

EXISTENCE AND UNIQUENESS OF AN OPTIMUM
in the Infinite—Horizon Portfolio—cum—Saving Model
with Semimartingale Investments †

by

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Abstract. The model considered here is essentially that formulated in the author's previous paper *Conditions for Optimality in the Infinite—Horizon Portfolio—cum—Saving Problem with Semimartingale Investments*, Stochastics and Stochastics Reports 29 (1990) 133–171. In this model, the vector process representing returns to investments is a general semimartingale. Processes defining portfolio plans are here required only to be predictable and non-negative. Existence of an optimal portfolio—cum—saving plan is proved under slight conditions of integrability imposed on the welfare functional; the proofs rely on properties of weak precompactness of portfolio and utility sequences in suitable L_p spaces together with dominated and monotone convergence arguments. Conditions are also obtained for the uniqueness of the portfolio plan generating a given returns process (i.e. for the uniqueness of the integrands generating a given sum of semimartingale integrals) and for the uniqueness of an optimal plan; here use is made of random measures associated with the jumps of a semimartingale.

Key words. Investment, portfolios, semimartingales, optimisation, weak compactness.

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1. INTRODUCTION

The object of this paper is to prove the existence and uniqueness of an optimal plan in a slightly modified version of the portfolio—cum—saving (PS) model formulated in [1]. The model considers planning for optimal saving and portfolio choice over an infinite horizon in continuous time by an investor who seeks to maximise welfare, defined as the expected integral of discounted utility. Both utility and the return to investment are subject to risk, the sources of risk and the investor's information structure being specified in a very general form; in particular, the vector process X representing log—returns to investment is only assumed to be a semimartingale with respect to a given filtration. Consumption and capital are constrained to be non—negative. Assets are divisible and can be traded at market prices without transaction costs and short sales are forbidden. In [1] we discussed conditions characterising an optimal plan but not whether such a plan exists. A variant of the model was considered in which the X —process is continuous and short sales are permitted, but this is left aside here apart from a brief remark. The only other substantial change in the definition of the model is that a portfolio plan π is defined here as a *predictable* process specifying the (non—negative) proportions of capital assigned to the available assets, whereas in [1] a portfolio plan was required to be adapted and continuous on the left with limits on the right (*collor*). The left continuity expressed the requirement that any jump $\Delta X(t)$ in the market should accrue to the portfolio $\pi(t) = \pi(t-)$ held immediately before t , not to the portfolio $\pi(t+)$ chosen at t . Bearing in mind that the predictable σ —algebra is generated by the adapted collor processes, the present definition does essentially the same job, while ensuring that limits of portfolio plans are portfolio plans. As shown in [11–12] in the special case where X has independent increments, an optimal plan with π collor cannot in general be expected to exist unless the characteristics of X

satisfy substantial conditions of smoothness, in particular the absence of fixed times of discontinuity. The theory of necessary and sufficient conditions for optimality with non-negative portfolios developed in [1] is substantially unaffected by the change of definition.

Existence of an optimal PS plan is proved here under the hypothesis that the welfare functional has a finite supremum and that there is a maximising utility sequence which is weakly precompact in a suitable L_1 space. An optimum exists if the undiscounted utility function is bounded, and separate sufficient conditions are given for the cases of negative, positive and logarithmic utility. In the case of a single asset (or a fixed portfolio plan) the method of proof is a modified version of that used in [2]. An extension of the one-asset proof has been applied in [3] to a finite-horizon PS model driven by Brownian motion, but to my knowledge the method has not previously been used in a PS model with general semimartingale investments.¹ The method used here to deal with convergence of portfolio plans — which makes use of weak sequential compactness in a suitable L_2 space — also appears to be new. Advantages of the present approach to existence include the (relatively) simple methods used and the generality of the assumptions about information and about the market and portfolio processes. In particular, the filtration need not be the smallest for which the X -process is optional; the question of the ‘completeness’ of the market does not arise; there need be no riskless security; no martingale representation or duality theory is required; and an infinite horizon is allowed.

Two questions concerning *uniqueness* are considered. The first relates to the conditions under which the portfolio plan generating a given returns process is unique — or, more or less equivalently, the conditions under which the sum of a finite number of semimartingale integrals uniquely determines the integrands. General results on this point do not appear to be available. It is well known that distinct portfolios can generate the

¹ Existence in the infinite-horizon case is also proved in [3] and in [10], but by different methods; the models considered are driven by Brownian motion with information defined by the ‘natural’ filtration.

same returns even in one-period models, and there are familiar conditions for portfolio uniqueness in this case, see [15] T.2.5. Sufficient conditions for portfolio uniqueness and uniqueness of equilibrium in models driven by Brownian motion are also known, see [16] and references cited there. Portfolio uniqueness when X is a process with independent increments and portfolio plans are required to be collar (except perhaps at fixed times of discontinuity of X) has been discussed in [11–12]. Here we shall investigate this question under the assumptions that X is a general semimartingale and portfolio plans are predictable and non-negative. The discussion involves the use of random measures associated with the jumps of X . This technique yields a simple treatment even in cases where X may have arbitrarily large jumps – in particular, where X is not a special semimartingale – and in cases where the paths of X may have countably many (arbitrarily small) jumps during a finite interval; such processes have sometimes been proposed as models of speculative prices. In elementary cases, the results obtained reduce to well-known conditions such as non-singularity of a returns matrix or positive definiteness of a covariance matrix. The second question relates to the uniqueness of optimal plans. Here the answers are classical: briefly, an optimal plan is unique if there is diminishing marginal utility and portfolio uniqueness.

The main features of the model together with some preliminary results are set out in Section 2; some material from [1] and [2] is included to keep the discussion self-contained. Section 3 investigates some convexity and convergence properties of feasible sets of consumption and portfolio plans needed for the existence theorems; the latter are then quickly proved in Section 4 and some criteria which are sufficient for existence are given. Uniqueness is considered in Section 5.

2. THE MODEL

Let $\mathcal{T} = [0, \infty)$ be the time domain with Borel sets \mathcal{B} and Lebesgue measure \mathbb{I} and let (Ω, \mathcal{A}, P) be a complete probability space with a filtration $\mathcal{A} = (\mathcal{A}_t; t \in \mathcal{T})$ satisfying the 'usual conditions' of right continuity and completeness, where $\mathcal{A} = \mathcal{A}_{\infty}$ while $\mathcal{A}_0 = \mathcal{A}_{0-}$ is generated by the P -null sets (so that an \mathcal{A}_0 -measurable variable is a.s. constant on Ω). \mathcal{A} represents the investor's *information structure* and P his *beliefs*. We define the products $\mathcal{S} = \Omega \times \mathcal{T}$, $\mathcal{A} \times \mathcal{B}$, $m = P \times \mathbb{I}$ and write $s = (\omega, t)$, $dm(s) = dP(\omega)dt$, $dt = d\mathbb{I}(t)$. Statements which hold apart from null sets of \mathcal{T} , Ω , \mathcal{S} are qualified by 'a.a.t.', 'a.s.', 'a.e.' respectively, the measures considered being \mathbb{I} , P , m or equivalent measures unless otherwise stated. In the product space \mathcal{S} we define in the usual way the σ -algebras \mathcal{H} , \mathcal{O} , \mathcal{P} of progressive, optional and predictable sets, as well as the corresponding classes of processes.² The following conventions apply unless we state or imply otherwise. Measures are by definition non-negative. All processes considered are assumed, or may easily be shown to be, at least progressively measurable, (but it would be possible to replace progressive by optional measurability throughout). Scalar r.v.s and processes take finite, or occasionally extended, real values, while vector r.v.s and processes are families with a finite number $\lambda = 1, \dots, \Lambda$ of components. The Euclidian norm in \mathbb{R}^Λ is written $\|\cdot\|$. For a scalar process ξ , $\xi > 0$ means $\xi(s) > 0$ for all s and $\xi \geq 0$ means $\xi(s) \geq 0$ for all s , modulo null sets, while similar notation for a vector process means that the condition applies to each component; analogous convention for r.v.s, with s replaced by ω . For $t \in \mathcal{T}$, E^t means $E(\cdot / \mathcal{A}_t)$; similarly E^ν , $E^{\nu-}$ for a stopping time ν .

² \mathcal{H} comprises (ω, t) sets H such that, for each $T \in \mathcal{T}$, the subset $H \cap \{\Omega \times [0, T]\}$ belongs to $\mathcal{A}_T \times \mathcal{B}_T$. \mathcal{O} is generated by intervals of the form $[\sigma, \tau[$ where σ, τ are stopping times, or equivalently by the right continuous, adapted processes. \mathcal{P} is generated by intervals $]\sigma, \tau]$ and sets $A \times \{0\}$ with $A \in \mathcal{A}_0$, or equivalently by the left continuous, adapted processes. We have $\mathcal{A} \times \mathcal{B} \supseteq \mathcal{H} \supseteq \mathcal{O} \supseteq \mathcal{P}$.

The terms positive, negative, increasing, decreasing have their strict meaning throughout, but \uparrow, \downarrow mean non-decreasing, non-increasing. Semimartingales, martingales, non-decreasing and finite variation processes will by definition be finite on \mathcal{T} and continuous on the right with limits on the left (*cor101*). A semimartingale ξ is called positive if $\xi(t) > 0$ and $\xi(t-) > 0$ on \mathcal{T} a.s. For any process we set $\xi(0-) = \xi(0)$, so that for stochastic (and pathwise) integrals we have $\int_{[0,T]} = \int_{(0,T]}$, which we often write as \int_0^T . Martingales, non-decreasing and finite variation processes satisfy $\xi(0) = 0$ unless otherwise stated. A process ξ is called *integrable on compacts* if $E|\xi_T| < \infty$ for each $T \in \mathcal{T}$, similarly *square-integrable on compacts* if $E\xi_T^2 < \infty$ for each $T \in \mathcal{T}$; but *locally integrable* means that $E|\xi(T_n)| < \infty$ for some sequence of stopping times $T_n \uparrow \infty$ a.s., etc. *Finite variation* means finite variation on compacts, the variation being written $|\xi|_T = \int_0^T |d\xi_t|$; thus integrable variation on compacts means $E|\xi|_T < \infty$ for each $T \in \mathcal{T}$, etc. Finally, if a given set Ξ of processes is specified, two elements ξ_1 and ξ_2 which are *indistinguishable*, i.e. which satisfy $\delta\xi(\omega, t) = 0$ for all $t \in \mathcal{T}$ a.s., where $\delta\xi = \xi_1 - \xi_2$, are identified and we write $\xi_1 \equiv \xi_2$ or $\delta\xi \equiv 0$. If $\delta\xi = 0$ a.e., we call the elements *similar* and write $\xi_1 \sim \xi_2$ or $\delta\xi \sim 0$, specifying the measure and σ -algebra in question if necessary; as explained in the footnote, we identify similar processes in certain cases.³

³ Identifications are usually defined so as to set up bijections among equivalence classes of processes associated with the same 'plan'. As will appear, there is for each 'plan' a consumption plan in natural units \bar{c} , a consumption plan in standardised units c , corresponding capital plans \bar{k}, k , a utility plan \bar{U} , a portfolio plan π , a returns process z and a log-returns process $x = \ln z$. Now \bar{k}, k, z, x are (*cor101* versions of) stochastic or ordinary integrals and are naturally defined as classes of indistinguishable processes; in these cases, $\xi_1 \sim \xi_2(m) \longleftrightarrow \xi_1 \equiv \xi_2$. On the other hand, \bar{c}, c, \bar{U} are progressive processes, and by identifying m -similar elements we obtain bijections among $\bar{c}, c, \bar{U}, \bar{k}, k, z$, and x classes. The details so far are straightforward if tedious and can usually be omitted. More care is needed in the case of portfolio plans, which are predictable processes with no specified properties of sample continuity. Suppose that π_i generates x_i , $i = 1, 2$. Distinct portfolios can generate the same returns even in a one-period model, so it is to be expected that in general $x_1 \equiv x_2$ does not imply $\pi_1 \sim \pi_2(m)$. Conversely, if the market can jump at

On the measurable space $(\mathcal{S}, \mathcal{H})$ we consider, in addition to m , a unitary measure μ , with the same null sets, defined by

$$d\mu(s) = d\mu(\omega, t) = q(\omega, t)dP(\omega)dt = q(s)dm(s), \quad \dots(2.1)$$

where q is a positive, finite, \mathcal{H} -measurable function satisfying $E \int_0^{\infty} q(\omega, t)dt = 1$; μ is called the *discount measure*, q the *discount density*. We denote by $\mathcal{L}_1 = L_1(\mathcal{S}, \mathcal{H}, \mu; \mathbb{R})$ the space of (classes of) progressive, μ -integrable, real-valued processes $\xi = \xi(s) = \xi(\omega, t)$ on \mathcal{S} with the norm $\int |\xi| d\mu$. When the domain is not specified, μ -integrals are taken over \mathcal{S} .

