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Ramsey-goodness—and otherwise

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Abstract

A celebrated result of Chvátal, Rödl, Szemerédi and Trotter states (in slightly weakened form) that, for every natural number Δ, there is a constant r_Δ such that, for any connected n-vertex graph G with maximum degree Δ, the Ramsey number $R(G, G)$ is at most $r_\Delta n$, provided n is sufficiently large.

In 1987, Burr made a strong conjecture implying that one may take $r_\Delta = \Delta$. However, Graham, Rödl and Ruciński showed, by taking G to be a suitable expander graph, that necessarily $r_\Delta > 2^{c\Delta}$ for some constant $c > 0$. We show that the use of expanders is essential: if we impose the additional restriction that the bandwidth of G be at most some function $\beta(n) = o(n)$, then $R(G, G) \le (2\chi(G) + 4)n \le (2\Delta + 6)n$, i.e., $r_\Delta = 2\Delta + 6$ suffices. On the other hand, we show that Burr's conjecture itself fails even for P_n^k, the kth power of a path P_n.

Brandt showed that for any c, if Δ is sufficiently large, there are connected n-vertex graphs G with $\Delta(G) \le \Delta$ but $R(G, K_3) > cn$. We show that, given Δ and H, there are $\beta > 0$ and n_0 such that, if G is a connected graph on $n \ge n_0$ vertices with maximum degree at most Δ and bandwidth at most βn, then we have $R(G, H) = (\chi(H) - 1)(n - 1) + \sigma(H)$, where $\sigma(H)$ is the smallest size of any part in any $\chi(H)$-partition of H. We also show that the same conclusion holds without any restriction on the maximum degree of G if the bandwidth of G is at most $\varepsilon(H) \log n / \log\log n$.

1 Introduction

Given two graphs G and H, the Ramsey number $R(G, H)$ is defined to be the smallest N such that, however the edges of K_N are coloured with red and blue, there exists either a red copy of G or a blue copy of H.

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In the 1980s, Burr [10] and Burr and Erdős [11] made various seemingly natural conjectures on the magnitudes of Ramsey numbers $R(G,H)$ in which one or both graphs is sparse. In the 1990s, Brandt [7] and Graham, Rödl and Ruciński [27] used expander graphs to give counterexamples to these conjectures. Our aim in this paper is to show that limiting the expansion of the graphs suffices to (almost) rescue the conjectures.

There are two slightly different sets of results within this paper. We are interested in $R(G,H)$ in the case when H is a (small) fixed graph, and G may be much larger, and we are also interested in the case when $H=G$. The results we prove have a similar flavour, and we use similar techniques. To start with, we think of H as a fixed graph.

A very simple general lower bound on the Ramsey number, given by Chvátal and Harary [15], is $R(G,H) \geq (\chi(H)-1)(|G|-1)+1$ for connected graphs G – here $|G|$ denotes the number of vertices of G. To see this, consider a two-colouring of the complete graph consisting of $\chi(H)-1$ disjoint red cliques each on $|G|-1$ vertices, with only blue edges between them. The red components are too small to contain G, and the chromatic number of the subgraph of blue edges is too small for H.

Burr and Erdős [11] defined a connected graph G to be *p-good* if $R(G,K_p)=(p-1)(|G|-1)+1$; in other words, if the Ramsey number is equal to the lower bound of Chvátal and Harary. A family $\mathcal{G}$ of graphs is defined to be *p-good* if there is some n_0 such that every $G\in\mathcal{G}$ with $|G|\geq n_0$ is p-good. Burr and Erdős were interested in the problem of finding families of graphs which are p-good for all p. Chvátal [14] showed that the family of trees is p-good for all p. Burr and Erdős [11] showed that for any k, the family of connected graphs with bandwidth at most k is p-good for all p (although the value of n_0 does increase with p). They made several conjectures regarding larger families of p-good graphs, many of which have been answered in a recent paper of Nikiforov and Rousseau [34]. One remaining open question is to determine whether the family of hypercubes is p-good for any $p\geq 3$.

As observed by Burr [9], the idea of the construction of Chvátal and Harary can be adapted to give a stronger lower bound in many cases. To explain this, we define a graph parameter: for any graph H of chromatic number $\chi(H)$, let $\sigma(H)$ be the minimum size of a colour class in a proper $\chi(H)$-colouring of H. Then we can add to Chvátal and Harary's construction a further red clique of size $\sigma(H)-1$, provided G is not too small.

Lemma 1 (Burr [9])**.** *For all graphs G and H, with G connected and $|G|=n>\sigma(H)$, we have*

$$R(G,H)\geq(\chi(H)-1)(n-1)+\sigma(H).$$

We say that a connected graph G is *H-good* if $R(G,H)=(\chi(H)-1)(|G|-1)+\sigma(H)$, and that a family of graphs $\mathcal{G}$ is *H-good* if all sufficiently large members of $\mathcal{G}$ are H-good. Finally, we call a graph class $\mathcal{G}$ *always-good* if $\mathcal{G}$ is H-good for every graph H. Burr [9] showed that, for all graphs G_1, the class of graphs homeomorphic to G_1 is always-good.

We mention two barriers to always-goodness. One necessary property for a family $\mathcal{G}$ to be always-good is that $\mathcal{G}$ does not contain arbitrarily large graphs G in which the maximum degree $\Delta(G)$ is nearly as large as $|G|$. An explicit version of this principle is illustrated by a construction of Brown [8] yielding, for every prime p, a (p^2+p+1)-vertex graph H_p with minimum degree $p+1$ containing no copy of $K_{2,2}$. Let Γ be the two-coloured complete graph obtained from H_p by colouring its edges blue and non-edges red. By definition, Γ does not contain either a blue copy of $K_{2,2}$ or a vertex of red-degree p^2. It follows that, if G is any graph on p^2+p vertices with $\Delta(G) \geq p^2$, then $R(G, K_{2,2}) \geq p^2+p+2$, which is strictly greater than $(\chi(K_{2,2})-1)(|G|-1)+\sigma(K_{2,2}) = p^2+p+1$, so G is not $K_{2,2}$-good. One can clearly obtain better bounds by using larger bipartite graphs in place of $K_{2,2}$.

