VALUATION AND MARTINGALE PROPERTIES OF SHADOW PRICES †

An Exposition

by

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Abstract

Concepts of asset valuation based on the martingale properties of shadow (or marginal utility) prices in continuous-time, infinite-horizon stochastic models of optimal saving and portfolio choice are reviewed and compared with their antecedents in static or deterministic economic theory. Applications of shadow pricing to valuation are described, including a new derivation of the Black-Scholes formula and a generalised net present value formula for valuing an indivisible project yielding a random income. Some new results are presented concerning (i) the characterisation of an optimum in a model of saving with an exogenous random income and (ii) the use of random time transforms to replace local by true martingales in the martingale and transversality conditions for optimal saving and portfolio choice.

Keywords: Valuation, Investment, Optimisation, Continuous Time, Martingales, Transversality, Time Change

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INTRODUCTION

It is well known that an optimal plan for saving and portfolio choice, over an infinite horizon in continuous time, can be characterised by means of the martingale and transversality properties of the associated shadow prices – the shadow price, or 'marginal utility price', of an asset or portfolio being defined as the product $y = z \cdot v$ of its returns or market price process $z = z(\omega,t)$ and the marginal utility process $v = v(\omega,t)$ calculated along an optimal plan. The present essay expounds and extends these properties, with special reference to their place in the economic theory of value and their applications to security and project valuation.

Regarding the theory of value, the essential points to be made at the outset are that the martingale properties of shadow prices may be regarded as generalisations to a dynamic stochastic setting of the principle of equi-marginal utility, while the transversality conditions correspond to the principles that an (unsatiated) consumer should spend the entire budget and that redundant resources attract zero prices. This foundation in fundamental concepts of economic theory lends a treatment of valuation based on marginal utility prices a remarkable degree of universality and transparency, and often affords insights which lead to simple and intuitively acceptable formulas.

The literature on valuation is fragmented, often proposing different theories and techniques for different types of assets and different situations. Thus there are theories for the valuation of 'underlying' securities, for derivatives and for indivisible projects, theories for complete and for incomplete markets, for continuous and for discrete time, for finite and for infinite horizons, and so forth. In general texts, the various models tend to be presented seriatim, perhaps in ascending order of mathematical difficulty or generality, with a profusion of mathematical methods and assumptions, often chosen for technical convenience in a particular discussion. Inevitably, the models are largely inspired by ideas from economic theory, but contact with these ideas (particularly in their older and simpler forms) is easily lost in the mass of technicalities. A substantial part of the recent literature restricts attention to the valuation of derivative securities in the setting of a complete market, taking as given the price processes of the 'underlying' securities. Interest centres on special mathematical methods which are convenient for this class of problems and contact with the general theory of value is minimal from the outset – the only major principles borrowed from it being (loosely speaking) that more money is better than less, and one or more forms of the 'law of one price'.[1]

The methodological stance of the present essay, briefly stated, is that the natural way to develop the subject of financial asset valuation is to start with the general concepts and principles of the economic theory of value, to classify particular problems and techniques according to criteria suggested by this theory, and to treat them, so far as is reasonable, as special cases within the general framework. It would also be desirable, where possible, to construct mathematical proofs so as to reflect underlying economic ideas, rather than simply to transplant the terminology and intuitions of physics. Naturally this is not a programme which can be implemented in a single paper – nor is it suggested that we must make a fresh start, since most of the material required exists in the present literature. The issues just raised concern rather the method and order of presentation of basic ideas. Section 1 below presents a review of some concepts of valuation along the lines suggested.

In order to keep the exposition within bounds, we shall concentrate attention on the valuation of assets within the framework of an optimal plan for an individual agent, (although some of the discussion can be re-interpreted as relating to the

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[1] Some practitioners even affect to despise more general theories of economics based on concepts of utility etc, and regard financial valuation as a 'stand-alone' branch of applied mathematics which finds its practical expression in so-called 'financial engineering'. Enthusiasm for methods of hedging and valuation of derivatives in complete markets, and for associated methods of computation, seems often to obscure the fact that these techniques do not provide a general theory of valuation and that they are liable to give at best only imprecise results when applied beyond their proper domain.

equilibrium of an exchange economy with identical 'representative' agents). The main topics will therefore be the conditions characterising an optimal consumption and investment plan for the agent, together with the valuation of assets within the plan and the valuation of changes. In support of our methodological position, it is noteworthy that the martingale and transversality properties of shadow prices which characterise an optimum, and the resulting formulas for valuation, can be proved fairly directly and generally by arguments which correspond to economic intuition. In particular, these general results have rather little to do with the type of market process (Wiener, Markov etc, although continuity does make a difference); or with the 'completeness' as distinct from 'perfection' of the market (although non-negativity conditions make a difference); or with the type of security to be valued (whether stock or bond, 'underlying' or 'derivative' etc).

Certain technical issues concerning martingale methods deserve notice at the outset. Economic valuation is forward-looking, the value of an asset being derived from the returns or services which it will render in future. For most problems concerning the valuation of *derivative* securities it is reasonable to restrict attention to a closed, finite time interval, since the returns from the securities up to a terminal date are usually specified in advance (as random variables or processes). A change of probability measure then transforms the (suitably discounted) market price or returns processes of the 'underlying' assets into uniformly integrable martingales, and the valuation problem for a derivative is solved via martingale representation, at least if the market is complete. By contrast, it is necessary in a *general* theory of valuation to treat time as infinitely extended (or perhaps as open-ended with an unpredictable *dies irae*) if arbitrary terminal valuations are to be avoided. For a reasonable degree of realism, it is also necessary to allow for incompleteness of inter-temporal markets in risks, even if the available markets are perfectly competitive. In this setting, the 'martingale measure' transformation is less useful, both because it is only 'local' (i.e.

defined only up to a stopping time) and because it is not unique. Instead, it is optimisation which (under assumptions to be recalled later) produces uniquely defined shadow price processes which are local martingales on $[0,\infty)$; the transversality conditions are expressed as limits taken along a sequence of stopping times which 'reduce' the shadow price processes to martingales. This holds even when markets are incomplete, though not necessarily when there are 'frictions'.

Working with *local* martingales is a nuisance, and besides, the mere information that a price process is a local martingale does not provide an adequate economic characterisation. We wish to identify suitable reducing times, not merely to know that they exist, and if possible to select times with an interesting economic interpretation, such as the first crossing times for shadow prices, or the first crossing times for capital or cumulative consumption denominated in suitable units. Going a step further, the possibility suggests itself of using a suitable family of times to define a random time change, such that the shadow price processes are transformed into true martingales and the transversality conditions are defined in terms of (transformed) 'clock' times. The problems which arise are quite subtle, partly because of the need for a reasonable economic interpretation, partly because of the joint dependence of the martingale and transversality conditions on the choice of stopping times and the need to ensure that the transformed conditions remain sufficient, as well as necessary, for optimality (which could fail, for instance, if the choice of a time transform which is not strictly increasing leads to a loss of relevant information). These issues are explored in Section 2 below and the Appendices, where several new results are proved. But when all is said and done, it as an inescapable feature of the economic problem that in general a martingale process defining shadow prices is not uniformly integrable on $[0,\infty]$, so that it is not possible to recover the entire process by taking conditional expectations of the limiting 'variable at infinity'.[2]

Turning now to an outline of the following Sections, ideas related to valuation in general and their relationship with properties of shadow prices are developed informally in Section 1.[3] Thereafter, we present in Section 2 a summary, mostly without proofs, of part of the mathematical theory concerning conditions characterising an optimal plan for saving, or for saving and portfolio choice, drawing largely on the author's previous work. We also present some new results, concerning (i) the effect of introducing into the saving model an exogenous, indivisible random income, such as a salary or revenue from a wholly-owned property, and (ii) the use of random time transforms to replace local by true martingales in the conditions for optimality; proofs are given in Appendices.

The remaining Sections present applications to simplified problems of valuation of more or less traditional kinds, which illustrate the advantages and difficulties of present methods. Section 3 deals with a simple portfolio-cum-saving (PS) model with initially just two 'long-lived' securities – a stock following geometric Brownian Motion and a bond – and considers the valuation of a derivative security such as a European call option; needless to say, this leads to (yet another) derivation of the Black-Scholes (BS) formula. The analysis helps to show to what extent the valuation formula results simply from martingale properties of shadow prices and what part is played by ancillary assumptions such as martingale representation and risklessness of the bond (either throughout its life or only at maturity). No direct use is made of arbitrage or replication arguments.

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[2] Thus, in the basic example of optimal saving with discounted CRRA utility and a Brownian market log-returns process, the shadow price process is an exponential martingale with limiting variable a.s. equal to zero, except in the case of logarithmic utility when the whole shadow price process is a.s. equal to a positive constant; see Foldes (1978a), Theorem 2 and Section 6, Remark (iii). Cf. also eqs. (3.13–14) below.

[3] Economists will no doubt find much of the discussion rather obvious, but I have been persuaded that a review of fundamental concepts from a traditional economic standpoint would be of interest to readers of a special issue on Computational Finance.

Section 4 considers a simplified problem of project evaluation, in which an investor who initially confronts an infinite-horizon problem of optimal saving with a single risky security has the opportunity at zero time of acquiring, in exchange for a lump sum, an *indivisible* random income stream (without the option of postponement). As is familiar from deterministic cost-benefit analysis, use of the net present value (NPV) formula for evaluating a project which is big enough to shift (market or shadow) prices leads to over-valuation if the prices used are those which prevail in the absence of the project, to under-valuation with the prices prevailing once the project is in place. We confirm an analogous conclusion in our continuous–time, stochastic set-up and show that the project may be correctly valued by a formula of Expected NPV type, with random discounting using a shadow price process obtained as the average of such processes corresponding to alternative hypothetical levels of project participation.

The literature related to the various topics is vast; a few references will be cited in individual Sections.

1. SOME CONCEPTS OF VALUATION

Many concepts and problems concerning valuation in continuous–time stochastic finance have analogues in traditional static or deterministic economic theory, and these similarities often afford insights pointing to solutions of technically complicated problems. In this Section, we take a few steps towards developing some common formulations, having in mind particularly the examples to be considered later. Some comments on the methodology of valuation are also offered.

Since theories of pricing for financial assets have developed largely as demand-side models with conditions of supply given exogenously, a suitable starting point for comparisons is afforded by the static theory of demand with pricing of a fixed supply. More precisely, we shall consider valuation mainly from the standpoint of an individual agent (consumer or investor), adopting broadly the 'quantity into price' approach as described by J.R. Hicks (1986), esp. Ch. IX., except that we accept unreservedly a 'cardinal' approach to utility. The agent has access to a market which is 'competitive', 'perfect' and 'in equilibrium' - at least for some commodities, and we say that these are *marketed*; for any remaining commodities, we assume for simplicity that no trade is possible, except perhaps for the offer of an 'indivisible' bundle of goods which must be accepted or rejected as a whole (house purchase, job offer, take-over etc). The market is not necessarily 'complete'. The precise meaning of the terms in quotation marks varies somewhat according to context. Briefly, the market is *complete* if all commodities (variables) appearing in the agent's utility function, or perfect substitutes for them, are marketed - commodities being distinguished, if appropriate, according to time and random state as well as physical or financial characteristics. The focus on individual decisions rather than market equilibrium or social welfare, and on a single agent rather than a company or public body, serves to limit the argument, but we shall sometimes extend the discussion informally to a wider setting.

Now *value* in economics means primarily *price*, which may be quoted in various units, e.g. money, corn, labour or utility. Thus *valuation* may be taken to mean the assignment of prices, having specified properties, to quantities of goods, or claims to goods such as financial securities. In some settings, valuation may refer to the solution of a set of pricing equations defined by a fully specified, closed model, such as a system of general equilibrium. However, in typical problems of financial theory such as project appraisal, evaluation of a job offer or the valuation of a new security, one is concerned rather with some *variation* from a given benchmark - usually an optimal plan when considering valuation by an individual agent, or an equilibrium when dealing with a market. We shall be largely concerned with situations of this latter type.

We start with the static theory of demand and consider an agent whose plan is optimal for some 'initial' or benchmark data - prices, constraints, list of goods marketed etc. Given a change in the data - call it a *perturbation* - the agent formulates a revised plan which is optimal for the perturbed data, and it is reasonable to define the (money) valuation of the variation of the plan as the associated variation of the agent's capital which would leave the agent just as well off as before. Following Hicks, it is usual to distinguish between the *equivalent* variation and the *compensating* variation, defined respectively as the valuation for goods to be acquired (maximum sum payable) and for goods to be disposed of (minimum sum receivable), the two valuations agreeing for a 'marginal' (strictly, infinitesimal) variation of the quantity of a divisible good. More generally, we shall speak of the *offsetting* variation for a given perturbation.

Some comments are in order. (i) It is important that the definition relates to changes from one *optimal* plan to another, since for a move between non-optimal plans the offsetting variation need bear no particular relationship to the market prices of the (marketed) goods involved.

(ii) In textbook discussions of (say) the compensating variation it is usual to consider, as an alternative to the sum of money which enables the agent to attain the previous level of welfare, the sum which permits the purchase of the previously optimal basket of goods. The latter calculation does not provide a basis for a general definition of valuation because it is inapplicable if some goods are not marketed.

(iii) Sometimes the valuation of a commodity (or security) is defined simply as the market price. Clearly this will not do for goods which are not marketed, or for a potential trade so large that it would shift the equilibrium price for marketed quantities. But even in the case of trades of marketed, divisible goods which are substantial for the individual agent but small for the market, there is an apparant paradox which should be clarified. We are accustomed by the usual text-book expositions to think of the valuation of (say) a non-marginal purchase as the equivalent variation represented by the usual indifference curve apparatus, or, under Marshallian assumptions, as a suitable area under the demand curve. This is clearly not the same as the market price. Are there then two conflicting definitions of valuation? Not if the definition of valuation as offsetting variation is interpreted carefully. Applied to the case under consideration it does yield the result that the valuation is the market price, since an agent would not pay more than that for a consignment which could be bought in the market, but would pay any lesser price since unwanted amounts could be re-sold at a profit. The geometric constructions in the text-books, if interpreted as defining valuation, are to be taken as referring to a situation where (at least hypothetically) the goods are not 'traded' (e.g. because the agent has no access to the market, or there are prohibitive transaction costs). Of course, there are many intermediate situations. The general lesson to be drawn from these apparently trivial remarks is that valuation must be conceived as relative to a specified set of opportunities, or a specified change in this set – which is really just a version of the old doctrine of opportunity cost.

To illustrate – and to prepare a format for the argument of later Sections – consider the familiar static model of demand. The agent maximises a utility function $u[c_0,c_1,...c_n]$ with standard properties (say, defined and finite for $0 \leq c_i < \infty$ all i, with derivatives $u_i > 0 > u_{ii}$) subject to a constraint $\Sigma_1^n p_i c_i = K$; here c_i, $i = 1,\dots,n$, are quantities of divisible, marketed goods with prices p_i, K is the capital, and c_0 the amount of an additional divisible good which is not marketed but with which the consumer is endowed at a level c_0^0. There is a unique optimum, satisfying

(1.1) $$u_i/p_i = u_K$$

for all (divisible) goods which are bought, i.e. for $c_i > 0$; here u_K is the 'marginal utility of money' (defined as the derivative of the *maximum* utility w.r.t. K). Eq.(1) expresses the celebrated *principle of equi-marginal utility*, see Jevons (1871). For divisible, traded goods which are not bought, i.e. $c_i = 0$, one has only

(1.2) $$u_i/p_i \leq u_K.$$ [4]

For c_0, which is consumed but not traded, equation (1) defines a 'shadow' price

(1.3) $$\tilde{p}_0 = u_0/u_K.$$

In all the above cases, the marginal valuation of a good c_i is obviously equal to the ratio u_i/u_K, which for traded goods that are bought agrees with the market price p_i. Thus, as one would expect, the valuations of marginal variations of quantities from an optimal plan can be calculated from the marginal utilities at the optimum. Similar remarks apply to valuations induced by small changes in parameter values; for example, if there is a small change in the price of a marketed good which is bought, one has the well-known Roy's equation

(1.4) $$\partial u/\partial p_i = -c_i \cdot u_K,$$

(where $\partial u/\partial p_i$ is the derivative of the maximum utility).

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[4] The first order conditions for the constrained maximisation problem are

(†) $c_i \geq 0,\ u_i - \lambda p_i \leq 0,\ c_i[u_i - \lambda p_i] = 0, \quad i = 1,..,n,$

where λ is a Lagrange multiplier and the derivatives are evaluated at the optimum. To obtain (1) and (2), it remains to show that $u_K = \lambda$. Now, with prices fixed, varying K yields $\Sigma_1^n p_i \cdot dc_i/dK = 1$. Also, differentiating the equality in (†) w.r.t. K, one finds that either $dc_i/dK = 0$ or $u_i = \lambda p_i$ in case $c_i = 0$. Thus, using (†),

$$u_K \doteq du/dK = u_0 \cdot dc_0/dK + \Sigma_1^n u_i \cdot dc_i/dK = \lambda \cdot \Sigma_1^n p_i \cdot dc_i/dK = \lambda.$$

So much is obvious enough. We shall consider analogues of the preceding relations later on, in connection with the valuation of securities. Note that, when comparing optima, the relations hold at each optimum. Now consider the case where $c_O^0 = 0$ in the 'benchmark' situation (assumed to be an optimum) and the agent is offered a *fixed*, *non-marginal* quantity c_O^1 of the good, the price $\bar{\mathfrak{v}}$ for the whole amount being less than the capital K. The offer must be accepted or rejected as a whole - a simple example of evaluation of an 'indivisible' project, even though we assume for simplicity that the good is physically divisible. Also assume for simplicity that the prices $p_1,\dots,p_n$ remain unchanged. Now there will be one optimal plan for the remaining goods if the offer is accepted, another if it is rejected, with marginal utilities respectively u_K^1, u_i^1, and u_K^0, u_i^0. It is easily shown, under standard assumptions, that the equivalent variation $\mathfrak{v}$ satisfies

$$(1.5) \qquad u_O^0/u_K^0 > \mathfrak{v}/c_O^1 > u_O^1/u_K^1 ,$$

so that a criterion of the form $\bar{\mathfrak{v}} \leq c_O^1 \cdot (u_O^0/u_K^0)$ provides only a necessary condition for acceptance of the offer, a criterion $\bar{\mathfrak{v}} \leq c_O^1 \cdot (u_O^1/u_K^1)$ only a sufficient condition. To derive a necessary and sufficient condition, one can of course compare the *total* utilities with and without the project and obtain $\mathfrak{v}$ as that price for which the two totals are just equal; then $\bar{\mathfrak{v}} \leq \mathfrak{v}$ defines the required criterion. It is however more convenient to have a criterion expressed in terms of marginal utilities (shadow prices) or, if possible, market prices. For this purpose, consider the hypothetical situation where a proportion α of c_O^1 can be bought for a price $\mathfrak{v}(\alpha)$, and suppose that for each $\alpha \in [0,1]$ there is an optimal plan $(\alpha c_O^1, c_1^\alpha,\dots,c_n^\alpha)$ satisfying

$$(1.6) \qquad \Sigma_1^n \, p_i^\alpha \cdot c_i^\alpha = K - \mathfrak{v}(\alpha),$$

with utility

$$(1.7) \qquad u^\alpha = u[\alpha c_O^1, c_1^\alpha,\dots,c_n^\alpha].$$

Assume for simplicity that $c_i^\alpha > 0$ for all i and all α. Differentiating (7) and (6) we have, with obvious notation, (1.8) $$du^\alpha/d\alpha = u_0^\alpha \cdot c_0^1 + \Sigma_1^n u_i^\alpha \cdot dc_i^\alpha/d\alpha$$

$$= u_0^\alpha \cdot c_0^1 + u_K^\alpha \cdot \Sigma_1^n p_i \cdot dc_i^\alpha/d\alpha = u_0^\alpha \cdot c_0^1 - u_K^\alpha \cdot \mathfrak{v}'(\alpha).$$

Setting

(1.8′) $$\mathfrak{v}'(\alpha) = c_0^1 \cdot (u_0^\alpha/u_K^\alpha)$$

for each α yields $du^\alpha/d\alpha = 0$, hence $u^1 = u^0$, and the equivalent variation is

(1.9) $$\mathfrak{v} = \mathfrak{v}(1) = c_0^1 \cdot \int_0^1 (u_0^\alpha/u_K^\alpha) d\alpha .$$ [5]

The integral appearing here may be interpreted as a shadow price per unit of c_0^1. In Section 4, we shall adopt an analogous procedure to obtain a project evaluation formula in a continuous-time, stochastic model.

Two special cases deserve mention. (i) If the quantity c_0^1 can be considered as 'marginal', say $c_0^1 = 1$, then (at an approximation) the derivatives can be evaluated at $\alpha = 0$ and (9) reduces to (3) with $\mathfrak{v}$ in place of $\tilde{p}_0$.

(ii) It may be that there is a marketed commodity, or a linear combination of such commodities, which provides a perfect substitute for commodity 0. More formally, suppose that one unit of good 0 is considered to be 'the same' as a collection $(\gamma_1,\ldots,\gamma_n)$ of the remaining goods, where the γ_i are constants, so that the utility function satisfies an identity of the form

(1.10) $$u[c_0,c_1,\ldots,c_n] \equiv u[0,c_1+\gamma_1 c_0,\ldots,c_n+\gamma_n c_0], \quad \text{hence}$$

(1.10′) $$u_0^\alpha = \Sigma_1^n \gamma_i \cdot u_i^\alpha, \quad 0 \le \alpha \le 1.$$

In this case, (9) yields

(1.11) $$\mathfrak{v} = c_0^1 \cdot \int_0^1 \Sigma_1^n \gamma_i (u_i^\alpha/u_K^\alpha) d\alpha = \Sigma_1^n \gamma_i \cdot p_i,$$

and the 'project' may be valued 'by replication'.

