},t+2})^2\right]+\mathscr{I}+\mathscr{O}_3. \quad \text{[A.20.26]}$$

An identical method to that used to deduce equation [A.20.26] from [A.20.24] yields:

$$\frac{1}{2}\mathbb{E}_t\left[((\mathsf{C}_{\mathrm{m},t+1}-\mathbb{E}_t\mathsf{C}_{\mathrm{m},t+1})+\beta(\mathbb{E}_{t+1}\mathsf{C}_{\mathrm{o},t+2}-\mathbb{E}_t\mathsf{C}_{\mathrm{o},t+2}))^2\right]$$
$$-\mathbb{E}_t\left[\left(\left(\hat{\mathsf{C}}^*_{\mathrm{m},t+1}-\mathbb{E}_t\hat{\mathsf{C}}^*_{\mathrm{m},t+1}\right)+\beta\left(\mathbb{E}_{t+1}\hat{\mathsf{C}}^*_{\mathrm{o},t+2}-\mathbb{E}_t\hat{\mathsf{C}}^*_{\mathrm{o},t+2}\right)\right)(\mathsf{C}_{\mathrm{m},t+1}+\beta\mathsf{C}_{\mathrm{o},t+2})\right]$$
$$=\frac{1}{2}\mathbb{E}_t\left[((\tilde{\mathsf{c}}_{\mathrm{m},t+1}-\mathbb{E}_t\tilde{\mathsf{c}}_{\mathrm{m},t+1})+\beta(\mathbb{E}_{t+1}\tilde{\mathsf{c}}_{\mathrm{o},t+2}-\mathbb{E}_t\tilde{\mathsf{c}}_{\mathrm{o},t+2}))^2\right]+\mathscr{I}+\mathscr{O}_3. \quad \text{[A.20.27]}$$

Substituting equation [A.20.23] into [A.20.21] and making use of [A.20.26] and [A.20.27] yields:

$$\hat{\omega}^*_t\Psi_t^{-(1-\frac{1}{\sigma})}u_t=\left(\mathsf{C}_{\mathrm{y},t}+\beta\mathbb{E}_t\mathsf{C}_{\mathrm{m},t+1}+\beta^2\mathbb{E}_t\mathsf{C}_{\mathrm{o},t+2}\right)+\left(\hat{\varphi}^*_t\mathsf{C}_{\mathrm{y},t}+\beta\mathbb{E}_t\hat{\varphi}^*_{t+1}\mathsf{C}_{\mathrm{m},t+1}+\beta^2\mathbb{E}_t\hat{\varphi}^*_{t+2}\mathsf{C}_{\mathrm{o},t+2}\right)$$
$$-\left(1-\frac{1}{\sigma}\right)\left(\hat{\mathsf{Y}}_t\mathsf{C}_{\mathrm{y},t}+\beta\mathbb{E}_t\hat{\mathsf{Y}}_{t+1}\mathsf{C}_{\mathrm{m},t+1}+\beta^2\mathbb{E}_t\hat{\mathsf{Y}}_{t+2}\mathsf{C}_{\mathrm{o},t+2}\right)+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)\left(\mathsf{C}^2_{\mathrm{y},t}+\beta\mathbb{E}_t\mathsf{C}^2_{\mathrm{m},t+1}+\beta^2\mathbb{E}_t\mathsf{C}^2_{\mathrm{o},t+2}\right)$$
$$+\frac{1}{\sigma}\left(\hat{\mathsf{c}}^*_{\mathrm{y},t}\mathsf{C}_{\mathrm{y},t}+\beta\mathbb{E}_t\hat{\mathsf{c}}^*_{\mathrm{m},t+1}\mathsf{C}_{\mathrm{m},t+1}+\beta^2\mathbb{E}_t\hat{\mathsf{c}}^*_{\mathrm{o},t+2}\mathsf{C}_{\mathrm{o},t+2}\right)-\frac{1}{2}\left(\alpha-\frac{1}{\sigma}\right)\beta^2\mathbb{E}_t\left[(\tilde{\mathsf{c}}_{\mathrm{o},t+2}-\mathbb{E}_{t+1}\tilde{\mathsf{c}}_{\mathrm{o},t+2})^2\right]$$
$$-\frac{1}{2}\left(\alpha-\frac{1}{\sigma}\right)\left(\frac{\beta}{1+\beta}\right)\mathbb{E}_t\left[((\tilde{\mathsf{c}}_{\mathrm{m},t+1}-\mathbb{E}_t\tilde{\mathsf{c}}_{\mathrm{m},t+1})+\beta(\mathbb{E}_{t+1}\tilde{\mathsf{c}}_{\mathrm{o},t+2}-\mathbb{E}_t\tilde{\mathsf{c}}_{\mathrm{o},t+2}))^2\right]+\mathscr{I}+\mathscr{O}_3. \quad \text{[A.20.28]}$$

The social welfare function $\mathscr{W}_{t_0}$ in [A.20.3] is obtained by summing over these terms. Doing this and collecting terms in consumption levels during the same time periods leads to the following expression for the social welfare function:

$$\mathscr{W}_{t_0} = \frac{1}{3}\sum_{t=t_0}^{\infty} \beta^{t-t_0}\mathbb{E}_{t_0-2}\Bigg[(\mathsf{C}_{\mathrm{y},t} + \mathsf{C}_{\mathrm{m},t} + \mathsf{C}_{\mathrm{o},t}) + \frac{1}{\sigma}\left(\hat{\mathsf{c}}^*_{\mathrm{y},t}\mathsf{C}_{\mathrm{y},t} + \hat{\mathsf{c}}^*_{\mathrm{m},t}\mathsf{C}_{\mathrm{m},t} + \hat{\mathsf{c}}^*_{\mathrm{o},t}\mathsf{C}_{\mathrm{o},t}\right) + \hat{\varphi}^*_t\left(\mathsf{C}_{\mathrm{y},t} + \mathsf{C}_{\mathrm{m},t} + \mathsf{C}_{\mathrm{o},t}\right)$$
$$- \left(1 - \frac{1}{\sigma}\right)\hat{\mathsf{Y}}_t\left(\mathsf{C}_{\mathrm{y},t} + \mathsf{C}_{\mathrm{m},t} + \mathsf{C}_{\mathrm{o},t}\right) + \frac{1}{2}\left(1 - \frac{1}{\sigma}\right)\left(\mathsf{C}^2_{\mathrm{y},t} + \mathsf{C}^2_{\mathrm{m},t} + \mathsf{C}^2_{\mathrm{o},t}\right)\Bigg]$$
$$- \frac{1}{2}\frac{1}{3}\left(\alpha - \frac{1}{\sigma}\right)\sum_{t=t_0-2}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0-2}\Bigg[\left(\frac{\beta}{1+\beta}\right)\left((\tilde{\mathsf{c}}_{\mathrm{m},t+1} - \mathbb{E}_t\tilde{\mathsf{c}}_{\mathrm{m},t+1}) + \beta(\mathbb{E}_{t+1}\tilde{\mathsf{c}}_{\mathrm{o},t+2} - \mathbb{E}_t\tilde{\mathsf{c}}_{\mathrm{o},t+2})\right)^2$$
$$+ \beta^2\left(\tilde{\mathsf{c}}_{\mathrm{o},t+2} - \mathbb{E}_{t+1}\tilde{\mathsf{c}}_{\mathrm{o},t+2}\right)^2\Bigg] + \mathscr{I} + \mathscr{O}_3, \quad \text{[A.20.29]}$$

where terms dated prior to t_0 can be included in $\mathscr{I}$ because these are predetermined and thus independent of monetary policy.