As in [1], a finite number of assets (or securities) indexed by $\lambda = 1, \dots, \Lambda$ are assumed to be available at all times, where $\Lambda \geq 2$ to avoid trivialities.. For each λ there is given a semimartingale x^λ with $x^\lambda(\omega, 0) = 0$ called the *log-returns* or *compound interest process* for λ , and the formula $z^\lambda = e^{x^\lambda}$ defines a positive semimartingale called the *returns* or *price process* for λ . The vector $X = (x^1, \dots, x^\Lambda)$ is called the *market log-returns process*. Decompositions of x^λ are written

$$x^\lambda = M^{\lambda c} + M^{\lambda d} + V^{\lambda c} + V^{\lambda d} = M^\lambda + V^\lambda, \quad \dots(2.2)$$

where $M^{\lambda c}$, $M^{\lambda d}$ are continuous and compensated jump local martingales respectively, $V^{\lambda c}$, $V^{\lambda d}$ are continuous and discontinuous processes of finite variation; all these processes vanish at $t=0$. In general only $M^{\lambda c}$ is uniquely defined, but a choice of any one of $M^{\lambda d}$, $V^{\lambda c}$ or $V^{\lambda d}$ fixes the remaining terms. Similarly, for vectors, $X = M + V$ etc. We also write $x^{\lambda c} = M^{\lambda c} + V^{\lambda c}$, $X^c = M^c + V^c$, $x^{\lambda d} = M^{\lambda d} + V^{\lambda d}$, $X^d = M^d + V^d$.

It is always possible to choose a decomposition such that, a.s. for all $t \in \mathcal{T}$ and each λ ,

$$\Delta x^\lambda(\omega, t) = 0 \text{ implies } \Delta M^\lambda(\omega, t) = -\Delta V^\lambda(\omega, t) = 0. \quad \dots(2.3)$$

To check this, suppose first that X is a special semimartingale, choose the 'canonical' decomposition $X = M + V$ with V predictable, and let (ν'_i) be a sequence of predictable

predictable times, $\pi_1 \sim \pi_2(m)$ does not imply $x_1 \equiv x_2$. We shall introduce below measures for which $\pi_1 \sim \pi_2$ implies $x_1 \equiv x_2$, also the converse under additional conditions.

stopping times which exhausts the jumps of V . Further, let $(\nu'_i) \uparrow \infty$ be a sequence of finite predictable stopping times which reduce M , see [9] VI.84, and let $\nu_i = \nu'_i \wedge \nu''_i$. If the assertion were false, there would be some $\nu_i = \nu$, some λ , and an event $A \in \mathcal{A}_{\nu}$ with $PA > 0$ such that $\Delta x_{\nu}^{\lambda} = 0$ and $\Delta M_{\nu}^{\lambda} = -\Delta V_{\nu}^{\lambda} \neq 0$ on A . Now $V_{\nu}^{\lambda} \in \mathcal{A}_{\nu-}$ since V is a predictable process of finite variation, so $\Delta M_{\nu}^{\lambda} \cdot I_A = -\Delta V_{\nu}^{\lambda} \cdot I_A \in \mathcal{A}_{\nu-}$, hence $M_{\nu-}^{\lambda} = E^{\nu-} M_{\nu}^{\lambda}$ a.s. on A . On the other hand, the martingale property of $(M_{t \wedge \nu})$ implies $M_{\nu-}^{\lambda} = E^{\nu-} M_{\nu}^{\lambda}$ a.s., hence $\Delta M_{\nu}^{\lambda} = M_{\nu}^{\lambda} - M_{\nu-}^{\lambda} = 0$ a.s. on A , a contradiction which yields the result. If X is a general semimartingale, $\bar{V}_T^{\lambda} = \sum_{t \leq T} [\Delta x_t^{\lambda} \cdot I_{\|\Delta x_t\| > 1}]$ defines a process \bar{V} of finite variation, $X - \bar{V}$ is special and so has a decomposition $M + \bar{V}$ with \bar{V} predictable, and $X - \bar{V}$ and \bar{V} cannot jump simultaneously. It then follows from the preceding argument that $X = M + (\bar{V} + \bar{V})$ is a decomposition satisfying (3).||

Associated with $x^{\lambda} = M^{\lambda} + V^{\lambda}$ are the 'angle brackets' process $\langle M^{\lambda c}, M^{\lambda c} \rangle$, also written as $\langle M^{\lambda c} \rangle$ or $\langle x^{\lambda c} \rangle$, and the 'square brackets' processes $[x^{\lambda}, x^{\lambda}] = [x^{\lambda}]$ and $[M^{\lambda}]$. Recall that $\langle M^{\lambda c} \rangle$ is the unique continuous non-decreasing process such that $(M^{\lambda c})^2 - \langle M^{\lambda c} \rangle$ is a local martingale. The various brackets are related by

$$\begin{aligned} [x^{\lambda}]_T &= \langle x^{\lambda c} \rangle_T + [x^{\lambda d}]_T, & [x^{\lambda d}]_T &= \sum_{t \leq T} (\Delta x_t^{\lambda})^2, \\ [M^{\lambda}]_T &= \langle M^{\lambda c} \rangle_T + [M^{\lambda d}]_T, & [M^{\lambda d}]_T &= \sum_{t \leq T} (\Delta M_t^{\lambda})^2. \end{aligned} \quad \dots(2.4)$$

For an arbitrary local martingale M^{λ} , the bracket $\langle M^{\lambda} \rangle$ may be undefined, but if M^{λ} is locally square-integrable then $\langle M^{\lambda} \rangle$ is the unique predictable non-decreasing process such that $(M^{\lambda})^2 - \langle M^{\lambda} \rangle$ is a local martingale, and $\langle M^{\lambda} \rangle$ is the compensator (dual predictable projection) of $[M^{\lambda}]$ so that $[M^{\lambda}] - \langle M^{\lambda} \rangle$ is a local martingale, see [9] VII.39–42. Further, if M^{λ} is a (true) martingale and is square-integrable on compacts, then $[M^{\lambda}]$ and $\langle M^{\lambda} \rangle$ are integrable on compacts and both $(M^{\lambda})^2 - \langle M^{\lambda} \rangle$ and $[M^{\lambda}] - \langle M^{\lambda} \rangle$ are martingales, see [8] I.4.50. Square brackets are defined for all semimartingales. For vectors, we sometimes write

$$\begin{aligned} [X]_T &= \Sigma_\lambda [x^\lambda]_T, & |V|_T &= \Sigma_\lambda |V^\lambda|_T, \\ [M]_T &= \Sigma_\lambda [M^\lambda]_T, & \langle M^c \rangle_T &= \Sigma_\lambda \langle M^{\lambda c} \rangle_T, & \langle M \rangle_T &= \Sigma_\lambda \langle M^\lambda \rangle_T \end{aligned} \quad \dots(2.4a)$$

etc. The brackets $\langle M^{\lambda c}, M^{\ell c} \rangle_T$ etc. are defined by 'polarisation', i.e.

$$4\langle M^{\lambda c}, M^{\ell c} \rangle = \langle M^{\lambda c} + M^{\ell c} \rangle - \langle M^{\lambda c} - M^{\ell c} \rangle \text{ etc; in particular,}$$

$$[x^\lambda, x^\ell]_T = \langle M^{\lambda c}, M^{\ell c} \rangle_T + \Sigma_{t \leq T} (\Delta x_t^\lambda \cdot \Delta x_t^\ell). \quad \dots(2.5)$$

We denote by $\langle \mathfrak{M}^c \rangle_T$ the (random) matrix with elements $\langle M^{\lambda c}, M^{\ell c} \rangle_T$; $\langle \mathfrak{M}^c \rangle_T - \langle \mathfrak{M}^c \rangle_S$ is symmetric non-negative definite for each pair $S < T$ from \mathcal{T} a.s., see [7] 3.46–47.

A *portfolio plan* π is now defined as a predictable vector process with components π^λ satisfying

$$0 \leq \pi^\lambda(\omega, t) \leq 1, \quad \lambda = 1, \dots, \Lambda, \quad \dots(2.6)$$

$$\Sigma_\lambda \pi^\lambda(\omega, t) = 1 \quad \dots(2.7)$$

for all (ω, t) , (subject to conventions for identification to be discussed below). The restriction $\pi \geq 0$ means that short sales are forbidden. Note that, since $\Delta X(0) = 0$, the vector $\pi(0)$ may be defined arbitrarily without affecting portfolio returns (see below).

The set of all portfolio plans is denoted by Π (corresponding to Π^+ in [1]).

Given a portfolio plan $\pi \in \Pi$, the *portfolio returns process* z^π generated by π is defined as the unique semimartingale satisfying the equation

$$z^\pi(T) = 1 + \int_0^T z^\pi(t-) \Sigma_\lambda \pi^\lambda(t) \frac{dz^\lambda(t)}{z^\lambda(t-)} \quad \dots(2.8)$$

for all $t \in \mathcal{T}$ a.s. It may be verified, as in [1] S.2, that $z^\pi(T)$ and $z^\pi(T-)$ are defined and positive for all $T \in \mathcal{T}$ a.s. Consequently the relation $z^\pi = e^{x^\pi}$ defines a semimartingale x^π on \mathcal{T} with $x^\pi(0) = 0$ called the *portfolio log-returns process generated by π* , or simply the *compound interest process for π* . Writing

$$\zeta^\lambda(T) = \int_0^T \frac{dz^\lambda(t)}{z^\lambda(t-)}, \quad \zeta^\pi(T) = \int_0^T \frac{dz^\pi(t)}{z^\pi(t-)}, \quad \zeta^\lambda(0) = \zeta^\pi(0) = 0, \quad \dots(2.9)$$

transforms (8) into the linear relation

$$\zeta^\pi(T) = \int_0^T \Sigma_\lambda \pi^\lambda(t) d\zeta^\lambda(t), \quad \dots(2.10)$$

and the change-of-variables formula yields

$$\zeta^\lambda(T) = x_T^\lambda + \frac{1}{2} \langle x^{\lambda c} \rangle_T + \Sigma_{t \leq T} [e^{\Delta x^\lambda(t)} - 1 - \Delta x_t^\lambda], \quad \dots(2.11)$$

the sum on the right converging absolutely for all T , a.s. The processes z^π, x^π, ζ^π define one another uniquely. Using these relations it is shown in [1] that x_T^π may be calculated explicitly as

$$\begin{aligned} x^\pi(\omega, T) &= x_T^\pi \\ &= \int \Sigma_\lambda \pi^\lambda dM^{\lambda c} + \int \Sigma_\lambda \pi^\lambda dM^{\lambda d} \\ &+ \int \Sigma_\lambda \pi^\lambda dV^{\lambda c} + \int \Sigma_\lambda \pi^\lambda dV^{\lambda d} \\ &+ \frac{1}{2} \int \Sigma_\lambda \pi^\lambda d\langle M^{\lambda c} \rangle - \frac{1}{2} \int \Sigma_\lambda \Sigma_\ell \pi^\lambda \pi^\ell d\langle M^{\lambda c}, M^{\ell c} \rangle \\ &+ \Sigma_{t \leq T} [\Delta x^\pi - \Sigma_\lambda \pi^\lambda \Delta x^\lambda], \end{aligned} \quad \dots(2.12)$$

the sum over t in the last term converging absolutely for all T a.s; here $\int = \int_0^T$, all variables and angle brackets on the right of the equation should have the subscript t , and

$$\Delta x_t^\pi = \ln \left[\Sigma_\lambda \pi_t^\lambda e^{\Delta x^\lambda(t)} \right]. \quad \dots(2.13)$$

The investor has an *initial capital* K_0 and no 'outside' income. Given a $\pi \in \Pi$ and x^π , we say that an (m -a.e. defined) progressive process \bar{c} is a π -feasible consumption plan in natural units, or simply a \bar{c} -plan financed by π , if it is non-negative and a.s.