This example can also be used to illustrate some effects of risk. Suppose now

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[5] If the prices $p_1,\ldots,p_n$ vary with α, there is an additional term

$$-\int_0^1 [\Sigma_1^n c_i^\alpha \cdot dp_i/d\alpha] d\alpha = \int_0^1 [(1/u_K^\alpha)\, \Sigma_1^n \cdot \partial u^\alpha / \partial p_i^\alpha \cdot dp_i^\alpha/d\alpha] d\alpha$$

on the right side of (9), the equality here being due to Roy's equation (4).

that utility depends on a random element ω, say $u = u[c_0,c_1,...,c_n;\omega]$, and that the agent maximises the expectation of this function, the actual value of utility being observed only after the quantities $c_1,...,c_n$ have been chosen. The calculations (1 - 4) are practically unchanged, except that marginal utilities u_i, u_K are replaced by their expectations. Regarding the 'indivisible' offer, suppose further that the quantity c_0^1 obtainable for the price $\bar{\mathfrak{v}}$ is also random, the amount again being revealed after all decisions have been taken. Now (7) is replaced by

$$\text{(1.7a)} \qquad Eu^\alpha = Eu[\alpha c_0^1(\omega), c_1^\alpha,...,c_n^\alpha; \omega].$$

On calculating $dEu^\alpha/d\alpha$ as in (8) and setting the result equal to zero, we obtain

$$\text{(1.8a)} \qquad \mathfrak{v}'(\alpha) = E[u_0^\alpha \cdot c_0^1]/E[u_K^\alpha] = E[c_0^1] \cdot E[u_0^\alpha]/E[u_K^\alpha] + Cov[c_0^1, u_0^\alpha]/E[u_K^\alpha]$$

(assuming that we may differentiate under E and that the covariance exists). On integrating, (9) is replaced by

$$\text{(1.9a)} \qquad \mathfrak{v} = E\left[c_0^1 \cdot \int_0^1 (u_0^\alpha/Eu_K^\alpha) d\alpha\right]$$
$$= E[c_0^1] \cdot \int_0^1 \left[[Eu_0^\alpha]/[Eu_K^\alpha]\right] d\alpha + \int_0^1 \left[Cov[c_0^1, u_0^\alpha]/E[u_K^\alpha]\right] d\alpha.$$

If the covariance terms vanish for each α - specifically, if the random variables $c_0^1(\omega)$ and $u[\beta,c_1,...,c_n;\omega]$ are independent for each $\beta > 0$ - then (9a) is like (9), with marginal utilities replaced by their expectations; (this also goes for the additional term in fn. 5, in case $p_1,...p_n$ vary with α). In case good 0 is marketed, the ratio $E[u_0]/E[u_K]$ can, of course, be replaced by the market price p_0; but, even then, valuation generally involves the utility function unless the covariance terms vanish.

Now reconsider the special cases mentioned earlier. (i) If c_0^1 is 'small', then $\mathfrak{v} \sim c_0^1 \cdot \mathfrak{v}'(0)$, cf. (8′) and (9), but otherwise matters stand as before.

(ii) If there is replication as in (1.10–10′), with the same coefficients γ_i for all ω, we get, using the condition $E[u_i^\alpha]/E[u_K^\alpha] = p_i$,

$$\text{(1.11a)} \qquad \mathfrak{v}'(\alpha) = [Ec_0^1] \cdot \Sigma_1^n \gamma_i \cdot p_i + \Sigma_1^n \gamma_i \cdot Cov[c_0^1, u_i^\alpha]/E[u_K^\alpha].$$

The covariance term vanishes if the independence assumption stated above is satisfied, and then $\mathfrak{v}$ can be calculated from market prices without involving utilities; *however* *replication alone is not in general enough to allow this.*

Speaking informally, this discussion suggests (as is indeed the case in general) that project evaluation with risk involves shadow prices dependent on marginal utilities in at least three ways:

(a) the use of ratios of (expected) marginal utilities in place of market prices where goods are not marketed;

(b) the use of ratios of (expected) marginal utilities where project returns and non–project utilities are stochastically dependent;

(c) the averaging of (utility or market) prices over alternative levels of project participation where the project does not qualify as 'small'.

The question arises whether cases (a) and (b) are really distinct. Exponents of a radically 'neo-classical' standpoint in value theory are apt to classify "all possible occurrences in the world which impinge upon utilities" as "commodities", see Arrow (1963) pp. 945–6, and in particular to regard uninsurable risks as non-marketable commodities. (In effect, the circumstances of the real world are classified as 'imperfections' relative to a benchmark defined by an idealised perfectly competitive economy - the Arrow-Debreu world - in which insurance markets exist for all dated contingencies). From this standpoint, the distinction between (a) and (b) depends on the extent to which markets for risk are available (or can be imagined). In principle, (b) can be subsumed in (a), all stochastic dependence between project and non-project benefits and costs being attributed to non-marketed (or 'shadow') commodities to which shadow prices may, under suitable assumptions, be assigned; of course, this may require quite an un-natural formulation of the appraisal problem. In the extreme case where all 'commodities' (all random effects) are priced in competitive markets, category (a) – which now includes (b) – is empty, while (c) remains in the sense that large projects are liable to shift the state-contingent prices so that evaluation still involves some averaging.

Repeated mention has been made of *shadow prices*. These may be defined loosely as numbers (or more generally functions or random processes) which have specified properties of prices but are not necessarily paid or even quoted in any market - whether because relevant markets are not available, or because the prices are quoted in units such as utils which are unsuitable for trade, or because they are mere mathematical constructs such as Lagrange multipliers generated by constraints or dual variables appearing in programming problems. The dividing line between market and shadow prices is not a hard and fast one, and it is often convenient to use the latter term loosely.

The definition of valuation as assignment of (say) a monetary equivalent to a specified departure from a benchmark allocation of resources has some important, if largely unavoidable limitations. A brief mention of these must suffice. There are conceptual problems involved in introducing (say) a new, hitherto unconsidered, opportunity into a model which was initially assumed to be fully specified and optimised. At a practical level, it is often impossible to model an existing operation and a new opportunity in the same degree of detail, and some crude compromises may be necessary. Setting such problems of formulation aside, we have noted that many transactions or projects, such as the purchase of a house or a car, are 'indivisible' rather than marginal. One resulting difficulty, even in a deterministic setting, is that, because of diminishing marginal utility, the valuations of such projects are in general not additive. The impact of successive projects on overall risk, interacting with diminishing marginal utility (risk aversion), also contributes to non-additivity. In this essay, we ignore problems due to a succession of projects; however, the essential difficulties are similar to those encountered when a single large project is superimposed on an existing optimum or equilibrium.

The difficulties due to indivisibilities are largely avoided in theories of (corporate or public) finance by invoking some sharing or spreading mechanism (joint stock, insurance, public ownership etc) which allows the impact of a large project to be regarded as marginal for each agent; see for example Arrow & Lind (1970). However, sharing arrangements do not dispose of the difficulty due to induced price shifts, and while risk-spreading may allow neglect of independent risks associated with a particular project or security it does not avoid accounting for contributions to systematic risk. Besides, sharing or spreading arrangements are limited by well-known factors such as moral hazard and transaction costs.

It is remarkable how much of the literature on valuation in finance is devoted to working out detailed conditions in which assets or projects can be valued on the basis of given market prices alone, without direct reference to the preferences of agents. The basic procedures are those mentioned above. First, assume some sharing or spreading mechanism to iron out indivisibilities. Second, appeal to some hypothesis of replication in (sufficiently) complete markets to justify valuing inputs and outputs (including risks) at market prices, which are assumed to remain unchanged. The essential assumption is that perfect substitutes for the inputs and outputs of a project to be appraised, or perfect substitutes for a security to be valued, are already priced in a competitive market, so that it is only necessary to invoke the 'law of one price' and the assumption that people prefer more wealth to less. The well-known justification of the net present value rule for project evaluation under certainty in the presence of a 'perfect' capital market (which here includes completeness at $t = 0$ of the market in debts at all future times) is an example; the project is simply regarded as a bundle of dated cash flows, each of which can be traded in a perfect market. The procedure for valuation in an Arrow–Debreu economy is obviously an extension of the same idea to dated contingent debts. Indeed, the criterion of profit maximisation in the elementary theory of the firm, which is often advanced as a postulate, is more properly regarded as an application of the same argument, since in the absence of suitably complete, competitive markets it cannot in general be shown that the interests of the owners of the firm or other beneficiaries are best served by maximising the money value of net output. The valuation of securities in the mean–variance theory, by reference only to a riskless rate of interest and a price of systematic risk, is another example of the same approach. Needless to add, the valuation of derivative securities by reference only to the prices of 'underlying' securities, using replication or no-arbitrage arguments, belongs to the same category.

The upshot of our review so far may be summarised as follows. The valuation (or evaluation) of a given change to a plan involves, in principle, a comparison between total utility or welfare before and after the change, and the calculation of a sum of money equivalent to that change. If optimality of plans is taken as given, the valuation of marginal changes in holdings of divisible, traded goods (or financial assets) may be read off from market prices, assuming these to be unaffected. However, in the case of changes in holdings of goods which cannot be traded, or which cannot be sufficiently divided (whether physically or by sharing costs and benefits), or which, even if divided, affect relative prices or the risk profile, valuation unavoidably involves consideration of utilities. It remains possible (at least in the situations considered here) to replace comparison of utility levels by valuation formulas involving sums of prices times quantities, but usually the 'prices' will be in some sense shadow prices which depend on marginal utilities in the optimal plan. This is naturally inconvenient for computation, since one's own utility function (even if it exists) is difficult to formulate while other people's functions are difficult to measure.

Turning now to continuous-time stochastic problems, we consider only models with a single (composite) consumption good for each (ω,t) but possibly several financial assets, the consumption good serving as the unit of account for relative prices at (ω,t) of the financial assets. No more will be said about valuation in the setting of complete markets. The concepts in that setting are essentially the same as in the static theory of value, subject to quoting all prices in terms of the consumption good at

$t = 0$ and replacing utility by expected utility (adjusted as necessary for impatience). Of course, there are technical problems when working with continuous time and probability densities, but they do not change the essential ideas concerning valuation.

We consider instead a stochastic, continuous-time model of optimal consumption and portfolio choice over an infinite horizon in which a number of 'long-lived' securities are traded continuously, as in Foldes (1990). The market is not necessarily complete, although it might be so in special cases, see Duffie & Huang (1985). The securities are continuously traded in perfectly competitive markets, with (strictly) positive returns processes $z^\lambda = z^\lambda(\omega,t)$, $\lambda = 1,\dots,\Lambda$, per unit of investment at zero time, (return being identified with price, unless otherwise stated, by assuming all dividends etc. to be instantaneously reinvested in the same security). The returns processes are modelled either as general semimartingales, in which case short sales (negative holdings) are excluded to ensure that portfolio returns stay positive, or as continuous semimartingales, and then we distinguish between cases with and without short sales.[6]

The investor is endowed with an initial capital K_0 and maximises 'welfare', defined as the expected integral of utility of consumption $\bar{c}$, $\bar{c}$ being expressed in suitable 'natural' units, (e.g. bushels, or a real-terms index of consumption expenditure). We refer to a problem of *optimal saving* when there is only one security

[6] It is usual to refer to z^λ as a 'money' price or returns process, but this is not strictly correct. The price $z^\lambda(\omega,t)$ is quoted in units of consumption goods at (ω,t), the securities being the carriers of value, i.e. of entitlement to consumption, across times and states. In the absence of complete markets, there need be no single unit of account, such as goods at $\tau = 0$, which can be transformed into goods at any specified (ω,t). Of course, consumption variables can be eliminated from quotations of relative prices by taking a given security or portfolio plan, say π, as numéraire, i.e. replacing $z^\lambda(\omega,t)$ by $z^\lambda(\omega,t)/z^\pi(\omega,t)$. It is usual to take as numéraire the riskless security if there is one; but note that in the present setting the proper definition of a riskless security is one whose payoff in terms of consumption (i.e. in 'real terms') is non-random. While costless storage of such a security – call it cash – may ensure that its own-rate of interest is non-negative, its real rate can be negative. Note also that, as is usual in models of mathematical finance, there is no transactions motive to hold cash. Thus, if there is a dated (say, zero-coupon) bond with a non-random cash value at redemption, its yield must be the same as the yield on cash if both are to be held.

or when the portfolio process is taken as given. As usual, consumption and capital are constrained to be non-negative (or equivalently, for an optimum, positive, since we assume $u'(0) = \infty$); these constraints are the source of most of the technical problems. Further details of the model are given in Section 2; for the moment we proceed informally, anticipating definitions and results to be given in Section 2 as required.

We assume throughout that a unique optimal plan exists, and for the moment consider only necessary conditions for optimality. An optimal plan is denoted $(\bar{c}^*,\pi^*)$, where consumption $\bar{c}$ is in 'natural' units and a portfolio plan π is specified in terms of the proportions $\pi^\lambda(\omega,t)$ of capital invested in the various securities. We write z^* for the returns process generated by π^* (or simply z if there is only one asset), also $c^* = \bar{c}^*/z^*$ and $k^* = \bar{k}^*/z^*$ for the 'standardised' optimal consumption and capital processes, and denote by $v = v(\omega,t)$ the marginal utility process evaluated along the optimal plan.

Before returning to the topic of valuation, we consider what is the proper analogue in this setting of the static principle of equi-marginal utility. Starting with the *problem of optimal saving*, we note that the random return at a time S from one unit invested at $t = 0$ is $z(\omega,S)$ in natural units of consumption, yielding approximately $z(\omega,S)v(\omega,S)$ in utility units if this return is consumed at S. Alternatively, if consumption is postponed to a later time T, the utility contribution is $z(\omega,T)v(\omega,T)$. Writing $y = z \cdot v$, the principle of equi–marginal utility in the case of certainty would yield $y_S = y_T$, hence $y_T = y_0 = v_0$ for all $T \in [0,\infty)$. In the case of risk, the investor does not know at S what the value of z_T will be, so that, bearing in mind that optimisation under risk requires consideration of *expected* utility, one would conjecture the correct analogue of (1.1) to be

$$y_S = E^S y_T, \tag{1.12}$$

where E^S represents conditional expectation relative to information at S. In other words, the conjecture is that the process $y = z \cdot v$ should be a martingale. One might even be tempted to think that an equation like (12) should hold for arbitrary finite stopping times S, T satisfying $S(\omega) \leq T(\omega)$ a.s. Unfortunately, neither assertion is correct in general. An equality (12) is a necessary condition for optimality if one considers *consumption* (or equivalently *depletion*) *times* S, T, where by definition the consumption time at the level $i \in [0,K_0)$ is the upcrossing time at the level i of the process $\int_0^T c^*(t)dt$, or the downcrossing time at $K_0 - i$ of the process k^*. In other words, if a random time transform is performed so that time is measured by the cumulative (standardised) consumption along the optimal plan, then the transform $\hat{y}$ of the y-process is a true martingale with respect to the transformed information structure. It is also true that (12) is a necessary condition if one considers *price times*, the time at the level $n > y_0$ being defined as the first upcrossing time by y of the level n (or the clock time n if that is earlier).

To prove (i.e. test) these assertions by variational methods, one argues that, if there are (possibly stochastic) intervals of the form $I_1 = [S,S')$, $I_2 = [T,T')$ with $S' < T$, such that, for ω in some set A measurable at S, the time average of y on I_1 is less than the corresponding average on I_2, then welfare can be increased by reducing consumption during I_1 and carrying forward the capital saved to augment consumption during I_2, contrary to the assumption that the 'star' plan is optimal. Taking conditional expectations at S and going to limits as the intervals become small (also using the fact that y can be chosen right continuous) one concludes that

$$y_S \geq E^S y_T \tag{1.13}$$

a.s. on A, i.e. y is a supermartingale. This argument works in all cases, even if the market process has jumps. In the opposite direction, one tries to argue that, if y is on average greater on I_1 than on I_2 etc, then welfare could be increased by consuming more in the earlier interval and less in the later, continuing until the variation of the capital stock is reduced to zero. This unfortunately is not always possible unless the intervals are suitably chosen, for example with end-points at consumption times. It is also true that the martingale equality holds for all pairs of (finite) clock times if the standardised optimal capital $k^*(\omega,\tau)$ is bounded away from zero at each clock time τ. In general, however, one can only say that the untransformed process y is a positive *local* martingale, i.e. that for each of a suitable sequence $\chi_n = \chi_n(\omega)$ of stopping times, tending a.s. to ∞, the process y stopped at χ_n is a true martingale.

In the case of the *portfolio-cum-saving* (PS) problem, i.e. when several securities are available, matters are more complicated. In this case we write $y^\lambda = z^\lambda \cdot v$ for the shadow price process associated with an individual security and $y^* = z^* \cdot v$ for the process associated with the optimal plan. The properties of y^* are like those found for y in the problem of optimal saving. An equi–marginal argument shows that in all cases the processes y^λ satisfy the supermartingale inequality

(1.14) $$y^\lambda_S \geq E^S y^\lambda_T \qquad S \leq T$$

for clock times, and consequently (by the Stopping Theorem) for stopping times also, see Foldes (1990); this inequality may be considered as analogous to (2). If optimal holdings of all securities are always positive, or if the market process is continuous and short sales are allowed, the processes y^λ are *local* martingales for each security λ; the (rather restrictive) conditions under which their transforms to consumption time are *true* martingales are considered in Appendix C.

Continuing for simplicity with a continuous market with short sales permitted for traded securities, the martingale property (12) may be expected to define shadow prices for securities with which the agent is endowed but which cannot be traded, more or less analogously with the way in which (3) defines shadow prices for untraded commodities. For example, given a security, such as an option, which pays a random variable z^0_T at some (possibly) random time T and nothing at other times, the equation

(1.15) $$y^0_S = E^S y^0_T, \quad \text{or equivalently} \quad z^0_S = E^S[z^0_T \cdot (v_T/v_S)] \qquad S \leq T,$$

may be expected to define shadow price processes for times $S \leq T$, in units of utility and in money (or rather, consumption). This further suggests interpreting the process

$(z_t^0;\ t \le T)$ as a price process which, if it were quoted in the market, would be consistent with holding the actual endowment of the untraded security in the optimal portfolio up to time T. There are, however, some reservations to this analogy, connected in particular with the distinction between local and true martingales. In Section 3 we shall investigate this topic in more detail, with special reference to the case of a European call option, pointing out in particular how the completeness of the market in the underlying securities, together with an explicit specification of their price processes, allows a formula like (15) to be put in a readily computable form (Black-Scholes formula).

It is clear that a formula like (15) can be interpreted as a formula for valuation, but now the concept of valuation needs to be interpreted more widely. A valuation need no longer be thought of as a price or sum of money (say, at $t = 0$), but may now be a suitable random variable or even a process. Recall that we said earlier that valuation might refer either to the solution of pricing equations relating to a situation defined by specified parameters, or to the calculation of an offsetting variation when comparing two situations with different parameter values. Using (15) to calculate z_S^0 for $S \le T$ when the variable z_T^0 is given may be interpreted in the former sense, so that the random variable z_S^0 defines the *valuation variable* of the security at S and the process $(z_t^0;\ t \le T)$ the *valuation process.* (The formula can also be interpreted as defining offsetting variables or processes corresponding to the introduction of a marginal quantity of the security, but enough has been said.) As this example shows, the general concept of valuation now becomes potentially very wide, and it is necessary in particular cases to specify the type of object (process, random variable or constant, money or corn, payment date etc) in terms of which a valuation is to be expressed. Calculating an offsetting payment at $t = 0$ remains a basic problem, and in Section 4 we shall derive an analogue for (9a), using this definition, in the case of an indivisible financial asset which is not marketable.

The return z_t^λ has been defined as the total value of the fund accrued at time t per unit of investment at zero time, with immediate reinvestment of any dividends etc in the same security. While this definition is convenient for proving martingale conditions of optimality, it is often preferable for valuation to derive a formula which expresses the *nominal* price of a share (i.e. the price of one nominal unit, without reinvestment) at a given date as the sum of the discounted future dividends up to a horizon date plus the discounted nominal price at the horizon. We shall merely outline a derivation of one such formula from the conditions for optimality.