First- and second-order accurate approximations of the goods-market clearing equation [2.16] are:

$$\frac{1}{3}\left(\mathsf{c}_{\mathrm{y},t} + \mathsf{c}_{\mathrm{m},t} + \mathsf{c}_{\mathrm{o},t}\right) = \mathscr{O}_2, \quad \text{and} \quad \frac{1}{3}\left(\mathsf{c}_{\mathrm{y},t} + \mathsf{c}_{\mathrm{m},t} + \mathsf{c}_{\mathrm{o},t}\right) + \frac{1}{3}\frac{1}{2}\left(\mathsf{c}^2_{\mathrm{y},t} + \mathsf{c}^2_{\mathrm{m},t} + \mathsf{c}^2_{\mathrm{o},t}\right) = \mathscr{O}_3, \quad \text{[A.20.30a]}$$

and by substituting the identities from [A.20.16] these equations become:

$$\frac{1}{3}\left(\mathsf{C}_{\mathrm{y},t} + \mathsf{C}_{\mathrm{m},t} + \mathsf{C}_{\mathrm{o},t}\right) = \mathsf{Y}_t + \mathscr{O}_2, \quad \text{and} \quad \text{[A.20.30b]}$$
$$\frac{1}{3}\left(\mathsf{C}_{\mathrm{y},t} + \mathsf{C}_{\mathrm{m},t} + \mathsf{C}_{\mathrm{o},t}\right) + \frac{1}{3}\frac{1}{2}\left(\mathsf{C}^2_{\mathrm{y},t} + \mathsf{C}^2_{\mathrm{m},t} + \mathsf{C}^2_{\mathrm{o},t}\right) = \mathsf{Y}_t - \frac{1}{2}\mathsf{Y}^2_t + \frac{1}{3}\left(\mathsf{C}_{\mathrm{y},t} + \mathsf{C}_{\mathrm{m},t} + \mathsf{C}_{\mathrm{o},t}\right)\mathsf{Y}_t + \mathscr{O}_3. \quad \text{[A.20.30c]}$$

Using equation [A.20.30c] and the identities in [A.20.16] it follows that:

$$\frac{1}{3}\left(\mathsf{C}_{\mathrm{y},t} + \mathsf{C}_{\mathrm{m},t} + \mathsf{C}_{\mathrm{o},t}\right) + \frac{1}{3}\frac{1}{2}\left(\mathsf{C}^2_{\mathrm{y},t} + \mathsf{C}^2_{\mathrm{m},t} + \mathsf{C}^2_{\mathrm{o},t}\right) - \frac{1}{3}\left(\mathsf{C}_{\mathrm{y},t} + \mathsf{C}_{\mathrm{m},t} + \mathsf{C}_{\mathrm{o},t}\right)\hat{\mathsf{Y}}_t$$
$$= \mathsf{Y}_t - \frac{1}{2}\mathsf{Y}^2_t + \left(\frac{1}{3}\left(\mathsf{c}_{\mathrm{y},t} + \mathsf{c}_{\mathrm{m},t} + \mathsf{c}_{\mathrm{o},t}\right) + \mathsf{Y}_t\right)(\mathsf{Y}_t - \hat{\mathsf{Y}}_t) + \mathscr{O}_3$$
$$= \hat{\mathsf{Y}}_t + (\mathsf{Y}_t - \hat{\mathsf{Y}}_t) - \frac{1}{2}\hat{\mathsf{Y}}^2_t + \frac{1}{2}(\mathsf{Y}_t - \hat{\mathsf{Y}}_t)^2 + \frac{1}{3}\left(\mathsf{c}_{\mathrm{y},t} + \mathsf{c}_{\mathrm{m},t} + \mathsf{c}_{\mathrm{o},t}\right)(\mathsf{Y}_t - \hat{\mathsf{Y}}_t) + \mathscr{O}_3$$
$$= -\Psi_t + \frac{1}{2}\Psi^2_t - \frac{1}{3}\left(\mathsf{c}_{\mathrm{y},t} + \mathsf{c}_{\mathrm{m},t} + \mathsf{c}_{\mathrm{o},t}\right)\Psi_t + \hat{\mathsf{Y}}_t - \frac{1}{2}\hat{\mathsf{Y}}^2_t + \mathscr{O}_3 = -\Psi_t + \mathscr{I} + \mathscr{O}_3, \quad \text{[A.20.31]}$$

where the final equality uses [A.20.30a] together with $\Psi_t = \mathscr{O}_2$ (from [A.20.5b]), and that $\hat{Y}_t$ is independent of monetary policy. Using [A.20.16] it can also be seen that:

$$-\frac{1}{2}\frac{1}{\sigma}\left(\mathsf{C}^2_{\mathrm{y},t} + \mathsf{C}^2_{\mathrm{m},t} + \mathsf{C}^2_{\mathrm{o},t}\right) + \frac{1}{\sigma}\hat{\mathsf{Y}}_t\left(\mathsf{C}_{\mathrm{y},t} + \mathsf{C}_{\mathrm{m},t} + \mathsf{C}_{\mathrm{o},t}\right) + \frac{1}{\sigma}\left(\hat{\mathsf{c}}^*_{\mathrm{y},t}\mathsf{C}_{\mathrm{y},t} + \hat{\mathsf{c}}^*_{\mathrm{m},t}\mathsf{C}_{\mathrm{m},t} + \hat{\mathsf{c}}^*_{\mathrm{o},t}\mathsf{C}_{\mathrm{o},t}\right)$$
$$= -\frac{1}{2}\frac{1}{\sigma}\left((\mathsf{C}^2_{\mathrm{y},t} - 2\hat{\mathsf{C}}^*_{\mathrm{y},t}\mathsf{C}_{\mathrm{y},t}) + (\mathsf{C}^2_{\mathrm{m},t} - 2\hat{\mathsf{C}}^*_{\mathrm{m},t}\mathsf{C}_{\mathrm{m},t}) + (\mathsf{C}^2_{\mathrm{o},t} - 2\hat{\mathsf{C}}^*_{\mathrm{o},t}\mathsf{C}_{\mathrm{o},t})\right)$$
$$= -\frac{1}{2}\frac{1}{\sigma}\left((\mathsf{C}_{\mathrm{y},t} - \hat{\mathsf{C}}^*_{\mathrm{y},t})^2 + (\mathsf{C}_{\mathrm{m},t} - \hat{\mathsf{C}}^*_{\mathrm{m},t})^2 + (\mathsf{C}_{\mathrm{o},t} - \hat{\mathsf{C}}^*_{\mathrm{o},t})^2\right) + \mathscr{I}$$
$$= -\frac{1}{2}\frac{1}{\sigma}\left((\tilde{\mathsf{c}}_{\mathrm{y},t} - (\hat{\mathsf{c}}^*_{\mathrm{y},t} - \mathsf{c}^*_{\mathrm{y},t}) - \Psi_t)^2 + (\tilde{\mathsf{c}}_{\mathrm{m},t} - (\hat{\mathsf{c}}^*_{\mathrm{m},t} - \mathsf{c}^*_{\mathrm{m},t}) - \Psi_t)^2 + (\tilde{\mathsf{c}}_{\mathrm{o},t} - (\hat{\mathsf{c}}^*_{\mathrm{o},t} - \mathsf{c}^*_{\mathrm{o},t}) - \Psi_t)^2\right) + \mathscr{I}$$
$$= -\frac{1}{2}\frac{1}{\sigma}\left(\tilde{\mathsf{c}}^2_{\mathrm{y},t} + \tilde{\mathsf{c}}^2_{\mathrm{m},t} + \tilde{\mathsf{c}}^2_{\mathrm{o},t}\right) + \mathscr{I} + \mathscr{O}_3, \quad \text{[A.20.32]}$$

where the third line uses that $\hat{\mathsf{C}}^*_{\mathrm{y},t}$, $\hat{\mathsf{C}}^*_{\mathrm{m},t}$, and $\hat{\mathsf{C}}^*_{\mathrm{o},t}$ depend on flexible-price output $\hat{Y}_t$ and are thus independent of monetary policy, while the final equality makes use of [A.20.5b] and [A.20.25]. Finally, observe that:

$$\hat{\varphi}^*_t\left(\mathsf{C}_{\mathrm{y},t} + \mathsf{C}_{\mathrm{m},t} + \mathsf{C}_{\mathrm{o},t}\right) = \hat{\varphi}^*_t\mathsf{Y}_t + \mathscr{O}_3 = \hat{\varphi}^*_t\hat{\mathsf{Y}}_t + \hat{\varphi}^*_t\Psi_t + \mathscr{O}_3 = \mathscr{I} + \mathscr{O}_3, \quad \text{[A.20.33]}$$

where the first equality uses [A.20.30a], the second uses [A.20.16], and the final one [A.20.5b] and that $\hat{Y}_t$

and $\hat{\varphi}^*$ are independent of monetary policy. Substituting [A.20.31], [A.20.32], and [A.20.33] into [A.20.29]:

$$\mathscr{W}_{t_0} = -\frac{1}{2}\sum_{t=t_0}^{\infty} \beta^{t-t_0}\mathbb{E}_{t_0-2}\left[2\Psi_t + \frac{1}{3}\frac{1}{\sigma}\left(\tilde{c}_{y,t}^2 + \tilde{c}_{m,t}^2 + \tilde{c}_{o,t}^2\right)\right]$$
$$-\frac{1}{2}\frac{1}{3}\left(\alpha - \frac{1}{\sigma}\right)\sum_{t=t_0-2}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0-2}\Bigg[\left(\frac{\beta}{1+\beta}\right)\left((\tilde{c}_{m,t+1} - \mathbb{E}_t\tilde{c}_{m,t+1}) + \beta(\mathbb{E}_{t+1}\tilde{c}_{o,t+2} - \mathbb{E}_t\tilde{c}_{o,t+2})\right)^2$$
$$+\beta^2\left(\tilde{c}_{o,t+2} - \mathbb{E}_{t+1}\tilde{c}_{o,t+2}\right)^2\Bigg] + \mathscr{I} + \mathscr{O}_3. \quad \text{[A.20.34]}$$

The results of Proposition 6 imply that:

$$\tilde{c}_{y,t} = -\gamma\phi\tilde{d}_t + \mathscr{O}_2, \quad \tilde{c}_{m,t} = -\gamma(1-\phi)\tilde{d}_t + \mathscr{O}_2, \quad \tilde{c}_{o,t} = \gamma\tilde{d}_t + \mathscr{O}_2, \quad \text{and } \mathbb{E}_t\tilde{d}_{t+1} = \lambda\tilde{d}_t + \mathscr{O}_2, \quad \text{[A.20.35]}$$

where $\tilde{d}_t \equiv d_t - d_t^*$, and thus:

$$\frac{1}{3}\left(\tilde{c}_{y,t}^2 + \tilde{c}_{m,t}^2 + \tilde{c}_{o,t}^2\right) = \frac{\gamma^2}{3}\left(1 + \phi^2 + (1-\phi)^2\right)\tilde{d}_t^2 + \mathscr{O}_3 = \frac{2\gamma^2}{3}\left(1 - \phi + \phi^2\right)\tilde{d}_t^2 + \mathscr{O}_3. \quad \text{[A.20.36]}$$

Using the results of [A.20.35], it follows that:

$$\tilde{c}_{o,t+2} - \mathbb{E}_{t+1}\tilde{c}_{o,t+2} = \gamma(\tilde{d}_{t+2} - \mathbb{E}_{t+1}\tilde{d}_{t+2}) + \mathscr{O}_2, \quad \text{and } (\tilde{c}_{m,t+1} - \mathbb{E}_t\tilde{c}_{m,t+1}) + \beta(\mathbb{E}_{t+1}\tilde{c}_{o,t+2} - \mathbb{E}_t\tilde{c}_{o,t+2})$$
$$= -\gamma(1-\phi)(\tilde{d}_{t+1} - \mathbb{E}_t\tilde{d}_{t+1}) + \gamma\beta(\mathbb{E}_{t+1}\tilde{d}_{t+2} - \mathbb{E}_t\tilde{d}_{t+1}) + \mathscr{O}_2,$$

and hence:

$$(\tilde{c}_{o,t+2} - \mathbb{E}_{t+1}\tilde{c}_{o,t+2})^2 = \gamma^2(\tilde{d}_{t+2} - \lambda\tilde{d}_{t+1})^2 + \mathscr{O}_3, \quad \text{and}$$
$$((\tilde{c}_{m,t+1} - \mathbb{E}_t\tilde{c}_{m,t+1}) + \beta(\mathbb{E}_{t+1}\tilde{c}_{o,t+2} - \mathbb{E}_t\tilde{c}_{o,t+2}))^2 = \gamma^2\left(1 - \phi - \beta\lambda\right)^2\left(\tilde{d}_{t+1} - \lambda\tilde{d}_t\right)^2 + \mathscr{O}_3.$$

These terms can be summed up as follows:

$$\sum_{t=t_0-2}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0-2}\left[\left(\frac{\beta}{1+\beta}\right)\left((\tilde{c}_{m,t+1} - \mathbb{E}_t\tilde{c}_{m,t+1}) + \beta(\mathbb{E}_{t+1}\tilde{c}_{o,t+2} - \mathbb{E}_t\tilde{c}_{o,t+2})\right)^2 + \beta^2\left(\tilde{c}_{o,t+2} - \mathbb{E}_{t+1}\tilde{c}_{o,t+2}\right)^2\right]$$
$$= \gamma^2\left(1 + \frac{(1-\phi-\beta\lambda))^2}{1+\beta}\right)\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0-2}\left[(\tilde{d}_t - \lambda\tilde{d}_{t-1})^2\right] + \mathscr{I} + \mathscr{O}_3, \quad \text{[A.20.37]}$$

where terms predetermined by time t_0 are independent of monetary policy and thus included in $\mathscr{I}$. The summation can be further simplified by expanding the bracket and making use of [A.20.35] to deduce:

$$\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0-2}\left[(\tilde{d}_t - \lambda\tilde{d}_{t-1})^2\right] = \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0-2}\left[\tilde{d}_t^2 - 2\lambda\tilde{d}_{t-1}\tilde{d}_t + \lambda^2\tilde{d}_{t-1}^2\right]$$
$$= (1+\beta\lambda^2)\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0-2}\tilde{d}_t^2 - 2\lambda\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0-2}\left[\tilde{d}_{t-1}\mathbb{E}_{t-1}\tilde{d}_t\right] = \left(1 - \beta\lambda^2\right)\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0-2}\tilde{d}_t^2 + \mathscr{I}.$$
[A.20.38]

Substituting equations [A.20.5a], [A.20.36], [A.20.37], and [A.20.38] into the expression for the social welfare function in equation [A.20.34] yields:

$$\mathscr{W}_{t_0} = -\frac{1}{2}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0-2}\left[\aleph\tilde{d}_t^2 + \varepsilon\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t)^2\right] + \mathscr{I} + \mathscr{O}_3, \quad \text{[A.20.39]}$$

where the coefficient $\aleph$ is given by:

$$\aleph = \frac{2}{3}\frac{\gamma^2}{\sigma}\left(1 - \phi + \phi^2\right) + \frac{\gamma^2}{3}\left(\alpha - \frac{1}{\sigma}\right)\left(1 - \beta\lambda^2\right)\left(1 + \frac{(1-\phi-\beta\lambda)^2}{1+\beta}\right). \quad \text{[A.20.40]}$$

This is clearly equivalent to the expression given in equation [5.12b].