The class of trees is K_p-good for all p but – for instance by the above argument – not always-good. The graph families considered by Nikiforov and Rousseau also contain graphs with such high degrees, and are thus not always-good.

A second barrier to always-goodness is strong vertex expansion. Burr and Erdős conjectured that, for any Δ and p, if n is sufficiently large, then any n-vertex graph G with $\Delta(G) \leq \Delta$ is p-good; Burr [10] made the natural strengthening to conjecture that for any Δ, the graph class $\{G\colon \Delta(G) \leq \Delta\}$ is always-good. However Brandt [7] showed that, for $\Delta \geq 168$, the family of all Δ-regular graphs is not even K_3-good; Nikiforov and Rousseau [34] reduced this degree requirement to 100. Both proofs relied upon the fact that such graphs can have strong vertex expansion properties. To be precise, Brandt proved the following result showing that Burr and Erdős' conjecture is already wrong by an arbitrarily large factor for $p = 3$.

Theorem 2 (Brandt [7])**.** *Let c be any constant. Then if Δ and n are sufficiently large, there exists an n-vertex graph G with $\Delta(G) \leq \Delta$ such that $R(G, K_3) > cn$.*

We show that Brandt's use of expander graphs is necessary: if $\mathcal{G}$ is a graph class with not only bounded maximum degree but also suitably limited expansion, then the more general conjecture of Burr is rescued.

We first state our results in terms of restricting the bandwidth of G.

Given a graph F, the kth power of F, denoted F^k, is the graph with vertex set $V(F)$ and edges between any two vertices whose distance in F is at most k. In particular, P_n^k is the kth power of the n-vertex path P_n. For any graph G on n vertices, the *bandwidth* of G, $\mathrm{bw}(G)$, is the smallest k such that G is a subgraph of P_n^k.

First we consider what happens if we bound the bandwidth of graphs G in the class $\mathcal{G}$, but do not further bound the degree. In this case, to show that $\mathcal{G}$ is always-good, it suffices to show that the class of graphs P_n^k is always-good. One may think of k as being fixed, but in fact our proof works provided k grows more slowly than $\log n/\log\log n$.

Theorem 3. *For each fixed graph H and natural number k, $R(P_n^k, H) = (\chi(H)-1)(n-1)+\sigma(H)$ whenever $n \geq (20k|H|)^{16k|H|}$.*

In particular, if $\kappa(n)$ is any function with $\kappa(n) = o(\log n/\log\log n)$, then the graph class $\mathcal{B}_\kappa = \{G : \mathrm{bw}(G) \leq \kappa(|G|)$ and G is connected$\}$ is always-good.

This result, even in the case where $\kappa(n)$ is the constant function k, includes the result of Burr and Erdős stating that $\mathcal{B}_k$ is p-good for all p, as well as the result of Burr that the class of graphs homeomorphic to any fixed G_1 is always-good.

If $\kappa(n) = n^\varepsilon$, for any fixed $\varepsilon > 0$, then the class $\mathcal{B}_\kappa$ is not always-good. To see this, note that $R(K_s, K_t) = \Omega(s^{t/2})$ for fixed t as $s \to \infty$, by a standard probabilistic argument [39]. Therefore also $R(P_n^s, K_t) = \Omega(s^{t/2})$, since P_n^s contains K_s, and so the class of connected graphs with bandwidth at most n^ε is not $K_{4/\varepsilon}$-good.

Our proof of Theorem 3 uses a method from [1], inspired by the Szemerédi Regularity Lemma [40]. This method yields a partition of $V(G)$ and an auxiliary graph G^* on the parts, but the partition arises from the direct use of Ramsey's theorem rather than an iterated refinement procedure, enabling us to obtain a somewhat reasonable bound on the size of n we need.

Our next theorem shows that, if we put an absolute bound on the maximum degree of graphs in our class, it is enough to impose any upper bound on the bandwidth that is sublinear in the order of the graph.

Theorem 4. *For every fixed Δ, and every function $\beta(n) = o(n)$, the graph class*

$$\mathcal{G}_{\Delta,\beta} = \{G : \Delta(G) \le \Delta, \mathrm{bw}(G) \le \beta(|G|), \text{ and } G \text{ is connected}\}$$

is always-good.

In other words, a class of connected graphs is always-good if the maximum degree of graphs in the class is bounded and, for any $\beta > 0$, all sufficiently large graphs G in the class have bandwidth at most $\beta|G|$.

Our proof of Theorem 4 follows the same lines as Theorem 3, using also an embedding method of Böttcher, Schacht and Taraz [6], which does involve the use of the Regularity Lemma.

As we now explain, Theorem 4 can be converted to a result where the expansion properties of the graph G are explicitly limited.

Böttcher, Pruessmann, Taraz and Würfl [5] define a graph G to be (b, ε)-*bounded* if, for every subgraph G' of G with $|G'| \ge b$, there exists a set $U \subset V(G')$ with $|U| \le |G'|/2$ and $|\Gamma(U) - U| \le \varepsilon|U|$. Here $\Gamma(U)$ denotes the neighbourhood of U in the graph G'. They proved the following theorem.