Dropping superscripts when no confusion arises, let p with $p(0) = 1$ be the positive semimartingale representing the nominal price process of one share of security λ and let D with $D(0) = 0$ be the predictable non-decreasing process representing cumulative dividends per share. For simplicity, it is assumed that $\Delta p_t = -\Delta D_t$, hence also $\Delta z_t = 0$, whenever $\Delta D_t > 0$, i.e. the nominal price falls by exactly the amount of any lump-sum dividend.[7] Taking into account the accrual of returns upon reinvested dividends, we have

(1.16) $$z_T^\lambda = p_T + z_T^\lambda \cdot \int_{(0,T]} (1/z_{t-}^\lambda) dD_t,$$

Alternatively, differentiating formally and simplifying,

(1.17) $$dz^\lambda / z_-^\lambda = dp/p_- + dD/p_-,$$

as one would expect. These formulas allow any relation involving z^λ to be expressed in terms of p and D. Now assume *either* that all returns processes are continuous with short sales allowed, *or* that optimal holdings of all securities are always positive. Then, according to the conditions for optimality (2.14) and (2.20) below, the processes

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[7] This assumption gets rid of the square bracket term between z^λ and $\int(1/z_-^\lambda)dD$ when differentiating (16) below, and also allows $1/z_-^\lambda$ to be replaced by $1/z^\lambda$ in (16) without altering the value of the integral. See also fn.10 for general conventions about processes. The assumption that z^λ is a positive semimartingale (which includes the condition $z_-^\lambda > 0$) ensures that the integral in (16) is defined and finite for $T < \infty$. It then follows from (16) that $\int_{(0,\infty)} (1/z_-^\lambda) dD \leq 1$.

y^λ and y^* are local martingales. Let T be a finite stopping time which 'reduces' y^λ, so that $y^\lambda_S = E^S y^\lambda_T$ for times $S \le T$. Setting $y^\lambda = z^\lambda \cdot v$ in this equation, using the expression for z^λ from (16), replacing $1/z^\lambda_{t-}$ by $1/z^\lambda_t$ and rearranging yields

$$p_S v_S = E^S\Big[p_T v_T + (z^\lambda_T v^\lambda_T - z^\lambda_S v_S)\int_{(0,S]} (1/z^\lambda_t) dD_t + (z^\lambda_T v_T)\int_{(S,T]} (1/z^\lambda_t) dD_t\Big].$$

The second term under E^S vanishes by the martingale property of y^λ, and the third term may be replaced by $\int_{(S,T]} (y^\lambda_t / z^\lambda_t) dD_t = \int_{(S,T]} v_t \cdot dD_t$ using a slightly modified version of the theorem on integration of a martingale w.r.t. a non-decreasing process, Elliott (1982) 7.16. This leaves

(1.18) $$p_S v_S = E^S\Big[p_T v_T + \int_{(S,T]} v_t \cdot dD_t\Big].$$

Cf. Duffie (1991) 4.3, where similar formulas are derived by other methods. Now let $T \uparrow \infty$ along a sequence of stopping times (χ_n) which reduce y^λ and also satisfy the conditions (2.14); under some additional assumptions, it follows from the transversality condition (2.14b) that the term $E^S[p_T v_T]$ in (18) goes to zero,[8] yielding a formula which expresses the nominal price p_S as the expected integral of future (utility-discounted) dividends up to infinity.

This brief derivation illustrates both the need to separate nominal prices from dividends and the importance of treating the time horizon as infinite (or at least as open with an unpredictable final time when all values vanish) if a satisfactory general

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[8] For example, suppose that for all t, a.s., the optimal nominal holdings q^i of all shares are non–negative while q^λ is bounded away from zero, say $q^\lambda \ge 1$. Write $\bar{k}^* = \Sigma_i[p^i q^i]$; then $p^\lambda v \le q^\lambda p^\lambda v \le \bar{k}^* v = k^* y^*$. If (χ_n) satisfies (2.14), then $E[k^*(\chi_n) y^*(\chi_n)] \to 0$ as $n \uparrow \infty$, so $E^S[p^\lambda(\chi_n) v(\chi_n)] \to 0$ a.s.

The requirement that $S \le (\chi_n) \uparrow \infty$ a.s., with a sequence (χ_n) which reduces y^λ and also satisfies (2.14), restricts the choice of S. Various conditions implying the existence of a suitable sequence may be inferred from results reported below. If y^λ and y^* are true martingales, S may be any bounded stopping time and the χ_n may be chosen as clock times. If S is bounded above by a price time of the form $\wp^*_n \wedge \wp^\lambda_n$, see (2.21) and Appendix A, then the χ_n may be chosen as price times of the same form. Again, if S is bounded above by a consumption time, and if $z^\lambda / z^* = y^\lambda / y^*$ is a bounded process, then the χ_n may be chosen as consumption times, see (2.16) and Appendix C.

theory of valuation is to be obtained. It is worthwhile to expend some rhetoric on the latter point. As explained by Irving Fisher (1930) p.14,

> 'The value of capital must be computed from the value of its estimated future net income, not *vice versa*';

and, as Fisher shows, a coherent theory of capital requires that *income* be defined essentially as *real consumption*; see also Fisher (1906). If the problem of optimal saving and portfolio choice is treated in a (closed) finite-horizon setting, the terminal assets must be valued more or less arbitrarily; and bearing in mind that a returns or price process constructed by taking conditional expectations of terminal values will automatically be a uniformly integrable martingale, the result will be a theory of valuation which hangs to a greater or lesser extent by its own bootstraps.

In this connection, it is of interest to comment on the distinction between the martingale properties of utility prices y^λ in our infinite-horizon model and the martingale properties of (suitably discounted) market prices under a transformed probability measure which are exhibited in many papers. Suppose either that optimal holdings are always positive or that the market is continuous with short sales allowed. The (local or true) martingale properties of our y^λ are relative to the *original* measure P (i.e. the one thought to operate in the market), they are *consequences of optimality*, under standard assumptions the processes are *unique* (whether or not the markets are complete), and in the case of 'long-lived' assets they are defined up to an *infinite horizon*, (or at least an *open horizon* in depletion time); but, typically, the processes are not uniformly integrable (so that it is not possible to reconstruct the whole utility-price process by 'working back from infinity').

By contrast, let z^Λ be the returns process to a particular asset or portfolio (e.g. a riskless bond, or an optimal portfolio), and consider the 'Λ-discounted' processes $\tilde{z}^\lambda = z^\lambda / z^\Lambda$. Given suitable assumptions of consistency, the $\tilde{z}^\lambda$ are (true, uniformly integrable) martingales on a *finite* (though possibly random) interval $[0,T]$ under a probability Q_T obtained by an absolutely continuous change of measure $dQ_T = L_T \cdot dP$, (where L_T is an $\mathscr{A}_T$-measurable positive random variable satisfying $E_P L_T = 1$); for a given vector process Z and a given choice of Λ, the measure yielding the desired property is unique if the market is complete but not in general.

If, assuming completeness, successively longer intervals $[0,\tau_n]$ are considered, with $\tau_n \to \infty$ a.s., a sequence $L_{\tau_n} = dQ_{\tau_n}/dP$ of $\mathscr{A}_{\tau_n}$- measurable random variables is required to obtain the martingale property on the successive intervals, and it can happen that the measures Q_{τ_n} converge to a limit measure Q_∞ which is concentrated on a P-null set; in short, this martingale property also is 'local'.

There is, of course, a connection between the two kinds of local properties. Suppose that an optimal plan is given, and that y^*, y^Λ and all y^λ are local martingales reduced by stopping times $\tau_n \to \infty$, so that $Ey^\lambda(\tau_n) = Ey^\Lambda(\tau_n) = Ey^*(\tau_n) = y_0$. Defining measures Q_{τ_n} on $\mathscr{A}_{\tau_n}$ by $dQ_{\tau_n} = [y^\Lambda(\tau_n)/y_0] \cdot dP$, the processes $\tilde{y}^\lambda = y^\lambda/y^\Lambda = z^\lambda/z^\Lambda = \tilde{z}^\lambda$, stopped at τ_n, are Q_{τ_n}-martingales; see Foldes (1990) S.7. for details. Conversely, the assumptions leading to the existence of 'martingale measures' Q_{τ_n} do not imply that the prices z^λ are compatible with either optimality or equilibrium for *specified* utilities, but under suitable assumptions it is possible to *construct* utilities for which the given prices have these properties.

There is therefore a loose equivalence between the two kinds of martingale characterisations of prices, although this is by no means transparent because of the different assumptions made in both kinds of models (regarding short sales, continuity, horizons, square-summability etc) and the difficult distinctions among the various criteria for consistency of a price system (freedom from arbitrage, no free lunch, viability, existence of a martingale measure); see Duffie (1991) for details. Ultimately the difference seems to lie in the different objectives of the theories (although it is difficult to make the distinction without over-stating the case). The kind of model considered here aims mainly at extending the economic theory of value to an inter-temporal, stochastic setting with incomplete markets; interest centres on the characterisation of optima and equilibia, and thus on the valuation of 'underlying' or 'long-lived' securities. The alternative approach centres on characterising systems of prices which are in some sense consistent but may otherwise be arbitrary, and on the valuation of derivative securities against the background of a consistent underlying system.

2. OUTLINE OF THE MATHEMATICAL THEORY

The model which forms the basis of the detailed discussion is that developed in Foldes (1990), see also Foldes (1978a&b) – partly because it is the one which I understand best, but mainly because it derives necessary and sufficient conditions for optimal saving and portfolio choice directly in terms of martingale and transversality conditions of shadow price processes.[9] The main features of the model will now be summarised. Proofs are omitted apart from a few points which need clarification, and some new results which are proved in the Appendices.

Let $\mathcal{T} = [0,\infty)$, equipped with its Borel sets and Lebesgue measure, be the time domain, and let $(\Omega, \mathcal{A}, \mathrm{P})$ be a complete probability space with a filtration $\mathfrak{A} = (\mathcal{A}_t;\ t \in \mathcal{T})$ satisfying the 'usual conditions' of right continuity and completeness, also $\mathcal{A} = \mathcal{A}_\infty$ while $\mathcal{A}_0 = \mathcal{A}_{0-}$ is generated by the P–null sets. $\mathfrak{A}$ represents the investor's information structure and P his beliefs. In the product space $\Omega \times \mathcal{T}$ we define the usual σ–algebras of progressive, optional and predictable sets, as well as the corresponding classes of processes. All processes considered are assumed, or may be shown to be, at least progressively measurable (with respect to $\mathfrak{A}$ unless otherwise stated, and we write simply 'process', or $\mathfrak{A}$-process in case of doubt). Unless otherwise stated, conventions concerning definitions and notation will be as in Foldes (1990).[10]

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[9] I have also retained the terminology of these papers, e.g. calling v the marginal utility process, rather than the state price process as in Duffie (1991).

[10] Thus 'positive' means 'strictly positive' etc. For a scalar process ξ, '$\xi > 0$' means '$\xi(\omega,t) > 0$ for all (ω,t)' – modulo null sets, see below – while similar notation for a vector process means that the condition applies to each component. Similarly for '$\xi \geq 0$' and for '$\xi = 0$'.

Processes of a given type (e.g. semimartingales, or consumption plans) which differ only on null progressive sets are identified; thus '$\xi^0 = \xi^1$' *for processes of that type* may be read as '$\xi^0(\omega,t) = \xi^1(\omega,t)$ for a.e. (ω,t)'. But note that, if membership of the type requires right continuity with left limits (corlol), or left continuity with right limits (collor), then '$\xi^0 = \xi^1$' is equivalent to '$\xi^0(\omega,t) = \xi^1(\omega,t)$ for all t, a.s.'. In particular, all semimartingales (including true, sub, super and local martingales) and all processes of finite variation will by definition be corlol, with $\xi(0-) = \xi(0)$ a.s.; a *positive* process ξ of one of these types is assumed to satisfy both $\xi(t) > 0$ and $\xi(t-) > 0$ on $\mathcal{T}$, a.s.

For $t \in \mathcal{T}$, E^t means $E(\cdot / \mathcal{A}_t)$; similarly P^t, also E^χ, $E^{\chi-}$ for a stopping time χ.

Before proceeding, it is useful to recall some facts about the martingale exponentials and logarithms of semimartingales (semis for short). Given a semi J with $J(0) = 0$ and a number $G_0 > 0$, there is a unique semi G with $G(0) = G_0$ such that G/G_0 is the 'mart-exp' $\mathscr{E}(J)$ of J, i.e. G/G_0 is the unique semi solution of the equation

(2.1) $$g(T) = 1 + \int_{[0,T]} g(t-)dJ(t).$$

Conversely, if G is a given *positive* semi, J is the 'mart–log' $\mathscr{L}(G)$ of G, i.e.

(2.2) $$J(T) = \int_{[0,T]} [1/G(t-)]dG(t).$$

If J is a local martingale, so is G, and conversely if $G > 0$, (but the term 'local' cannot be omitted). If J is continuous, so is G, and conversely if $G > 0$; in this case,

(2.3) $$G_T = G_0 \cdot \exp\{J_T - \tfrac{1}{2}[J,J]_T\}.$$

see Doléans-Dade (1970) p.186, also Jacod (1979) VI.1. As regards notation, if a positive semi is denoted z (with some affix), then $\mathscr{L}(z)$ will be denoted ζ (with the same affix); but if the positive semi is denoted y, then $\mathscr{L}(y)$ will be denoted η.

Recall also that, for two semis G and H, the 'square bracket process' or 'quadratic cross variation' $[G,H]$ may be defined as that finite variation process which for each $t \in \mathscr{T}$ satisfies

$$[G,H]_t = G_0H_0 + \text{p–lim} \sum_{0\le k< 2^n} [G(t^n_{k+1}) - G(t^n_k)]\cdot[H(t^n_{k+1}) - H(t^n_k)]$$

as $n \to \infty$, where $t^n_k = t\delta^n_k$, $\delta^n_k = k2^{-n}$, $n = 1,2,\ldots$, $k = 0,1,\ldots$

If both G and H are continuous with local martingale parts M and N respectively (where $M_0 = G_0$, $N_0 = H_0$), then

$$[G,H] = [M,N] = \langle M,N\rangle$$

is the unique continuous finite variation process such that $MN - [M,N]$ is a continuous local martingale vanishing at t=0; see Dellacherie & Meyer (1980) VII.39–44, VIII.20.

Some notes on local martingales and time changes are also in order; the definitions which follow take account of the special assumptions of our model. A corlol process y is called a *local martingale* (relative to $\mathfrak{A}$) if there exists a sequence (χ_n) of stopping times, $\chi_n(\omega) \uparrow \infty$ a.s., such that, for each n, the stopped process $y^n = (y(t \wedge \chi_n); t \in \mathscr{T})$ is a uniformly integrable (*u.i.*) martingale. Then each χ_n is said to *reduce* y, and (χ_n) is a *reducing* or *fundamental sequence* for the local martingale y, see Dellacherie & Meyer (1980) VI.27 e.s. A given local martingale may admit various fundamental sequences. A non-negative local martingale y is a supermartingale, hence converges a.s. to a limiting variable $y(\infty)$.

Sometimes a family $\chi = (\chi_i)$ of stopping times is defined for an index i taking values in a *real* interval $\mathscr{I} = [0,I)$ with $I \leq \infty$, such that $\chi = (\chi(\omega,i))$, regarded as a process, is right continuous and non-decreasing and takes values in $[0,\infty]$, with $\chi(0) = 0$ and $\chi(i) \uparrow \infty$ as $i \uparrow I$. Let $\breve{\mathscr{A}}_i$ denote the σ-algebra of events at χ_i; the family $\breve{\mathfrak{A}} = (\breve{\mathscr{A}}_i ; i \in \mathscr{I})$ satisfies the 'usual conditions'. Then χ (together with $\breve{\mathfrak{A}}$) can be regarded as defining a *time change.* If $\xi = (\xi_t; t \in \mathscr{T})$ is an $\mathfrak{A}$-process, its *transform* under χ is the $\breve{\mathfrak{A}}$-process $\breve{\xi} = (\breve{\xi}_i; i \in \mathscr{I})$, where $\breve{\xi}_i = \xi(\chi_i)$; if ξ admits an a.s. limiting variable ξ_∞ we set $\breve{\xi}_i = \xi_\infty$ when $\chi_i = \infty$, and also define the limiting variable $\breve{\xi}_I = \xi_\infty$.

In particular, if Ψ is a right continuous, non-decreasing process, with $\Psi(0) = 0$ and $\Psi(\infty) \leq \infty$, a time change χ may be defined by

$$\chi(i) = \inf\{t: \Psi(t) > i\} \text{ for } i \in \mathscr{I}, \text{ setting } \chi(i) = \infty \text{ if } \Psi(\infty) \leq i.$$

Conversely, given a time change χ, we can define Ψ as an inverse time change by

$$\Psi(t) = \inf\{i: \chi(i) > t\} \text{ for } t \in \mathscr{T}.$$

In all cases considered here, Ψ will be continuous and χ right continuous; if Ψ is strictly increasing, then χ is continuous, (including continuity at values of i for which $\chi(i) = \infty$).

In the sequel, we shall encounter situations where there is a corlol $y > 0$ admitting a limiting variable $y(\infty)$, and a time change χ, such that *every sequence* $(\chi_i) = (\chi_{i(n)})$ with $\chi_i \uparrow \infty$ a.s. is fundamental for y. We extend the usual terminology and say that χ is *fundamental* for y. In this case, for each $i = i(n) < I$,

$y^i = (y(t \wedge \chi_i);\ t \in \mathscr{T})$ is a u.i. $\mathfrak{A}$-martingale, which implies (by the Stopping Theorem) that $\breve{y}^i = (\breve{y}(j \wedge i);\ j \in \mathscr{J})$ is a u.i. $\breve{\mathfrak{A}}$-martingale, so that $\breve{y} = (\breve{y}_i : i \in \mathscr{J})$ is an $\breve{\mathfrak{A}}$-martingale, which need not be u.i. (A converse proposition applies if χ is defined by a strictly increasing process Ψ, but if Ψ is only non-decreasing the paths $i \mapsto \chi(i)$ may 'jump across' intervals of $\mathscr{T}$, and then martingale properties of $\breve{y}$ imply nothing about y on these intervals.) For more on time changes, see Dellacherie & Meyer (1980) VI.56 and references given there, also Meyer (1966) Ch.IV, Jacod (1979) Ch.X.

Returning to the model, it is assumed that a finite number of assets (also called securities) indexed by $\lambda = 1,\dots,\Lambda$ is available at all times. For each λ there is given a positive semimartingale z^λ with $z^\lambda(\omega,0) = 1$ called the (market) *returns* or *price process* for λ; thus $z^\lambda(\omega,t)$ represents the value at t in state ω of one unit of capital invested at zero time in asset λ (with instantaneous reinvestment of dividends etc in the same asset), the return being measured in suitable 'natural' units. The formula $\zeta^\lambda = \mathscr{L}(z^\lambda)$ then defines the mart–log returns process for λ. We write $Z = (z^1,\dots,z^\Lambda)$, $\mathscr{L}(Z) = (\zeta^1,\dots,\zeta^\Lambda)$ for the corresponding vector processes. In case z^λ is continuous, ζ^λ has a unique decomposition

$$\zeta^\lambda = M^\lambda + A^\lambda, \qquad M^\lambda(0) = A^\lambda(0) = 0,$$

where M^λ is a continuous local martingale and A^λ is a continuous process of (locally) finite variation.[11]

A *portfolio plan* π is defined as a finite vector process with components π^λ, $\lambda = 1,\dots,\Lambda$, which is defined for $t \geq 0$, adapted and left continuous with right limits (hence predictable and locally bounded) and which satisfies

[11] Here I have simplified the notation of Foldes (1990) by suppressing the process $x^\lambda = \ln\{z^\lambda\}$ with its decomposition $x^\lambda = M^\lambda + V^\lambda$. To match up formulas in the continuous case, note (dropping the superscript) that $z = \exp\{x\}$ implies $dz = z\{dx + \frac{1}{2}d\langle M,M\rangle\} = z\{dM + dV + \frac{1}{2}d\langle M,M\rangle\}$, so that $A = V + \frac{1}{2}d\langle M,M\rangle = V + \frac{1}{2}d[M,M]$. If $M = \sigma W$ with W Wiener, $\sigma > 0$, and $V = mt$, then setting $m = \mu - \frac{1}{2}\sigma^2$ we have $z_t = \exp\{x_t\} = \exp\{mt + \sigma W_t\}$, $\zeta_t = \mu t + \sigma W_t$. Cf. eq. (3.1) below.

$$\Sigma_\lambda \pi^\lambda(\omega,t) = 1 \quad \text{for all } (\omega,t), \qquad \pi^\lambda(0) = \pi^\lambda(0+);$$

the π^λ represent the proportions of the investor's wealth invested in asset λ. We denote by Π^0 the set of all such plans, and by Π^+ the subset of non-negative elements ('no short sales'). The set of all π which are admissible in a particular problem is denoted by Π, and it is assumed as in Foldes (1990) that Π is Π^0 or Π^+ if Z is continuous but $\Pi = \Pi^+$ if Z has jumps. Sometimes we write simply Π when it is not necessary to specify which case is considered.

Given the mart–log processes ζ^λ for the individual securities, the corresponding process ζ^π for the portfolio plan π may be defined simply by

$$(2.4) \qquad \zeta^\pi(T) = \int_{[0,T]} \Sigma_\lambda \, \pi^\lambda(t)\, d\zeta^\lambda(t),$$

and then the *portfolio returns process* z^π for π is given by $z^\pi = \mathscr{E}(\zeta^\pi)$ – see Foldes (1990) for details. It may be checked that z^π always remains positive if Z is continuous or if $\Pi = \Pi^+$.

Suppose now that the investor has an initial capital $K_0 > 0$ and no outside income. Given a portfolio plan π, we say that a (progressive) process $\bar{c}$ is a π-feasible *consumption plan in natural units*, or simply a $\bar{c}$*-plan*, if it is non-negative and a.s. Lebesgue integrable on finite intervals and if the equation

$$(2.5) \qquad \bar{k}^\pi(T) - K_0 = \int_{[0,T]} \bar{k}^\pi(t-)\, d\zeta^\pi(t) - \int_0^T \bar{c}(t)dt$$

is solved by one and only one semimartingale $\bar{k}$ and this solution is a.s. non–negative on $\mathscr{T}$; then $\bar{k}^\pi$ is called the *capital plan in natural units* corresponding to $\bar{c}$. The solution is given by

$$(2.6) \qquad \bar{k}^\pi(T) = z^\pi(T)\left[K_0 - \int_0^T [\bar{c}(t)/z^\pi(t)]dt\right].$$

Defining new processes k^π and c^π, called *capital and consumption plans in* π*-standardised units*, by

$$k^\pi_t = \bar{k}^\pi_t / z^\pi_t, \quad c^\pi_t = \bar{c}^\pi_t / z^\pi_t,$$

and dropping the superscript π for the moment, the preceding equation becomes

(2.7) $$k(\tau) = K_0 - \int_0^\tau c(t)dt.$$

Now, since $c \geq 0$, the condition that $k(\tau) \geq 0$ for all τ, a.s., is equivalent to

(2.8) $$\int_0^\infty c(t)dt \leq K_0 \quad \text{a.s.}$$

This condition does not depend on the choice of π, so that we may define the set $\mathscr{C}$ of *consumption plans in standardised units* simply as the set of non–negative, progressive processes $c = (c_t)$ satisfying (8). A portfolio-consumption plan, or portfolio-cum-saving (PS) plan, in standardised units can then be defined as a pair (c,π) with $c \in \mathscr{C}$, $\pi \in \Pi$; the corresponding plan $(\bar{c},\pi)$ in natural units is obtained by setting $\bar{c}^\pi = c \cdot z^\pi$.