Since $\varepsilon > 1$ and $\kappa > 0$, the coefficient on the squared inflation surprise is necessarily positive. For the loss function to be convex, it is necessary to establish that the coefficient $\aleph$ on the squared debt gap is always positive. First, write the formula for $\aleph$ from [A.20.40] as follows (making use of [A.20.36]):

$$\aleph = \frac{\gamma^2}{3}\left(\frac{1}{\sigma}\left((1+\phi^2+(1-\phi)^2)-(1-\beta\lambda^2)\left(1+\frac{(1-\phi-\beta\lambda)^2}{1+\beta}\right)\right)+\alpha(1-\beta\lambda^2)\left(1+\frac{(1-\phi-\beta\lambda)^2}{1+\beta}\right)\right). \quad \text{[A.20.41]}$$

Since $0<\beta<1$ and $|\lambda|<1$, the coefficient of α is positive, so it is sufficient to prove that $\aleph$ would be positive if α were zero. To do this, it must be established that:

$$\left(1+\phi^2+(1-\phi)^2\right)-(1-\beta\lambda^2)\left(1+\frac{(1-\phi-\beta\lambda)^2}{1+\beta}\right)>0. \quad \text{[A.20.42]}$$

Given the range of possible values for β and λ, it must be the case that $0<1-\beta\lambda^2<1$, hence to establish that [A.20.42] holds it is sufficient to show:

$$\phi^2+(1-\phi)^2>\frac{(1-\phi-\beta\lambda)^2}{1+\beta}. \quad \text{[A.20.43]}$$

The proof of Proposition 5 shows that ϕ, θ, and λ from [A.4.1c], [A.4.1d], and [A.4.1e] are such that $\theta=\lambda+\beta^{-1}\phi$. The inequality [A.20.43] is therefore equivalent to $(1+\beta)(\phi^2+(1-\phi)^2)>(1-\beta\theta)^2$, and to the following by expanding the brackets:

$$\left(\beta+2\beta\theta-2(1+\beta)\phi+\phi^2\right)+\left((1+2\beta)\phi^2-\beta^2\theta^2\right)>0.$$

Since $\phi>0$, the inequality above is equivalent to:

$$\left(\frac{\beta}{\phi^2}+\frac{2\beta\theta}{\phi^2}-\frac{2(1+\beta)\theta}{\phi}+1\right)+\left((1+2\beta)-\left(\frac{\beta\theta}{\phi}\right)^2\right)>0. \quad \text{[A.20.44]}$$

Defining $\Phi\equiv\gamma/\sigma$, the expressions for χ and ϕ from [A.4.1a] and [A.4.1c] and $\theta=\lambda+\beta^{-1}\phi$ imply that:

$$\frac{1}{\phi}=\frac{\chi^{-1}+\beta\Phi}{\beta+\beta\Phi},\quad \theta=\frac{(1+\beta(1+\lambda))\Phi}{\chi^{-1}+\beta\Phi},\quad \text{and}\quad \frac{\beta\theta}{\phi}=\frac{\Phi}{1+\Phi}\left(1+\beta(1+\lambda)\right). \quad \text{[A.20.45]}$$

Making use of the expression for $1/\phi$ from [A.20.45] leads to:

$$\frac{\beta}{\phi^2}+\frac{2\beta\theta}{\phi^2}-\frac{2(1+\beta)}{\phi}+1=\frac{(\chi^{-2}-2(1+\beta)\chi^{-1}+\beta)+\beta\Phi(2\chi^{-1}+2\chi^{-1}\lambda+\beta\Phi-2\beta+2\beta\lambda\Phi+\Phi)}{\beta(1+\Phi)^2}. \quad \text{[A.20.46]}$$

The formula for χ from [A.4.1a] implies that:

$$\chi^{-2}-2(1+\beta)\chi^{-1}+\beta=\beta\Phi(1+\beta-\chi^{-1}-\beta\Phi),$$

which can be substituted into [A.20.46] to obtain:

$$\frac{\beta}{\phi^2}+\frac{2\beta\theta}{\phi^2}-\frac{2(1+\beta)}{\phi}+1=\frac{\Phi}{(1+\Phi)^2}\left((1-\beta)+(1+2\lambda)\chi^{-1}+(1+2\beta\lambda)\Phi\right). \quad \text{[A.20.47]}$$

Using the expression for $\beta\theta/\phi$ from [A.20.45] and equation [A.20.47], the inequality [A.20.44] becomes:

$$\frac{(1-\beta)\Phi+(1+2\beta)(1+2\Phi)+(1+2\lambda)\chi^{-1}\Phi+(1-\beta^2(1+\lambda)^2)\Phi^2}{(1+\Phi)^2}>0.$$

This inequality is satisfied if and only if:

$$(1+2\beta)+\left((1-\beta)+(1+2\lambda)\chi^{-1}\right)\Phi+\Phi\mathscr{A}(\Phi)>0,\quad \text{where}\quad \mathscr{A}(\Phi)\equiv 2(1+2\beta)+(1-\beta^2(1+\lambda)^2)\Phi. \quad \text{[A.20.48]}$$

Since $0<\beta<1$ and $\lambda>-1/2$ according to Lemma 2, $\mathscr{A}(\Phi)>0$ for all valid $\Phi\equiv\gamma/\sigma$ values is sufficient to demonstrate that the inequality holds.

Lemma 2 establishes that λ is increasing in Φ. For low values of Φ, λ is negative, but satisfies $\lambda>-1/2$. Hence for sufficiently low Φ, λ is such that $\beta^2(1+\lambda)^2<1$ and hence $\mathscr{A}(\Phi)>0$. If Φ goes beyond the point where $\beta^2(1+\lambda)^2=1$ then $\mathscr{A}(\Phi)$ is decreasing in Φ in this range. Proposition 1 demonstrates that $\beta\Phi<(1+\beta)$ is a necessary condition for a unique equilibrium. Therefore to show that $\mathscr{A}(\Phi)>0$ for all valid parameter values it is sufficient to establish that $\mathscr{A}((1+\beta)/\beta)>0$. Substituting $\Phi=(1+\beta)/\beta$ into the expression for λ from [A.4.1e] shows that $\lambda = \beta$, and thus from the definition of $\mathscr{A}(\Phi)$ in [A.20.48]:

$$\mathscr{A}\left(\frac{1+\beta}{\beta}\right) = 2(1+2\beta) + \left(1-\beta^2(1+\beta)^2\right)\left(\frac{1+\beta}{\beta}\right).$$

This expression can be factorized as follows:

$$\mathscr{A}\left(\frac{1+\beta}{\beta}\right) = \frac{1}{\beta}\left((1-\beta^2)(1+3\beta+4\beta^2) + (1-\beta)\beta^4\right),$$

which proves that $\mathscr{A}((1+\beta)/\beta) > 0$ since $0 < \beta < 1$. Since [A.20.43] is sufficient to prove $\aleph > 0$ for all valid parameters, and as the inequalities [A.20.43] and [A.20.48] are equivalent, it is shown that $\aleph > 0$. This completes the proof.

A.21 Proof of Proposition 14

Starting from time t_0, the Pareto weights are those supporting the complete-markets equilibrium with financial markets open for securities paying off at time t_0 and later. The calculation of the debt gap $\tilde{\mathsf{d}}_t$ then depends on the starting point t_0 since $\mathsf{d}^*_{t_0-1} = \mathsf{d}_{t_0-1}$, hence $\tilde{\mathsf{d}}_{t_0-1} = 0$.