Theorem 5 (Böttcher, Pruessmann, Taraz and Würfl [5])**.** *For any $\Delta \ge 1$ and $\beta_1 > 0$, there exist $\varepsilon > 0$, $\beta_2 > 0$ and n_0 such that, whenever $n \ge n_0$, every $(\beta_2 n, \varepsilon)$-bounded n-vertex graph G with $\Delta(G) = \Delta$ has $\mathrm{bw}(G) \le \beta_1 n$.*

We call a graph class $\mathcal{G}$ *non-expanding on large subsets* if for any $\beta, \varepsilon > 0$ the following is true. There exists n_0 such that if $G \in \mathcal{G}$ has $n \ge n_0$ vertices, then G is $(\beta n, \varepsilon)$-bounded.

An immediate corollary of Theorem 4, together with Theorem 5, is the following.

Corollary 6. *Given Δ, let $\mathcal{G}$ be a class of connected graphs of maximum degree Δ which is non-expanding on large subsets. Then $\mathcal{G}$ is always-good.*

Corollary 6 is best possible in the following sense. Brandt's method [7] can be adapted easily to show that, for any sufficiently large Δ and $\beta > 0$, if n and p are sufficiently large, and G is an n-vertex graph with $\Delta(G) \leq \Delta$ which does possess a subgraph G' on at least βn vertices with strong expansion properties (for example: if G' is a typical Δ-regular graph), then G is not p-good.

There is another sense in which Brandt's family of counterexamples is the simplest possible. He showed that the class of connected graphs with maximum degree at most Δ is not H-good for $H = K_3$. On the other hand, this class of graphs is H-good for every *bipartite* H, as observed by Burr, Erdős, Faudree, Rousseau and Schelp.

Theorem 7 (Burr, Erdős, Faudree, Rousseau and Schelp [12])**.** *For each fixed Δ, let $\mathcal{D}_\Delta$ be the class of connected graphs with maximum degree at most Δ. Then $\mathcal{D}_\Delta$ is H-good for every bipartite graph H.*

We note that there is a natural extension of the notion of H-goodness to the multicolour setting. For graphs $H_1, \ldots, H_r$, we say that a connected graph G is $(H_1, \ldots, H_r)$*-good* when there are integers W and Z (depending on $(H_1, \ldots, H_r)$ but not on G) such that $R(G, H_1, \ldots, H_r) = W(|G| - 1) + Z$. We say that a graph class $\mathcal{G}$ is *multicolour-always-good* when, for every $r \geq 2$ and every collection of graphs $H_1, \ldots, H_r$, G is $(H_1, \ldots, H_r)$-good for all sufficiently large $G \in \mathcal{G}$. In Section 5, we discuss the problems of finding W and Z, and prove the following theorem.

Theorem 8. *If $\mathcal{G}$ is any always-good class of graphs, then $\mathcal{G}$ is multicolour-always-good.*

We now turn our attention to the case where $G = H$, and H is again of bounded maximum degree. Burr [10] conjectured that, for each fixed Δ, if H is a sufficiently large connected graph with maximum degree at most Δ, then H is itself H-good, i.e.,

$$R(H, H) = (\chi(H) - 1)(|H| - 1) + \sigma(H).$$

In his paper, Burr warns that this conjecture "may be too bold", and indeed so it proved.

Burr's conjecture would imply that, for each fixed Δ, $R(H, H) \leq \Delta|H|$, whenever H is a sufficiently large graph with maximum degree Δ. Chvátal, Rödl, Szemerédi and Trotter [16] proved that some result along these lines is true: for every Δ, there is some constant r such that, whenever H has maximum degree Δ, $R(H, H) \leq r|H|$.

For each fixed Δ, let $r_\Delta = \liminf_{n \to \infty} \max\{R(H, H)/n : H$ is a connected graph on n vertices with maximum degree at most $\Delta\}$. So the result of Chvátal, Rödl, Szemerédi and Trotter is that r_Δ is finite for all Δ, and Burr's conjecture would imply that $r_\Delta \leq \Delta$.

The question of determining the rate of growth of r_Δ was addressed by Graham, Rödl and Ruciński [27], who proved the following theorem, giving bounds in both directions.

Theorem 9 (Graham, Rödl and Ruciński [27])**.** *There exist constants $c, c' > 0$ such that the following hold.*

(i) Whenever H is an n-vertex graph with $\Delta(H) \le \Delta$, $R(H,H) \le 2^{c'\Delta \log^2 \Delta} n$.

(ii) For each sufficiently large n, there exists a bipartite n-vertex graph H with $\Delta(H) \le \Delta$ and $R(H,H) > 2^{c\Delta} n$.

Theorem 9 implies that $2^{c\Delta} \le r_\Delta \le 2^{c'\Delta \log^2 \Delta}$, and in particular that Burr's conjecture is false. The proof of the lower bound in Theorem 9 relies upon a (probabilistic) construction of a graph H with maximum degree Δ and good expansion properties.

Recently, Fox and Sudakov [25] established the alternative upper bound $R(H,H) \le 2^{c\Delta(H)\chi(H)}|H|$, for some explicit constant c. In particular, if H is bipartite, this matches the form of the lower bound in Theorem 9. The result for bipartite graphs was obtained independently by Conlon [17], and very recently Conlon, Fox and Sudakov [18], improving on Theorem 9, showed that there is a constant c'' such that $r_\Delta \le 2^{c''\Delta \log \Delta}$.

We show that the use of expansion in the lower bound is necessary – that is, when both maximum degree and expansion are restricted, the Ramsey number may be bounded above by a function linear in both n and Δ. In fact, we will prove something slightly stronger: when expansion is appropriately restricted, the Ramsey number is primarily controlled by the chromatic number of H, not the maximum degree, as in Burr's conjecture.

Observe that simply requiring H to fail some global expansion condition will not suffice to bound $R(H,H)$ below $2^{c\Delta}n$. To see this, take some large Δ and n, let H' be an $(n/10)$-vertex graph with $\Delta(H') \le \Delta$ and $R(H',H') > 2^{c\Delta}n/10$, and form H by adding $9n/10$ isolated vertices to H'. The new graph H is a poor expander, yet $R(H,H) > 2^{c\Delta-4}n$. It follows that, as before, we need to restrict the expansion of all large subgraphs of H, or equivalently the bandwidth of H.