Next, it is assumed that the investor's aim is to maximise a *welfare functional*

(2.9) $$\varphi(c,\pi) = E \int_0^\infty u[\bar{c}(\omega,t)]e^{-\varrho t}dt, \text{ where } \bar{c} = c \cdot z^\pi.$$

The function u is defined and twice continuously differentiable on $[0,\infty]$ and takes values in $[-\infty,\infty]$, with $u'' < 0 < u'$ on $(0,\infty)$ and $u'(0) = \infty$.[12] Following Terence Gorman, we refer to $u(\cdot)$ as the *felicity function*, to $e^{-\varrho t}$ with $\rho > 0$ as the *impatience function* and to $u(\cdot)e^{-\varrho t}$, $u'(\cdot)e^{-\varrho t}$ as the *utility* and *marginal utility functions* at t. The *utility* and *marginal utility plans* corresponding to (c,π) are the *processes*

$$u[\bar{c}(\omega,t)]e^{-\varrho t} \text{ and } u'[\bar{c}(\omega,t)]e^{-\varrho t}, \text{ where } \bar{c}(\omega,t) = c(\omega,t)z^\pi(\omega,t).$$

The domain of the functional φ is taken to be $\mathscr{C} \times \Pi$. It is always assumed that for each feasible plan the positive part of the double integral in (9) is finite, and further that the supremum φ^* of the functional is finite. The PS problem is to maximise φ on $\mathscr{C} \times \Pi$ if possible. A plan (c^*,π^*) or $(\bar{c}^*,\pi^*)$ is *optimal* if $\varphi(c^*,\pi^*) = \varphi^*$; we also write $\bar{c}^* = c^* \cdot z^*$, $z^* = z^{\pi^*}$. In this Paper, we usually take the existence of optima for granted without special comment. The 'star' notation is reserved for a distinguished

[12] The functional has been specified in a more restrictive form than in Foldes (1990), mainly in order to avoid certain questions of measurability. In particular, 'state dependence' of the form $u[\bar{c}(\omega,t);\omega,t]$ is excluded here, although it is consistent with validity of the martingale conditions for optimality. Also, for neatness, we now write u, φ rather than $\bar{u}$, $\bar{\varphi}$.

plan which is either optimal or a candidate for optimality. For this plan, *we assume, usually without special mention*, that $c^* > 0$, and also that there is $\alpha_0 \in (0,1]$ such that

(2.10) $$\varphi(c^*-\alpha c^*,\pi^*) > -\infty \quad \text{for} \quad 0 \le \alpha \le \alpha_0.$$

The marginal utility plan evaluated along a star plan is written for short as v, i.e.

(2.11) $$v(\omega,t) = u'[\bar{c}^*(\omega,t)]e^{-\varrho t}, \quad \bar{c}^* = c^* \cdot z^*.$$

If π is fixed, the problem of maximising $\varphi(.,\pi)$ on $\mathscr{C}$ is equivalent to a *problem of optimal saving* with a single security. In this case we say that c^* is *π–optimal* (or equivalently that $\bar{c}^* = c^* \cdot z^\pi$ is π–optimal) if $\varphi(.,\pi)$ attains a finite supremum on $\mathscr{C}$ at c^*.

Let v be as in (11), and for arbitrary $\pi \in \Pi$ define a process y^π, called the *shadow price process* (or *marginal utility price process*) associated with π, by

(2.12) $$y^\pi(\omega,t) = v(\omega,t)z^\pi(\omega,t).$$

In particular, $y^\lambda = v \cdot z^\lambda$ defines the shadow price process for asset λ. If $\eta^\pi = \mathscr{L}(y^\pi)$, then (in abridged notation)

(2.12a) $$\eta^\pi = \int dy^\pi/y^\pi_- = \int d(z^\pi v)/(z^\pi v)_- = \zeta^\pi + \int dv/v_- + [\int dv/v_-, \zeta^\pi]$$

using (12) and integration by parts. Consequently, taking account of (4),

(2.12b) $$\eta^\pi(T) = \int_{[0,T]} \Sigma_\lambda \, \pi^\lambda(t) \, d\eta^\lambda(t).$$

If $\pi = \pi^*$, we write $y^* = v \cdot z^*$ (but if there is only one security we sometimes write just $y = v \cdot z$). Thus, for arbitrary π,

$$y^\pi/y^* = z^\pi/z^* = \mathscr{E}(\zeta^\pi)/\mathscr{E}(\zeta^*).$$

Note that in these definitions the marginal utility process v is always evaluated along the star plan. Note also that $0 < \bar{c}^* < \infty$ implies $0 < v < \infty$, and then $0 < z^\pi, z^* < \infty$ implies $0 < y^\pi, y^* < \infty$. The assumption that $u'(0) = \infty$ ensures that $\bar{c}^* > 0$ if $\bar{c}^*$ is π^*–optimal, hence also $c^* > 0$, so that $k^*(\omega,.)$ is strictly decreasing on $\mathscr{T}$, a.s. Usually we write $y^*(0)$ as y_0; we have $y^\pi(0) = y_0$ for every π and $y_0 = v(0) = v_0$.

Given the star plan and another plan (c,π) or $(\bar{c},\pi)$ with $\bar{c} = c\cdot z^{\pi}$, we write $\delta\bar{c} = \bar{c} - \bar{c}^*$, $\delta c = c - c^*$, $\delta\bar{k} = \bar{k} - \bar{k}^*$, $\delta k = k - k^*$, $\delta\pi = \pi - \pi^*$ etc, noting that

$$\delta\bar{c} = c\cdot z^{\pi} - c^*\cdot z^* = \delta c\cdot z^{\pi} + c^*(z^{\pi} - z^*), \qquad \delta\bar{c}\cdot v = \delta c\cdot y^{\pi} + c^*(y^{\pi} - y^*).$$

We denote by $D\varphi = D\varphi(\bar{c}^*,\pi^*;\ \delta\bar{c},\delta\pi)$, or $D\varphi(c^*,\pi^*;\ \delta c,\delta\pi)$, the (Gâteaux) directional derivative of φ at $(\bar{c}^*,\pi^*)$ in the direction of the (feasible) variation $(\delta\bar{c},\delta\pi)$ or $(\delta c,\delta\pi)$; explicitly

(2.13) $$D\varphi = E\int_0^{\infty} \delta\bar{c}\cdot v\cdot dt = E\int_0^{\infty} [\delta c\cdot y^{\pi} + c^*(y^{\pi} - y^*)]dt.$$

The plan $(\bar{c}^*,\pi^*)$ is optimal iff $D\varphi \le 0$ for each variation; then, for each variation,

(2.13a) $$0 \ge D\varphi(c^*,\pi^*;\ \delta c,\delta\pi) \ge D\varphi(c^*,\pi^*;\ -c^*,0] = -E\int_0^{\infty} c^*\cdot y^*\ dt,$$

and it follows from from (10) that the last term is finite, see Foldes (1990) eqs.(4.1–3), (4.6–9) and (2.32). Thus each $D\varphi$ is finite. In particular, setting $\delta c = 0$, we have

(2.13b) $$0 \le E\int_0^{\infty} c^*\cdot y^{\pi}\cdot dt \le E\int_0^{\infty} c^*\cdot y^*\cdot dt < \infty,$$

so that each $c^*\cdot y^{\pi}$ is (product) integrable.

If $\pi = \pi^*$ *is fixed, or if there is only one asset,* we have simply

(2.13c) $$D\varphi = E\int_0^{\infty} \delta\bar{c}\cdot v\cdot dt = E\int_0^{\infty} \delta c\cdot y^*\cdot dt,$$

and $c^* \in \mathscr{C}$ is π^*-optimal iff $D\varphi \le 0$ for all $\delta c = c - c^*$ with $c \in \mathscr{C}$. Necessary and sufficient conditions for c^* to be π^*-optimal may be given in martingale form as follows:

(2.14a) y^* is an $\mathfrak{A}$-local martingale reduced by some sequence (χ_n) of stopping times, increasing a.s. to ∞ as $n \uparrow \infty$, such that

(2.14b) $$\lim_n E[y^*(\chi_n)k^*(\chi_n)] = 0,$$

see Prop.1 of Foldes (1990); bear in mind the implicit conditions $c^* > 0$ and (10). Note that these conditions do not require z^* to be continuous. Since y^* is a positive local martingale and k^* is positive and decreasing, the a.s. limits $y^*(\infty)$ and $k^*(\infty)$ exist and, by Fatou's Lemma, (14b) *implies* $y^*(\infty)\cdot k^*(\infty) = 0$ a.s., or

(2.14b′) $$y^*(\infty) = 0 \text{ if } k^*(\infty) > 0, \text{ a.s.}$$

Also note that, given (14a), the following is *equivalent* to (14b):

(2.15) $$E\int_0^{\infty} y^*(t)c^*(t)dt = y_0 \cdot K_0.$$

Indeed, given (14a–b), we have

$$E\int_0^{\infty} y^*(t)c^*(t)dt = \lim_n E\int_0^{\chi_n} y^*(t\wedge\chi_n)c^*(t)dt = \lim_n E\left[y^*(\chi_n)\cdot\int_0^{\chi_n} c^*(t)dt\right]$$

$$= \lim_n E\,\{y^*(\chi_n)[K_0 - k^*(\chi_n)]\} \;=\; y_0\cdot K_0\,,$$

using the theorem on integration of a u.i. martingale with respect to a non-decreasing process, Elliott (1982) 7.16, then (7) and $Ey^*(\chi_n) = y_0$. Conversely, given (14a) and (15), the preceding calculation implies (14b). It is to be expected on economic grounds that (14b′) and (15) – which do not involve the sequence (χ_n) – should be satisfied by an optimal plan. The former condition says that (standardised) capital which remains unused 'at infinity' has a zero price (in utils), while the latter says that the total value (price × quantity) of capital is consumed over time. The question remains of finding sequences (χ_n), with reasonable economic interpretations, for which (14) is satisfied.

It follows from the results of Foldes (1978a) that one may take any sequence of the form

(2.16a) $$\chi_n(\omega) = \tau_i(\omega) \quad \text{with } i = i(n) \uparrow K_0 \text{ as } n \uparrow \infty,$$

where τ_i is the *consumption time at the level* i defined by

(2.16b) $$\tau_i(\omega) \;=\; \inf\{T\colon \textstyle\int_0^T c^*(\omega,t)dt \;> i\} \wedge \infty, \quad 0 \le i < K_0;$$

for example, set $i(n) = K_0(1-2^{-n})$. Explicitly, since $c^* > 0$, $\tau_i(\omega)$ is both the *upcrossing time* and the *arrival time* of $\int_0^T c^*(t)dt$ at the level i, except that $\tau_i(\omega) = \infty$ if this level is never reached. We denote by $\hat{\mathcal{A}}_i$ the σ–algebra of events at τ_i. Equivalently, we may consider the *depletion time at the level* i defined by

(2.17) $$\nu_i(\omega) \;=\; \inf\{T\colon k^*(\omega,T) < K_0 - i\} \wedge \infty, \quad 0 \le i < K_0.$$

On present assumptions, the concepts of consumption time and depletion time coincide, but later we shall need to distinguish between them. The family $\tau = (\tau_i;\, 0 \le i < K_0)$ defines a time change, which together with the filtration

$\hat{\mathfrak{A}} = (\hat{\mathscr{A}}_i)$ will be called here the *transformation to consumption time* – rather than to depletion time as in Foldes (1978a&1990). If $\xi = (\xi_t;\ t \in \mathscr{T})$ is an $\mathfrak{A}$–process admitting an a.s. limiting variable ξ_∞, its *transform under* τ is denoted $\hat{\xi} = (\hat{\xi}_i;\ 0 \le i < K_0)$, where $\hat{\xi}_i = \xi[\tau_i]$, with $\hat{\xi}_i = \xi_\infty$ if $\tau_i = \infty$. If the times (χ_n) appearing in (14a–b) are chosen as in (16), then (14b′) is *equivalent* to (14b) in the presence of (14a). Indeed, given (14b′) and (16), we have

$$y(\tau_i) = 0 \text{ if } \tau_i = \infty \text{ and } k^*(\tau_i) = K_0 - i \text{ if } \tau_i < \infty, \quad \text{hence}$$

$$E\{y^*(\tau_i)\cdot k^*(\tau_i)\} = (K_0-i)\cdot Ey^*(\tau_i) = (K_0-i)y_0 \longrightarrow 0 \quad \text{as } i \uparrow K_0.$$

Thus the use of consumption or depletion times allows the transversality condition to be given in 'pointwise form at infinity' in both the sufficient and the necessary conditions of optimality. The conditions (14) may also be written as

(2.18a) $\hat{y}^*$ is a (true) $\hat{\mathfrak{A}}$-martingale,

(2.18b) $\lim_i E\{\hat{y}^*(i)\cdot\hat{k}^*(i)\} = 0, \quad i = i(n) \uparrow K_0.$

Further, if $k^*(.,\mathrm{T}) = K_0 - \int_0^{\mathrm{T}} c^*(.,t)dt$ is a.s. bounded away from zero for each T, (so that each τ_i has a positive lower bound), then (14) may be replaced by

(2.19a) y^* is a (true) $\mathfrak{A}$–martingale;

(2.19b) $\lim_{\mathrm{T}} E[y(\mathrm{T})k^*(\mathrm{T})] = 0, \quad \mathrm{T} \uparrow \infty,$

see Foldes (1978a) T.6.

Turning now to the case where both c *and* π *are variable*, let (c^*,π^*) be a distinguished plan satisfying $c^* > 0$ and (10) and further assume *either* that Z is continuous and $\Pi = \Pi^0$ *or* that $\pi^* > 0$ (in which case Π may be Π^0 or Π^+). Then (c^*,π^*) is optimal iff

(2.20a) c^* is π^*–optimal, *and*

(2.20b) y^λ is a local martingale for each λ, or equivalently

(2.20b′) y^π is a local martingale for each π;

see Foldes (1990) S.3 and Appendices A & C below for further details, also Foldes (1990) for discussion of cases with $\Pi = \Pi^+$ and $\pi^* \ge 0$.

Three new extensions of the preceding results may be sketched here.

A. *Price Times.* Returning to the case of a single security (or a fixed portfolio plan), a possible choice of times satisfying (14a–b) is to take for (χ_n) a sequence of *price times*

(2.21) $$\wp_n = n \wedge \inf\{t: y(t) > n\}, \quad y_0 \leq n \uparrow \infty.$$

The use of price times to characterise an optimum can be extended to the PS model; see Appendix A for details.

B. *Saving Model with Income Process.* Again with a single security (z not required to be continuous), suppose that the investor also has an exogenous, non–marketable income, e.g. a salary, or rent from a real property. To model this, let $\bar{s} = \bar{s}(\omega,t) \geq 0$ be a progressive process, a.s. integrable on compacts of $\mathcal{T}$, called the *income process* in natural units, and replace the term $-\int_0^T \bar{c}(t)dt$ in the equation of accumulation (5) by $\int_0^T [\bar{s}(t)-\bar{c}(t)]dt$. Setting $s(t) = \bar{s}(t)/z(t)$ to define the income process in standardised units, and assuming that this also is a.s. integrable on compacts, (7) becomes

(2.22) $$k(\tau) = K_0 + \int_0^T [s(t)-c(t)]dt.$$

An optimal plan (c^*,k^*) must again satisfy $c^* > 0$ for a.e. (ω,t), (since otherwise one can construct a variation δc with $D\varphi = \infty$, contrary to optimality). For an optimum, and more generally for a star plan, we continue to assume that $c^* > 0$ and that (10) holds, so that

(2.23) $$E \int_0^\infty y(t)c^*(t)dt < \infty,$$

and also assume that

(2.24) $$E \int_0^\infty [v(t)\bar{s}(t)]dt = E \int_0^\infty [y(t)s(t)]dt < \infty.$$

A solvency constraint, whether of the form $k(\tau) \geq 0$ or (say) $E^T[\liminf_{t\to\infty} k(t)] \geq 0$ a.s., now defines a separate constraint for each τ, so that one cannot pass from (22) to an analogue of (8). Indeed, the saving and PS problems with exogenous income are in general much more complicated than the corresponding problems without; see El Karoui & Jeanblanc-Piqué (1998) for a recent treatment of a related model and a survey of the literature.[13] Nevertheless, matters are simplified if, at every time and state, there is a positive conditional probability of the exogenous income drying up in the immediate future, at least for a while. More formally, we assume that, for every $\tau \in \mathcal{T}$, there exists a (strictly) positive random variable $h_\tau = h_\tau(\omega)$ such that

(2.25) $$P^{\tau}[s(t) = 0 \text{ for } \tau < t < \tau + h_\tau] > 0 \text{ a.s.}$$

This condition together with $u'(0) = \infty$ ensures that capital is always maintained at a positive level along an optimal plan; (if not, there will be a progressive (ω,t)-set of positive product measure on which $c^* = 0$). Thus, in discussing optimality, we may as well restrict attention to plans satisfying $k(\tau) \geq 0$ on $\mathcal{T}$ a.s.

The statement and derivation of the conditions for optimality need some modification because in general a feasible process k need no longer be everywhere decreasing or bounded or converge at ∞ (and these difficulties do not seem to be excluded even in the case of an optimal plan). Given a star plan, the conditions (14) are still sufficient for optimality, but characterisation of suitable fundamental sequences and necessity present new problems. We confine attention to modified conditions involving depletion times. Let

(2.26) $$\Gamma(\omega,\tau) = \sup\{K_0 - k^*(\omega,t) \colon t \leq \tau\} = \sup\{\textstyle\int_0^t [c^*(\theta) - s(\theta)]d\theta \colon t \leq \tau\};$$

this process, called the *depletion process,* is progressive, absolutely continuous and non-decreasing with values in $[0,K_0]$ and $\Gamma(0) = 0$. Then k^* may be replaced by $K_0 - \Gamma$ in (17) without changing the definition of the depletion time ν_i; thus

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[13] The present model differs significantly, as to both specification and techniques, from the works surveyed. Without attempting a detailed comparison, it should be noted that all the publications considered model securities markets as diffusions (rather than general semimartingales), with the income process appearing either as an additional diffusion or as a process adapted to the natural filtration of the Brownian motion driving the securities market. In the latter case the 'income process...is spanned by the market assets and therefore is not a source of new uncertainty', El Karoui & Jeanblanc-Piqué (1998) p.413; such an assumption largely nullifies the economic relevance of the results, at least in so far as the model addresses optimisation with a stochastic rather than a sure wage.

(2.27a) $\nu_i(\omega) = \inf\{t: \Gamma(\omega,t) > i\}, \quad 0 \leq i < K_0,$ with

(2.27b) $\nu_i = \infty$ iff $\Gamma(\omega,\infty) \leq i$, and

(2.27c) $k^*(\nu_i) = K_0 - \Gamma(\nu_i) = K_0 - i$ iff $\nu_i < \infty$.

Note that Γ can be constant on certain intervals, so that ν_i remains the first downcrossing time for k^* at the level $K_0 - i$ but can no longer be called the first arrival time. The conditions for optimality (14a–b) remain necessary and sufficient if one takes for (χ_n) any sequence (ν_i) with $i = i(n) \uparrow K_0$ and replaces k^* by $K_0 - \Gamma$. The equivalence of (14b′) and (14b) stands with the same replacement. Furthermore, the conditions (14), as now amended, imply (15) with c^* replaced by $c^* - s$. It is also true that y is a true martingale if $K_0 - \Gamma$ is bounded away from zero for each t. See Appendix B for explicit statements of these results and some proofs.

Conditions analogous to (18) can also be obtained. Denoting by $\check{\mathscr{A}}_i$ the σ-algebra at ν_i, the family $\nu = (\nu_i;\ 0 \leq i < K_0)$ can be regarded as a time change, which together with the filtration $\check{\mathfrak{A}} = (\check{\mathscr{A}}_i)$ is the *transformation to depletion time.* If ξ is an $\mathfrak{A}$–process admitting an a.s. limiting variable ξ_∞, its *transform* under ν is the $\check{\mathfrak{A}}$-process $\check{\xi} = (\check{\xi}_i;\ 0 \leq i < K_0)$ where $\check{\xi}_i = \xi[\nu_i]$, and $\check{\xi}_i = \xi_\infty$ if $\nu_i = \infty$. The conditions for optimality may then be written as in (18) with $\hat{y}, \hat{\mathfrak{A}}, \hat{k}^*$ replaced by $\check{y}, \check{\mathfrak{A}}, K_0 - \check{\Gamma}$.

The theory is simplified if it is assumed that

(2.28) $\int_0^\infty s(t)dt < \infty$ a.s.