\bar{c} -integrable on compacts of \mathcal{T} and if the semimartingale \bar{k} solving (strongly) the equation

$$\bar{k}(T) - K_0 = \int_0^T \bar{k}(t-) e^{-x^\pi(t-)} d e^{x^\pi(t)} - \int_0^T \bar{c}(t) dt \quad \dots(2.14)$$

is a.s. non-negative on \mathcal{T} ; then \bar{k} is called the *capital plan in natural units* corresponding to \bar{c} . As shown in [1], (14) has a unique semimartingale solution defined on the whole of \mathcal{T} a.s., which is given explicitly by

$$\bar{k}(T) = e^{x^\pi(T)} \left[K_0 - \int_0^T \bar{c}(t) e^{-x^\pi(t-)} dt \right]. \quad \dots(2.15)$$

The set of all \bar{c} -plans which are π -feasible is denoted $\bar{\mathcal{C}}^\pi$ and the set of all \bar{c} -plans which

are π -feasible for some $\pi \in \Pi$ is $\bar{\mathcal{C}}$. A (feasible) *portfolio-cum-consumption plan* – or simply a *PS plan* – in natural units is a pair (\bar{c}, π) such that $\bar{c} \in \bar{\mathcal{C}}^\pi$ and $\pi \in \Pi$.

The investor's aim is to maximise a *welfare functional* of the form

$$\bar{\varphi}(\bar{c}) = E \int_0^\infty \bar{u}[\bar{c}(\omega, t); \omega, t] q(\omega, t) dt = \int_{\mathcal{C}} \bar{u}[\bar{c}(s); s] d\mu(s). \quad \dots(2.16)$$

The *utility function* $\bar{u} = \bar{u}(C; \omega, t)$ is defined for $0 \leq C \leq \infty$, $\omega \in \Omega$, $t \in \mathcal{T}$, and takes values in $[-\infty, \infty]$. Considered as a function of all its variables, it is $\mathcal{B}_{[0, \infty]} \times \mathcal{H}$ measurable. For fixed (ω, t) , \bar{u} is continuous, concave and increasing in C (the continuity at $C=0$ and $C=\infty$ being one-sided). As usual, \bar{u} may be selected from a family of functions differing only as to scale and origin; thus we may assume $\bar{u} \leq 0$ if utility is bounded above, $\bar{u} \geq 0$ if utility is bounded below.⁴

It follows from a Measurability Lemma for processes, [6] p.503, that for $\bar{c} \in \bar{\mathcal{C}}$ the *utility plan* defined (m-a.e.) by

$$\bar{U}(\cdot) = \bar{U}_{\bar{c}}(\cdot) = \bar{u}[\bar{c}(\cdot, \cdot); \cdot, \cdot] \quad \dots(2.17)$$

is \mathcal{H} -measurable. The domain of the functional $\bar{\varphi}$ is taken to be $\bar{\mathcal{C}}$; it is always assumed (or inferred from other assumptions) that for each \bar{c} in this set the positive part of the double integral in (16) is finite, and further that the supremum φ^* of the functional is finite. The *PS problem* is to maximise $\bar{\varphi}$ on $\bar{\mathcal{C}}$ (if possible). A PS plan (\bar{c}^*, π^*) is called *optimal* if $\bar{\varphi}(\bar{c}^*) = \varphi^*$ (and φ^* is finite).

Let us for the moment fix π , write $x = x^\pi$, $z = z^\pi$, and consider the problem of

⁴ The definition (16) differs from that in [1] because of the appearance of q in the integrand, but this is not a point of substance since the distinction between \bar{u} and q is largely arbitrary; the reason for introducing q is that we want to integrate with respect to a unitary measure. Note also that the marginal utility function $\bar{u}'(C; \omega, t)$ is not needed here, and that no special conditions are imposed on the limits of $\bar{u}(C)/C$ as $C \rightarrow 0$ and $C \rightarrow \infty$. It would be possible for present purposes to replace 'progressive' by 'predictable' in the definition of a \bar{c} -plan, but this would be inconvenient for the theory of necessary and sufficient conditions for optimality, where one wants the process $\bar{u}'(\bar{c})q e^{x^\pi}$ defined by an optimal plan to be a right continuous local martingale – see [1].

maximising $\bar{\varphi}$ on $\bar{\mathcal{E}}^\pi$. Given an element \bar{c} and corresponding \bar{k} , we introduce new processes c, k by the definition

$$c(\omega, t) = \bar{c}(\omega, t)e^{-x^\pi(\omega, t)}, \quad k(\omega, t) = \bar{k}(\omega, t)e^{-x^\pi(\omega, t)}. \quad \dots(2.18)$$

The solution (15) of (14) then reduces to

$$k(T) = K_0 - \int_0^T c(t)dt, \quad \dots(2.19)$$

and the requirement that $\bar{k}(T) \geq 0$ on \mathcal{F} a.s. is equivalent to

$$\int_0^T c(\omega, t)dt \leq K_0 \quad \text{a.s.} \quad \dots(2.20)$$

We call c and k the *consumption and capital plans in π -standardised units* — or simply the c and k plans — corresponding to \bar{c} and \bar{k} . It is clear that a c -plan can be defined directly as a progressive process $c = c(\omega, t) \geq 0$ which satisfies (20); this definition does not involve k , which can be defined by (19) if required. We denote by \mathcal{E} the set of all c -plans; an advantage of working with \mathcal{E} as the feasible set is that it does not depend on the choice of π . Given any π , each $c \in \mathcal{E}$ defines an element $\bar{c} = ce^{x^\pi} \in \bar{\mathcal{E}}^\pi$, and every $\bar{c} \in \bar{\mathcal{E}}^\pi$ can be obtained in this way from some c and π . Thus a *PS plan* can be specified either as a pair (\bar{c}, π) or as a pair (c, π) . In the latter case, the set of all plans is simply $\mathcal{E} \times \Pi$, and it is sometimes convenient to write the functional (16) in the form

$$\varphi(c, \pi) = E \int_0^T \bar{u}[c(t)e^{x^\pi(t)}; t]q_t dt = \int_{\mathcal{E}} \bar{u}[c(s)e^{x^\pi(s)}; s]d\mu(s). \quad \dots(2.21)$$

The problem of maximising $\bar{\varphi}$ on $\bar{\mathcal{E}}^\pi$ is clearly equivalent to that of maximising $\varphi(\cdot, \pi)$ on \mathcal{E} , which in turn is essentially the same as the problem of optimal saving with a single asset studied in [2]. We say that c^* is π -optimal, or equivalently that $\bar{c}^* = c^*e^{x^\pi}$ is π -optimal, if $\varphi(\cdot, \pi)$ attains its supremum on \mathcal{E} at c^* and $\varphi(c^*, \pi)$ is finite.

The existence theory of Sections 3–4 below can be simplified by a preliminary transformation of the probability measure. It is known — see [9] VII.58, 63 bis & 98c, also [13] 7.3 — that there exists a probability Q equivalent to P , with dQ/dP bounded,

such that, under Q , X has a decomposition $M + V$ such that, for each λ , V^λ is predictable, each M^λ is a martingale and $E(M_T^\lambda)^2$, $E(|V^\lambda|_T)$ are finite for each $T \in \mathcal{T}$; (one can also obtain $E(|V^\lambda|_T)^2 < \infty$, but this is inessential here). In general Q is not unique, so we make a convenient choice once and for all. Replacing P by Q and q_t by $q_t E_Q^t(dP/dQ)$ leaves unchanged the measure μ defined as in (2.1), and since the set of feasible plans is unaffected the values of the functional $\bar{\varphi}$ or φ are also unchanged. *Until further notice we shall assume, without changing the notation, that these replacements have been made.* When working with the transformed P , we consider only the canonical decomposition of X . Since in this case M and V as well as M^c are uniquely defined, the same is true of $M^d = M - M^c$, V^c and V^d ; moreover (3) holds.

Now recall that a non-decreasing process ξ which is integrable on compacts defines a σ -finite 'Doléans' measure \mathfrak{d}_ξ on $(\mathcal{S}, \mathcal{H})$ by setting

$$\mathfrak{d}_\xi(\eta) = E \int_0^\infty \eta_t d\xi_t \quad \dots(2.22)$$

for non-negative processes η and then extending to ξ -integrable processes. If η_1 and η_2 are processes for which \mathfrak{d}_ξ is defined, the condition $\mathfrak{d}_\xi((\delta\eta)^2) = 0$, where $\delta\eta = \eta_1 - \eta_2$, is equivalent to $\eta_1 \sim \eta_2 (\mathfrak{d}_\xi)$, and then we call the processes *similar for \mathfrak{d}_ξ* , or simply for ξ . We shall be mainly concerned with the case where the integrand η is predictable and bounded. It is a standard result that ξ has a compensator, defined as the non-decreasing predictable process ξ^P such that $\xi - \xi^P$ is a martingale, and for non-negative or ξ -integrable, predictable η we have $\mathfrak{d}_{\xi^P}(\eta) = \mathfrak{d}_\xi(\eta)$. Vector processes η_1 and η_2 are here considered to be similar if $\|\delta\eta\| \sim 0$, where $\|\delta\eta\|$ denotes the process $(\|\delta\eta\|_t : t \in \mathcal{T})$; thus $\eta \sim 0$ means $\|\eta\| \sim 0$. A measure \mathfrak{d}_ξ can also be defined for locally integrable, non-decreasing ξ ; in this case, $\xi - \xi^P$ is only a local martingale, see [7] 1(d-e), [9] VI.79-80.

Examples of such measures already encountered are $m = \mathfrak{d}_t$ and $\mu = \mathfrak{d}_{\int Q}$. Under our transformed P the processes $[X]$, $|V^c|$, $|V|$, $[M]$ and $\langle M \rangle$ are integrable on compacts and we may consider the three measures

$$\mathbb{P}[V^c] + [X] = \mathbb{P}[V^c] + \langle M^c \rangle + [X^d], \quad \mathbb{P}[M] + |V|, \quad \mathbb{P}\langle M \rangle + |V|. \quad \dots(2.23)$$

The first and middle terms are clearly equivalent, taking into account (3); also, the middle and last coincide on \mathcal{P} since $|V|$ is predictable and $\langle M \rangle = [M]^P$.