We shall show that, if the degree of H is at most Δ, H is sufficiently large, and the bandwidth of H is at most $\beta|H|$ for some small constant β, then $R(H,H) \le (2\chi(H)+4)|H|$. Thus imposing a restriction on the bandwidth of H almost rescues Burr's conjecture.

Our first task in this direction is to investigate the Ramsey numbers of powers of paths.

In Section 6, we consider the Ramsey numbers $R(P_n^k, P_n^k)$ and $R(C_n^k, C_n^k)$. Gerencsér and Gyárfás [26] showed that $R(P_n, P_n) = n - 1 + \sigma(P_n)$, and (for $n \ge 5$) Faudree and Schelp [24] and Rosta [36] showed that $R(C_n, C_n) = (\chi(C_n) - 1)(n-1) + \sigma(C_n)$, matching the lower bounds in Lemma 1, and in Burr's conjecture. It is natural to ask whether this continues to hold (for sufficiently large n) for each k: for powers of paths, this would mean that $R(P_n^k, P_n^k) = (k + \frac{1}{k+1})n + O(1)$.

In Section 6, we give a construction showing that this is not the case. For convenience, we state the result when n is a multiple of $k+1$.

Theorem 10. *For $k \ge 2$, and n a multiple of $k+1$, we have*

$$R(C_n^k, C_n^k),\ R(P_n^k, P_n^k) \ge (k+1)n - 2k.$$

This shows in particular that even bounding the bandwidth of H by a constant does not suffice to rescue Burr's conjecture.

We suspect that the inequality above is tight, at least for powers of paths. We have not been able to show this, but we offer the following upper bounds, which differ from the lower bounds by a multiplicative factor slightly greater than 2.

Theorem 11. *For any $k \geq 2$, we have*

$$R(P_n^k, P_n^k) \leq \left(2k + 2 + \frac{2}{k+1}\right) n + o(n)\,,$$

and

$$R(C_n^k, C_n^k) \leq \left(2\chi(C_n^k) + \frac{2}{\chi(C_n^k)}\right) n + o(n)\,.$$

Using Theorem 11, together with the embedding method of Böttcher, Schacht and Taraz [6], we prove the following result.

Theorem 12. *Given $\Delta \geq 1$, there exist n_0 and β_1 such that, whenever $n \geq n_0$ and H is an n-vertex graph with maximum degree at most Δ and $\mathrm{bw}(H) \leq \beta_1 n$, we have $R(H, H) \leq (2\chi(H) + 4)n$.*

As before, we can use Theorem 5 to convert the hypothesis of sublinear bandwidth to a condition on the expansion of all large subgraphs.

Corollary 13. *For any $\Delta \geq 1$, there exist n_0, β_2 and ε such that, whenever $n \geq n_0$ and H is a $(\beta_2 n, \varepsilon)$-bounded n-vertex graph with maximum degree at most Δ, we have $R(H, H) \leq (2\chi(H) + 4)n$.*

One might hope to show that, under the conditions of Theorem 12 or its corollary, $R(H, H) \leq (\chi(H) + C)n$. In order to prove this, a first step would be to show such a bound for the case $H = P_n^k$, but there are likely to be additional difficulties in the general case. One asymptotically sharp result in this direction has been proved.

Theorem 14 (Sárközy, Schacht and Taraz [37])**.** *For every $\gamma > 0$ and Δ, there exist $\beta > 0$ and n_0 such that, whenever $n \geq n_0$ and H is an n-vertex bipartite graph with maximum degree at most Δ, $\mathrm{bw}(H) \leq \beta n$, and parts of size t_1 and t_2 (where $t_1 \leq t_2$), we have*

$$R(H, H) \leq (1 + \gamma) \max(2t_1 + t_2, 2t_2)\,.$$

A final observation is that combining the Four Colour Theorem [2, 3] and another result of Böttcher, Pruessmann, Taraz and Würfl [5], namely that the bandwidth of every n-vertex planar graph of maximum degree Δ is bounded by $15n/\log_\Delta n$, we obtain, as a corollary to Theorem 12, the following.

Corollary 15. *For every Δ there exists n_0 such that, whenever $n \geq n_0$ and H is an n-vertex planar graph with maximum degree Δ, we have $R(H, H) \leq 12n$.*

2 A version of the blow-up lemma

In our proofs, we need an embedding lemma, similar in style to the Blow-up Lemma of Komlós, Sárközy and Szemerédi [31]. That result could be used as it stands, but using an alternative approach allows us to obtain significantly better bounds on the sizes of the graphs to which our results apply.

Instead of considering '(ε, δ)-super-regular' pairs of sets (as in the original Blow-up Lemma), where there are relatively few but well distributed edges, we will be interested in pairs of vertex sets within two-coloured complete graphs which do not contain a red $K_{s,s}$ for some s. By the Kövari-Sós-Turán theorem [32], this condition strongly limits the number and distribution of red edges. We give two forms.

Theorem 16 (Kövári, Sós and Turán [32])**.**

(a) For all $s, n \in \mathbb{N}$ with $n \geq s^2$, any n-vertex graph which does not contain $K_{s,s}$ has at most $2n^{2-\frac{1}{s}}$ edges.

(b) Let G be a bipartite graph, with parts X and Y, which does not contain a copy of $K_{s,s}$. If $2\big(s/|Y|\big)^{\frac{1}{s}} \leq p \leq 1$, then at most $2s/p$ vertices in X have degree greater than $p|Y|$.

We can now state and prove our embedding lemma.