Then, for every feasible plan, (22), $k \geq 0$ and $c \geq 0$ imply

(2.29) $\int_0^\infty c(t)dt \leq K_0 + \int_0^\infty s(t)dt < \infty, \quad k(\infty) = K_0 + \int_0^\infty [s(t)-c(t)]dt < \infty,$ a.s.,

so that, a.s., k is of finite variation on $[0,\infty]$ and $k(\infty)$ exists as a finite limit. Conditions (14a&b) remain necessary and sufficient with (χ_n) any sequence (ν_i) such that $i = i(n) \uparrow K_0$, even without replacing k^* by $K_0 - \Gamma$; but then (14b′) is not equivalent to (14b). See also Appendix B, Remark I.[14]

C. *Consumption Times for PS Model.* For the PS model, it was shown in Foldes (1990), as part of the proof of the conditions (20), that, for each portfolio plan π, the process $\hat{y}^{\pi}$, defined as the transform of y^{π} to consumption (= depletion) time, is an $\hat{\mathfrak{A}}$–supermartingale; (this holds whether or not Z is continuous, for $\Pi = \Pi^0$ or $\Pi = \Pi^+$). If Z is continuous and $\Pi = \Pi^0$, it is shown in Appendix C that $\hat{y}^{\pi}$ is actually an $\hat{\mathfrak{A}}$–martingale for every π satisfying certain boundedness conditions. It is enough if the process z^{π}/z^* is bounded; this is too restrictive even where $\ln z^{\pi}$ is a Brownian motion, but is reasonable for some applications to derivatives. Some more general, though less transparent, conditions are also given.

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[14] The discussion of the model with income can be modified to allow negative values of s, provided that $K_0 + \int_0^T s(t)dt$ is a.s. positive and bounded away from zero on $\mathcal{T}$; this modification is useful if (say) s is the cash flow from an investment project.

3. DERIVATION OF BLACK-SCHOLES FORMULA

A rather comprehensive survey of alternative derivations of the BS formula has been prepared by Andreasen et al. (1996), saving us a great deal of labour. These authors distinguish eight methods, six of which are based on arbitrage or hedging/ replication arguments with continuous trading in a frictionless market. The remaining two involve maximising the expectation of a utility function. One of these replaces continuous trading by the assumption of a representative investor with power utility. The other is based on a finite-horizon, continuous-time, Merton-style CAPM. First order conditions for portfolio optimality are obtained by dynamic programming, and it is then shown that, if a European call option is introduced in zero net supply, and its price in equilibrium is a function of time and current stock price only, then that price must satisfy the BS p.d.e. Reference is also made to some papers which consider the robustness of the BS formula.

Mention should also be made here of recent work on option valuation in continuous time with portfolio constraints or market frictions, for example transaction costs. If it is required in these situations to obtain precise formulas for valuation rather than mere bounds (which in many practical situations may be so wide as to be almost useless), it is usually necessary to maximise a *specified* expected utility or other criterion functional. It is then of interest to show that the BS formula is obtained in the limit as the transaction costs are eliminated; see Barles & Soner (1998) for a recent contribution to this approach, with a survey of earlier work. The BS formula thus comes to serve as a benchmark for valuation in utility-based models. In this connection, a derivation of the formula which appeals directly to martingale properties of marginal utilities may be instructive.

We begin with the simplest special case of the PS model which is suitable for deriving the BS formula, and then comment informally on the consequences of relaxing the assumptions. Given $(\Omega, \mathscr{A}, P)$ and $\mathscr{T}$ as before, let $W = (W_t\colon t \in \mathscr{T})$ be a standard

Wiener process on $(\Omega, \mathscr{A}, P)$, (i.e. an a.s. continuous process with stationary independent increments and W_1 a standard normal variable), and let $\mathfrak{A} = (\mathscr{A}_t)$ be the augmented natural filtration of W. Suppose initially that there are only two securities, a stock numbered 1 and a bond numbered n, with market price and mart-log price processes of the form

(3.1) $$z_t^1 = \exp\{(\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\}, \qquad \zeta_t^1 = \mu t + \sigma W_t,$$

(3.2) $$z_t^n = \exp\{rt\}, \qquad \zeta_t^n = rt,$$

for $t \in \mathscr{T}$, where μ, $\sigma > 0$ and $r > 0$ are constants. These prices cannot be influenced by the individual investor. Assuming the existence of an optimal PS plan, let v be the marginal utility process along this plan. Dropping superscripts and abridging the notation we have, for each security or portfolio, a shadow price process $y = z \cdot v$ and a mart-log process

(3.3) $$\eta = \int dy/y = \int d(zv)/zv = \zeta + \int dv/v + [\int dv/v, \zeta],$$

where $\zeta = \int dz/z$, cf. (2.12a). (Note: writing v and y rather than v- and y- in the denominators takes into account that v is continuous; this can be proved directly but will also be confirmed by the calculation which follows.) Explicitly,

(3.4) $$\eta_t^1 = \mu t + \sigma W_t + \int_{(0,t]} dv/v + [\int dv/v, \sigma W]_t,$$

(3.5) $$\eta_t^n = rt + \int_{(0,t]} dv/v;$$

(there is no bracket term in (5) because ζ^n is deterministic). Now, by optimality, the processes y_t^1 and y_t^n are local martingales, therefore the same is true of η_t^1 and η_t^n, and being local martingales on the Wiener filtration these processes are continuous and representable as stochastic integrals. We choose to represent η_t^n :

(3.6) $$\eta_t^n = \int_{(0,t]} f_s \, dW_s,$$

with some adapted $f_s = f(\omega, s)$ satisfying $\int_0^t f_s^2 \, ds < \infty$ a.s. on $\mathscr{T}$. Then, combining (5) and (6),

(3.7) $$\int_{(0,t]} dv/v = \int_{(0,t]} f_s \, dW_s - rt, \quad \text{hence}$$

(3.8) $$[\int dv/v, \sigma W]_t = [\int f \, dW, \sigma W]_t = \sigma \int_0^t f_s \, ds.$$

Subtracting (5) from (4), using (8) to evaluate the bracket and rearranging, we have

(3.9) $$\eta^1_t - \eta^n_t - \sigma W_t = (\mu - r)t + \sigma\int_0^t f_s\,ds.$$

Now both sides vanish identically, because the left side is a continuous local martingale while the right side is of finite variation, with initial values of zero. Consequently

(3.10) $$\eta^1_t - \eta^n_t = \sigma W_t \quad \text{and} \quad f_t = (r-\mu)/\sigma\,;$$

in particular, the 'price of risk process' $-f$ is a constant. Referring to (7),

(3.11) $$\int_{(0,t]} dv/v = fW_t - rt = [(r-\mu)/\sigma]W_t - rt, \quad \text{hence}$$

(3.12) $$v(t) = v_0\cdot \mathscr{E}(\textstyle\int dv/v)_t = v_0\cdot\exp\{fW_t - \tfrac{1}{2}f^2t - rt\} = v_0\cdot e^{-rt}\cdot \mathscr{E}\{fW\}_t\,.$$

This result is remarkable in that the marginal utility *process* v is determined, up to a scale factor, irrespective of the marginal utility *function* u'; this is analogous to the fact that, in a deterministic dynamic model, the ratio of marginal utilities of consumption at different times is equated to the corresponding compound interest factor. Before going further, let us note that the processes y^1 and y^n, which are local martingales by optimality, are actually true martingales. Substituting into (4) from (11) and (8), then calculating y^1 as $v_0\cdot \mathscr{E}(\eta^1)$, also y^n as $v_0\cdot \mathscr{E}(\eta^n)$ from (6), and simplifying, we obtain

(3.13) $$y^1_t = v_0\cdot \mathscr{E}\{\textstyle\int (\sigma + f)dW\}_t,$$

(3.14) $$y^n_t = v_0\cdot \mathscr{E}\{\textstyle\int f\,dW\}_t,$$

which are true martingales (but not uniformly integrable on $\mathscr{T}$).

It is also of interest to see whether y^* is a true martingale. By (2.12b) we have

(3.15) $$\int_{(0,t]} dy^*/y^* = \eta^*_t = \int_{(0,t]} [\pi^{1*}_s\cdot d\eta^1_s + (1-\pi^{1*}_s)d\eta^n_s];$$

substituting for $d\eta^1$, then for $\int dv/v$, and simplifying yields

(3.16) $$\eta^*_t = \int_{(0,t]} (f + \sigma\pi^{1*}_s)dW_s, \quad y^*_t = v_0\cdot \mathscr{E}\{\eta^*\}_t,$$

and y^* is a true martingale if (for example) π^{1*} is bounded on compacts, or if $E\exp\{\frac{1}{2}\int_0^T (\pi^{1*}_s)^2\,dt\} < \infty$ for $T < \infty$. Some such condition is satisfied for the usual utility functions, but is not essential for the derivation of the BS formula.

We now modify the model by introducing a European call option (numbered 0) on the stock, with expiry at a clock time $\mathfrak{s} > 0$ and exercise price $C > 0$; of course, the return at $\mathfrak{s}$ from one unit of the option is

$$(3.17) \qquad z^0_{\mathfrak{s}} = (z^1_{\mathfrak{s}} - C)^+ = (z^1_{\mathfrak{s}} - C)\mathfrak{I}_{\{z^1_{\mathfrak{s}} \geq C\}}.$$

The problem is to determine a price process $\{z^0_t\,; t \leq \mathfrak{s}\}$ for the option which is consistent with portfolio optimality. We assume that market prices of the stock and bond cannot be influenced by the individual investor and, for simplicity, that the availability of the option alters neither these prices nor the marginal utility process v – say, because the option, being redundant, is not actually traded, or being traded does not alter the 'fundamentals' of the market. However, the formulas which follow remain valid if the price and marginal utility processes change, provided that the processes considered are those which obtain in the presence of the option.

Assuming continued optimality, there should – subject to Remark II below – be a shadow (marginal utility) price process

$$(3.18) \qquad y^0_t = z^0_t \cdot v_t,$$

defined for times $t \leq \mathfrak{s}$, which is at least a local martingale, where $z^0_{\mathfrak{s}}$ satisfies (17). Since $0 \leq z^0 \leq z^1$, we have $0 \leq y^0 \leq y^1$, and since y^1 is a uniformly integrable martingale on $[0,\mathfrak{s}]$ the same is true of y^0. Consequently, for $t \leq \mathfrak{s}$,

$$(3.19) \qquad y^0_t = E^t y^0_{\mathfrak{s}}, \quad \text{i.e.} \quad z^0_t = E^t[(v_{\mathfrak{s}}/v_t)\cdot z^0_{\mathfrak{s}}], \quad \text{or, using (12),}$$

$$(3.20) \qquad z^0_t = e^{-r(\mathfrak{s}-t)}\, E^t[\,\mathcal{E}\{f(W_{\mathfrak{s}} - W_t)\}\cdot z^0_{\mathfrak{s}}].$$

Substituting from (17),

$$(3.21) \qquad \begin{aligned} z^0_t &= E^t\{(v_{\mathfrak{s}}/v_t)\cdot(z^1_{\mathfrak{s}} - C)^+\} \\ &= e^{-r(\mathfrak{s}-t)}\, E^t\Big\{e^{f[W(\mathfrak{s})-W(t)]-\frac{1}{2}f^2(\mathfrak{s}-t)}\cdot\Big[(z^1_{\mathfrak{s}} - C)\mathfrak{I}_{\{z^1_{\mathfrak{s}} \geq C\}}\Big]\Big\}. \end{aligned}$$

The usual transformations using the properties of the normal distribution – see Neftci (1996) Ch.15 – now yield the BS formula.

REMARK I. Despite its intuitive appeal, the part of the preceding argument relating to a European call option does not fit formally into the theory of Foldes (1990), since it is assumed there that all securities exist for all time and have strictly positive returns processes. It is not difficult, although tedious, to extend the theory to allow for securities with finite life and a positive probability of zero return from some time onward, but for present purposes the following remarks will suffice. To deal with the problem of finite life, the process z^0 can be defined formally for $t \geq \mathfrak{s}$ by setting

$$(3.22) \qquad z^0_t = z^0_{\mathfrak{s}} \cdot z^*_t / z^*_{\mathfrak{s}}, \quad t \geq \mathfrak{s},$$

corresponding to the assumption that the proceeds $z^0_{\mathfrak{s}}$ are invested from $\mathfrak{s}$ onward in an optimal portfolio of 'long–lived' securities; we now assume such an extension without special mention. Regarding the problem of zero return, as long as one is concerned only with valuing a European call option in the 'geometric Brownian' model one can simply replace (17) by

$$(3.17a) \qquad z^0_{\mathfrak{s}}(\epsilon) = (z^1_{\mathfrak{s}} - C)\mathfrak{I}_{\{z^1_{\mathfrak{s}} \geq C\}} + \epsilon \cdot \mathfrak{I}_{\{z^1_{\mathfrak{s}} < C\}}$$

with a small $\epsilon > 0$, carry out the transformations and then let $\epsilon \to 0$.

Remark II. As emphasised earlier, the fact that the y–process for any security is a local martingale (given continuity of the market process and $\Pi = \Pi^0$) is a consequence of optimality as such and does not depend on special assumptions of the BS world such as geometric Brownian motion, martingale representation or the availability of a riskless security. Thus, given any security (not necessarily a derivative) with, say, a payoff $z^0_{\mathfrak{s}} > 0$ at a stopping time $\mathfrak{s}$ and nothing at other times, it is tempting to argue directly that (19) must hold by optimality even if we cannot pass to a more specific form such as (20) or (21) without further hypotheses. Indeed, it seems that the main role of the riskless security and martingale representation is to allow the marginal utility process v, which is not directly observable, to be replaced in the valuation formula by processes depending on parameters which can be more readily estimated.

While this is essentially correct, it must be borne in mind that optimality (or equilibrium) as such requires only that the process y^0 be a *local* martingale, which does not imply the valuation formula (19) unless $\mathfrak{s}$ is a time which reduces y^0. (Note that we stopped to check this in the argument leading to the BS formula). In the absence of such information, we can say only that there exists a sequence (χ_n) of stopping times, increasing a.s. to ∞, such that each stopped process $(y^0_{t\wedge\chi_n})$ is a uniformly integrable martingale, yielding a valuation formula

$$y^0_{t\wedge\chi_n} = E^{t\wedge\chi_n} y^0_{\mathfrak{s}\wedge\chi_n}, \quad t \in \mathscr{T}, \quad n = 1,2,...$$

REMARK III. It is known that the BS formula remains valid in suitable cases if the filtration is larger than the natural filtration of the stock, see Babbs and Selby (1998), also if bond prices are random but there is for each $\mathfrak{s}$ a bond maturing at $\mathfrak{s}$ and riskless at that date, see Merton (1973). The following examples illustrate the effects of such features on the methods of the present Section.

(a) Suppose first that $\mathfrak{A} = (\mathscr{A}_t)$ is extended to be the (augmented) natural filtration of the pair (W,B) of standard Wiener processes, which for simplicity we take to be independent; in other respects, the model is unaltered. The argument up to (5) is unaltered, but the martingale representation (6) is replaced by

(3.6a) $$\eta^n_t = \int_{[0,t]} f_s \, dW_s + \int_{[0,t]} g_s \, dB_s,$$

where f and g are adapted processes with $\int_0^T f_s^2 \cdot ds$ and $\int_0^T g_s^2 \cdot ds$ finite on $\mathscr{T}$, a.s. Now (7) becomes

(3.7a) $$\int_{[0,t]} dv/v = \int_{[0,t]} f_s \, dW_s + \int_{[0,t]} g_s \, dB_s - rt,$$

but (8) remains unchanged by virtue of the independence of B and W. Consequently (9) and (10) also stand; thus f has the same constant value as before, but the process g remains undetermined. In (11) there is an additional term $\int g \, dB$ as in (7a), so that (12) becomes

(3.12a) $$v(t) = v_0 \cdot \mathscr{E}(\textstyle\int dv/v)_t = v_0 \cdot e^{-rt} \cdot \mathscr{E}\{fW\}_t \cdot \mathscr{E}\{gB\}_t ,$$

taking the independence into account. Once again the marginal utility process is determined up to a scale factor; there is now scope for the process g to depend on the utility function, but in general this process cannot be calculated without solving the whole PS problem. Note also that the representation (6a) allows us to infer only that $\mathscr{E}\{gB\}$ is a *local* martingale, cf. Remark II above. Consequently it cannot in general be asserted that the processes $y^1 = z^1 \cdot v$ and $y^n = z^n \cdot v$ are true martingales. Nevertheless, this property obtains in most reasonable cases and we now assume it. The argument then proceeds much as before, except that the expression under E^t in (20) and (21) is multiplied by $\mathscr{E}\{\int_{(t,\mathfrak{s}]} g_s dB_s\}$; but since this term is orthogonal to the remaining terms under E^t and its conditional expectation is unity, it may be discarded. The BS formula is then obtained as before.

(b) We now consider informally an example where the bond matures at $\mathfrak{s}$ and is riskless at maturity, but before $\mathfrak{s}$ its price depends on the process B. The filtration remains as in (a). Now (1) remains as before, but the expression for ζ^n in (2) is replaced by

(3.2b) $$\zeta^n_t = -r(\mathfrak{s}-t) - \beta(\mathfrak{s}-t)B_t, \quad t \leq \mathfrak{s},$$

where β is a constant. We also set $z^n_{\mathfrak{s}} = 1$, and in this example abandon the convention that $z^n_0 = 1$; instead, $z^n_0 = \exp\{-r\mathfrak{s}\}$. For later reference we note that

(3.23) $$\beta(\mathfrak{s}-t)B_t = \beta\int_{[0,t]} (\mathfrak{s}-s)dB_s - \beta\int_0^t B_s ds, \quad t \leq \mathfrak{s},$$

the second term on the right being of finite variation.

Now the formula for η^1 remains as in (4), but (5) is replaced by

(3.5b) $$\eta^n_t = -r(\mathfrak{s}-t) - \beta(\mathfrak{s}-t)B_t + \int_{[0,t]} dv/v - [\int dv/v, \beta\int(\mathfrak{s}-s)dB_s]_t;$$

the bracket process has been rewritten using (23), omitting the finite variation term. The martingale representation of η^n is written as in (3.6a); equating this to (5b) and rearranging, we get (in abridged notation)

(3.7b) $$\int dv/v = \int f dW + \int g dB + \beta(\mathfrak{s}-t)B_t + r(\mathfrak{s}-t) + [\int dv/v, \beta\int(\mathfrak{s}-s)dB_s].$$

The formula (8) for the bracket $[\int dv/v, \sigma W]$ remains valid, though perhaps with a new process f, while the bracket in (5b) is calculated from (7b) and (23) as

(3.8b) $$[\int dv/v, \beta\int(\mathfrak{s}-s)dB_s]_t = [\int g\, dB + \beta\int(\mathfrak{s}-s)dB, \beta\int(\mathfrak{s}-s)dB_s]_t$$
$$= \int_0^t\{\beta g_s(\mathfrak{s}-s) + \beta^2(\mathfrak{s}-s)^2\}ds.$$

Substituting (8b) into (7b),

$$(3.7b') \quad \int dv/v \;=\; \int f dW + \int g dB + \beta(\varsigma - t)B + r(\varsigma - t) + \int_0^t \{\beta g_s(\varsigma - s) + \beta^2(\varsigma - s)^2\} ds.$$

Subtracting η^n from η^1 once again, separating the martingale terms from the finite variation terms and noting that both sides of the resulting equation vanish identically we obtain, using the evaluations (8) and (8b) of the brackets,

$$(3.10b) \qquad \eta_t^1 - \eta_t^n - \sigma W_t - \beta(\varsigma - t)B_t \;=\; 0,$$

$$(\mu - r)t + r\varsigma + \sigma \int_0^t f_s\, ds + \int_0^t \{\beta g_s(\varsigma - s) + \beta^2(\varsigma - s)^2\} ds \;=\; 0$$

for all $t \leq \varsigma$, a.s., or, letting t vary with ς fixed,

$$(3.10b') \qquad \mu - r + \sigma f_t + \beta g_t(\varsigma - t) + \beta^2(\varsigma - t)^2 \;=\; 0.$$

This equation may be used to eliminate either g or f from (7b′) and other formulas above, but in order to evaluate one of these functions explicitly we need either to solve the PS problem as a whole or to introduce an additional assumption. Assume that (10b) holds for each maturity ς (at least in a suitable interval $[0, \varsigma^\dagger]$), with the same process $B(t)$ but possibly different processes $f(t;\varsigma)$, $g(t;\varsigma)$. (This corresponds loosely to supposing that there is a bond which is riskless at maturity for each ς, a variant of the formulation in Merton (1973); this idea cannot be formalised in our model as it stands since there is no provision for a continuum of securities, but we adopt the stated assumption as an illustration.) Then, eliminating the last integral in (10b) from previous work we get

$$(3.7b'') \qquad \int dv/v \;=\; \int f dW + \int [g + \beta(\varsigma - t)] dB - \mu t \;- \sigma \int f_s\, ds,$$

$$(3.4b) \qquad \eta^1 \;=\; \int [f + \sigma] dW + \int [g + \beta(\varsigma - t)] dB,$$

$$(3.6b) \qquad \eta^n \;=\; \int f dW + \int g\, dB.$$

Since these formulas must hold for $t \in [0, \varsigma]$, for each fixed $\varsigma \in [0, \varsigma^\dagger]$, and since W and B are independent, it follows that f and $g + \beta(\varsigma - t)$ do not depend on the date ς, although we may have $g = g(t;\varsigma)$. Reverting to (10b′), we get

$$(3.24) \qquad \mu - r + \sigma f = 0; \qquad g(t;\varsigma) = -\beta(\varsigma - t),$$

so that f again has the constant value given in (10). On substituting into (7b″), we again have (11), hence (12), and the rest of the derivation proceeds as in the case of a deterministic bond.