It follows from the definitions of the stochastic and Stieltjes integrals that portfolio plans π_1 and π_2 which are similar for all of $|V^c|$, $\langle M^c \rangle$ and $[X^d]$ define indistinguishable log-returns x_1 and x_2 (see Section 5 below for details). It might therefore seem natural to identify portfolios which are similar for one of the measures (23), say for

$\mathbb{P}\langle M \rangle + |V|$. However, such portfolios can still be different in an everyday sense, because sets which are null for $\langle M \rangle + |V|$ — e.g. intervals during which the market is closed — can have positive measure for μ , i.e. in ‘real time’. We therefore prefer to identify portfolio plans which are similar for $\langle M \rangle + |V|$ and for μ . Also, it is more convenient for the existence theory to work with equivalent measures which are bounded on \mathcal{P} . For this purpose we note that there is a predictable process f taking values in $(0,1]$ such that

$$\int_0^\infty f_t \cdot d(\langle M \rangle_t + |V|_t) \leq 1 \quad \text{a.s.}^5 \quad \dots(2.24)$$

Thus we may define bounded measures \tilde{n}, n on $(\mathcal{S}, \mathcal{P})$ by

$$\tilde{n} = \mathbb{P} \int f d\langle M \rangle + \int f d|V|, \quad n = \frac{1}{2}(\tilde{n} + \mu). \quad \dots(2.25)$$

Explicitly, we have

$$n(\eta) = \int_{\mathcal{S}} \eta \cdot dn = \frac{1}{2} E \int_0^\infty \eta_t f_t \cdot d(\langle M \rangle_t + |V|_t) + \frac{1}{2} E \int_0^\infty \eta_t q_t \cdot dt \quad \dots(2.26)$$

for non-negative or bounded predictable η , and $n(\mathcal{S}) = n(I_{\mathcal{S}}) \leq 1$. We denote by $\mathcal{L}_2 = L_2(\mathcal{S}, \mathcal{P}, n; \mathbb{R}^\Lambda)$ the space of (classes of n -similar) predictable \mathbb{R}^Λ -valued processes Y satisfying

$$\int_{\mathcal{S}} \|Y\|^2 dn < \infty, \quad \|Y_t\|^2 = \Sigma_\lambda (Y_t^\lambda)^2, \quad \dots(2.27)$$

and consider Π as a subset of \mathcal{L}_2 .

⁵ It is enough to consider the predictable times $\tau_i = \inf\{t: \langle M \rangle_t + |V|_t \geq i\}$, $i = 1, 2, \dots$, with $\tau_i = \infty$ permitted, and to set $f = \sum_i 2^{-i} I_{[\tau_{i-1}, \tau_i]}$, see [9] VI.86–87.

3. PROPERTIES OF FEASIBLE SETS

In this Section we establish some properties of the sets \mathcal{C} , Π and $\bar{\mathcal{C}}$ which are needed for the existence and uniqueness proofs.

The set \mathcal{C} . By definition, a process $c = c(s)$ belongs to \mathcal{C} iff it is progressive, non-negative and satisfies the inequality (2.20) a.s. Obviously \mathcal{C} is non-empty and convex. If $c \in \mathcal{C}$ and c' is a progressive process such that $0 \leq c' \leq c$ a.e. (for m or μ) then $c' \in \mathcal{C}$. If (c_n) is a sequence from \mathcal{C} and

$$c_*(s) = \liminf_n c_n(s) \quad \text{a.e.}(m), \quad \dots(3.1)$$

then c_* is progressive non-negative and Fatou's Lemma implies

$$\int_0^\infty c_*(\omega, t) dt \leq \liminf_n \int_0^\infty c_n(\omega, t) dt \leq K_0, \quad \dots(3.2)$$

so that $c_* \in \mathcal{C}$; we call this the *lower closure property* for \mathcal{C} .

If there is only one asset, or a fixed portfolio plan π with $z = e^{x^\pi}$ is given, we may write $c = \bar{c}/z$ and

$$u[c] = u[c(s); s] = \bar{u}[\bar{c}(s); s], \quad \varphi(c) = \int u[c(s); s] d\mu(s) = \bar{\varphi}(\bar{c}), \quad \dots(3.3)$$

cf.(2.16), (2.18) and (2.21). Conditions for the existence of a π -optimal element of \mathcal{C} can then be obtained as in [2] Section 3, with minor adjustments for changes of definitions and notation; (in particular, c, g, \mathcal{C} in [2] correspond to \bar{c}, c, \mathcal{C} here). Alternatively, one can proceed directly to Theorem 1 below, which in the case of a single asset is just the Existence Lemma in [2] with a slightly modified proof. Either way, the argument rests on the fact that, if (c_n) is a maximising sequence from \mathcal{C} with $U_n = u(c_n)$ converging weakly in \mathcal{L}_1 , one can construct a new sequence (\tilde{c}_m) from \mathcal{C} with $c_* = \liminf_m \tilde{c}_m$ such that

$$\varphi(c_*) \geq \lim_n \int u[c_n] d\mu = \varphi^*, \quad \dots(3.4)$$

and since c_* is in \mathcal{C} by lower closure it is optimal. This suggests a similar approach to

the existence problem when π is variable, with $\bar{c}, \bar{u}, \bar{\varphi}$ in place of c, u, φ ; however the properties of $\bar{\mathcal{E}}$ corresponding to those of \mathcal{E} are harder to establish.

The set Π . Obviously this set is non-empty and convex. Let π_1, π_2 be elements of Π generating x_1, x_2 , and let $\tilde{\pi}_\alpha$ be defined n-a.e. by

$$\tilde{\pi}_{\alpha t} = \alpha \pi_{1t} + (1-\alpha) \pi_{2t}, \quad 0 \leq \alpha \leq 1; \quad \dots(3.5)$$

then $\tilde{\pi}_\alpha$ is in Π and generates some x_α . Referring to (2.12–13), it is seen that for a pair $S < T$ from \mathcal{T} , or more generally for a pair of finite stopping times such that $S < T$ a.s., the function assigning to each α the random variable $x_{\alpha T} - x_{\alpha S}$ is a.s. finite and concave in α . More precisely, the first five terms on the right of (2.12) are linear in α ; the term $-\frac{1}{2} \int_{(S,T]} \Sigma_\lambda \Sigma_\ell \tilde{\pi}_\alpha^\lambda \tilde{\pi}_\alpha^\ell d\langle M^{\lambda c}, M^{\ell c} \rangle$ is concave because $\langle \mathfrak{M}^c \rangle_T - \langle \mathfrak{M}^c \rangle_S$ is non-negative definite; and the last term is also concave by (2.13) because the log function is concave. It follows that, a.s.,

$$x_{\alpha T} - x_{\alpha S} \geq \alpha(x_{1T} - x_{1S}) + (1-\alpha)(x_{2T} - x_{2S}). \quad \dots(3.6)$$

Let (π_n) be an arbitrary sequence from Π converging n-a.e. to some π_* ; clearly $\pi_* \in \Pi$. Let x_n be generated by π_n , x_* by π_* , and consider the convergence properties of (x_n) . Changing the π_n on an n-null set of \mathcal{S} if necessary – which does not alter the x_n – it may be assumed that $\pi_n \rightarrow \pi_*$ for all (ω, t) . It then follows from the dominated convergence theorem (d.c.t.) for stochastic integrals – see [7] 2.72–4 or [13] 2.3.4 – and (2.12–13) above that $x_n \rightarrow x_*$ in probability, uniformly on compacts of \mathcal{T} , i.e. for each $T \in \mathcal{T}$,

$$\sup_{t \leq T} |x_n(t) - x_*(t)| \rightarrow 0 \quad \text{in probability as } n \rightarrow \infty, \quad \dots(3.7)$$

and selecting subsequences if necessary (henceforth s.s.i.n.) we may assume that

$$x_n(t) \rightarrow x_*(t) \quad \text{a.s.,}$$

first for $t \leq T$ and then, letting $T = 1, 2, \dots$, on the whole of \mathcal{T} . So, s.s.i.n.,

$$x_n(s) \rightarrow x_*(s) \quad \text{a.e.(m).}$$

To justify the application of the d.c.t. in detail, consider separately the convergence of the terms on the right of (2.12), setting $\pi = \pi_n$. In the first two terms we may consider each λ separately and take as 'dominated integrands' the bounded predictable functions π_n^λ ; the required convergence of the martingale terms $\int \pi_n^\lambda dM^\lambda$ then follows (for instance) from [7] 2.73. The next four terms contain only Stieltjes integrals of the bounded integrands π_n^λ and $\pi_n^\lambda \cdot \pi_n^\ell$ with respect to integrators of finite variation, and the passage to the limit follows from [7] 2.72, i.e. essentially from the ordinary d.c.t. The last term is non-negative because $\sum \pi^\lambda e^{\Delta x^\lambda} \geq e^{\sum \pi^\lambda \Delta x^\lambda}$ (convexity inequality); also, using $\ln z \leq z-1$, $0 \leq \pi^\lambda \leq 1$ and $\sum \pi^\lambda = 1$, it is bounded above, uniformly with respect to π_n , by the non-negative sum $\sum_{t \leq T} \sum_\lambda [e^{\Delta x^\lambda(t)} - 1 - \Delta x_t^\lambda]$, which converges absolutely on \mathcal{T} a.s. according to (2.11), and the passage to the limit again follows from [7] 2.72. This completes the proof of (7).

Now consider Π as a subset of \mathcal{L}_2 .⁶ Since $n(\mathcal{S}) \leq 1$, the norms $(\int \|\pi\|^2 dn)^{\frac{1}{2}}$ are uniformly bounded by 1 for all $\pi \in \Pi$. Consequently Π is weakly sequentially compact in the reflexive space \mathcal{L}_2 , i.e. every sequence (π_n) from Π contains a subsequence converging weakly to a (predictable) process $\pi_* \in \mathcal{L}_2$, [4] II.3.28; here weak convergence means $\int \sum_\lambda (\pi_n^\lambda - \pi_*^\lambda) f^\lambda dn \rightarrow 0$ for every $f \in \mathcal{L}_2$. Denoting the subsequence again by (π_n) , there is a sequence $(\tilde{\pi}_m)$ of convex combinations,

$$\tilde{\pi}_m = \sum_{j=1}^{J_m} \beta_{jm} \pi_{m+j}, \quad \sum_{j=1}^{J_m} \beta_{jm} = 1, \quad m = m_1, m_2, \dots, \quad \dots(3.8)$$

⁶ In the argument which follows, the space $\mathcal{L}_2 = L_2(\mathcal{S}, \mathcal{P}, n; \mathbb{R}^\Lambda)$ could be replaced by the corresponding L_1 space, with the norm $\int \sum_\lambda |\pi_t^\lambda| dn$. Conditions for weak sequential compactness in L_1 spaces are recalled in Section 4 below.

converging to π_* in the norm of \mathcal{L}_2 .⁷ The norm-convergent sequence $(\tilde{\pi}_m)$ also converges in n -measure, [4] III 3.6–7, and s.s.i.n. converges n -a.e. to the same limit.

Clearly $\pi_* \in \Pi$, and if $\tilde{\pi}_m, \pi_*$ generate \tilde{x}_m, x_* we may assume s.s.i.n. that

$$\tilde{\pi}_m \rightarrow \pi_* \text{ a.e.}(n), \quad \tilde{x}_m \rightarrow x_* \quad t \in \mathcal{T} \text{ a.s.} \quad \dots(3.9)$$

It further follows from the discussion leading to (6) above that

$$\tilde{x}_m(t) \geq \sum_{j=1}^{J_m} \beta_{jm} x_{m+j}(t) \quad t \in \mathcal{T} \text{ a.s.}, \quad \dots(3.10)$$

hence, by the convexity of the exponential function,

$$e^{-\tilde{x}_m(t)} \leq \sum_{j=1}^{J_m} \beta_{jm} e^{-x_{m+j}(t)} \quad t \in \mathcal{T} \text{ a.s.} \quad \dots(3.11)$$

The set $\bar{\mathcal{E}}$. By definition, a process $\bar{c} = \bar{c}(s)$ belongs to $\bar{\mathcal{E}}$ iff it is progressive non-negative and there is a $\pi \in \Pi$ such that $c = \bar{c}e^{-x^\pi}$ satisfies (2.20) a.s. If $\bar{c} \in \bar{\mathcal{E}}$ and \bar{c}' is a progressive process such that $0 \leq \bar{c}' \leq \bar{c}$ a.e., then $\bar{c}' \in \bar{\mathcal{E}}$; moreover, if π finances \bar{c} , then π also finances \bar{c}' . (Proof: it is always possible to consume less!). As regards convexity, let \bar{c}_j be elements of $\bar{\mathcal{E}}$ financed by $\pi_j, j = 1, 2$, let x_j, z_j be generated by π_j , and let \bar{k}_j correspond to (\bar{c}_j, π_j) . A process of the form

$$\bar{c}_\alpha = \alpha \bar{c}_1 + (1-\alpha) \bar{c}_2, \quad 0 \leq \alpha \leq 1, \quad \dots(3.12)$$

is obviously progressive non-negative, and it is intuitively clear that \bar{c}_α can be financed as follows: divide the capital K_0 into two funds in proportions $\alpha_1 = \alpha, \alpha_2 = 1-\alpha$, invest $\alpha_j K_0$ in π_j and use the return to finance \bar{c}_j . The only point requiring proof is