Lemma 17. *Let t, s, r and d be natural numbers, with $s \geq d^2$. Suppose that the edges of a complete graph G with vertex set V are coloured red and blue. Let $V_1, \ldots, V_t$ be disjoint subsets of V, each of size at most s. Define a graph G' on disjoint vertex sets $V_1', \ldots, V_t'$, where $|V_i'| = \max\left(|V_i| - \left\lfloor 4r^2 s^{\frac{2r-1}{2r}}(d+1)\right\rfloor, 0\right)$ for each i, by putting edges between all vertices in V_i' and V_j' whenever there is no red $K_{r,r}$ between V_i and V_j in G. If H is any subgraph of G' with maximum degree d, then G contains a blue copy of H.*

Proof. Let G^{blue} be the spanning subgraph of G whose edges are the blue edges of G.

If $r = 1$, then G' is isomorphic to a subgraph of G^{blue}, and the result is trivially true. We will assume from now on that $r \geq 2$.

Let $p = 4r^2 s^{-1/2r}$: then, for each i, $|V_i| - |V_i'| \leq p(d+1)s$. Note that, if $p \geq \frac{1}{d+1}$, then each set V_i' is empty and there is nothing to prove, so we can assume $p < \frac{1}{d+1}$. By Theorem 16(b), if X and Y are vertex sets within a pair (V_i, V_j) that does not contain a red $K_{r,r}$, and $|Y| \geq \frac{\sqrt{s}}{2^r r^{2r-1}}$, then at most $2r/p$ vertices in X have red-degree greater than $p|Y|$.

Choose an embedding $\psi : V(H) \to V(G')$. Let $V(H) = \{x_1, x_2, \ldots\}$. We will successively choose vertices $\phi(x_1), \phi(x_2), \ldots \in V(G) = V(G^{\text{blue}})$ which give an embedding ϕ of H into G^{blue}. For each $x_i \in H$, set $A_{x_i,1} = V_j$, where V_j' is the part of G' containing $\psi(x_i)$.

The set $A_{x_i,t}$ is called the *allowed set* of x_i at time t; we invariably choose $\phi(x_t)$ to be within its allowed set at time t. We maintain two properties. First, if $x_i x_j \in E(H)$ and x_i has been embedded, then the allowed set of x_j is entirely within the blue-neighbourhood of

x_i. Second, if, at time t, x_i has not yet been embedded, then its allowed set has size larger than $ps/2 = 2r^2 s^{\frac{2r-1}{2r}}$: this quantity is definitely larger than the $\frac{\sqrt{s}}{2^r r^{2r-1}}$ required to apply Theorem 16. At time 1, the first condition is trivially satisfied, and the second is true by the choice of the sizes of the V_i'.

At time t, we choose a vertex $\phi(x_t) \in A_{x_t,t}$ which is blue-adjacent to at least $(1-p)|A_{x_\ell,t}|$ of the vertices of $A_{x_\ell,t}$ for each $\ell > t$ with x_ℓ adjacent to x_t. This is possible since, by Theorem 16(b), for each of the at most d neighbours of x_t not yet embedded, at most $2r/p$ vertices in $A_{x_t,t}$ fail to be blue-adjacent to $(1-p)|A_{x_\ell,t}|$ of the vertices of $A_{x_\ell,t}$, and $|A_{x_t,t}| \geq ps/2 > d\frac{2r}{p}$ by the choice of s.

Having chosen $\phi(x_t)$, for each $\ell > t$ we set $A_{x_\ell,t+1}$ equal to $A_{x_\ell,t} - \{\phi(x_t)\}$ if $x_t x_\ell \notin E(H)$, and equal to $A_{x_\ell,t} \cap \Gamma_{\text{blue}}(\phi(x_t))$ if x_ℓ is adjacent to x_t. It is clear that the allowed sets maintain the first property. If x_i is a vertex not yet embedded, with $\psi(x_i) \in V_j'$, then there are two reasons why a vertex $v \in V_j$ should not be in $A_{x_i,t+1}$: first, it might not be blue-adjacent to one of the at most d embedded neighbours of x_i, and second, it might be the image under ϕ of some preceding vertex (in V_j') of H. Thus we have

$$|A_{x_i,t+1}| \geq (1-p)^d |V_j| - |V_j'| > (1-pd)|V_j| - (|V_j| - p(d+1)s) \geq \frac{ps}{2},$$

so that the allowed sets maintain both the required conditions. It follows that this algorithm successfully embeds H into G^{blue}. $\square$

3 Powers of paths versus general graphs

The aim of this section is to prove Theorem 3, stating that $R(P_n^k, H) = (\chi(H)-1)(n-1) + \sigma(H)$ whenever n is sufficiently large in terms of k and $|H|$, and therefore that the family $\mathcal{B}_k$ of graphs of bandwidth at most k is always-good.

First we need to give a stability version of an old theorem of Erdős [20] stating that $R(P_n, K_\ell) = (\ell-1)(n-1)+1$.

Lemma 18. *Given $\ell \geq 2$, $0 \leq \alpha < 1/2$, $0 \leq \varepsilon < (1-\alpha)/\ell$, and $n \geq 1 + 1/\varepsilon$, the following is true. If G is a two-coloured graph on $(\ell-1-\alpha)(n-1)$ vertices, in which every vertex is adjacent to all but at most $\varepsilon(n-1)$ vertices of G, containing neither a red copy of P_n nor a blue copy of K_ℓ, then we can partition $V(G)$ into $\ell-1$ parts each containing at most $n-1$ vertices, such that every edge of G within a part is red, and every edge of G between different parts is blue.*

Proof. We prove the statement by induction on ℓ. The case $\ell = 2$ is trivial. Suppose $\ell \geq 3$, and that the statement is true for smaller values of ℓ.