(c) Still leaving the filtration as in (a), suppose now that the 'bond' is replaced by a second 'stock', so that (2) becomes

(3.2c) $$z_t^n = \exp\{r-\tfrac{1}{2}\gamma^2)t + \gamma B_t\}, \quad \zeta_t^n = rt + \gamma B_t,$$

for $t \in \mathscr{T}$, where $\gamma > 0$ is a constant.[15] Once again, (4) stands but (5) becomes

(3.5c) $$\eta_t^n = rt + \gamma B_t + \int dv/v + [\![\int dv/v, \gamma B]\!].$$

Representing η^n as in (6a), equating the resulting expressions and rearranging we have

(3.7c) $$\int dv/v = \int f dW + \int (g-\gamma) dB - rt - [\![\int dv/v, \gamma B]\!].$$

Once again, (8) remains valid. Forming the bracket with γB on both sides of (7c) yields

(3.8c) $$[\![\int dv/v, \gamma B]\!] = \gamma \int (g-\gamma) dt, \quad \text{hence}$$

(3.7c′) $$\int dv/v = \int f dW + \int (g-\gamma) dB - rt + \gamma \int (g-\gamma) dt.$$

Subtracting η^n from η^1 and equating the martingale and finite variation parts yields

(3.10c) $$\eta^1 - \eta^n + \sigma W + \gamma B = 0, \quad \mu - r + \sigma f - \gamma(g-\gamma) = 0,$$

for all t, a.s. We can then eliminate either f or g from (7c′) etc and obtain a valuation formula using (17–19), but once again it will involve an undetermined process which may not be independent of W. Note that difficulties of this kind are effectively assumed away in those derivations of the BS formula which assume from the outset that a valuation formula (solution of a suitable p.d.e.) exists which depends only on z_t^1 and t. In our terms this means, loosely speaking, that $\int dv/v + rt$ is a functional of the Wiener process W only, hence has a unique (up to null sets) representation as an Itô integral with respect to W, see Dudley (1977), so that $g = \gamma$ and the derivation of the BS formula proceeds as if B were absent.

[15] The resulting problem is related to those analysed by Margrabe (1978) and Grabbe (1983). I am indebted to Michaël Selby for this remark.

4. A FORMULA FOR PROJECT EVALUATION

As a second illustration of shadow pricing, we consider a classic problem of project evaluation in a continuous–time stochastic setting. We take as 'benchmark' the model of optimal saving with a single, continuously traded asset with returns process z (which is not required to be continuous). The investor is offered the option of an 'indivisible' project yielding a random cash flow $\bar{s} = \bar{s}(\omega,t)$, or $s = \bar{s}/z$ in z-standardised units, on payment of an initial capital sum $\bar{\mathfrak{V}}$. The situation resulting from acceptance is assumed to satisfy the conditions stated under B at the end of Section 2, (with K_0 replaced by $K_0 - \bar{\mathfrak{V}}$ and consequential changes in some definitions and notation); in particular, s is taken to be non–negative and to satisfy (2.25) and (2.28). Unlike some related contributions such as Constantinides (1978), we do not assume a market structure enabling the project to be priced 'in the market' without reference to utility.[16]

The 'obvious' project criterion is the change in the maximum value of the welfare functional resulting from acceptance, but we seek an equivalent criterion of the 'expected net present value' type, using shadow prices to value the cash flows, in the hope that this will yield greater insight.

Although the proposed project must actually be accepted or rejected as a whole, we extend the notation to allow hypothetical situations in which a proportion $\alpha \in [0,1]$ of the cash flow s is receivable and an initial payment of $\mathfrak{V}(\alpha) \in [0,K_0)$ is required. The theory of Section 2 is assumed to apply, with suitable values of parameters and adjustments to definitions, for each value of α considered, and details will be spelled

[16] It has been suggested that this Section might be presented as a contribution to the currently active literature on 'real options'. As summarised by Chirinko (1996), reviewing Dixit and Pindyck (1993), this literature develops 'the idea that irreversible investment, combined with ongoing uncertainty and timing flexibility, may have a substantial impact on the investment decision rule used by profit-maximising firms'. While the investment considered here is irreversible, we do not consider timing flexibility, and the criterion is a 'welfare' functional instead of profit. Our analysis is thus closer in spirit to the older literature on public project evaluation or cost-benefit analysis, as viewed from the standpoint of welfare economics.

out only occasionally. In particular we assume that, for each such α, an optimal plan exists, with a finite value of the welfare functional, and that assumptions corresponding to (2.10) and (2.23–24) are satisfied. We distinguish among functions etc relating to different values of α by means of superscripts. Thus $\bar{c}^\alpha$ and $c^\alpha = \bar{c}^\alpha/z$ will denote the optimal consumption plan, in natural and standardised units respectively, and φ^α the maximum welfare; (we drop the stars). So

$$(4.1) \qquad \varphi^\alpha = E\int_0^\infty u[\bar{c}_t^\alpha]e^{-\varrho t}dt = E\int_0^\infty u[c_t^\alpha \cdot z_t]e^{-\varrho t}dt,$$

with equation of accumulation (in standardised units, writing $k^\alpha = \bar{k}^\alpha/z$)

$$(4.2) \qquad k^\alpha(T) = K_0 - \mathfrak{V}(\alpha) + \int_0^T [\alpha s_t - c_t^\alpha]dt.$$

Initially we consider just the values $\alpha = 1$ and $\alpha = 0$, with $\mathfrak{V}(0) = 0$ and an *arbitrary* value $\bar{\mathfrak{V}} = \mathfrak{V}(1) < K_0$, writing the corresponding value of φ^α as $\bar{\varphi}^\alpha = \varphi^\alpha(K_0 - \alpha\bar{\mathfrak{V}})$; the 'obvious' criterion for project acceptance is then $\bar{\varphi}^1 \geq \bar{\varphi}^0$. Because of the concavity of the utility function, we have

$$(4.3) \qquad \bar{\varphi}^1 - \bar{\varphi}^0 \leq E\int_0^\infty \left\{u_t'^0 \cdot z_t[c_t^1 - c_t^0]\right\} e^{-\varrho t}\, dt = E\int_0^\infty \left\{y_t^0 \cdot [c_t^1 - c_t^0]\right\} dt,$$

using abridged notation; here $u_t'^0 = u'(\bar{c}_t^0)$, and $y_t^0 = u_t'^0 \cdot z_t \cdot e^{-\varrho t}$ is the shadow price process corresponding to $\alpha = 0$. Using (2) with $\alpha = 1$ and $\alpha = 0$ we have

$$(4.4) \qquad k^1(T) - k^0(T) = -\bar{\mathfrak{V}} + \int_0^T [s_t + c_t^0 - c_t^1]dt.$$

Since y^0 is a local martingale reduced by the times $\nu_i = \nu_i^0$, $0 \leq i < K_0$, the theorem on integration of a martingale w.r.t. a non-decreasing processs allows the right side of (3) to be rewitten as

$$(4.5) \qquad \lim_{i \uparrow K_0} E\left\{y^0(\nu_i^0) \cdot \int_0^{\nu_i^0} [c_t^1 - c_t^0]\right\} dt,$$

taking into account (2.8) and (2.28–29). Substituting from (4) and noting that $E\, y^0(\nu_i^0) = y_0^0$, this becomes

$$(4.6)\qquad -y_0^0\cdot\bar{\mathfrak{V}} + \lim_{i\uparrow K_0} E\Big\{y^0(\nu_i^0)\Big[k^0(\nu_i^0) - k^1(\nu_i^0) + \int_0^{\nu_i^0} s_t\,dt\Big]\Big\}.$$

Now, referring to (2.27) and (2.14b′), also Prop.B in Appendix B, we have

$$k^0(\nu_i^0) = K_0 - i \ \text{ if } \ \nu_i^0 < \infty, \qquad y^0(\nu_i^0) = 0 \ \text{ if } \ \nu_i^0 < \infty,$$

and it follows from (2.28–9) that the term in (6) multiplying $y^0(\nu_i^0)$ stays finite. Since $Ey^0(\nu_i^0) = y_0^0$, the term in (6) involving k^0 vanishes as $i\uparrow K_0$. Dropping the non-positive term in k^1, using again the fact that y^0 is reduced by the times (ν_i^0), letting $i\uparrow K_0$, $\nu_i^0\uparrow\infty$ and collecting results we get

$$(4.7)\qquad \bar{\varphi}^1 - \bar{\varphi}^0 \le -y_0^0\cdot\bar{\mathfrak{V}} + E\Big\{\int_0^\infty y_t^0\cdot s_t\,dt\Big\}.$$

A sufficient condition for project rejection is that the right side of (7) be negative, hence a *necessary condition for acceptance* is that it be positive (or zero), i.e. that

$$(4.8)\qquad \bar{\mathfrak{V}} \le E\Big\{\int_0^\infty (y_t^0/y_0^0)\cdot s_t\,dt\Big\}.$$

An analogous calculation, evaluating u' at $\bar{c}^1 = c^1\cdot z$ with initial capital $K_0 - \bar{\mathfrak{V}}$, yields a necessary condition for rejection, or a *sufficient condition for acceptance*, of the form

$$(4.9)\qquad \bar{\mathfrak{V}} \le E\Big\{\int_0^\infty (y_t^1/y_0^1)\cdot s_t\,dt\Big\}.$$

These inequalities will be recognised as a continuous-time, stochastic version of the result that a project large enough to shift prices – in this case, marginal utility prices – will be overvalued if appraised using the prices prevailing in the absence of the project, and undervalued by prices prevailing in its presence. (Admittedly, the analysis here is oversimplified by consideration of a one-asset model, so that portfolio rearrangement is neglected.) In the cost-benefit setting, it has often been suggested that correct evaluation requires the use of suitable intermediate or average 'shadow' prices. We proceed to make this idea precise in the present setting; but note that, while the preceding calculations are readily justified and details have been omitted only for brevity, the calculations which follow involve some analytic assumptions which, while entirely reasonable, require further investigation.[17]

We now consider values of α in the whole interval [0,1], so that in addition to the feasible situations there are hypothetical ones in which a proportion α of the cash flow s can be obtained for an initial investment $\mathfrak{V}(\alpha)$. Also, instead of fixing this investment arbitrarily, we choose $\mathfrak{V}(\alpha)$ for each α as the equivalent variation for the cash flow $\alpha \cdot s$, i.e. the payment which, given maximising behaviour, leaves the investor's welfare the same as at $\alpha = 0$. Thus, if $\bar{\mathfrak{V}}$ is the investment actually required, the necessary and sufficient condition for acceptance will be

(4.10) $$\bar{\mathfrak{V}} \leq \mathfrak{V}(1).$$

In this setting, (1) and (2) apply for each α, but now

(4.11) $$\varphi^{\alpha} = \varphi^{0} \qquad \text{for } 0 \leq \alpha \leq 1.$$

Assuming that it is permissible to differentiate (1) and (2) as follows, we obtain, with obvious notation

(4.12) $$0 = E\int_0^{\infty} \left\{u_t'^{\alpha} \cdot z_t \cdot e^{-\varrho t}[dc_t^{\alpha}/d\alpha]\right\} dt = E\int_0^{\infty} \left\{y_t^{\alpha} \cdot [dc_t^{\alpha}/d\alpha]\right\} dt,$$

(4.13) $$dk^{\alpha}(\tau)/d\alpha = -\mathfrak{V}'(\alpha) + \int_0^{\tau} [s_t - dc_t^{\alpha}/d\alpha]\, dt.$$

To make a transition similar to that from (3) to (5) above, we must amend the definition of the depletion times to allow for the fact that, for project participation at level α, the net initial capital is only $K_0 - \mathfrak{V}(\alpha) = K_0^{\alpha}$. By analogy with (2.17), let

(4.14) $$\nu_i^{\alpha}(\omega) = \inf\{t: k^{\alpha}(t) < K_0 - i\} \wedge \infty, \quad 0 \leq i - \mathfrak{V}(\alpha) < K_0^{\alpha}.$$

Since y^{α} is a local martingale reduced by the times ν_i^{α}, (12) may be written, taking

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[17] Given the assumptions already stated, the following conditions are sufficient for the remaining calculations. The function $\mathfrak{V}(\alpha)$ defined in the next paragraph is to exist (an extension of the assumption of the existence of optima). Also $dc^{\alpha}(\omega,t)/d\alpha$ and $dk^{\alpha}(\omega,t)/d\alpha$, defined as derivatives of the *optimal* processes c^{α} and k^{α} w.r.t. α (when $\mathfrak{V}(\alpha)$ is chosen to satisfy (11) below) are to exist as (progressive) processes for $0 \leq \alpha \leq 1$. The differentiation of the integrals in (1) and (2) to obtain the integrals in (12) and (13) is then justified by dominated convergence if the processes $dc^{\alpha}/d\alpha$ and $e^{-\varrho t} \cdot du[c^{\alpha} \cdot z]/d\alpha = y^{\alpha} \cdot dc^{\alpha}/d\alpha$ are dominated by product integrable processes. Existence of $\mathfrak{V}'(\alpha)$ in (13) then follows from the existence of the other terms. The integrability of $dc^{\alpha}/d\alpha$ is also used in passing from (12) to (15).

into account (13), as

$$(4.15)\qquad 0 = \lim_{i \uparrow K_0} E\left\{y^{\alpha}(\nu_i^{\alpha})\cdot\int_0^{\nu_i^{\alpha}} [dc_t^{\alpha}/d\alpha]\right\} dt$$

$$= -\mathfrak{V}'(\alpha)\cdot y_0^{\alpha} + \lim_{i \uparrow K_0} E\left\{y^{\alpha}(\nu_i^{\alpha})\cdot\left[-dk^{\alpha}(\nu_i^{\alpha})/d\alpha + \int_0^{\nu_i^{\alpha}} s_t\, dt\right]\right\}.$$

Now, for each α and $i \geq \mathfrak{V}(\alpha)$, we have

$$(4.16)\qquad \textit{either}\ k^{\alpha}(\nu_i^{\alpha}) = K_0 - i, \quad \textit{or}\ \nu_i^{\alpha} = \infty \ \text{and}\ \hat{y}_i^{\alpha} = 0.$$

Thus, on going to the limit, the term involving k^{α} vanishes, and on rearranging there remains simply

$$(4.17)\qquad \mathfrak{V}'(\alpha) = E\left[\int_0^{\infty} (y_t^{\alpha}/y_0^{\alpha})\cdot s_t\, dt\right],$$

or, integrating over α, interchanging the order of integration and setting

$$(4.18a)\qquad \bar{y}_t = \int_0^1 (y_t^{\alpha}/y_0^{\alpha})\, d\alpha,$$

$$(4.18b)\qquad \mathfrak{V}(1) = E\left[\int_0^{\infty} \bar{y}_t\cdot s_t\, dt\right].$$

The idea of valuation using 'suitable' average prices or discount rates thus turns out to be correct in the present case. The argument illustrates both the application of shadow pricing and the usefulness of the concept of depletion time. Actually calculating the values of the $\bar{y}_t$ is, of course, a different matter. In general, we have so far progressed only from a project criterion involving 'welfare' or total utility to one involving (random) marginal utility.

APPENDICES

In these Appendices we give outline proofs of the new results asserted under headings A – C in Section 2. These proofs build on the theory in Foldes (1978a&1990), and the reader wishing to follow them in detail will need to refer to these papers.

APPENDIX A: PRICE TIMES

Consider the saving model with a single security, z not required to be continuous and no income process. The price times $\wp_n$ may be defined as in (2.21) for *real* $n \in [y_0,\infty)$. (Note that $\wp_n > 0$ for $n > y_0$ since we assume $y_0 = y(0+)$.) Referring to the conditions for optimality (2.14), we wish to prove

PROPOSITION A: A plan (c^*,k^*) satisfying $c^* > 0$ and (2.10) is optimal iff

(A.1) y is an $\mathfrak{A}$-local martingale reduced by any sequence $(\wp_n)$ with $y_0 \leq n \uparrow \infty$, and

(A.2) $\lim_n E\{y(\wp_n)\cdot k^*(\wp_n)\} = 0$ for any such sequence.

The *sufficiency* proof follows the same lines as similar proofs in Foldes (1978a&1990) and is omitted; (see also Appendix B below for a more complicated argument of the same kind).

Turning to *necessity*, let c^* be optimal; since $u'(0) = \infty$, we know that $c^* > 0$ and $k^* > 0$ up to null sets.

Local Martingale Condition. We know from the results of Foldes (1978a) that y is a positive local martingale (reduced by any sequence (τ_i) of consumption times with $i \uparrow K_0$), with a limiting variable $y(\infty) \geq 0$ satisfying $y(\omega,\infty) = 0$ if $k^*(\omega,\infty) > 0$. On the other hand, y is reduced by any sequence of times $(\wp_n)$ with $n \uparrow \infty$, see for instance Meyer (1976) IV.4 bis, Dellacherie & Meyer (1980) VII.8,12,13. The necessity of (1) follows.

Transversality. Now let (c,k) be any plan such that $\delta k(t) < 0$ for all $t > 0$, a.s., and consider the following calculation

(A.3) $$D\varphi(c^*, \delta c) = E\int_0^\infty y.\delta c.dt = -E\int_0^\infty y.d(\delta k) = -\lim_n E\int_0^{\varrho_n} y\cdot d(\delta k)$$
$$= \lim_n E\Big\{y(\wp_n)\cdot[k^*(\wp_n) - k(\wp_n)]\Big\}.$$

The first equality is (2.13c); the second uses $\delta c = -\delta\dot{k} = -d(\delta k)/dt$, and then the passage to the limit is justified because $\wp(n) \uparrow \infty$ a.s. while the product integral defining $D\varphi$ converges under (2.10). The last equality uses integration of a martingale with respect to a non–decreasing process, noting that δk may be written as the difference between two such (finite) processes on the bounded interval $[0,\wp_n]$, while y is uniformly integrable on this interval; also $\delta k(0) = 0$. Now $y > 0$ and $k^*-k > 0$ for $t > 0$, yielding $D\varphi \geq 0$, hence $D\varphi = 0$ by optimality. This leaves

(A.4) $$D\varphi = 0 = \lim_n E\{y(\wp_n)[k^*(\wp_n)-k(\wp_n)]\}.$$

It remains to get rid of the term $k(\wp_n)$ in (4) and so obtain (2). This step is trivial in a discrete–time model, where one can choose for k the plan which consumes all capital at $t = 0$, and could be made so in the present model by allowing free disposal of a 'lump' of capital. Without such an assumption it is not in general possible to construct a feasible k such that $k(\wp_n) = 0$ a.s. for some fixed n. We therefore need a limiting argument.

The first step is to define a family of plans $(c^\vartheta,k^\vartheta;\ \theta > 0)$ by setting

(A.5) $$k^\vartheta(\omega,t) = k^*(\omega,t)e^{-\vartheta t},$$

hence (in abridged notation)

(A.6) $$c^\vartheta = -\dot{k}^\vartheta = (\theta k^* - \dot{k}^*)e^{-\vartheta t} = (\theta k^* + c^*)e^{-\vartheta t} > 0.$$

Clearly the plans are feasible and each k^ϑ is decreasing on $\mathscr{T}$. If we now replace k by k^ϑ in (4), then (2) will follow if it is shown that, for $\theta \uparrow \infty$, $n \uparrow \infty$,

(A.7) $$\lim_\theta \lim_n E\{y(\wp_n)k^\vartheta(\wp_n)\} = 0.$$

Turning to the second step, note that we may regard the family of stopping times $(\wp_n;\ y_0 \leq n < \infty)$ as defining a (right continuous) time change, the associated filtration $\tilde{\mathfrak{A}} = (\tilde{\mathscr{A}}_n) = (\mathscr{A}_{\varrho_n})$ satisfying the usual conditions. It follows readily that the transform $\tilde{y}$ of y, defined for $y(0) \le n < \infty$ by $\tilde{y}(n) = y(\wp_n)$, is an $\tilde{\mathfrak{A}}$-martingale and that the transforms $\tilde{k}^*(n) = k^*(\wp_n)$, $\tilde{k}^\vartheta = k^\vartheta(\wp_n)$ are decreasing and bounded above by K_0, all these processes being positive and collor. We may now define the limiting variables $\wp(\infty) = \infty$, $\tilde{y}(\infty) = y(\infty)$, $\tilde{k}(\infty)$ and $\tilde{k}^\vartheta(\infty)$, where $\tilde{k}^\vartheta(\infty) = 0$ by (5). It follows that the product

$$h^\vartheta(n) = \tilde{y}(n) \cdot \tilde{k}^\vartheta(n)$$

defines a collor, positive $\tilde{\mathfrak{A}}$-supermartingale with $h^\vartheta(\infty) = 0$. We consider h^ϑ in some interval $[n_0, \infty)$ with $n_0 > y_0$. Write

(A.8) $$H^\vartheta(n) = Eh^\vartheta(n), \quad n_0 \le n < \infty, \qquad H^\vartheta(\infty) = \lim_n H^\vartheta(n),$$

noting that $H^\vartheta(\infty)$ exists as a limit because $H^\vartheta(\cdot)$ is non-increasing and that $H^\vartheta(.)$ is right continuous because h^ϑ is sample right continuous, see Meyer (1966) VI.4. Thus

(A.9) $$y_0 K_0 \ge H^\vartheta(n) \ge H^\vartheta(\infty) \ge Eh^\vartheta(\infty) = 0,$$

bearing in mind that $0 \le h^\vartheta(n) < \tilde{y}(n) \cdot K_0$ and $Ey(n) = y(0)$ for $n_0 \le n < \infty$.

Now the third step. On letting $\theta \uparrow \infty$, $h^\vartheta(n) \downarrow 0$ for each n, so that (by dominated convergence) we have

$$\lim_\vartheta H^\vartheta(n) = 0, \qquad n_0 \le n < \infty;$$

this assertion also holds for $n = \infty$ because of (9). Obviously, the left–hand limits $H^\vartheta(n-) \downarrow 0$ also as $\theta \uparrow \infty$. Thus, for any sequence $\theta \uparrow \infty$, the functions $H^\vartheta(n)$ are a sequence of positive functions on $[n_0, \infty]$, collor, decreasing simply to zero together with the left limits $H^\vartheta(n-)$; it then follows from a generalisation of Dini's Theorem, Dellacherie & Meyer (1980) VII.2 (Lemma), that the convergence is uniform. This uniformity justifies the following interchange of limits, which completes the proof:

$$0 = \lim_n \lim_\vartheta H^\vartheta(n) = \lim_\vartheta \lim_n H^\vartheta(n). \|$$

The proof yields the following

COROLLARY A: If (c^*, k^*) is optimal, then

(A.10) $\tilde{y}$ is an $\tilde{\mathfrak{A}}$-martingale, and

(A.11) $\lim_n E\{\tilde{y}(n) \cdot \tilde{k}^*(n)\} = 0$ for any sequence of positive reals $n \uparrow \infty$.