The Lagrangian for the problem of minimizing [5.12a] subject to the two constraints in [5.13] is

$$\mathcal{L}_{t_0} = \frac{1}{2}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\aleph\tilde{\mathsf{d}}_t^2 + \varepsilon\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t)^2\right] + \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\daleth_t\left\{\lambda\tilde{\mathsf{d}}_t - \mathbb{E}_t\tilde{\mathsf{d}}_{t+1}\right\}\right]$$
$$+\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\beth_t\left\{\mathsf{i}_{t-1} - \tilde{\mathsf{d}}_t - \beta^{-1}\phi\tilde{\mathsf{d}}_{t-1} - \mu\pi_t - (1-\mu)\mathbb{E}_{t-1}\pi_t - \mathsf{r}^*_t\right\}\right], \quad \text{[A.21.1]}$$

where the Lagrangian multipliers are $\daleth_t$ and $\beth_t$ (each scaled by β^{t-t_0} for convenience). The first-order conditions with respect to each of the endogenous variables $\tilde{\mathsf{d}}_t$, π_t, and i_t at $t \geq t_0$ are:

$$\aleph\tilde{\mathsf{d}}_t + \lambda\daleth_t - \beta^{-1}\daleth_{t-1} - \beth_t - \phi\mathbb{E}_t\beth_{t+1} = 0; \quad \text{[A.21.2a]}$$
$$\varepsilon\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t) - \mu\beth_t - (1-\mu)\mathbb{E}_{t-1}\beth_t = 0; \quad \text{[A.21.2b]}$$
$$\mathbb{E}_t\beth_{t+1} = 0. \quad \text{[A.21.2c]}$$

There is no constraint corresponding to the Lagrangian multiplier $\daleth_{t_0-1}$, hence $\daleth_{t_0-1} = 0$.

Taking the expectation of equation [A.21.2a] at time $t+1$ conditional on time t information, multiplying both sides by β, and using [A.21.2c] to eliminate terms in $\mathbb{E}_t\beth_{t+1}$ yields:

$$\daleth_t = \beta\lambda\mathbb{E}_t\daleth_{t+1} + \beta\aleph\mathbb{E}_t\tilde{\mathsf{d}}_{t+1}.$$

An expression for $\daleth_t$ can be obtained by repeated forward substitution of this equation and using [5.13] to deduce that $\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell} = \lambda^\ell\tilde{\mathsf{d}}_t$:

$$\daleth_t = \beta\aleph\sum_{\ell=1}^{\infty}(\beta\lambda)^{\ell-1}\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell} = \beta\lambda\aleph\left\{\sum_{\ell=1}^{\infty}(\beta\lambda^2)^{\ell-1}\right\}\tilde{\mathsf{d}}_t = \frac{\beta\lambda}{1-\beta\lambda^2}\aleph\tilde{\mathsf{d}}_t.$$

This equation holds for all $t \geq t_0$. Using the formula for $\daleth_t$, it follows that for all $t \geq t_0 + 1$:

$$\aleph\tilde{\mathsf{d}}_t + \lambda\daleth_t - \beta^{-1}\daleth_{t-1} = \frac{\aleph}{1-\beta\lambda^2}(\tilde{\mathsf{d}}_t - \lambda\tilde{\mathsf{d}}_{t-1}).$$

Since $\daleth_{t_0-1} = 0$ and $\tilde{\mathsf{d}}_{t_0-1} = 0$, this equation also holds for $t = t_0$. Substituting this result into [A.21.2a], using [A.21.2c], and noting [5.13] implies $\mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1}$, yields an expression for $\beth_t$:

$$\beth_t = \frac{\aleph}{1-\beta\lambda^2}(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t), \quad \text{[A.21.3]}$$

which is valid for all $t \geq t_0$.

Taking conditional expectations of [A.21.2b] at $t = t_0$ using period t_0-1 information implies $\mathbb{E}_{t_0-1}\beth_{t_0} = 0$. Together with [A.21.2c], it follows that $\mathbb{E}_{t-1}\beth_t = 0$ for all $t \geq t_0$. Making use of this finding, a further expression for $\beth_t$ can be obtained by dividing both sides of [A.21.2b] by μ:

$$\beth_t = \frac{\varepsilon\kappa}{\mu}(\pi_t - \mathbb{E}_{t-1}\pi_t).$$

Using this equation to eliminate $\beth_t$ from [A.21.3] implies:

$$\frac{\aleph}{1-\beta\lambda^2}(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t) = \frac{\varepsilon\kappa}{\mu}(\pi_t - \mathbb{E}_{t-1}\pi_t) = 0,$$

which yields the first-order condition [5.14].

Suppose monetary policy achieves the target [5.15] with $\hat{\varpi}$ and $\varpi^\dagger$ as defined in [5.15] and [4.19] respectively. This implies that

$$(\mathsf{P}_t - \mathbb{E}_{t-1}\mathsf{P}_t) + \hat{\varpi}\varpi^\dagger(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t) = -\hat{\varpi}\varpi^\dagger(\mathsf{d}^*_t - \mathbb{E}_{t-1}\mathsf{d}^*_t). \qquad \text{[A.21.4]}$$

Using the definition of the debt gap $\tilde{\mathsf{d}}_t = \mathsf{d}_t - \mathsf{d}^*_t$ the above equation can be written as:

$$\hat{\varpi}\varpi^\dagger(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t) = (\mathsf{P}_t - \mathbb{E}_{t-1}\mathsf{P}_t) + \hat{\varpi}\varpi^\dagger(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t) + \hat{\varpi}\varpi^\dagger(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t). \qquad \text{[A.21.5]}$$

For the case of a general debt maturity parameter μ, Proposition 12 shows that the unexpected component $\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t$ of the debt-to-GDP ratio satisfies equation [4.19]. Multiplying both sides of that equation by $\hat{\varpi}$ yields:

$$\hat{\varpi}\varpi^\dagger(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t) = -\hat{\varpi}(\mathsf{P}_t - \mathbb{E}_{t-1}\mathsf{P}_t) - \hat{\varpi}\varpi^\dagger(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t),$$

and then substituting into [A.21.5]:

$$\hat{\varpi}\varpi^\dagger(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t) = (1-\hat{\varpi})(\mathsf{P}_t - \mathbb{E}_{t-1}\mathsf{P}_t).$$

Since $\mathsf{P}_t - \mathbb{E}_{t-1}\mathsf{P}_t = \pi_t - \mathbb{E}_{t-1}\pi_t$, this equation is equivalent to:

$$\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = \frac{1}{\varpi^\dagger}\left(\frac{1-\hat{\varpi}}{\hat{\varpi}}\right)(\pi_t - \mathbb{E}_{t-1}\pi_t).$$

Using the formulas for $\varpi^\dagger$ and $\hat{\varpi}$ from [4.19] and [5.15], it can be seen that the coefficient of $\pi_t - \mathbb{E}_{t-1}\pi_t$ in the equation above is the same as that in the first-order condition [5.14].

Finally, consider the case where TFP (and hence GDP) is described by the stochastic process [4.10]. If monetary policy achieves the target $\mathsf{P}_t + \hat{\varpi}\varpi^\dagger\varpi^*\mathsf{Y}_t = 0$ (where ϖ^* is as defined in [4.11]) then the following equation holds:

$$(\mathsf{P}_t - \mathbb{E}_{t-1}\mathsf{P}_t) + \hat{\varpi}\varpi^\dagger\varpi^*(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t) = 0. \qquad \text{[A.21.6]}$$

Given the stochastic process [4.10], equation [4.11] must hold. Multiplying both sides of that equation by $\hat{\varpi}\varpi^\dagger$ implies:

$$\hat{\varpi}\varpi^\dagger\varpi^*(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t) = \hat{\varpi}\varpi^\dagger(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t) + \hat{\varpi}\varpi^\dagger(\mathsf{d}^*_t - \mathbb{E}_{t-1}\mathsf{d}^*_t).$$

Substituting this into [A.21.6] yields equation [A.21.4]. This completes the proof.