Let G satisfy the conditions of the lemma, and let P be a maximal red path in G, so we have $|P| < n$. Let u be the first vertex of P, and set $X = \Gamma(u) \setminus P$. By maximality of P, every vertex of X is blue-adjacent to u. It follows that $G[X]$ is a two-coloured graph containing neither a red copy of P_n nor a blue copy of $K_{\ell-1}$, in which every vertex is adjacent to all but at most $\varepsilon(n-1)$ vertices of X. Since we have

$$|X| \geq d(u) - (|P|-1) \geq (\ell-1-\alpha)(n-1) - \varepsilon(n-1) - (n-2) > (\ell-2-\alpha-\varepsilon)(n-1)$$

and $\varepsilon < (1-\alpha-\varepsilon)/(\ell-1)$, it follows by induction that we can partition X into $\ell-2$ parts, $X = X_1 \cup \cdots \cup X_{\ell-2}$, such that any edge within a part is red, while any edge between different parts is blue. For convenience, we assume that the sets $X_1, \ldots, X_{\ell-2}$ are in increasing order of size. Each part contains at most $n-1$ vertices, and therefore the smallest part size $|X_1|$ is at least $(1-\alpha-\varepsilon)(n-1)$.

Since $1-\alpha-3\varepsilon > 0$, we have $\delta(G[X_i]) > |X_i|/2$ for each i, and therefore by Dirac's theorem $G[X_i]$ is Hamiltonian for each i. Now observe that, for each i,

$$|X_i| + |P| \geq |G| - \varepsilon(n-1) - (\ell-3)(n-1) \geq (2-\alpha-\varepsilon)(n-1) > (1+2\varepsilon)(n-1).$$

It follows that, if P' is a red path in $G[P]$ covering all but at most $\varepsilon(n-1)$ vertices of P, then an endvertex of P' cannot send any red edges to any set X_i.

The first vertex u of P has at most $\varepsilon(n-1)$ non-neighbours in G (including itself), so it must be adjacent to at least one vertex v among the last $\varepsilon(n-1)$ vertices of P. This vertex v is the endvertex of the red path from u to v following P, which covers all but at most $\varepsilon(n-1)$ vertices of P. It follows that v has no red neighbours in any set X_i.

Since $|X_1| > 2\varepsilon(n-1)$, we can find in X_1 a common neighbour x_1 of the vertices u and v; similarly in X_2 we can find a common neighbour of u, v and x_1, and so on until we find ℓ vertices forming a clique in G. Since there is no blue K_ℓ in G, and since all the other edges are blue, it must be the case that uv is red. It now follows that every vertex in P is the endvertex of a path covering all but at most $\varepsilon(n-1)$ vertices of P.

For each $1 \leq i \leq \ell-2$, let Y_i be the red component of G containing X_i, and let $Y_{\ell-1}$ be the set consisting of the remaining vertices of G. Since G contains no red P_n, the Y_i are all distinct, and $P \subset Y_{\ell-1}$. We claim that this partition satisfies the desired properties. By definition, there are no red edges between any pair of parts. Since every edge in a part can be extended to a copy of K_ℓ in G by choosing greedily one vertex from each other part, every edge within a part must be red. Finally, since each $G[Y_i]$ has minimum degree at least $|Y_i| - \varepsilon(n-1) > |Y_i|/2$, each component contains a spanning path by Dirac's theorem: as G contains no P_n, this implies that $|Y_i| < n$ for each i. □

In the next lemma, we need only the special case of Lemma 18 where G is complete – in this case, the proof above can be streamlined. We will make use of the full version of the lemma later.

Our next aim is to prove a similar stability result for $R(P_n^k, H)$. We do not need the full strength of the following result in order to establish the value of $R(P_n^k, H)$, but it is necessary for the proof of Theorem 4 later.

Lemma 19. *Let H be a graph, k a natural number, and ε a positive constant at most $1/2|H|^2$. Set $n_0 = (1/\varepsilon)(200k^2|H|/\varepsilon)^{4k|H|}$. If $n \geq n_0$ and G is a two-coloured complete* *graph on at least $(\chi(H)-1)n-n/6$ vertices, which contains neither a red copy of P_n^k nor a blue copy of H, then there is a partition $V(G)=V_1\cup\cdots\cup V_{\chi(H)-1}\cup L$ with the following properties.*

- $|L|\le\varepsilon n$.
- $2n/3\le|V_i|<n$ *for each* i.
- *For each* i *and* $v\in V_i$, v *has at most* $\varepsilon|V_i|$ *blue neighbours in* V_i.
- *For each* i *and* j, *no vertex in* V_i *has more than* $\varepsilon|V_j|$ *red neighbours in* V_j.

Proof. Suppose we are given a graph H, a natural number k, and a positive ε at most $1/2|H|^2$. We now choose $s=\lceil(128k^2|H|/\varepsilon)^{4k}\rceil$ and note that $n_0\ge(3/\varepsilon)(2s)^{|H|}$.

Take any $n\ge n_0$, and let G be a two-coloured complete graph on $(\chi(H)-1)n-n/6$ vertices which contains neither a red copy of P_n^k nor a blue copy of H.

Since $R(K_s,H)\le R(K_s,K_{|H|})\le\binom{s+|H|}{|H|}\le(2s)^{|H|}\le\varepsilon n/3$ by the Erdős-Szekeres bound [23], and G contains no blue copy of H, it follows that any $(\varepsilon n/3)$-vertex set in G contains a red copy of K_s. Thus we can partition $V(G)$ into disjoint s-vertex red cliques $Q_1,Q_2,\dots,Q_M$ and a leftover set L_1 with $|L_1|\le\varepsilon n/3$.