REMARK. The use of price times to characterise an optimal plan can be extended to the PS model as follows. Suppose *either* that Z is continuous and $\Pi = \Pi^{o}$, *or* that $\pi^* > 0$. For each security λ, define the times $\wp_n^\lambda = n \wedge \{y^\lambda > n\}$, rewrite $\wp_n$ as $\wp_n^*$ and set $\wp_n^\nabla = \wp_n^* \wedge \wp_n^1 \wedge \ldots \wedge \wp_n^\Lambda$. Then Prop. A, with $\wp_n^\nabla$ in place of $\wp_n$ and $y = y^*$, still characterises π^*-optimality of a plan (c^*,k^*), and in the conditions for optimality (2.20a&b) one can choose a sequence $(\wp_n^\nabla)$ as fundamental for all the y^λ as well as for y^*. However, with this choice, (2.20b) cannot in general be replaced by (2.20b′) in a set of necessary conditions.

APPENDIX B: DEPLETION TIMES FOR MODEL WITH INCOME

Consider the model with a (positive) income process defined in Section 2B, and refer to (2.26–27) for definitions of Γ and ν_i. We wish to prove

PROPOSITION B: A plan (c^*,k^*), satifying $c^* > 0$ and (2.10), is optimal iff

(B.1) y is an $\mathfrak{A}$-local martingale reduced by any sequence $\{\nu_i\}$ with $i = i(n) \uparrow K_0$, and

(B.2) $\lim_i E\{y(\nu_i)[K_0 - \Gamma(\nu_i)]\} = 0$ for any such sequence.

Condition (2) may be replaced by one of the following:

(B.3) $\lim_i \{y(\nu_i)[K_0 - \Gamma(\nu_i)]\} = 0$ a.s.;

(B.4) $y(\omega,\infty) = 0$ if $\Gamma(\omega,\infty) < K_0$, a.s.

Notation. For brevity, we write

$y^i_t = y^i(t) = y(t\Lambda\nu_i)$, $\psi^i_t = \psi^i_t(\omega) = \mathfrak{I}\{t\colon t < \nu_i(\omega)\}$, $N_i = N_i(\omega) = \mathfrak{I}\{\omega\colon \nu_i(\omega) < \infty\}$.

Equivalences. We first check the equivalence of (2), (3) and (4) in the presence of (1). Given that y is a positive local martingale, hence a supermartingale, and that $K_0 - \Gamma$ is monotonic and ≥ 0, the a.s. limiting variables $y(\infty)$ and $K_0 - \Gamma(\infty)$ exist and are ≥ 0. Since $\nu_i \uparrow \infty$ a.s. as $i \uparrow K_0$, (2) implies (3) by Fatou's Lemma, and then (3) and (4) are clearly equivalent. To see that (4) implies (2), note that $\nu_i < \infty$ implies $K_0 - \Gamma(\nu_i) = K_0 - i$ by (2.27), whereas $\nu_i = \infty$ implies $\Gamma(\infty) \leq i < K_0$, hence $y(\nu_i) = y(\infty) = 0$ by (4). Then (2) follows from

(B.5) $$0 \leq E\{y(\nu_i)[K_0 - \Gamma(\nu_i)]\} = E\{N_i \cdot y(\nu_i) \cdot (K_0 - i)\} = (K_0 - i)Ey(\nu_i)$$
$$= (K_0 - i)y_0 \longrightarrow 0 \text{ as } i \uparrow K_0.$$

Sufficiency. The proof follows the same lines as in Foldes (1978a&1990), with some complications because capital plans in standardised units need not be decreasing or bounded or converge at $t = \infty$. To avoid difficulties with such terms as $y(\nu_i) \cdot \delta k(\nu_i)$ when $\nu_i = \infty$, we give an elementary argument rather than rely directly on the theorem on integration of a martingale with respect to a non-decreasing process.

PROOF. By assumption, $\varphi(c^*)$ is finite. If (c,k) is an alternative plan (with $c \geq 0$ and $k \geq 0$), we have to show that $D\varphi \leq 0$. and we may assume that $\varphi(c) > -\infty$. Then $\varphi(c) - \varphi(c^*) = E \int_0^\infty [u(\bar{c}) - u(\bar{c}^*)]e^{-\varrho t} \cdot dt$ is finite. By concavity,

$$[u(\bar{c}) - u(\bar{c}^*)]e^{-\varrho t} \leq y \cdot \delta c, \text{ so } \varphi(c) - \varphi(c^*) \leq E \int_0^\infty y \cdot \delta c \cdot dt,$$

and it suffices to show that the last expression is ≤ 0. Also, the fact that $y \cdot \delta c$ is bounded below by a product integrable process justifies the interchanges of the order of integration in what follows. We have

$$\text{(B.6)}\quad E\int_0^{\nu_i} y_t \cdot \delta c_t \cdot dt = -E\int_0^{\nu_i} y_t \cdot \delta\dot{k}_t \cdot dt = -E\int_0^\infty y_t^i \cdot \delta\dot{k}_t \cdot \psi_t^i \cdot dt$$

$$= -E\int_0^\infty [E^t y(\nu_i)] \cdot \delta\dot{k}_t \cdot \psi_t^i \cdot dt = -\int_0^\infty E\{[E^t y(\nu_i)] \cdot \delta\dot{k}_t \cdot \psi_t^i\} dt$$

$$= -\int_0^\infty E\{y(\nu_i) \cdot \delta\dot{k}_t \cdot \psi_t^i\} dt = -E\int_0^{\nu_i} y(\nu_i) \cdot \delta\dot{k}_t \cdot dt = -E\{N_i \cdot y(\nu_i)\int_0^{\nu_i} \delta\dot{k}_t \cdot dt\}$$

$$= -E\{N_i \cdot y(\nu_i) \cdot \delta k(\nu_i)\} \leq E\{N_i \cdot y(\nu_i) \cdot k^*(\nu_i)\} = E\{N_i \cdot y(\nu_i) \cdot [K_0 - \Gamma(\nu_i)]\}$$

$$= E\{y(\nu_i) \cdot [K_0 - \Gamma(\nu_i)]\} \longrightarrow 0.$$

The first equation follows from (2.22), the second is just a change of notation. The third uses the fact that y_t^i is a u.i. martingale by (1). The fourth uses Fubini's Theorem, the fifth properties of conditional expectation, the sixth Fubini again. *Note that in the resulting (seventh) term, the values* $y(\nu_i)$ *with* $\nu_i = \infty$, *which must equal zero, multiply only finite numbers.* The seventh equation excludes these vanishing terms and takes $y(\nu_i)$ outside the time integral, which is then rewritten as $\delta k(\nu_i)$. The inequality drops the term in $-k(\nu_i)$, allowing $k^*(\nu_i)$ with $\nu_i < \infty$ to be replaced by $K_0 - \Gamma(\nu_i)$, see (2.27), and since $K_0 - \Gamma$ is bounded we may reinstate the terms with $\nu_i = \infty$ under the expectation sign and go to the limit as $i \uparrow K_0$, which is zero by (2).||

COROLLARY B1. The Conditions (B.1–2) imply

$$\text{(B.7)}\quad E \int_0^\infty y(t)[c^*(t) - s(t)] \cdot dt = y_0 \cdot K_0.$$

PROOF. With notation as in the preceding proof, set $\dot{k} = 0$, $k = K_0$, $c = s$, hence $\delta c = \dot{k}^* = s - c^*$ for all t. Substituting in (6) up to the ninth term and reversing the signs, one has

$$\text{(B.8)} \qquad E\int_0^{\nu_i} y_t \cdot (c_t^* - s_t) \cdot dt \;=\; E\{N_i \cdot y(\nu_i) \cdot [K_0 - k^*(\nu_i)]\}.$$

Now $K_0 - k^*(\nu_i) = \Gamma(\nu_i) = i$ for $\nu_i < \infty$ and $E\{N_i \cdot y(\nu_i)\} = Ey(\nu_i) = y_0$, and it remains to let $i \uparrow K_0$. The convergence of the right side of (8) implies the convergence of the left side, and the limit may be written as on the left of (7) by virtue of (2.23–24).‖

Necessity. As noted in Section 2, optimality and (2.25) ensure that $c^* > 0$ and $k^* > 0$. Regarding the *consumption* times τ_i as now defined for *all* $i \geq 0$, it may be shown as in Foldes (1990) S.6 that $\hat{y}$ is a (positive) $\hat{\mathfrak{A}}$-supermartingale and hence that y is an $\mathfrak{A}$-supermartingale, also that a limiting variable $y(\infty) \geq 0$ exists. We wish now to show that y is a local martingale reduced by any sequence of *depletion* times ν_i, $i = i(n) \uparrow K_0$ (hence that $\check{y}$ is an $\check{\mathfrak{A}}$–martingale), and that the transversality condition (2) holds. A difficulty arises from the fact that now the sample paths of k^* need not be everywhere decreasing. To deal with this, we first prove a Depletion Lemma which shows that 'upward excursions' of k^* contribute nothing to the integral $E\int y \cdot \dot{k}^* \cdot dt$, i.e. to the total value of investment, and so can be neglected when calculating certain directional derivatives; this Lemma may be of some economic interest.

(i) *Properties of Depletion Times.* For fixed ω, let

$$\text{(B.9)} \qquad B(\omega) \;=\; \{t \in \mathscr{T}\colon t = \nu(\omega, i) \text{ for some } i \in [0, K_0)\}$$

i.e. $B(\omega)$ comprises those times t which are depletion times. Bearing in mind that the sample functions of $K_0 - k^*$ are absolutely continuous, it is clear that $K_0 - k^* = \Gamma^*$ on B, more precisely that $K_0 - k^*(\nu_i) = \Gamma(\nu_i) = K_0 - i$ if $\nu_i < \infty$ – see (2.27c) – while $K_0 - k^*(t) \leq \Gamma(t) \leq K_0 - i$ for $t \leq \nu_i$. Further, $K_0 - k^*(t) = \Gamma(t) > K_0 - i$ for t in some right neighbourhood of ν_i, i.e. each such t is again a first upcrossing time of some level $\imath > i$ and so belongs to B. Thus $B(\omega)$ is the union of a finite or infinite sequence of disjoint half-open intervals of positive length of the form

(B.10) $\quad [\alpha_n(\omega), \beta_n(\omega)), \qquad n = 1,2,...$

where $\beta_n = \infty$ may occur. In particular, $\alpha_1 = \nu(0) \geq 0$, and a strict inequality cannot be ruled out. The set $\{(\omega,t): t \in B(\omega)\}$ is obviously progressive. Further, since for fixed ω we have $\Gamma = K_0 - k^*$ on B while Γ is constant on each complementary interval $[\beta_n, \alpha_{n+1})$, it follows that for a.a. $t \in \mathscr{T}$ the derivative $\gamma(t) = d\Gamma(t)/dt$ is defined and may be chosen so as to satisfy

(B.11) $\quad \gamma(t) = -\dot{k}^*(t) > 0 \ \text{ for } t \in B, \qquad \gamma(t) = 0 \text{ for } t \notin B.$

DEPLETION LEMMA. Let $0 \leq \text{I} < i < K_0$ and $A \in \mathscr{A}_I$, and let $\mathfrak{I}_{B^c}(\omega,t)$ be the indicator function of the set $\mathscr{T} - B(\omega)$; then

(B.12) $$\int_A \left[\int_{\nu_I}^{\nu_i} y(t)\dot{k}^*(t)\mathfrak{I}_{B^c}(t)\, dt \right] dP = 0;$$

(B.12′) $$E\left[\int_0^{\nu_0} y(t)\dot{k}^*(t)\mathfrak{I}_{B^c}]\, dt \right] = 0.$$

For brevity we prove only (12), the proof of (12′) being similar.

PROOF. (a) We may assume $\nu_I < \infty$ for $\omega \in A$. We first show that the left side of (12) is not positive. Construct a variation $\delta k = k - k^*$ by setting (in abridged notation)

(B.13) $$-\delta c = \delta\dot{k} = -\dot{k}^*, \quad k = K_0 - \Gamma, \quad \text{hence } c = c^* + \delta c = s - \dot{k}^* - \delta\dot{k} = s \geq 0,$$
$$\text{for } t \in B^c \cap [\nu_I, \nu_i), \ \omega \in A,$$

and $\delta\dot{k} = 0$, $k = k^*$ otherwise. (Thus k cuts off any excursions of k^* above $K_0 - \Gamma$ occuring between ν_I and ν_i, but otherwise coincides with k^*). Since $k \geq 0$, $c \geq 0$, this is feasible. On computing $D\varphi$ from (2.13c) we have the expression on the left of (12), which by optimality must be ≤ 0.

(b) To establish the opposite inequality, one would like to set $\delta\dot{k}(t) = \epsilon\dot{k}^*(t)$, $\epsilon > 0$, for $t \in B^c \cap [\nu_I, \nu_i)$, $\omega \in A$, and $\delta\dot{k} = 0$ otherwise, (i.e. k would blow up any excursions of k^* above $K_0-\Gamma$ occurring between ν_I and ν_i but would otherwise coincide with k^*). Unfortunately the corresponding value

$$c = c^* + \delta c = c^* - \delta\dot{k} = c^* - \epsilon\dot{k}^* = s - (1+\epsilon)\dot{k}^*$$

could become negative, and we therefore define k on each component interval $[\beta,\alpha)$ of $B^c\cap[\nu_I,\nu_i)$ as the solution of the o.d.e.

(B.14) $$\dot{k} = s \text{ if } s \geq (1+\epsilon)\dot{k}^*, \quad \dot{k} = 0 \text{ if } s < (1+\epsilon)\dot{k}^*,$$

with the initial condition $k(\beta) = k^*(\beta)$; note that $c = 0$ when $\dot{k} = s$. It can be checked that the solution curve $k(t)$ lies above $k^*(t)$ throughout (β,α) but not above $k^*(t) + \epsilon[k^*(t) - k^*(\beta)]$, and rejoins k^* at α if $\alpha < \infty$. Since $k \geq k^* > 0$ and $c \geq s \geq 0$, the variation is feasible, and we have $\delta c = -\delta\dot{k} \geq -\epsilon\dot{k}^*$ on each interval $[\beta,\alpha)$, hence on $B^c\cap[\nu_I,\nu_i)$; it follows by optimality and (2.13c) that

(B.15) $$0 \geq D\varphi = -\int_A \int_{\nu_I}^{\nu_i} y\cdot\delta\dot{k}\cdot\mathfrak{I}_{B^c}\cdot dt \geq -\epsilon\int_A \int_{\nu_I}^{\nu_i} y\cdot\dot{k}^*\cdot\mathfrak{I}_{B^c}\cdot dt. \|$$

Corollary B.2. Let $0 \leq I < i < K_0$. Then, for $A \in \mathscr{A}_I$,

(B.16) $$\int_A \left[\int_{\nu_I}^{\nu_i} y(t)\dot{k}^*(t)dt\right] dP \leq 0,$$

the inequality being strict unless $\nu_I = \infty$ a.s. on A.

Proof. This inequality follows from the preceding Lemma (which shows that the contribution to the time integral made by B^c vanishes) and from the facts that $\dot{k}^* < 0$ on the interior of B while $y(t) > 0$. ||

(ii) *Local martingale condition.* For each fixed $i \in [0,K_0)$, we define the stopped process

(B.17) $$y^i = (y[t\wedge\nu_i];\ t\in\mathscr{T});$$

we have to show that y^i is a u.i. $\mathfrak{A}$-martingale. Since each $t\wedge\nu_i$ is an $\mathfrak{A}$-stopping time and y is a positive corlol supermartingale, it follows by the Stopping Theorem that y^i has the same properties, Dellacherie & Meyer (1980) VI.12; and by the convergence theorem, ibid. VI.6, $y^i(t)$ converges a.s. to a finite limiting variable

$y^i(\infty) = y(\nu_i-)$. The supermartingale inequality and predictability of ν_i then yield

(B.18) $$y(0) = y^i(0) \geq Ey^i(t) \geq Ey^i(\infty) = Ey(\nu_i-) \geq Ey(\nu_i), \qquad t\in\mathscr{T}.$$

In the paragraphs which follow we shall show that

(B.19) $$y(0) \leq Ey(\nu_i);$$

this will imply first that y^i is a martingale, since then $Ey^i(t) = y(0)$ for all $t\in\mathscr{T}$, and second that the $y^i(t)$ are uniformly integrable because they are positive and we have the relations $y^i(t) \to y^i(\infty)$ a.s. and $Ey^i(t) \to Ey^i(\infty)$, $t \to \infty$, see Meyer (1966) II.21.

PROOF OF (19). For fixed $i\in[0,K_0)$, choose numbers $\theta\in(0,K_0-i)$ and $h > 0$ and define a new plan $c = c^*+\delta c$, $k = k^*+\delta k$ in three phases (a), (b), (c).

(a) The first phase is defined for $0 \leq t < h\wedge\nu_i$ by

(B.20) $$\delta c = \theta/h = -\delta\dot{k}, \text{ hence } c = c^* + \delta c > 0 \text{ and}$$
$$k(t) = k^*(t)-(\theta/h)t > K_0-i-(\theta/h)t > 0, \text{ because } (\theta/h)t < \theta < K_0-i,$$

so that the variation is feasible. The value of k at the end of the first phase satisfies:

(B.21) $$k(h\wedge\nu_i) = k(\nu_i) = k^*(\nu_i) - (\theta/h)\nu_i = K_0 - i - (\theta/h)\nu_i \quad \text{if } \nu_i \leq h,$$

(B.22) $$k(h\wedge\nu_i) = k(h) = k^*(h) - \theta \geq K_0 - i - \theta \qquad \text{if } h < \nu_i.$$

The first line holds because $\nu_i < \infty$ implies $\nu_i \in B$, hence $k^*(\nu_i) = K_0 - i$. The second line follows from (20).

(b) In case $h < \nu_i$, a second phase is defined by

(B.23) $$\delta c = 0, \quad \dot{k} = \dot{k}^*, \quad \text{hence}$$
$$k(t) = k(h) + k^*(t) - k^*(h) = k^*(t) - \theta \geq K_0-i-\theta > 0, \qquad h \leq t \leq \nu_i.$$

The construction so far guarantees that $c \geq 0$, $k > 0$, $\delta\dot{k} \leq 0$ and $\delta k \leq 0$ for $t \leq \nu_i$. In particular, (21) and (23) yield

(B.24) $$k(\nu_i) = k^*(\nu_i) - (\theta/h)(h\wedge\nu_i) \geq K_0 - i - \theta,$$

so that

$$\text{(B.25)} \qquad \delta k(\nu_i) = -(\theta/h)\nu_i \quad \text{if } \nu_i \leq h$$
$$\delta k(\nu_i) = -\theta \quad \text{if } h \leq \nu_i.$$

(c) If $\nu_i < \infty$, a third phase is defined by

$$\text{(B.26)} \qquad \delta c = \dot{k}^*, \quad \text{hence } c(t) = s(t) \geq 0, \quad \dot{k}(t) = 0, \quad \nu_i \leq t < \rho,$$

where ρ is the first time in B such that $k(t) = k^*(t)$ if this exists, with $\rho = \infty$ otherwise. For $t \geq \rho$, we set $k(t) = k^*(t)$. Feasibility is obvious, since $k(t) = k(\nu_i) \geq K_0 - i - \theta$ on $[\nu_i, \rho)$ implies $k(t) \geq 0$. Explicitly, it follows from (25) and the definition of depletion times that

$$\text{(B.27)} \qquad \rho = \nu_I \text{ with } I = i + (\theta/h)\nu_i \text{ if } \nu_i \leq h; \qquad \rho = \nu_{i+\theta} \text{ if } h \leq \nu_i.$$

This completes the construction of k. We now substitute into the formula (2.13c) for $D\varphi$ and, noting that $\delta c = -\delta\dot{k} \geq \theta/h$ in phase (a), $\delta c = 0$ in phase (b), and $\delta c = \dot{k}^*$ in phase (c), obtain

$$\text{(B.28)} \qquad 0 \geq D\varphi \geq E\left[(\theta/h)\int_0^{h\wedge\nu_i} y\cdot dt + \int_{\nu_i}^{\rho} y\cdot\dot{k}^*\cdot dt\right].$$

According to (27), ρ may take one of the two values ν_I and $\nu_{i+\theta}$, and clearly $\nu_I \leq \nu_{i+\theta}$; nevertheless we may replace ρ by $\nu_{i+\theta}$ in the second integral in (28) without disturbing the inequality. This follows from the fact that $\dot{k}^* < 0$ for a.a. $t \in B$, whereas the Depletion Lemma shows that B^c makes a zero contribution to the integral. On dividing the resulting inequality by θ and rearranging we have

$$\text{(B.29)} \qquad E\left[(1/h)\int_0^{h\wedge\nu_i} y\cdot dt\right] \leq -E\left[(1/\theta)\int_{\nu_i}^{\nu_{i+\theta}} y\cdot\dot{k}^*\cdot dt\right].$$

We consider separately the two sides of this inequality. The random variables on the left are dominated as $h \downarrow 0$ because

$$\text{(B.30)} \qquad (1/h)\int_0^{h\wedge\nu_i} y\cdot dt \leq (1/h)\int_0^h y\cdot dt \quad \text{and} \quad E\left[(1/h)\int_0^h y dt\right] \leq y(0),$$

the second inequality being due to the supermartingale property of y. On passing to the limit under E and taking into account the right continuity of y, it is seen that the left side of (29) tends to y(0).