⁷ More precisely, there is for each integer n an integer m_n , a set of integers $j = 1, \dots, J_n$

and a set of non-negative numbers $(\beta_{1n}, \dots, \beta_{J_n n})$ satisfying $\sum_{j=1}^{J_n} \beta_{jn} = 1$ such that

$m_{n+1} > m_n + J_n$ and $\pi_* - \sum_{j=1}^{J_n} \beta_{jn} \pi_{m_n+j} \rightarrow 0$ in norm as $n \rightarrow \infty$, see [4] V.3.14, also [5] pp. 54&91, for details. Since $m_n \rightarrow \infty$, we may replace β_{jn} by β_{jm} , π_{m_n+j} by π_{m+j} , J_n by J_m , where $m = m_n$.

that such a scheme corresponds to some portfolio plan as the term is defined here. It may be checked that an appropriate $\pi = \pi_\alpha$ is obtained by setting, for each λ and each s ,

$$\pi_\alpha^\lambda = [\alpha_1 \pi_1^\lambda \bar{k}_{1-} + \alpha_2 \pi_2^\lambda \bar{k}_{2-}] / [\alpha_1 \bar{k}_{1-} + \alpha_2 \bar{k}_{2-}], \quad \dots(3.13)$$

where $\bar{k}_{i-} = \bar{k}_i(t-)$, whenever the denominator is positive, with an arbitrary choice otherwise; (this last qualification was omitted in [1] (4.4–5)). Note that in general the process $\tilde{\pi}_\alpha$ defined in (5) does not finance \bar{c}_α .

We next consider lower closure. Let (\bar{c}_n, π_n) be a sequence of PS plans and let \bar{c}_* be defined by

$$\bar{c}_*(s) = \liminf_n \bar{c}_n(s) = \lim_{n \rightarrow \infty} \inf_{j \geq n} \bar{c}_j(s) \quad \text{a.e.}(m); \quad \dots(3.14)$$

we want to show that $\bar{c}_* \in \bar{\mathcal{C}}$. Now, for each n , the process $\hat{\bar{c}}_n$ defined by

$$\hat{\bar{c}}_n(s) = \inf_{j \geq n} \bar{c}_j(s) \quad \text{a.e.}(m).$$

is progressive non-negative and satisfies $0 \leq \hat{\bar{c}}_n \leq \bar{c}_n$, hence is in $\bar{\mathcal{C}}$ and is financed by π_n . Since $\hat{\bar{c}}_n \uparrow \bar{c}_*$, we may from the outset assume w.l.o.g. that $\bar{c}_n \uparrow \bar{c}_*$.

It remains to show that there is a $\pi_* \in \Pi$ such that (\bar{c}_*, π_*) is a PS plan.

Obviously \bar{c}_* is progressive non-negative, so we need only show that there is a $\pi_* \in \Pi$ generating an x_* such that

$$\int_0^\infty \bar{c}_*(t) e^{-x_*(t)} dt \leq K_0 \quad \text{a.s.} \quad \dots(3.15)$$

— see(2.20). By definition, we have

$$\int_0^\infty \bar{c}_n(t) e^{-x_n(t)} dt \leq K_0 \quad \text{a.s.} \quad \dots(3.16)$$

Referring to the discussion of Π , let $\pi_n \rightarrow \pi_*$ weakly in \mathcal{L}_2 and define $\tilde{\pi}_m$ as in (8) so that (9)–(11) hold, s.s.i.n. The following calculation shows that the process $x_* = \lim \tilde{x}_m$

satisfies (15):

$$\begin{aligned}
& \int_0^\infty \bar{c}_*(t) e^{-x_*(t)} dt \\
= & \int_0^\infty [\lim_m \bar{c}_m(t)] e^{-\lim_m \tilde{x}_m(t)} dt \\
= & \int_0^\infty \lim_m \left[\bar{c}_m(t) e^{-\tilde{x}_m(t)} \right] dt \\
\leq & \lim_m \int_0^\infty \bar{c}_m(t) e^{-\tilde{x}_m(t)} dt \\
\leq & \lim_m \sum_{j=1}^{J_m} \beta_{jm} \int_0^\infty \bar{c}_m(t) e^{-x_{m+j}(t)} dt \\
\leq & \lim_m \sum_{j=1}^{J_m} \beta_{jm} \int_0^\infty \bar{c}_{m+j}(t) e^{-x_{m+j}(t)} dt \\
\leq & K_0. \tag{3.17}
\end{aligned}$$

The first equation follows from the definitions of the limits \bar{c}_* , π_* and the fact that (\bar{c}_m) is a subsequence of (\bar{c}_n) . The second equation follows because the semimartingales \tilde{x}_m and \tilde{x}_* are finite. The first inequality results from Fatou's Lemma since all variables are non-negative and then the next inequality follows from (11). The third inequality follows because $(\bar{c}_n) \uparrow$ with n , hence $(\bar{c}_m) \uparrow$ with m . The last inequality then follows from (16) and the second equation in (8).||

4. EXISTENCE THEOREMS

THEOREM 1. Let φ^* be finite, and suppose that there is a maximising sequence (\bar{c}_n) from $\bar{\mathcal{C}}$ such that the corresponding sequence (\bar{U}_n) defined by

$$\bar{U}_n(s) = \bar{u}[\bar{c}_n(s), s] \tag{4.1}$$

is weakly precompact in \mathcal{L}_1 . Then there is a PS plan (\bar{c}_*, π_*) such that $\bar{\varphi}(\bar{c}_*) = \varphi^*$.

PROOF. To say that (\bar{U}_n) is weakly precompact in \mathcal{L}_1 is to say that there is a subsequence converging weakly to some \bar{U}_* in \mathcal{L}_1 . Denoting the subsequence also by

(\bar{U}_n) , weak convergence to \bar{U}_* means $\int (\bar{U}_n - \bar{U}_*) d\mu \rightarrow 0$ for every $f \in \mathcal{L}_\omega$; an equivalent condition is that the norms $\int |\bar{U}_n| d\mu$ are uniformly bounded and that $\int_H \bar{U}_n d\mu \rightarrow \int_H \bar{U}_* d\mu$ for every $H \in \mathcal{H}$, see [4] IV.8.7. If \bar{U}_n converges weakly to \bar{U}_* , there exists a sequence of convex combinations

$$\tilde{U}_m = \sum_{j=1}^{J_m} \alpha_{jm} \bar{U}_{m+j}, \quad \sum_{j=1}^{J_m} \alpha_{jm} = 1, \quad m = m_1, m_2, \dots, \quad \dots(4.2)$$

converging to \bar{U}_* in norm and s.s.i.n. for μ -a.e. s . We have

$$\int \bar{U}_*(s) d\mu = \lim_m \int \tilde{U}_m(s) d\mu = \lim_n \int \bar{U}_n(s) d\mu = \varphi^* \quad \dots(4.3)$$

by the definitions of weak and norm convergence and the fact that (\bar{U}_n) is maximising.

Using the non-negative constants α_{jm} we define a sequence (\tilde{c}_m) by

$$\tilde{c}_m = \sum_{j=1}^{J_m} \alpha_{jm} \bar{c}_{m+j}; \quad \dots(4.4)$$

then $\tilde{c}_m \in \bar{\mathcal{C}}$ by convexity, and the concavity of \bar{u} implies (in abridged notation)

$$\bar{u}[\tilde{c}_m] = \bar{u}\left[\sum_{j=1}^{J_m} \alpha_{jm} \bar{c}_{m+j}\right] \geq \sum_{j=1}^{J_m} \alpha_{jm} \bar{u}[\bar{c}_{m+j}] = \tilde{U}_m. \quad \dots(4.5)$$

Define

$$\bar{c}_* = \lim_m \tilde{c}_m; \quad \dots(4.6)$$

as shown above, we have $\bar{c}_* \in \bar{\mathcal{C}}$, and using first the monotonicity and continuity of \bar{u} ,

then (5), then $\tilde{U}_m \rightarrow \bar{U}_*$ we obtain

$$\bar{u}(\bar{c}_*) = \lim_m \bar{u}(\tilde{c}_m) \geq \lim_m \tilde{U}_m = \bar{U}_* \text{ a.e.} \quad \dots(4.7)$$

Integrating and taking account of (3) yields

$$\int \bar{u}(\bar{c}_*) d\mu \geq \int \bar{U}_* d\mu = \varphi^*. \quad \dots(4.8)$$

Since φ^* is the supremum of the utility integral on $\bar{\mathcal{C}}$ we have equality in (8) and \bar{c}_* is optimal. ||

This result reduces the existence problem to the search for explicit conditions which imply (a) the finite supremum condition and (b) the existence of a weakly precompact maximising sequence. A sufficient condition for (b) is that the set of all utility plans — see (2.17) — is weakly sequentially compact in \mathcal{L}_1 , i.e. that every infinite sequence from the

set is weakly precompact in \mathcal{L}_1 . We recall that, if $\mu(\mathcal{S}) < \infty$, a necessary and sufficient condition for a set \mathcal{F} in \mathcal{L}_1 to be weakly sequentially compact is that the functions $f \in \mathcal{F}$ are uniformly μ -integrable. An equivalent condition is that for $f \in \mathcal{F}$ the norms $\int |f| d\mu$ are uniformly bounded and the indefinite integrals are uniformly continuous, i.e. that for every $\epsilon > 0$ there is a $\delta > 0$ such that, for every $f \in \mathcal{F}$, the conditions $H \in \mathcal{H}$ and $\mu(H) < \delta$ imply $\int_H |f| d\mu < \epsilon$. It is sufficient if there is an $f_0 \in \mathcal{L}_1$ such that, for each $f \in \mathcal{F}$, $|f| \leq |f_0|$ a.e., and a fortiori if all $|f|$ are uniformly bounded (by a constant). It is also sufficient if, for some $\epsilon > 0$,

$$\sup \left[\int |f(s)|^{1+\epsilon} d\mu(s) : f \in \mathcal{F} \right] < \infty. \quad \dots(4.9)$$

For these results, see [9]A II.17–25, [4] Ch.IV, also [2] S.3.

Before turning to examples, we state separately an existence theorem for the case of negative utility.

THEOREM 2. If $\bar{u} \leq 0$ and there is an element $\bar{c}_+ \in \bar{\mathcal{C}}$ such that

$$\bar{\varphi}(h\bar{c}_+) > -\infty \quad \text{for each } h \in (0,1], \quad \dots(4.10)$$

then there is a PS plan (\bar{c}^*, π^*) such that $\bar{\varphi}(\bar{c}^*) = \varphi^*$.

PROOF. This is essentially that for the one-asset model given in [2] (3.10–18) and will not be repeated in detail. Aside from the change of notation mentioned earlier, it is only necessary to replace standardised units by natural units; (thus assertions about $g, u, U, \varphi, \bar{c}_\alpha, \mathcal{G}$, in [2] are to be replaced here by corresponding assertions about $\bar{c}, \bar{u}, \bar{U}, \bar{\varphi}, \bar{\tilde{c}}_\alpha, \bar{\mathcal{C}}$, with minor changes of wording). The argument uses the concavity of $\bar{\mathcal{C}}$, which was discussed at (3.12–13) above. Note that the hypothesis of Theorem 2 implies that φ^* is finite. The proof proceeds by assuming that there is no maximising sequence (\bar{c}_n) from $\bar{\mathcal{C}}$ such that the corresponding sequence (\bar{U}_n) is weakly precompact in \mathcal{L}_1 and deriving a contradiction; thus Theorem 2 appears as a corollary of Theorem 1.⁸

⁸ I am grateful to Costis Skiadas for pointing out an error in an earlier attempt at a simplified, direct proof of Theorem 2.