Let $m=\lceil n/s\rceil$. Observe that the number M of cliques is at least

$$\begin{aligned}\frac{(\chi(H)-1)n-n/6-\varepsilon n/3}{s}&\ge(\chi(H)-1)(m-1)-\frac{m}{6}-\frac{\varepsilon m}{3}\\&\ge\left(\chi(H)-1-\frac{1}{3}\right)(m-1).\end{aligned}$$

We say that two red cliques Q_i and Q_j, $i\ne j$, are *red-adjacent* if the induced bipartite graph $G[Q_i,Q_j]$ contains a red $K_{2k,2k}$, and *blue-adjacent* otherwise. This gives us an auxiliary two-coloured complete graph G^* whose nodes are the M red cliques.

Suppose there is a red-adjacent path $Q_{j_1}Q_{j_2}\cdots Q_{j_m}$ on m vertices in G^*. We claim that the $sm\ge n$ vertices in these m cliques of G can then be covered by a red kth power of a path. Since each consecutive pair of cliques on the path is red-adjacent in G^*, we can find vertex-disjoint red copies of $K_{k,k}$ between each consecutive pair of cliques; now we construct a copy of P_{sm}^k by traversing the sequence of cliques in order, using the copies of $K_{k,k}$ to step from one clique to the next. Therefore there is no red copy of P_m in G^*.

If G^* contains a blue-adjacent clique with vertex set $\{Q_{j_1},\dots,Q_{j_{\chi(H)}}\}$, then we can apply Lemma 17, with $t=\chi(H)$, $d=|H|-1$, $r=2k$, and the given value of s, to the sets $V_i=Q_{j_i}$. Each vertex set V_i' has size $s-\lfloor16k^2s^{1-1/4k}|H|\rfloor\ge s/2\ge|H|$ (since $s\ge(64k^2|H|)^{4k}$), and so the auxiliary graph G' contains a copy of H, and therefore by Lemma 17 there is a blue copy of H in G. Therefore there is no blue copy of $K_{\chi(H)}$ in G^*.

Thus G^* is a two-coloured complete graph on at least $(\chi(H)-1-\frac{1}{3})(m-1)$ vertices, with neither a red P_m nor blue $K_{\chi(H)}$. By Lemma 18, applied with $\alpha=1/3$, G^* must consist of $\chi(H)-1$ red cliques $C_1^*, \dots, C_{\chi(H)-1}^*$, each with between $2m/3$ and $m-1$ nodes, joined entirely by blue edges. These red cliques in G^* correspond to red *clusters* $C_1, \dots, C_{\chi(H)-1}$ in G, where each cluster contains between $2n/3$ and $n-1$ vertices.

Our plan is to show that we can form the required sets V_j by removing a small number of vertices from each cluster C_j, placing these in the leftover set. Since we only remove vertices from the clusters, the resulting V_j will all have at most $n-1$ vertices. As long as the leftover set contains at most $\varepsilon n \le n/6$ vertices, it follows that each V_i contains at least

$$(\chi(H)-1)n - \frac{n}{6} - (\chi(H)-2)(n-1) - \frac{n}{6} > \frac{2n}{3}$$

vertices.

Consider a clique Q in the cluster C_i. Let Q' be a clique in the cluster C_j, where $j \neq i$. Then QQ' is a blue-adjacent edge of G^*: by definition there is no red $K_{2k,2k}$ between Q and Q' in G. By Theorem 16(a) the number of red edges between Q and Q' is at most $2s^{2-\frac{1}{2k}} \le \varepsilon^2 s^2/6(\chi(H))^2$ – since $s \ge (12|H|^2/\varepsilon^2)^{2k}$. It follows that there are at most

$$\frac{\varepsilon^2|C_i||C_j|}{6(\chi(H))^2}$$

red edges between C_i and C_j in G. In particular, at most $\varepsilon|C_i|/3(\chi(H))^2$ vertices of C_i can have more than $\varepsilon|C_j|/2$ red neighbours in C_j.

For each i, let V_i' be the set of those vertices of C_i which have at most $\varepsilon|C_j|/2$ red neighbours in C_j for each $j \neq i$. We have $|C_i| - |V_i'| \le \varepsilon|C_i|/3\chi(H)$ for each i, and so the set $L_2 = \displaystyle\bigcup_{i=1}^{\chi(H)-1} (C_i \setminus V_i')$ of discarded vertices has size at most $\varepsilon n/3$.

Suppose that V_1' contains more than $\varepsilon^2|V_1'|^2/6\chi(H)$ blue edges. This number is at least $2|V_1'|^{2-1/|H|}$, since $|V_1'| \ge |C_1|/2 \ge n/3$, and we comfortably have $n \ge 3(12|H|/\varepsilon^2)^{|H|}$. Thus, by Theorem 16(a), there is a blue copy H_1 of the bipartite graph $K_{|H|,|H|}$ in V_1'.

We now show that such a blue bipartite graph inside V_1' can be extended to a blue copy of the $\chi(H)$-partite graph $K_{|H|,\dots,|H|}$ in G, by taking a suitable set of $|H|$ vertices from each other V_j'.

The number of vertices in V_2' which send red edges to any vertex of H_1 is at most $2|H|\varepsilon|C_2|/2 \le |V_2'| - |H|$; in particular, there are $|H|$ vertices of V_2' which each send blue edges to every vertex of H_1. Thus we have a blue copy H_2 of the tripartite graph $K_{|H|,|H|,|H|}$ in $V_1' \cup V_2'$.

Repeating this argument for each $V_3', \dots, V_r'$ successively, using that, at each stage, $(\chi(H)-1)|H|\varepsilon|C_j|/2 \le |V_j'| - |H|$ – this follows because $|V_j'| \ge |C_j|/2$ and $\varepsilon|H|^2 \le 1/2$ – we find eventually a blue copy $H_{\chi(H)-1}$ of the $\chi(H)$-partite graph $K_{|H|,\dots,|H|}$, as claimed. This graph contains H, which is a contradiction.