A.22 Proof of Proposition 15

The loss function

With Pareto weights $\Omega^*_t = 1$ supporting the complete-markets equilibrium (irrespective of the level of output), $\beta = \delta$ according to [2.5] with $\sigma = 1$, and utility function [A.6.1], the welfare function in [3.3] becomes:

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0-2}\left[\frac{1}{3}\sum_{t=t_0-2}^{\infty}\beta^{t-t_0}\left\{\log C_{\mathrm{y},t} + \beta\log C_{\mathrm{m},t+1} + \beta^2\log C_{\mathrm{o},t+2} - \frac{H^\eta_{\mathrm{y},t}}{\eta\Theta^{\eta-1}_{\mathrm{y}}} - \beta\frac{H^\eta_{\mathrm{m},t+1}}{\eta\Theta^{\eta-1}_{\mathrm{m}}} - \beta^2\frac{H^\eta_{\mathrm{o},t+2}}{\eta\Theta^{\eta-1}_{\mathrm{o}}}\right\}\right].$$

Changing the order of summation leads to the following expression:

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0-2}\left[\frac{1}{3}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left\{(\log C_{\mathrm{y},t} + \log C_{\mathrm{m},t} + \log C_{\mathrm{o},t}) - \frac{1}{\eta}\left(\frac{H^\eta_{\mathrm{y},t}}{\Theta^{\eta-1}_{\mathrm{y}}} + \frac{H^\eta_{\mathrm{m},t}}{\Theta^{\eta-1}_{\mathrm{m}}} + \frac{H^\eta_{\mathrm{o},t}}{\Theta^{\eta-1}_{\mathrm{o}}}\right)\right\}\right] + \mathscr{I}, \qquad \text{[A.22.1]}$$

where $\mathscr{I}$ denotes terms independent of monetary policy, which here includes terms predetermined as of time t_0. Using the definitions $c_{j,t} \equiv C_{j,t}/Y_t$ and $\tilde{Y}_t \equiv Y_t/\hat{Y}_t^*$ and the steady-state values $\bar{c}_j = 1$ and $\bar{\tilde{Y}} = 1$, it follows that the terms in consumption can be written as:

$$\frac{1}{3}\left(\log C_{\mathrm{y},t} + \log C_{\mathrm{m},t} + \log C_{\mathrm{o},t}\right) = \tilde{\mathsf{Y}}_t + \frac{1}{3}(\mathsf{c}_{\mathrm{y},t} + \mathsf{c}_{\mathrm{m},t} + \mathsf{c}_{\mathrm{o},t}) + \mathscr{I}, \quad \text{[A.22.2]}$$

where terms in $\hat{Y}_t^* = A_t$ are included in $\mathscr{I}$ because TFP A_t is independent of monetary policy. A second-order accurate approximation of the resource constraint [2.16] is:

$$\frac{1}{3}\left(\mathsf{c}_{\mathrm{y},t} + \mathsf{c}_{\mathrm{m},t} + \mathsf{c}_{\mathrm{o},t}\right) = -\frac{1}{2}\frac{1}{3}\left(\mathsf{c}_{\mathrm{y},t}^2 + \mathsf{c}_{\mathrm{m},t}^2 + \mathsf{c}_{\mathrm{o},t}^2\right) + \mathscr{O}_3, \quad \text{[A.22.3]}$$

where $\mathscr{O}_3$ denotes third- and higher-order terms. Combining the results of Proposition 2 (with $\alpha = 1$ and $\sigma = 1$) and Proposition 6 leads to the following expressions for the consumption ratios:

$$\mathsf{c}_{\mathrm{y},t} = -\gamma\phi\mathsf{d}_t + \mathscr{O}_2, \quad \mathsf{c}_{\mathrm{m},t} = -\gamma(1-\phi)\mathsf{d}_t + \mathscr{O}_2, \quad \text{and} \;\; \mathsf{c}_{\mathrm{o},t} = \gamma\mathsf{d}_t + \mathscr{O}_2. \quad \text{[A.22.4]}$$

By substituting these into [A.22.3], equation [A.22.2] can be written as:

$$\frac{1}{3}\left(\log C_{\mathrm{y},t} + \log C_{\mathrm{m},t} + \log C_{\mathrm{o},t}\right) = \tilde{\mathsf{Y}}_t - \frac{1}{2}\frac{\gamma^2}{3}\left(1 + \phi^2 + (1-\phi)^2\right)\mathsf{d}_t^2 + \mathscr{I} + \mathscr{O}_3. \quad \text{[A.22.5]}$$

Writing $H_{j,t}^{\eta} = H_{j,t}H_{j,t}^{\eta-1}$ and noting that equation [A.6.3] implies $(H_{j,t}/\Theta_i)^{\eta-1} = w_{j,t}/C_{j,t}$, the terms in the disutility of labour at time t from [A.22.1] can be written as follows using the definition $c_{j,t} \equiv C_{j,t}/Y_t$:

$$\frac{H_{\mathrm{y},t}^{\eta}}{\Theta_{\mathrm{y}}^{\eta-1}} + \frac{H_{\mathrm{m},t}^{\eta}}{\Theta_{\mathrm{m}}^{\eta-1}} + \frac{H_{\mathrm{o},t}^{\eta}}{\Theta_{\mathrm{o}}^{\eta-1}} = \frac{1}{Y_t}\left(\frac{w_{\mathrm{y},t}H_{\mathrm{y},t}}{c_{\mathrm{y},t}} + \frac{w_{\mathrm{m},t}H_{\mathrm{m},t}}{c_{\mathrm{m},t}} + \frac{w_{\mathrm{o},t}H_{\mathrm{o},t}}{c_{\mathrm{o},t}}\right).$$

Using equation [A.6.11] together with the aggregate production function from [A.6.10], the wage-bill subsidy $s = \varepsilon^{-1}$, and real marginal cost $k_t = w_t/A_t$:

$$\frac{H_{\mathrm{y},t}^{\eta}}{\Theta_{\mathrm{y}}^{\eta-1}} + \frac{H_{\mathrm{m},t}^{\eta}}{\Theta_{\mathrm{m}}^{\eta-1}} + \frac{H_{\mathrm{o},t}^{\eta}}{\Theta_{\mathrm{o}}^{\eta-1}} = \left(\frac{k_t\Psi_t}{(1-\varepsilon^{-1})}\right)\left(\frac{\Theta_{\mathrm{y}}}{c_{\mathrm{y},t}} + \frac{\Theta_{\mathrm{m}}}{c_{\mathrm{m},t}} + \frac{\Theta_{\mathrm{o}}}{c_{\mathrm{o},t}}\right). \quad \text{[A.22.6]}$$

Substituting the expression for real marginal cost from [A.6.13] and the parameterization of Θ_{y}, Θ_{m}, and Θ_{o} from [2.6], and using the definition of the output gap $\tilde{Y}_t \equiv Y_t/\hat{Y}_t^*$ with $\hat{Y}_t^* = A_t$:

$$\frac{H_{\mathrm{y},t}^{\eta}}{\Theta_{\mathrm{y}}^{\eta-1}} + \frac{H_{\mathrm{m},t}^{\eta}}{\Theta_{\mathrm{m}}^{\eta-1}} + \frac{H_{\mathrm{o},t}^{\eta}}{\Theta_{\mathrm{o}}^{\eta-1}} = \Psi_t^{\eta}\tilde{Y}_t^{\eta}\left(c_{\mathrm{y},t}^{\frac{1-\beta\gamma}{3}} c_{\mathrm{m},t}^{\frac{1+(1+\beta)\gamma}{3}} c_{\mathrm{o},t}^{\frac{1-\gamma}{3}}\right)\left(\frac{1-\beta\gamma}{c_{\mathrm{y},t}} + \frac{1+(1+\beta)\gamma}{c_{\mathrm{m},t}} + \frac{1-\gamma}{c_{\mathrm{o},t}}\right). \quad \text{[A.22.7]}$$