On the right side of (29), we may by the Depletion Lemma restrict the time integral to $B \cap [\nu_i, \nu_{i+\theta})$, and on this set $-\dot{k}^*(t)dt$ may, according to (11), be replaced by $\gamma(t)dt = d\Gamma(t)$; on the other hand, $\gamma(t) = 0$ for $t \notin B$, so the whole expression on the right of (29) is equal to

(B.31) $$(1/\theta)E\int_{\nu_i}^{\nu_{i+\theta}} y(t)\cdot d\Gamma(t).$$

Since $\Gamma(t)$ is non-decreasing and absolutely continuous and $\nu(\iota) = \inf\{t: \Gamma(t) > \iota\}$, a (pathwise) change of variable gives

(B.32) $$\int_{\nu_i}^{\nu_{i+\theta}} y(t)\cdot d\Gamma(t) = \int_{i\wedge\Gamma(\infty)}^{(i+\theta)\wedge\Gamma(\infty)} y[\nu(\iota)]\cdot d\iota \le \int_i^{i+\theta} y[\nu(\iota)]d\iota,$$

where $y[\nu(\iota)] = y(\infty)$ in case $\iota \ge \Gamma(\infty)$, i.e. in case $\nu(\iota) = \infty$. Now the Stopping Theorem implies that $Ey(\nu_\iota) \le Ey(\nu_i)$ for $i \le \iota < i+\theta$, and it follows from (32) that (31) cannot exceed $Ey(\nu_i)$. On collecting results and referring to (29) we have $y(0) \le Ey(\nu_i)$, which completes the proof of (19) and hence of (1).

(iii) *Transversality.* It follows from $Ey(\nu_i) = y(0)$ and the argument following (29) that equality must hold a.s. in (32), in other words that $y(\infty) = 0$ if $\Gamma(\infty) \le \iota \in [i, i+\theta)$, and since i and θ are arbitrary subject to $0 \le i < i+\theta < K_0$ it follows that, a.s.,

(B.33) $$y(\omega,\infty) = 0 \text{ if } \Gamma(\infty) < K_0.$$

This in turn implies (2), since then

(B.34) $$E\{y(\nu_i)[K_0 - \Gamma(\nu_i)]\} = E\{N_i\cdot y(\nu_i)[K_0 - \Gamma(\nu_i)]\} = (K_0 - i)E\{y(\nu_i)\}$$
$$= (K_0 - i)y(0) \to 0 \qquad i \uparrow K_0.\|$$

The foregoing also yields

COROLLARY B.3. If (c^*, k^*) is optimal, then

(B.35) $\check{y}$ is an $\check{\mathfrak{A}}$-martingale,

(B.36) $\check{y}(\omega,i) = 0$ if $\check{\Gamma}(\omega,\infty) < K_0$, a.s.,

or equivalently

(B.36′) $\lim_i E\{N_i\cdot\check{y}(i)\cdot\check{k}^*(i)\} = 0$ for any sequence $i \uparrow K_0$.

REMARK I. If it is assumed that $\int_0^\infty s(t)dt < \infty$ a.s., then, for every feasible plan a.s., k is of finite variation on $[0,\infty]$ and $k(\infty)$ exists as a finite limit, see (2.28–29). Then, since $K_0 - \Gamma(\nu_i) = k^*(\nu_i)$ when $\nu_i < \infty$ whereas $y(\nu_i) = 0$ when $\nu_i = \infty$, it is clear that (2) and (3) imply the corresponding conditions with $k^*(\nu_i)$ in place of $K_0 - \Gamma(\nu_i)$. The converse also holds since $0 \leq K_0 - \Gamma \leq k^*$. However, it seems that one cannot exclude the possibility that $k^*(\infty) > K_0 - \Gamma(\infty)$ on a set of positive probability, so that in (4) one cannot replace $\Gamma(\infty) < K_0$ by $k^*(\infty) > 0$.

It can also be shown that, under (2.28), (7) is equivalent to (2) in the presence of (1). It suffices to review the proof of Cor. B1 and note that now the convergence of the left side of (8) implies that of the right side, and the result follows because $k^*(\infty)$ now exists as a finite limit.

REMARK II. Suppose that the optimal plan is such that the paths of Γ are uniformly bounded away from K_0 on $\mathcal{T}$ – more precisely, that there exists a positive, non-decreasing, right continuous function $\Gamma^+(t)$ on $\mathcal{T}$ such that, a.s.,

(B.37) $$\Gamma(\omega,t) \leq \Gamma^+(t) < K_0 \quad \text{for all } t \in \mathcal{T}.$$

Then y is a true martingale and $E\{y(t)\cdot\Gamma(t)\} \to 0$ as $t \to \infty$; these conditions, together with $c^* > 0$ and (2.10), are also sufficient for optimality (even if Γ is not bounded away from K_0). This proposition is analogous to Theorem 6 of Foldes (1978a) and proof is omitted.

APPENDIX C: CONSUMPTION TIMES FOR PS MODEL

We consider the PS model with a continuous market returns process Z and $\Pi = \Pi^0$, i.e. short sales are permitted. The question at issue is whether, in the conditions for optimality (2.20) with (2.14), one can take the consumption times as fundamental for some or all of the local martingales y^π. Sufficiency is straightforward, and we concentrate on necessity. The best result so far is

PROPOSITION C. Let (c^*,π^*) be an optimal plan. The process $\hat{y}^\pi$, defined as the transform of y^π to consumption time, is an $\hat{\mathfrak{A}}$-martingale for each portfolio plan π satisfying (C.20) below, and a fortiori for each π satisfing the boundedness condition (C.2).

As a preliminary point, let (c^*,π^*) be just a *distinguished* plan satisfying $c^* > 0$ and (2.10), and recall Prop.7 of Foldes (1990) in the following, corrected, form:

> If $\Pi = \Pi^0$, if y^π is a local supermartingale for every $\pi \in \Pi^0$ *and $y*$ is a local martingale*, then y^π is a local martingale for every $\pi \in \Pi^0$.

The italicised words were omitted in Foldes (1990)[1].

It follows readily that, under the stated hypotheses – and therefore under (2.20) – $\hat{y}^\pi$ is a *local* $\hat{\mathfrak{A}}$-martingale for every π. This can be shown by noting that the proof of the (corrected) Prop.7 works equally well for the y-processes transformed to consumption time; unfortunately, the method of proof does not identify the stopping times which reduce the $\hat{y}^\pi$. Hence the need for a more refined argument.

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[1] The second paragraph of the proof given in Foldes (1990) should be amended to read as follows:

To be explicit, suppose that ∂^1 does not vanish. Let $\tau > 0$ be a predictable time such that $\partial^1_\tau < 0$ with positive probability. Choose a number $p < 0$ and let π' be a portfolio policy defined by $\pi'^1 = p1 + (1-p)\pi^{*1}$, $\pi'^\lambda = (1-p)\pi^{*\lambda}$ for $\lambda \neq 1$. Then $\partial^{\pi'}_\tau = p\partial^1_\tau + (1-p)\partial^*_\tau = p\partial^1_\tau$, which takes positive values with positive probability, contrary to $\partial^{\pi'} \leq 0$.□

Returning to the case of an *optimal* plan (c^*,π^*), we know from the proof of Theorem 2 of Foldes (1990) that every y^π is an $\mathfrak{A}$-supermartingale and every $\hat{y}^\pi$ is an $\hat{\mathfrak{A}}$-supermartingale; further, since c^* is π^*-optimal, it follows from Theorem 5 of Foldes (1978a) that $\hat{y}^*$ is a (true) $\hat{\mathfrak{A}}$-martingale. Thus, in order to prove Prop.C, it suffices to show that

(C.1) For every $\pi \in \Pi^0$ satisfying (C.20), $E\hat{y}^\pi(0) \geq E\hat{y}^\pi(\iota)$ for every $\iota \in [0,K_0)$.

(Cf. the discussion of (B.19).) Note that $E\hat{y}^\pi(0) = \hat{y}^\pi(0) = \hat{y}^*(0) = y^*(0) = Ey^*(0)$.

We consider a fixed π throughout. The procedure will be to reverse, so far as possible, the proof of the supermartingale property given in Foldes (1990), so that ideally a phase of increased consumption, financed by sales of the portfolio π, will be followed by a neutral phase and then by a phase of increased savings, used to buy in π, until capital has caught up with the star plan. The argument is simplified here in that the first phase can start at zero time and the variation need not be restricted to a subset of Ω, but it is now more difficult to satisfy the non-negativity constraints. Since the stopping routines needed to ensure feasibility in the general case are liable to make the notation impenetrable, *we give the proof first under the following boundedness condition:*

(C.2) For some finite number $\alpha > 1$, we have $z^\pi_t / z^*_t < \alpha$ on $\mathcal{T}$, a.s.

Choose numbers h, ι, κ such that

(C.3) $0 < h < \iota < \iota + \kappa < K_0,$

then ϵ such that

(C.4a) $0 < \epsilon < \kappa/\alpha,$ and

(C.4b) $0 < \epsilon < (K_0 - \iota - \kappa)/\alpha,$ hence also $\epsilon < (K_0 - \iota)/\alpha < (K_0 - h)/\alpha.$

Now bear in mind that τ_i denotes the consumption time at the level i, and if $\tau_i < \tau_j$ with $0 \leq i < j < K_0$ then, a.s.,

$$\int_{\tau(i)}^{\tau(j)} c^*(t)dt = k(\tau_i) - k(\tau_j) \leq j - i,$$

with equality if $\tau_j(\omega) < \infty$. We define a variation $\delta\bar{c} = \bar{c} - \bar{c}^*$ by setting

(C.5) $$\delta\bar{c}_t/\bar{c}^*_t = \begin{cases} +(\epsilon/h)(z^\pi_t/z^*_t) & 0 \le t < \tau_h \quad \text{(spending phase)} \\ 0 & \tau_h \le t < \tau_I \quad \text{(neutral phase)} \\ -(\epsilon/H)(z^\pi_t/z^*_t) & \tau_I \le t < \tau_{I+H} \quad \text{(saving phase)} \end{cases}$$

and $\delta\bar{c}_t = 0$ for $t \ge \tau_{I+H}$. Note that $\bar{c}$ and $\bar{c}^*$ are in natural units and that $\bar{c}^*/z^* = c^* = -dk^*/dt$. The extra spending is financed by sales of π, and the extra saving is used to buy in this portfolio; thus, as in Foldes (1990) eq.(4.20), we have

(C.6) $$\delta\bar{k}_T = -z^\pi_T \cdot \int_0^T [\delta\bar{c}(t)/z^\pi(t)]dt, \quad T \in \mathcal{T}.$$

To check feasibility of (5): During the *spending phase*, clearly $\bar{c}_t = \bar{c}^*_t + \delta\bar{c}_t > 0$, (bearing in mind that $\bar{c}^*_t > 0$ always), Also

(C.7a) $$-\delta\bar{k}_T = (\epsilon/h)z^\pi_T \cdot \int_0^T c^*(t)dt \le \epsilon z^\pi_T,$$

(C.7b) $$\bar{k}_T = \bar{k}^*_T + \delta\bar{k}_T = z^*_T(k^*_T - \epsilon z^\pi_T/z^*_T) \ge z^*_T(K_0 - h - \epsilon\alpha) > 0$$

for $T \le \tau_h$, using (2–4) and noting that $\int_0^T c^*(t)dt = K_0 - k^*_T \le h$ for $T \le \tau_h$.

For the *neutral phase*, $\bar{c}_t = \bar{c}^*_t > 0$, and

(C.8a) $$-\delta\bar{k}_T/z^\pi_T = -\delta\bar{k}_{\tau(h)}/z^\pi_{\tau(h)} = (\epsilon/h) \cdot \int_0^{\tau(h)} c^*(t)dt \le \epsilon,$$

(C.8b) $$\bar{k}_T = \bar{k}^*_T + \delta\bar{k}_T = z^*_T(k^*_T - \epsilon z^\pi_T/z^*_T) \ge z^*_T(K_0 - I - \epsilon\alpha) > 0$$

reasoning as above. (If $\tau(h) < \infty$, there is equality throughout (8a).)

For the *saving phase,*

(C.9a) $$\bar{c}_t = \bar{c}^*_t[1 - (\epsilon/H)(z^\pi_t/z^*_t)] > \bar{c}^*_t[1 - (\epsilon/H)\alpha] > 0$$

by (4a). Next,

(C.9b) $$-\delta\bar{k}_T/z^\pi_T = -\delta\bar{k}_{\tau(I)}/z^\pi_{\tau(I)} + \int_{\tau(I)}^T (\delta\bar{c}_t/z^\pi_t)dt \le \epsilon - (\epsilon/H) \cdot \int_{\tau(I)}^T c^*(t)dt \le \epsilon$$

since $\int_{\tau(I)}^T c^*(t)dt \le H$ for $T \le \tau_{I+H}$, and then, using (2–4) again,

(C.9c) $$\bar{k}_T = \bar{k}^*_T + \delta\bar{k}_T \ge z^*_T \cdot k^*_T - \epsilon z^\pi_T + (\epsilon/H)z^\pi_T \cdot \int_{\tau(I)}^T c^*(t)dt$$
$$\ge z^*_T(k^*_T - \epsilon z^\pi_T/z^*_T) > z^*_T(K_0 - I - H - \epsilon\alpha) > 0.$$

So the whole variation is feasible. For $T = \tau_{I+H} < \infty$, (9.c) yields

(C.9d) $$\bar{k}(\tau_{I+H}) \ge \bar{k}^*(\tau_{I+H}) - \epsilon z^\pi(\tau_{I+H})[1-(1/H)\int_{\tau(I)}^{\tau(I+H)} c^*(t)dt] = \bar{k}^*(\tau_{I+H}),$$

so that $\bar{k}$ catches up with $\bar{k}^*$ at or before τ_{I+H} if this time is finite. (The saving phase of the variation (5) could be stopped at the catching-up time if this occurs before τ_{I+H}, but the notation is simpler if this is not done.)

The rest of the argument then proceeds as in Foldes (1990&78a). Substituting from (5) into the formula

(C.10) $$D\varphi = E \int \delta\bar{c}\cdot v\cdot dt$$

– see (2.13), writing

(C.11) $$y^{\pi} = z^{\pi}\cdot v,$$

dividing by ϵ and rearranging, we have

(C.12) $$(1/h)E\int_0^{\tau(h)} y_t^{\pi}\cdot c_t^*\cdot dt \le (1/H)E\int_{\tau(I)}^{\tau(I+H)} y_t^{\pi}\cdot c_t^*\cdot dt,$$

or, transforming to consumption time,

(C.13) $$(1/h)E\int_0^{h} \hat{y}_i^{\pi}\cdot di \le (1/H)E\int_I^{I+H} \hat{y}_i^{\pi}\cdot di.$$

Letting $h \downarrow 0$, $H \downarrow 0$, and bearing in mind that we already know that $\hat{y}^{\pi}$ is a right continuous, positive supermartingale with $E\hat{y}^{\pi}(0) = y(0) < \infty$, it follows as in the proof of Theorem 2 of Foldes (1990) that in the limit we have

(C.14) $$y(0) = E\hat{y}^{\pi}(0) \le E\hat{y}^{\pi}(I), \quad 0 \le I < K_0,$$

so that in fact there is equality and $\hat{y}^{\pi}$ is a (true) $\hat{\mathfrak{A}}$ - martingale.

We now drop (C.2) *and review the proof.* For given $\alpha > 1$, (fixed for the time being), define a stopping time

(C.15) $$\varsigma_\alpha = \varsigma(\alpha,\pi;\omega) = \inf\{t: z_t^{\pi}/z_t^* > \alpha\} \wedge \infty;$$

(note that ς_α cannot be finite a.s., since this would contradict optimality of the star plan). For brevity, we sometimes write $\bar{k}(\varsigma_\alpha) = \bar{k}_\alpha$ etc. Continuing to choose ϵ in accordance with (4), replace each τ_i in (5) by $\tau_i \wedge \varsigma_\alpha$. The verification of feasibility up to (9c) remains valid with these replacements. It is not true in general that $\bar{k}$ catches up with $\bar{k}^*$ at or before $\tau_{I+H} \wedge \varsigma_\alpha$ if $\varsigma_\alpha < \tau_{I+H}$. We therefore define a *continuation phase* of the variation by setting

(C.16) $\bar{c}_t/\bar{c}^*_t = \bar{k}_\alpha/\bar{k}^*_\alpha$ for $t > \varsigma_\alpha$ if $\varsigma_\alpha < \tau_{I+H}$, i.e.

$$\delta\bar{c}_t = \bar{c}_t - \bar{c}^*_t = \bar{c}^*_t[\bar{k}_\alpha/\bar{k}^*_\alpha - 1] = \bar{c}^*_t \cdot \delta\bar{k}_\alpha/\bar{k}^*_\alpha\,;$$

obviously this is feasible. (The variation could be stopped at the catching-up time if this occurs before $\tau_{I+H} \wedge \varsigma_\alpha$, but once again we leave this aside for simplicity.) Reviewing the proof for the bounded case, and noting that $z^\pi_\alpha = \alpha\cdot z^*_\alpha$ by continuity if $\varsigma_\alpha < \infty$, we find that

(C.17) $$-\,\delta\bar{k}_\alpha \le \epsilon\cdot z^\pi_\alpha = \epsilon\cdot\alpha\cdot z^*_\alpha \quad \text{if } \varsigma_\alpha < \tau_{I+H}\,, \quad \text{hence}$$
$$-\,\delta\bar{k}_\alpha/\bar{k}^*_\alpha \le \epsilon\cdot\alpha\cdot z^*_\alpha/\bar{k}^*_\alpha = \epsilon\cdot\alpha/k^*_\alpha\,.$$

Calculating the directional derivative as above, (12) is now replaced by

(C.18) $$(1/h)E\int_0^{\tau_h\wedge\varsigma_\alpha} y^\pi_t\cdot c^*_t\cdot dt \le (1/H)E\int_{\tau_I\wedge\varsigma_\alpha}^{\tau_{I+H}\wedge\varsigma_\alpha} y^\pi_t\cdot c^*_t\cdot dt$$
$$-\,(1/\epsilon)E\Big[\mathfrak{I}\{0 < \varsigma_\alpha < \tau_{I+H}\}\cdot\int_{\varsigma_\alpha}^{\infty} \bar{c}^*_t(\delta\bar{k}_\alpha/\bar{k}^*_\alpha)v_t\cdot dt\Big].$$

Taking into account the integrability conditions mentioned earlier, the term on the left and the first term on the right converge as $\alpha\uparrow\infty$ to the corresponding terms in (12). Thus, in order to complete the proof, we want the second (or 'nuisance') term on the right to vanish, as $\alpha\uparrow\infty$, for an arbitrary choice of I+H. This term is positive, and using the upper bound for

(C.19) $$-\delta\bar{k}_\alpha/\bar{k}^*_\alpha$$

supplied by (17), a sufficient condition is

(C.20) $$\lim_{\alpha\to\infty} E\Big[\alpha\cdot\mathfrak{I}\{0 < \varsigma_\alpha < \tau_{I+H}\}\cdot(1/k^*_\alpha)\cdot\int_{\varsigma_\alpha}^{\infty} \bar{c}^*_t\cdot v_t\cdot dt\Big] = 0 \text{ for } 0 < I+H < K_0.$$

If z^π/z^* is bounded, the indicator function in this expression vanishes for α large enough, so that the condition is satisfied. Note also that, as $\alpha\uparrow\infty$, so that $\varsigma_\alpha\uparrow\infty$, both the indicator function and the integral in (20) decrease pointwise to zero. However, the condition is not very convenient and its economic meaning is unclear. To get a more transparent sufficient condition, note that

(C.21) $$\int_{\varsigma_\alpha}^{\infty} \bar{c}_t^* \cdot v_t \cdot dt \;=\; \int_{\varsigma_\alpha}^{\infty} y_t^* \cdot c_t^* \cdot dt \;\leq\; \sup\{y^*(\omega,t)\colon t \geq \varsigma_\alpha\} \cdot k_\alpha^*.$$

Thus, applying first the Schwarz inequality to the expectation in (20) and then a maximal inequality, it is found to be sufficient if both

(C.22a) $$\sup\,\{E(y_t^*)^2\colon t \in \mathscr{T}\} \;<\; \infty \quad \text{and}$$

(C.22b) $$\lim_{\alpha \mapsto \infty} [\alpha \cdot P\{0 < \varsigma_\alpha < \tau_i\}] \;=\; 0 \quad \text{for } 0 < i < K_0.$$

REMARK I. This Appendix has assumed a continuous market process Z with short sales permitted. However, continuity of Z as such plays no part in the proof when (C.2) applies, and in this case the argument extends to discontinuous Z and $\Pi = \Pi^+$, provided that $\pi^* > 0$ and that holdings within π^* of securities comprising π are sufficient to allow the spending phase (with small ϵ) to be financed by sales of π.

REMARK II.[2] A much shorter argument can be given when (C.2) applies. Assume *either* that Z is continuous with $\Pi = \Pi^0$, *or* that $\pi^* > 0$ with Π^0 or Π^+. The conditions (2.20) apply. Also, we know from earlier work, see (2.16)–(2.20), that y^* is a local martingale reduced by any sequence (τ_i) with $i \uparrow K_0$. If $z^\pi < \alpha z^*$, then $y^\pi < \alpha y^*$; and since, for each $i \in [0,K_0)$, y^* stopped at τ_i is a u.i. martingale, the same is true of y^π stopped at τ_i. So $\hat{y}^\pi$ is an $\hat{\mathfrak{A}}$-martingale.||

[2] Esprit de l'escalier: to be added to Journal version in proof.

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