It remains to give some examples of explicit criteria which imply that the hypotheses of Theorems 1 or 2 are satisfied. When considering applications of these criteria, it should be borne in mind that Theorems 1 and 2 involve the measure μ but not P and q separately. Since μ is unaltered by the transformation of P and q introduced in Section 2, the criteria may be applied using either the original P and q or the transforms; the former choice is usually the natural one. Also, for given P , the split of the integrand in $\bar{\varphi}$ into the factors \bar{u} and q is largely arbitrary and may be chosen for convenience.

Bounded Utility. It follows from either Theorem 1 or Theorem 2 that *it is sufficient for the existence of an optimal plan if \bar{u} is a bounded function.*

Negative Utility. We apply Theorem 2. Suppose that there is a $b > 1$ such that

$$0 \geq \bar{u}(C, s) \geq (1-b)^{-1} C^{1-b} \quad \text{for all } (C, s); \quad \dots(4.11)$$

then, if (\bar{c}, π) or (c, π) with $c = \bar{c}e^{-x^\pi}$ is a PS plan, we have

$$\varphi(c, \pi) \geq (1-b)^{-1} E \int_0^\infty c(t)^{1-b} e^{(1-b)x^\pi(t)} q(t) dt. \quad \dots(4.12)$$

If, for given $\pi \in \Pi$, the integral

$$\mathfrak{N}^\pi = \int_0^\infty \left[E \{ e^{(1-b)x^\pi(t)} q(t) \} \right]^{1/b} dt \quad \dots(4.13)$$

converges, a short calculation shows that the non-random function

$$c(t) = c(0) \left[E \{ e^{(1-b)x^\pi(t)} q(t) \} \right]^{1/b}, \quad c(0) = K_0 / \mathfrak{N}^\pi, \quad \dots(4.14)$$

is in \mathcal{C} and that $\varphi(hc, \pi) = \bar{\varphi}(h\bar{c}) > -\infty$ for $h \in (0, 1]$, cf. [11] S.4. In particular, if $q(\omega, t) = \rho e^{-\rho t}$ with some $\rho > 0$, it is sufficient for existence if there is one asset λ for which

$$\int_0^\infty \left[E e^{(1-b)x^\lambda(t)} \right]^{1/b} e^{-(\rho/b)t} dt < \infty. \quad \dots(4.15)$$

Note that $Ee^{(1-b)x^\pi(t)}$ is for each t a value of the bilateral Laplace Transform of a random variable; thus more detailed conditions can be obtained in special cases, notably if X has independent increments, see [12].

Positive Utility. Suppose now that $\bar{u} \geq 0$; here we apply Theorem 1, modifying the argument in [2] 3.21–26 to allow for portfolios. According to (9) above, every feasible utility sequence is weakly precompact in \mathcal{L}_1 if there is an $\epsilon > 0$ such that

$$\sup \left[\int [\bar{u}(\bar{c}(s), s)]^{1+\epsilon} d\mu(s) : \bar{c} \in \bar{\mathcal{C}} \right] < \infty, \quad \dots(4.16)$$

and this condition also ensures that $\varphi^* < \infty$. In particular, the condition is satisfied if there is a $b \in (0, 1)$ such that

$$0 \leq \bar{u}(C; s) \leq (1-b)^{-1} C^{1-b} \quad \text{for all } (C, s), \quad \dots(4.17)$$

and further a $\beta \in (0, b)$ such that

$$\sup \left[E \int_0^\infty \bar{c}(t)^{(1-\beta)} q(t) dt : \bar{c} \in \bar{\mathcal{C}} \right] < \infty. \quad \dots(4.18)$$

For a more explicit condition which is sufficient for (18) to hold, use $\bar{c} = ce^{x^\pi}$, then

Hölder's inequality, then $\int_0^\infty c_t dt \leq K_0$ with $K_0 = 1$ to obtain

$$\begin{aligned} E \int_0^\infty \bar{c}_t^{1-\beta} q_t dt &= E \int_0^\infty \{c_t e^{x^\pi(t)}\}^{1-\beta} q_t dt \\ &\leq \left[E \int_0^\infty c_t dt \right]^{1-\beta} \cdot \left[E \int_0^\infty \{e^{(1-\beta)x^\pi(t)} q_t\}^{1/\beta} dt \right]^\beta \\ &\leq \left[E \int_0^\infty \{e^{(1-\beta)x^\pi(t)} q_t\}^{1/\beta} dt \right]^\beta. \end{aligned} \quad \dots(4.19)$$

Thus it is sufficient for existence if the supremum of the last expression taken over all $\pi \in \Pi$ is finite for some $\beta \in (0, b)$, and a fortiori if

$$\int_0^\infty \sup_\pi E \left[\{e^{(1-\beta)x^\pi(t)} q_t\}^{1/\beta} \right] dt < \infty. \quad \dots(4.20)$$

Once again, more precise conditions can be given in special cases, see [12].

Logarithmic Utility. Conditions for positive and negative utilities can be combined to deal with cases where utility is unbounded above and below in the same way as in the one-asset case — see [2] S.3 — and we shall not go over the ground again. It is however worthwhile to deal separately with the case of logarithmic utility. Let

$$\bar{u}(C; \varsigma) = \ell n C \quad \text{for all } (C, \varsigma), \quad \dots(4.21)$$

so that

$$\varphi(c, \pi) = E \int_0^\infty (\ell n c_t + x_t^\pi) q_t dt, \quad \dots(4.22)$$

and assume for simplicity that q is non-random. The functional (22) has a finite supremum iff

$$E \int_0^\infty (\ell n c_t) q_t dt \quad \dots(4.23)$$

has a finite supremum on \mathcal{C} and

$$E \int_0^\infty (x_t^\pi) q_t dt = \int x^\pi(\varsigma) d\mu(\varsigma) \quad \dots(4.24)$$

has a finite supremum on Π . The search for a maximum of (23) on \mathcal{C} may be confined to non-random functions, and it may be shown that this problem has a solution iff

$$\int_0^\infty |\ell n q_t| q_t dt < \infty, \quad \dots(4.25)$$

see [11] S.6. The processes $(x^\pi: \pi \in \Pi)$ are uniformly μ -integrable, and an element of Π which maximises (24) exists, if there is some $\beta > 1$ such that

$$\sup \left[E \int_0^\infty |x_t^\pi|^\beta q_t dt : \pi \in \Pi \right] < \infty; \quad \dots(4.26)$$

in fact, $|x_t^\pi|^\beta$ can be replaced in (26) by $(x_t^{\pi+})^\beta + |x_t^{\pi-}|$, where $x^{\pi+}, x^{\pi-}$ denote the positive and negative parts of x^π . We omit further details.

Short Sales, Continuous X. Conditions of optimality were derived in [1] for a version of the PS model where X is sample continuous and short sales are allowed, i.e. (2.6) is omitted from the definition of a (collor) portfolio plan π . The question arises whether the existence proofs given above also extend to a modified version of the model as defined here, with X

continuous and (2.6) omitted from the definition of a (predictable) π . A review of the preceding argument and relevant passages in [1] shows that the set of feasible π must be further restricted if stochastic integrals such as those in (2.8–12) are to be well defined, if an arbitrary sequence (π_n) is to be weakly precompact in \mathcal{L}_2 , and if the dominated convergence argument leading to (3.7) is to be valid. It is sufficient if the choice of π is restricted to a set satisfying a condition of the form $\|\pi(s)\| \leq g(s)$, n-a.e., where g is a predictable, real-valued process satisfying $\int \mathcal{G} g^2 dn < \infty$ (or a condition of the form $\sum_\lambda |\pi^\lambda(s)| \leq g(s)$ with g predictable, locally bounded and satisfying $\int \mathcal{G} g(s) dn < \infty$, see footnote 6 above). Limited short sales can also be permitted in certain cases where X has jumps, for example if the jumps are uniformly bounded in absolute value.

5. UNIQUENESS

Associated with any plan are processes \bar{c} , π , $x=x^\pi$, $z=e^x$, $c=\bar{c}e^{-x}$, $\bar{U}=\bar{u}(\bar{c})$ etc. As mentioned above — see fn. 3 — the uniqueness of the correspondence among suitably defined equivalence classes of these processes is straightforward, except in the case of the relation between π and x^π . In this Section, we shall consider under what conditions, and for what definitions of equivalence among portfolio plans, the element π generating a given returns process x is unique. Thereafter, we shall briefly consider conditions for uniqueness of the processes \bar{c} , π , x , c defining an optimal plan.

Let π_1, π_2 again be elements of Π generating x_1, x_2 and write $\delta\pi = \pi_1 - \pi_2$, $\delta x = x_1 - x_2$ etc.; thus $\Delta(\delta x)_t = \Delta x_{1t} - \Delta x_{2t} = \delta(\Delta x_t)$, while $\langle(\delta x)^c\rangle$ denotes the angle bracket of the continuous martingale part of δx . In the present discussion we do not impose in advance any convention defining equivalence among elements of Π (except as usual that indistinguishable processes are identified). Since some of the conditions occurring below involve the distributions of the variables X_t , it is now preferable to work

with the original (untransformed) versions of P and q . Thus M^c and M^d are now only local martingales, V^c and V^d processes (locally) of finite variation.

We begin by noting some conditions on $\delta\pi$ which are *sufficient* for $\delta x \equiv 0$. It is enough if all three of the following hold, a.s. for all $T \in \mathcal{T}$, for each λ , for some decomposition of X satisfying (2.3):

$$\int_0^T (\delta\pi_t^\lambda)^2 d\langle M^{\lambda c} \rangle_t = 0, \quad \dots(5.1)$$

$$\int_0^T (\delta\pi_t^\lambda)^2 d|V^{\lambda c}|_t = 0, \quad \dots(5.2)$$

$$\sum_{t \leq T} (\delta\pi_t^\lambda)^2 (e^{\Delta x^\lambda(t)} - 1)^2 = 0. \quad \dots(5.3)$$

This assertion follows from (any standard version of) the definition of the integral of a bounded predictable process with respect to a semimartingale, but for completeness we outline a proof. Write out an expression for δx_T as a function of $\delta\pi$, π_1 , π_2 from (2.12–13) and consider it term by term. We know that

$$E(\int_0^T \delta\pi_t^\lambda dM_t^{\lambda c})^2 \leq E \int_0^T (\delta\pi_t^\lambda)^2 d\langle M^{\lambda c} \rangle_t, \quad \dots(5.4)$$

and then it follows from (1) that $\int_0^T \delta\pi_t^\lambda dM_t^{\lambda c}$ vanishes a.s. for each T , hence vanishes a.s. on \mathcal{T} by continuity. The fact that $\int \delta\pi^\lambda d\langle M^{\lambda c} \rangle$ and $\int \delta\pi^\lambda dV^{\lambda c}$ vanish follows easily from (1) and (2) by the Schwarz inequality for Stieltjes integrals. Next,

$$\begin{aligned} \left[\int_0^T \Sigma_\lambda \Sigma_\ell (\pi_1^\lambda \pi_1^\ell - \pi_2^\lambda \pi_2^\ell) d\langle M^{\lambda c}, M^{\ell c} \rangle \right]^2 &= \left[\Sigma_\lambda \Sigma_\ell \int_0^T \delta\pi^\lambda (\pi_1^\ell + \pi_2^\ell) d\langle M^{\lambda c}, M^{\ell c} \rangle \right]^2 \\ &\leq \Sigma_\lambda \int_0^T (\delta\pi^\lambda)^2 d\langle M^{\lambda c} \rangle \cdot \Sigma_\ell \int_0^T (\pi_1^\ell + \pi_2^\ell)^2 d\langle M^{\ell c} \rangle = 0 \end{aligned} \quad \dots(5.5)$$

by rearrangement, then the Kunita–Watanabe inequality [9] VII.54, and (1). This disposes of the continuous terms in δx , i.e. those with superscript c .