Now note the following second-order accurate approximations:

$$\tilde{Y}_t^{\eta} = 1 + \eta\tilde{\mathsf{Y}}_t + \frac{\eta^2}{2}\tilde{\mathsf{Y}}_t^2 + \mathscr{O}_3, \quad \Psi_t^{\eta} = 1 + \eta\Psi_t + \frac{\eta^2}{2}\Psi_t^2 + \mathscr{O}_3, \quad \Psi_t = \frac{\varepsilon\kappa}{2}(\pi_t - \mathbb{E}_{t-1}\pi_t)^2 + \mathscr{O}_3,$$

$$c_{\mathrm{y},t}^{\frac{1-\beta\gamma}{3}} c_{\mathrm{m},t}^{\frac{1+(1+\beta)\gamma}{3}} c_{\mathrm{o},t}^{\frac{1-\gamma}{3}} = 1 + \left(\frac{1-\beta\gamma}{3}\mathsf{c}_{\mathrm{y},t} + \frac{1+(1+\beta)\gamma}{3}\mathsf{c}_{\mathrm{m},t} + \frac{1-\gamma}{3}\mathsf{c}_{\mathrm{o},t}\right)$$
$$+ \frac{1}{2}\left(\frac{1-\beta\gamma}{3}\mathsf{c}_{\mathrm{y},t} + \frac{1+(1+\beta)\gamma}{3}\mathsf{c}_{\mathrm{m},t} + \frac{1-\gamma}{3}\mathsf{c}_{\mathrm{o},t}\right)^2 + \mathscr{O}_3, \quad \text{and} \quad \frac{1}{c_{j,t}} = 1 - \mathsf{c}_{j,t} + \frac{1}{2}\mathsf{c}_{j,t}^2 + \mathscr{O}_3,$$

where the expression for Ψ_t follows from $\Psi_t = (\varepsilon\kappa/2)\check{\mathsf{p}}_t^2 + \mathscr{O}_3$ derived from [5.9b] together with $\check{\mathsf{p}}_t = -(\pi_t - \mathbb{E}_{t-1}\pi_t) + \mathscr{O}_2$ from [5.10]. Using these results in [A.22.7] leads to:

$$\frac{1}{3}\frac{1}{\eta}\Psi_t^{\eta}\tilde{Y}_t^{\eta}\left(c_{\mathrm{y},t}^{\frac{1-\beta\gamma}{3}} c_{\mathrm{m},t}^{\frac{1+(1+\beta)\gamma}{3}} c_{\mathrm{o},t}^{\frac{1-\gamma}{3}}\right)\left(\frac{1-\beta\gamma}{c_{\mathrm{y},t}} + \frac{1+(1+\beta)\gamma}{c_{\mathrm{m},t}} + \frac{1-\gamma}{c_{\mathrm{o},t}}\right)$$
$$= \frac{\varepsilon\kappa}{2}(\pi_t - \mathbb{E}_{t-1}\pi_t)^2 + \tilde{\mathsf{Y}}_t + \frac{\eta}{2}\tilde{\mathsf{Y}}_t^2 + \frac{1}{2}\frac{1}{\eta}\left(\frac{1-\beta\gamma}{3}\mathsf{c}_{\mathrm{y},t}^2 + \frac{1+(1+\beta)\gamma}{3}\mathsf{c}_{\mathrm{m},t}^2 + \frac{1-\gamma}{3}\mathsf{c}_{\mathrm{o},t}^2\right)$$
$$- \frac{1}{2}\frac{1}{\eta}\left(\frac{1-\beta\gamma}{3}\mathsf{c}_{\mathrm{y},t} + \frac{1+(1+\beta)\gamma}{3}\mathsf{c}_{\mathrm{m},t} + \frac{1-\gamma}{3}\mathsf{c}_{\mathrm{o},t}\right)^2 + \mathscr{I} + \mathscr{O}_3. \quad \text{[A.22.8]}$$

Equation [A.6.15] gives the following first-order accurate approximation:

$$\left(\frac{1-\beta\gamma}{3}\mathsf{c}_{\mathrm{y},t} + \frac{1+(1+\beta)\gamma}{3}\mathsf{c}_{\mathrm{m},t} + \frac{1-\gamma}{3}\mathsf{c}_{\mathrm{o},t}\right) = -\nu\mathsf{d}_t + \mathscr{O}_2, \quad \text{[A.22.9]}$$

where the coefficient ν defined in [A.6.15] satisfies:

$$\nu = \frac{1}{3}\left(\gamma\phi(1-\beta\gamma) + \gamma(1+(1+\beta)\gamma)(1-\phi) - \gamma(1-\gamma)\right). \quad \text{[A.22.10]}$$

Note that by using [A.22.9] and the expressions for the consumption ratios in [A.22.4]:

$$\left(\frac{1-\beta\gamma}{3}c_{y,t}^2 + \frac{1+(1+\beta)\gamma}{3}c_{m,t}^2 + \frac{1-\gamma}{3}c_{o,t}^2\right) - \left(\frac{1-\beta\gamma}{3}c_{y,t} + \frac{1+(1+\beta)\gamma}{3}c_{m,t} + \frac{1-\gamma}{3}c_{o,t}\right)^2$$
$$= \frac{1-\beta\gamma}{3}(c_{y,t}+\nu d_t)^2 + \frac{1+(1+\beta)\gamma}{3}(c_{m,t}+\nu d_t)^2 + \frac{1-\gamma}{3}(c_{o,t}+\nu d_t)^2 + \mathscr{O}_3$$
$$= \left(\frac{1-\beta\gamma}{3}(-\gamma\phi+\nu)^2 + \frac{1+(1+\beta)\gamma}{3}(-\gamma(1-\phi)+\nu)^2 + \frac{1-\gamma}{3}(\gamma+\nu)^2\right)d_t^2 + \mathscr{O}_3. \quad \text{[A.22.11]}$$

The coefficient of d_t^2 can be rewritten as follows using the formula for ν from [A.22.10]:

$$\left(\frac{1-\beta\gamma}{3}(-\gamma\phi+\nu)^2 + \frac{1+(1+\beta)\gamma}{3}(-\gamma(1-\phi)+\nu)^2 + \frac{1-\gamma}{3}(\gamma+\nu)^2\right)$$
$$= \frac{1}{3}\left(\gamma^2(1+\phi^2+(1-\phi)^2) + \gamma^3((1+\beta)(1-\phi)^2 - \beta\phi^2 - 1) - \nu^2\right). \quad \text{[A.22.12]}$$

Combining equations [A.22.1], [A.22.5], [A.22.8], [A.22.11], and [A.22.12] implies an expression for welfare:

$$\mathscr{W}_{t_0} = -\frac{1}{2}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0-2}\left[\aleph d_t^2 + \epsilon\kappa(\pi - \mathbb{E}_{t-1}\pi_t)^2 + \eta\tilde{Y}_t^2\right] + \mathscr{I} + \mathscr{O}_3,$$

where the coefficient $\aleph$ is as defined in [A.6.20]. This confirms the loss function given in equation [A.6.19].

Optimal monetary policy

Setting up the Lagrangian for minimizing loss function [A.6.19] subject to the constraints [A.6.18a]–[A.6.18c]:

$$\mathcal{L}_{t_0} = \frac{1}{2}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\aleph d_t^2 + \epsilon\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t)^2 + \eta\tilde{Y}_t^2\right] + \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\daleth_t\left\{\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t) - \eta\tilde{Y}_t + \nu d_t\right\}\right]$$
$$+ \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\urcorner_t\left\{\lambda d_t - \mathbb{E}_t d_{t+1}\right\} + \beth_t\left\{i_{t-1} - d_t - \beta^{-1}\phi d_{t-1} - \pi_t - \tilde{Y}_t + \tilde{Y}_{t-1} - \hat{r}_t^*\right\}\right].$$

[A.22.13]

The first-order conditions of [A.22.13] with respect to the endogenous variables π_t, d_t, $\tilde{Y}_t$, and i_t are:

$$\aleph d_t + \nu\daleth_t + \lambda\urcorner_t - \beta^{-1}\urcorner_{t-1} - \beth_t - \phi\mathbb{E}_t\beth_{t+1} = 0; \quad \text{[A.22.14a]}$$
$$\epsilon\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t) + \kappa(\daleth_t - \mathbb{E}_{t-1}\daleth_t) - \beth_t = 0; \quad \text{[A.22.14b]}$$
$$\eta\tilde{Y}_t - \eta\daleth_t - \beth_t + \beta\mathbb{E}_t\beth_{t+1} = 0; \quad \text{[A.22.14c]}$$
$$\beta\mathbb{E}_t\beth_{t+1} = 0. \quad \text{[A.22.14d]}$$

Substituting [A.22.14d] into equation [A.22.14c] and solving for $\daleth_t$:

$$\daleth_t = \tilde{Y}_t - \frac{1}{\eta}\beth_t.$$

The Phillips curve [A.6.18a] implies $\tilde{Y}_t = (\kappa/\eta)(\pi_t - \mathbb{E}_{t-1}\pi_t) + (\nu/\eta)d_t$, which can be combined with the equation above to obtain:

$$\daleth_t = \frac{\kappa}{\eta}(\pi_t - \mathbb{E}_{t-1}\pi_t) + \frac{\nu}{\eta}d_t - \frac{1}{\eta}\beth_t. \quad \text{[A.22.15]}$$

Taking expectations of the above equation at time $t+1$ conditional on period-t information and making use of equation [A.22.14d]:

$$\mathbb{E}_t\daleth_{t+1} = \frac{\nu}{\eta}\mathbb{E}_t d_{t+1}. \quad \text{[A.22.16]}$$

Considering equation [A.22.14a] at time $t+1$, multiplying both sides by β, and taking the period-t conditional expectation leads to:

$$\daleth_t = \beta\lambda\mathbb{E}_t\daleth_{t+1} + \beta\mathbb{E}_t\left[\aleph\mathsf{d}_{t+1} + \nu\gimel_{t+1}\right],$$

where equation [A.22.14d] has been used to note that $\mathbb{E}_t\beth_{t+1} = 0$. Substituting from equation [A.22.16] implies:

$$\daleth_t = \beta\lambda\mathbb{E}_t\daleth_{t+1} + \beta\left(\aleph + \frac{\nu^2}{\eta}\right)\mathbb{E}_t\mathsf{d}_{t+1}.$$

Solving forwards and using [A.6.18c] to deduce that $\mathbb{E}_t\mathsf{d}_{t+\ell} = \lambda^\ell\mathsf{d}_t$ yields the following expression for $\daleth_t$:

$$\daleth_t = \left(\aleph + \frac{\nu^2}{\eta}\right)\left(\frac{\beta\lambda}{1-\beta\lambda^2}\right)\mathsf{d}_t. \qquad \text{[A.22.17]}$$

Now substituting the expressions for $\gimel_t$ and $\daleth_t$ from [A.22.15] and [A.22.17] into [A.22.14a] and using [A.22.14d] to set $\mathbb{E}_t\beth_{t+1} = 0$:

$$\aleph\mathsf{d}_t + \nu\left(\frac{\kappa}{\eta}(\pi_t - \mathbb{E}_{t-1}\pi_t) + \frac{\nu}{\eta}\mathsf{d}_t - \frac{1}{\eta}\beth_t\right) + \left(\aleph + \frac{\nu^2}{\eta}\right)\left(\frac{\beta\lambda}{1-\beta\lambda^2}\right)\left(\lambda\mathsf{d}_t - \frac{1}{\beta}\mathsf{d}_{t-1}\right) - \beth_t = 0.$$

Simplifying this equation yields:

$$\left(1 + \frac{\nu}{\eta}\right)\beth_t = \left(\frac{\aleph + \frac{\nu^2}{\eta}}{1-\beta\lambda^2}\right)(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t) + \frac{\kappa\nu}{\eta}(\pi_t - \mathbb{E}_{t-1}\pi_t), \qquad \text{[A.22.18]}$$

where equation [A.6.18c] has been used to write $\lambda\mathsf{d}_{t-1} = \mathbb{E}_{t-1}\mathsf{d}_t$.

Equating the unexpected components of both sides of [A.22.15] and using [A.22.14d] implies:

$$\gimel_t - \mathbb{E}_{t-1}\gimel_t = \frac{\kappa}{\eta}(\pi_t - \mathbb{E}_{t-1}\pi_t) + \frac{\nu}{\eta}(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t) - \frac{1}{\eta}\beth_t. \qquad \text{[A.22.19]}$$

Substituting this into the first-order condition [A.22.14b] leads to the following equation:

$$\left(1 + \frac{\kappa}{\eta}\right)\beth_t = \kappa\left(\left(\varepsilon + \frac{\kappa}{\eta}\right)(\pi_t - \mathbb{E}_{t-1}\pi_t) + \frac{\nu}{\eta}(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t)\right). \qquad \text{[A.22.20]}$$

Multiplying both sides of [A.22.18] by $(1+\kappa/\eta)$ and both sides of [A.22.20] by $(1+\nu/\eta)$ allows $\beth_t$ to be eliminated:

$$\left(1 + \frac{\kappa}{\eta}\right)\left(\left(\frac{\aleph + \frac{\nu^2}{\eta}}{1-\beta\lambda^2}\right)(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t) + \frac{\kappa\nu}{\eta}(\pi_t - \mathbb{E}_{t-1}\pi_t)\right)$$
$$= \kappa\left(1 + \frac{\nu}{\eta}\right)\left(\left(\varepsilon + \frac{\kappa}{\eta}\right)(\pi_t - \mathbb{E}_{t-1}\pi_t) + \frac{\nu}{\eta}(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t)\right),$$

and simplifying this equation leads to:

$$\left(\left(\frac{\aleph + \frac{\nu^2}{\eta}}{1-\beta\lambda^2}\right)\left(1 + \beta\lambda^2\frac{\kappa}{\eta}\right) + (\aleph - \nu)\frac{\kappa}{\eta}\right)(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t) = \kappa\left(\varepsilon + \frac{\kappa}{\eta} + (\varepsilon - 1)\frac{\nu}{\eta}\right)(\pi_t - \mathbb{E}_{t-1}\pi_t).$$

This confirms the first-order condition [A.6.21].

Achieving the target $\mathsf{P}_t + \hat{\varpi}\mathsf{Y}_t = 0$ (where $\hat{\varpi}$ is as defined in [A.6.22]) implies:

$$(\mathsf{P}_t - \mathbb{E}_{t-1}\mathsf{P}_t) + \hat{\varpi}(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t) = 0. \qquad \text{[A.22.21]}$$

Multiplying both sides of [4.8] by $\hat{\varpi}$ and using the definition of nominal GDP $\mathsf{M}_t = \mathsf{P}_t + \mathsf{Y}_t$:

$$\hat{\varpi}(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t) = -\hat{\varpi}(\mathsf{P}_t - \mathbb{E}_{t-1}\mathsf{P}_t) - \hat{\varpi}(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t).$$

Substituting this into equation [A.22.21], noting $\mathsf{P}_t - \mathbb{E}_{t-1}\mathsf{P}_t = \pi_t - \mathbb{E}_{t-1}\pi_t$, and rearranging:

$$\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t = \left(\frac{1-\hat{\varpi}}{\hat{\varpi}}\right)(\pi_t - \mathbb{E}_{t-1}\pi_t).$$

The expression for $\hat{\varpi}$ in [A.6.22] shows this implies [A.6.21]. This completes the proof.

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The Centre for Economic Performance Publications Unit
Tel 020 7955 7673 Fax 020 7404 0612
Email info@cep.lse.ac.uk Web site http://cep.lse.ac.uk