Now note that (3) says that, a.s., $\delta\pi_t^\lambda = 0$ whenever $e^{\Delta x^\lambda(t)} \neq 1$, or equivalently whenever $\Delta x_t^\lambda = \Delta M_t^\lambda + \Delta V_t^\lambda \neq 0$; hence, by (2.3),

$$\delta\pi_t^\lambda = 0 \text{ whenever } \Delta M_t^\lambda \neq 0 \text{ and whenever } \Delta V_t^\lambda \neq 0. \quad \dots(5.6)$$

Thus the terms $\sum_{t \leq T} \delta\pi_t^\lambda \Delta x_t^\lambda$ and $\int_0^T \delta\pi_t^\lambda dV_t^{\lambda d} = \sum_{t \leq T} \delta\pi_t^\lambda \Delta V_t^\lambda$ vanish. Next, $\int \delta\pi^\lambda dM^{\lambda d}$ is by definition the compensated jump local martingale L^λ satisfying $L^\lambda(0) = 0$ and $\Delta L_T^\lambda = \delta\pi_T^\lambda \cdot \Delta M_T^\lambda$, and since $\delta\pi^\lambda \cdot \Delta M^\lambda \equiv 0$ we have $L^\lambda \equiv 0$. Finally,

$$\Delta(\delta x)_t = \ln \left[\sum_{\lambda} \pi_{1t}^{\lambda} e^{\Delta x^{\lambda}(t)} \right] - \ln \left[\sum_{\lambda} \pi_{2t}^{\lambda} e^{\Delta x^{\lambda}(t)} \right] \quad \dots(5.7)$$

by (2.13), and on separating the sums over indices with $e^{\Delta x^{\lambda}(t)} = 1$ from those with $e^{\Delta x^{\lambda}(t)} \neq 1$ and using $\sum_{\lambda} \pi_{1t}^{\lambda} = \sum_{\lambda} \pi_{2t}^{\lambda} = 1$ it is seen that this expression also vanishes. ||

If X is a special semimartingale and the canonical decomposition is considered, the processes $\langle M^c \rangle$, $|V^c|$ and $[X^d]$ are locally integrable and the measures $\mathfrak{d}_{\langle M^c \rangle}$, $\mathfrak{d}_{|V^c|}$ and $\mathfrak{d}_{[X^d]}$ may be defined as in Section 2. It is clear that $\delta\pi \sim 0$ for all three measures iff (1), (2) and (3) above hold a.s. for all T and λ . This confirms the assertion in Section 2 that, under the transformed probability measure introduced there, $\delta\pi \sim 0$ for all three measures implies $\delta x \equiv 0$. If X is a general semimartingale, we fix a convenient decomposition of X satisfying (2.3) and work with this from now on. Now $\langle M^c \rangle$ and $|V^c|$ are locally integrable so that $\mathfrak{d}_{\langle M^c \rangle}$ and $\mathfrak{d}_{|V^c|}$ are defined, and $\delta\pi \sim 0$ for these two measures iff (1) and (2) hold a.s. for all T and λ . However $|V^d|$, hence also $|V|$, $[X]$ and $[X^d]$, may fail to be locally integrable, leaving $\mathfrak{d}_{[X^d]}$ and indeed all the measures in (2.23) and (2.25) undefined, (and in this case the same will be true for every decomposition of X , see [9] VII.25). To obtain conditions expressed in terms of predictable measures which fill the gap, we first rewrite (3) as

$$\int_{[0,T]} \int_{\Xi} \left[\delta\pi_t^{\lambda} \{e^{\xi^{\lambda}} - 1\} \right]^2 \mathbb{F}(d\xi, dt) = 0, \quad \dots(5.8)$$

where $\mathbb{F} = \mathbb{F}(\omega; d\xi, dt)$ is the integer-valued random measure associated with the jumps of X , the inner integral being taken over $\xi = (\xi^1, \dots, \xi^{\Lambda})$ in Ξ , an auxiliary 'space of jumps' which is a copy of \mathfrak{R}^{Λ} , see [7] 3.22 or [8] II.1.16. Since $\delta\pi$ is predictable and the double integral vanishes identically, a condition equivalent to (8) is obtained if \mathbb{F} is replaced

therein by its compensator $F = F(\omega; d\xi, dt)$, see [7] 3.15, [8] II.1.8.⁹ Further, F can be factorised in the form

$$F(\omega; d\xi, dt) = f(\omega; d\xi, t) dG(\omega, t), \quad \dots(5.9)$$

where $f_t(\cdot) = f(\omega; \cdot, t)$ is for each (ω, t) a measure on the Borel sets of Ξ which does not charge the origin, $f(\cdot; B, \cdot)$ is for each Borel set a predictable process, and G is a predictable, non-decreasing process satisfying $G(0) = 0$ and $EG_t < \infty$ for each t . Thus the measure ν_G is well defined, and (3) or equivalently (8) holds on \mathcal{T} a.s. for each λ iff

$$\int_{[0, T]} \Sigma_\lambda \left[(\delta\pi_t^\lambda)^2 \int_{\Xi} \{e^{\xi^\lambda} - 1\}^2 f(d\xi, t) \right] dG(t) = 0 \quad \dots(5.10)$$

on \mathcal{T} a.s. A fortiori, (3) and (8) hold if $\delta\pi \sim 0$ (ν_G).

Note further that G can be decomposed as $G^c + G^d$, where each component has the properties of G stated above and G^c is continuous, G^d purely discontinuous. We define $J = \{(\omega, t): \Delta G^d(\omega, t) \neq 0\}$ – up to indistinguishability – and note that J is the predictable support of F , see [7] 3.24–25. J is thin (i.e. each section J_ω contains at most

⁹ \mathbb{F} and F are (non-negative) random measures in the sense that, for each ω , $\mathbb{F}(\omega; \cdot, \cdot)$ and $F(\omega; \cdot, \cdot)$ are measures on the Borel sets of $\Xi \times \mathcal{T}$. Roughly speaking, \mathbb{F} is constructed by placing a unit mass (delta-function) at the point (ω, ξ, t) if $\Delta X(\omega, t) = \xi \neq 0$; (if $\Delta X(\omega, t) = 0$, ξ is replaced by a point $\gamma \notin \Xi$). Denoting by \mathcal{B}^Λ the Borel sets of Ξ , we define optional and predictable σ -algebras in $(\Omega \times \mathcal{T}) \times \Xi$ by $\mathcal{O} \times \mathcal{B}^\Lambda$, $\mathcal{P} \times \mathcal{B}^\Lambda$. Then \mathbb{F} is an optional measure in the sense that, if $W = W(\omega, \xi, t)$ is an optional non-negative or $(F \times P)$ -integrable function, the process $(W * \mathbb{F})_T = \int_{[0, T]} \int_{\Xi} W \cdot d\mathbb{F}$ is optional in the usual sense; similarly, F is a predictable measure. The measures \mathbb{F} and F are related by the following facts: if $W \geq 0$ is predictable, $E(W * \mathbb{F})_\omega = E(W * F)_\omega$; and if W is predictable and such that $|W| * F$, or equivalently $|W| * F$, is locally P -integrable, then the process $W * F$ is the compensator of $W * \mathbb{F}$, i.e. $\int_{[0, T]} \int_{\Xi} W \cdot d(F - \mathbb{F})$ is a local martingale. For a survey, see [14].

a countable set) and so is a null set for G^c . Consequently $\delta\pi \sim 0$ for G iff $\delta\pi \sim 0$ for both G^c and G^d .

To sum up so far, it is sufficient for $\delta x \equiv 0$ if $\delta\pi \sim 0$ for all of $\langle M^c \rangle$, $|V^c|$, G^c and G^d . This criterion is certainly not necessary. On the other hand, it is complicated by the absence of any particular relationship among the null sets for the various measures. To simplify a little, without serious loss for applications, we shall from now on *assume* that G^c as well as the processes $\langle M^{\lambda c}, M^{\ell c} \rangle$ and $V^{\lambda c}$ for all λ, ℓ have absolutely continuous sample functions with (predictable, m-a.e. defined) derivatives $g, \sigma^{\lambda \ell}, v^{\lambda}$. Then the preceding criterion for $\delta x \equiv 0$ can be replaced by the simpler (but even stronger) *sufficient condition*

$$\delta\pi \sim 0 \quad \text{for} \quad m + \mathfrak{D}_{G^d}. \quad \dots(5.11)$$

We now seek conditions under which, conversely, $\delta x \equiv 0$ implies (11). First note some equations which follow from $\delta x \equiv 0$ without further assumptions. We have

$$[\delta x]_T \equiv \langle (\delta x)^c \rangle_T + \sum_{t \leq T} (\Delta \delta x_t)^2 \equiv 0, \quad \dots(5.12)$$

and $\langle (\delta x)^c \rangle$ vanishes with $[\delta x]$, see [7] 2.23; hence, a.s. for each $T \in \mathcal{T}$,

$$\langle (\delta x)^c \rangle_T = \int_0^T \sum_{\lambda} \sum_{\ell} \delta\pi_t^{\lambda} \delta\pi_t^{\ell} \sigma_t^{\lambda \ell} dt = 0. \quad \dots(5.13)$$

Also, using (7) and $\sum_{\lambda} \delta\pi_t^{\lambda} = 0$, the condition that $\sum_{t \leq T} (\Delta \delta x_t)^2$ vanishes can be written

$$\sum_{t \leq T} \left[\sum_{\lambda} \delta\pi_t^{\lambda} \{e^{\Delta x^{\lambda}(t)} - 1\} \right]^2 = \int_{[0, T]} \int_{\Xi} \left[\sum_{\lambda} \delta\pi_t^{\lambda} \{e^{\xi^{\lambda}} - 1\} \right]^2 \mathbb{F}(d\xi, dt) = 0. \quad \dots(5.14)$$

Since the double integral vanishes, so does its compensator, and we may again replace \mathbb{F} by F . Factorising F as in (9), then replacing dG_t by $gdt + dG_t^d$, we obtain two integrals each of which must vanish on \mathcal{T} a.s., i.e.

$$\int_{[0, T]} \int_{\Xi} \left[\sum_{\lambda} \delta\pi_t^{\lambda} \{e^{\xi^{\lambda}} - 1\} \right]^2 f(d\xi, t) g(t) dt = 0, \quad \dots(5.15)$$

$$\int_{[0, T]} \int_{\Xi} \left[\sum_{\lambda} \delta\pi_t^{\lambda} \{e^{\xi^{\lambda}} - 1\} \right]^2 f(d\xi, t) dG_t^d(t) = 0. \quad \dots(5.16)$$

Note that in general (15–16) are weaker than (10).

We now consider additional assumptions. It is clear from (13) that $\delta\pi \sim 0$ (m) if

$$[\sigma_t^{\lambda\ell}] \text{ is a positive definite matrix} \quad \dots(5.17)$$

a.e.(m). Alternatively, $\delta\pi \sim 0$ (m) may be inferred from further assumptions about the jumps of X . Suppose that

$$g(t) > 0 \quad \dots(5.18)$$

a.e.(m). Then (15) leads to

$$\Sigma_\lambda \delta\pi_t^\lambda (e^{\xi^\lambda} - 1) = 0 \quad \text{for } f_t\text{-almost all } \xi \in \Xi \quad \dots(5.19)$$

a.e.(m). For given (ω, t) , (19) says that the vector $\delta\pi(\omega, t)$ must be orthogonal in $\Xi = \mathfrak{X}^\Lambda$ to f_t -almost all vectors $e^\xi - 1$, or equivalently to the subspace $\{f_t\}$ generated by vectors $e^\xi - 1$ with ξ in the support of the measure f_t . A sufficient condition for this to imply $\delta\pi(\omega, t) = 0$ is that

$$\{f_t\} = \mathfrak{X}^\Lambda. \quad \dots(5.20)$$

Thus $\delta\pi \sim 0$ (m) if (18) and (20) hold a.e.(m). These conditions imply that, during any interval of the form $A \times (S, T]$ with $A \in \mathcal{A}_S$, $S < T$ in \mathcal{T} , there is for each asset a positive probability of a jump in price occurring at some totally inaccessible time.

A separate condition is needed to ensure that $\delta\pi \sim 0$ for G^d . Bearing in mind that $dG^d(\omega, t) = \Delta G(\omega, t) > 0$ iff $(\omega, t) \in J$, it is seen that since (16) holds for all T a.s. we have (19) for $(\omega, t) \in J$, and then the result follows if (20) holds for $(\omega, t) \in J$. This can also be expressed in another way. Letting (ν_i) be a sequence of predictable times with disjoint graphs such that $J = \cup [\nu_i]$, (14) implies

$$\Sigma_{\nu_i \leq T} \left[\Sigma_\lambda \delta\pi^\lambda(\nu_i) \{e^{\Delta x^\lambda(\nu_i)} - 1\} \right]^2 = 0 \quad \dots(5.21)$$

for all T a.s.; thus the big bracket vanishes a.s. for each $i=1,2,\dots$. Writing $\nu_i = \nu$, taking conditional expectations with respect to $\mathcal{A}_{\nu-}$, and noting that $\delta\pi(\nu) \in \mathcal{A}_{\nu-}$ since $\delta\pi$ is predictable, we obtain

$$\Sigma_\lambda \delta\pi^\lambda(\nu) E^{\nu-} [e^{\Delta x^\lambda(\nu)} - 1] = 0 \quad \text{a.s.} \quad \dots(5.22)$$

This implies $\delta\pi_\nu = 0$ a.s. if, a.s., the subspace generated by the support of the distribution

of $\exp(\Delta X_\nu) - 1$ conditional on \mathcal{A}_ν is \mathfrak{H}^Λ . To sum up we have

THEOREM 3. Suppose that the processes $V^{\lambda c}$, $\langle M^{\lambda c}, M^{\ell c} \rangle$ for all λ, ℓ and G^c are absolutely continuous. Then $\pi_1 \sim \pi_2$ ($m + \mathfrak{D}_{gd}$) implies $x_1 \equiv x_2$. The converse implication also holds if one of the following is satisfied:

$$(17) \text{ holds a.e.}(m), \text{ and } (20) \text{ holds a.e.}(\mathfrak{D}_{gd}); \quad \dots(5.23)$$

$$(18) \text{ and } (20) \text{ hold a.e.}(m), \text{ and } (20) \text{ holds a.e.}(\mathfrak{D}_{gd}). \quad \dots(5.24)$$

REMARK. Conditions (17) and (20) can be refined. Let $\{1\}$ denote the subspace of \mathfrak{H}^Λ generated by 1 and $\{1\}^*$ its orthogonal complement; the relation $\Sigma_\lambda \delta\pi^\lambda = 0$ implies $\delta\pi_t \in \{1\}^*$. Dropping the subscript t when convenient, let σ^λ denote the vector in \mathfrak{H}^Λ with co-ordinates $\sigma^{\lambda\ell}$, $\{\sigma_t\} = \{\sigma\}$ the subspace generated by the vectors σ^λ , and consider $B(\xi) = \Sigma_\lambda \Sigma_\ell \xi^\lambda \xi^\ell \sigma^{\lambda\ell}$ as a form on \mathfrak{H}^Λ . Since $B \geq 0$, the subspace $N = \{\xi: B(\xi) = 0\}$ is precisely the set where B attains its minimum, from which it follows that $N = \{\xi: (\xi, \sigma^\lambda) = 0 \text{ each } \lambda\} = \{\sigma\}^*$. Now (13) implies that, a.e.(m), $B(\delta\pi) = 0$, hence $\delta\pi \in \{\sigma\}^* \cap \{1\}^*$, which in turn implies $\delta\pi = 0$ if $\{\sigma\}^* \cap \{1\}^* = \{0\}$, i.e. if

$$\{\sigma_t\} \text{ is a } (\Lambda-1)\text{-dimensional subspace not containing } 1. \quad \dots(5.17a)$$

Next, consider $\{f_t\} = \{f\}$. As noted earlier, (19) implies $\delta\pi \in \{f\}^*$, hence $\delta\pi \in \{f\}^* \cap \{1\}^*$, which in turn implies $\delta\pi = 0$ if $\{f\}^* \cap \{1\}^* = \{0\}$, i.e. if

$$\{f_t\} \text{ is a } (\Lambda-1)\text{-dimensional subspace not containing } 1. \quad \dots(5.20a)$$

Replacing (17) and (20) by (17a) and (20a) allows conditions for portfolio uniqueness to be given in case there is a riskless asset; we omit further details.

Conditions (23–24), suitably restricted to random intervals, also allow the concavity inequality (3.6) to be replaced by a strict inequality. For brevity, we verify only the following form of this assertion, which is needed below: *if $x_1 \neq x_2$ and either (23) or (24) is satisfied, there is a $T \in \mathcal{T}$ and a set $A \in \mathcal{A}_T$ with $P_A > 0$ such that*

$$x_{\alpha t} > \alpha x_{1t} + (1-\alpha)x_{2t} \quad \text{for } t \geq T, \quad \omega \in A, \quad 0 < \alpha < 1. \quad \dots(5.25)$$

It is only necessary to prove the inequality for $t=T$, since (3.6) does the rest. Now, if $x_1 \neq x_2$, there is some $T \in \mathcal{T}$ and $C = \{\omega: \delta x(\omega, T) \neq 0\} \in \mathcal{A}_T$ with $P(C) > 0$. It follows from (2.12) and the 'local character' of the stochastic integral, [9] VIII.23, that there is a predictable set $H = \{(\omega, t): t \leq T \text{ \& } \delta \pi(\omega, t) \neq 0\}$ whose sections H_ω are a.s. non-empty for $\omega \in C$, and for given $\omega \in C$ we have either $l(H_\omega) > 0$, or $\delta \pi(\omega, t) \neq 0$ for some $t \in J_\omega \cap H_\omega$. Let C^i, C^p denote the subsets of C for which these conditions are respectively satisfied, and note that at least one of them has positive probability. Suppose that $P(C^i) > 0$. Then, if (17) holds a.e.(m), the term $-\frac{1}{2} \int_{H_\omega} \Sigma_\lambda \Sigma_\ell \tilde{\pi}_\alpha^\lambda \tilde{\pi}_\alpha^\ell \sigma^{\lambda\ell} dt$ is strictly concave for $\omega \in C^i$, and taking into account the concavity of the remaining terms in (2.12) this implies (25) with $A = C^i$ a.s. Alternatively, if (18) and (20) hold a.e.(m), the set $C^{id} = \{\omega \in C^i: \Sigma_{t \in H_\omega \cap J_\omega^c} (\delta \Delta x_t)^2 > 0\} = \{\omega \in C^i: \int_0^T \int_\Xi [\Sigma_\lambda \delta \pi_\lambda (e^{\xi^\lambda} - 1)]^2 f(d\xi, t) g(t) dt > 0\}$ has positive probability, and for $\omega \in C^{id}$ we have $\Delta x_{1t} \neq \Delta x_{2t}$ for some $t \leq T$. It then follows from (2.12) and the strict concavity of the log function that $\Delta x_{\alpha t} > \alpha \Delta x_{1t} + (1-\alpha) \Delta x_{2t}$, and taking into account that all terms in (2.12) are concave this implies $x_{\alpha T} > \alpha x_{1T} + (1-\alpha)x_{2T}$ for $\omega \in C^{id}$, proving (25) with $A = C^{id}$. Finally, if $P(C^p) > 0$, the assumption that (20) holds for $(\omega, t) \in J$ implies that $C^p = \{\omega \in C: \Sigma_{t \in H_\omega \cap J_\omega} (\delta \Delta x_t)^2 > 0\}$, i.e. $\omega \in C^p$ means that $\Delta x_{1t} \neq \Delta x_{2t}$ for some $t \leq T$, $t \in J_\omega$; then the strict concavity of the log function again implies (25) with $A = C^p$.||

Finally, we consider uniqueness properties of *optimal* plans. Let (\bar{c}_1, π_1) and (\bar{c}_2, π_2) now be optimal, with x_1, x_2 generated by π_1, π_2 and $c_1 = \bar{c}_1 e^{-x_1}$, $c_2 = \bar{c}_2 e^{-x_2}$. We have $\bar{\varphi}(\bar{c}_1) = \bar{\varphi}(\bar{c}_2) = \varphi^*$. Let $(\bar{c}_\alpha, \pi_\alpha)$ be as in (3.12–13); this plan is feasible, so $c_\alpha = \bar{c}_\alpha e^{-x_\alpha} \in \mathcal{C}$; It follows from the concavity of $\bar{u}(\cdot; s)$ and the definition

of $\bar{\varphi}$ that

$$\bar{\varphi}(\bar{c}_\alpha) \geq \alpha \bar{\varphi}(\bar{c}_1) + (1-\alpha) \bar{\varphi}(\bar{c}_2) = \varphi^* ; \quad \dots(5.26)$$

thus, if $\bar{u}(\cdot; s)$ is *strictly* concave a.e. (law of diminishing marginal utility) we have

$\bar{c}_1 \sim \bar{c}_2$ (m), since otherwise the above inequality becomes strict, contrary to optimality. A

similar argument using (3.3) shows that, for a *fixed* π , a π -optimal element of \mathcal{E} is

uniquely defined a.e. if $\bar{u}(\cdot; s)$ is strictly concave a.e. From now on we *assume* this strict

concavity and denote by $\bar{c}_* = \bar{c}_1 = \bar{c}_2 = c_1 e^{-x_1} = c_2 e^{-x_2}$ the unique optimal element of

$\bar{\mathcal{E}}$. Obviously $c_1 \sim c_2$ (m) iff $x_1 \equiv x_2$. Define $\tilde{\pi}_\alpha$ as in (3.5) with some fixed $\alpha \in (0,1)$,

let $\tilde{\pi}_\alpha$ generate x_α , and define processes $c_\alpha, \tilde{c}_\alpha$ by

$$c_\alpha = \alpha c_1 + (1-\alpha) c_2, \quad \tilde{c}_\alpha = c_\alpha e^{x_\alpha}; \quad \dots(5.27)$$

then $c_\alpha \in \mathcal{E}$, so $\tilde{c}_\alpha \in \bar{\mathcal{E}}$. We have, for a.e. (ω, t) ,

$$\begin{aligned} \tilde{c}_\alpha &= c_\alpha e^{x_\alpha} \geq \{\alpha c_1 + (1-\alpha) c_2\} e^{\alpha x_1 + (1-\alpha) x_2} \\ &= \bar{c}_* \{\alpha e^{(1-\alpha)(x_1-x_2)} + (1-\alpha) e^{\alpha(x_1-x_2)}\} \geq \bar{c}_*; \end{aligned} \quad \dots(5.28)$$

the first inequality results from (3.6), the second from the convexity of the exponential

function. Since \bar{c}_* is optimal, the inequalities must in fact be equalities a.e., and

$(c_\alpha, \pi_\alpha, x_\alpha)$ is optimal for each α . Now assume one or other of the portfolio uniqueness

conditions (23), (24) — or some variant implying that the strict inequality (25) holds for

some $T \in \mathcal{T}$ and some $A \in \mathcal{A}_T$ with $P_A > 0$ whenever $x_1 \neq x_2$. It follows from (25) that

the first inequality in (28) is strict for $\omega \in A$ and $t \geq T$, contrary to the result obtained

previously, so that in fact $x_1 \equiv x_2$, $\pi_1 \sim \pi_2$ (m + \mathcal{D}_{gd}). To sum up, we have

THEOREM 4. Suppose that $\bar{u}(\cdot; s)$ is strictly concave a.e.(m) and that either (5.23) or (5.24)

is satisfied. If (\bar{c}_j, π_j) with associated processes $x_j, c_j = \bar{c}_j e^{-x_j}$ are optimal plans,

$j = 1, 2$, then $\bar{c}_1 \sim \bar{c}_2$ (m), $c_1 \sim c_2$ (m), $x_1 \equiv x_2$, $\pi_1 \sim \pi_2$ (m + \mathcal{D}_{gd}).

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