It follows that V_1' contains at most $\varepsilon^2|V_1'|^2/6\chi(H)$ blue edges, and thus we can delete a set of at most $2\varepsilon|V_1'|/3\chi(H)$ vertices of V_1' to obtain a set V_1 such that, for each $v \in V_1$, v has at most $\varepsilon|V_1'|/2$ blue neighbours in V_1.

By symmetry, for each $2 \le i \le \chi(H) - 1$, one may remove $2\varepsilon|V_i'|/3\chi(H)$ vertices from V_i' to obtain a set V_i such that each $v \in V_i$ has at most $\varepsilon|V_i'|/2$ neighbours in V_i. The set $L_3 = \bigcup_{i=1}^{\chi(H)-1} (V_i \setminus V_i')$ of vertices discarded in this step is again of size at most $\varepsilon n/3$.

Now set $L = L_1 \cup L_2 \cup L_3$, so $|L| \le \varepsilon n$. Note also that each set V_i has size at least $|C_i|/2$, so every vertex in each V_i has at most $\varepsilon|V_i|$ blue neighbours in V_i, and at most $\varepsilon|V_j|$ red neighbours in each other V_j.

Therefore the partition $V(G) = V_1 \cup \cdots \cup V_{\chi(H)-1} \cup L$ is as desired. □

Given a graph G possessing a partition as in Lemma 19, one can easily find (by the Sauer-Spencer Theorem [38]) in V_i a red copy of any graph on $|V_i|$ vertices with maximum degree at most $1/2\varepsilon$. However we would like to find a red copy of P_n^k, and our method of proof only gives sets V_i of size $(1-\varepsilon)n$. So, in order to establish the exact value of $R(P_n^k, H)$, we will have to find a way either to incorporate the vertices of the leftover set L into the sets V_i, or to show that, when this is not possible, G contains a blue copy of H. To do this, we give an embedding lemma based on the Sauer-Spencer Theorem; we shall apply it in the case where F is the red graph with vertex set consisting of one of the sets V_i together with some vertices of L, and $J = P_n^k$.

Lemma 20. *Given a natural number $\Delta \ge 1$ and any $0 < \varepsilon < 1/(\Delta^2 + 4)$, let F be an n-vertex graph in which every vertex has degree at least $3\Delta\varepsilon n$, and all but at most εn vertices have degree at least $(1 - 2\varepsilon)n$. Let J be any n-vertex graph with $\Delta(J) \le \Delta$. Then $J \subset F$.*

Proof. Let $P \subset V(F)$ be those vertices of F with degree less than $(1 - 2\varepsilon)n$. Since $|P| \le \varepsilon n < n/(\Delta^2 + 1)$, we can find a set $I \subset V(J)$ with $|I| = |P|$, and such that no two vertices of I are either adjacent or have any common neighbour in J (we simply choose vertices satisfying the conditions greedily).

Let $\phi : I \to P$ be any bijection from I to P. We construct now a partial embedding ϕ' of I together with all its neighbours $\Gamma(I)$ into F, extending ϕ. We do this by taking an enumeration $\{x_1, \ldots, x_m\}$ of $\Gamma(I)$ and, for each i in turn, choosing a vertex $y_i \in V(F)$ to be $\phi'(x_i)$ with the following properties.

First, we require that $y_i \notin \phi'(I \cup \{x_1, \ldots, x_{i-1}\})$. At most $|I \cup \Gamma(I)| \le (\Delta + 1)\varepsilon n$ vertices of F fail to satisfy this condition.

Second, for any vertex v of $I \cup \{x_1, \ldots, x_{i-1}\}$ which is adjacent to x_i, y_i must be adjacent to $\phi'(v)$. Observe that, in J, there is exactly one vertex of I adjacent to x_i and at most $\Delta - 1$ other vertices (not in I) adjacent to x_i. It follows that at most $(n - 3\Delta\varepsilon n) + (\Delta - 1)2\varepsilon n$ vertices of F fail to satisfy this condition.

Since $n > (\Delta + 1)\varepsilon n + (n - 3\Delta\varepsilon n) + (\Delta - 1)2\varepsilon n$, we will never become stuck, and the desired extension ϕ' exists.

Now let ψ be a bijection from $V(J)$ to $V(F)$ extending ϕ' and such that $|\{e \in E(J) : \psi(e) \notin E(F)\}|$ is minimised. We claim that the number of such 'bad' edges is in fact zero;

that is, ψ is an embedding of J into F, as desired.

Suppose this were false: then there is an edge ab of J such that $\psi(ab) \notin E(F)$. Because ψ extends ϕ', and ϕ' is an embedding, at least one of a and b, say b, is not in $I \cup \Gamma(I)$.

Because $b \notin I \cup \Gamma(I)$, every neighbour v of b in J satisfies $d(\psi(v)) \geq (1-2\varepsilon)n$. In particular, there are at least $(1-2\Delta\varepsilon)n$ vertices of F which are adjacent to $\psi(v)$ for every neighbour v of b.

The number of vertices of J which have a neighbour in $\psi^{-1}\big(V(F) \setminus \Gamma(\psi(b))\big)$ is at most $2\Delta\varepsilon n$, since $|V(F) \setminus \Gamma(\psi(b))| \leq 2\varepsilon n$ and J has maximum degree Δ.

Since $(1-2\Delta\varepsilon)n - 2\Delta\varepsilon n > 0$, there is a vertex c of J such that $\psi(c)$ is adjacent to $\psi(v)$ for each neighbour v of b and such that c has no neighbours in $\psi^{-1}(V(F) \setminus \Gamma(\psi(b)))$; that is, for each neighbour v of c, $\psi(v)$ is a neighbour of $\psi(b)$.

Now let $\psi' : V(J) \to V(F)$ be defined as follows.

$$\psi'(x) = \begin{cases} \psi(c) & \text{if } x = b, \\ \psi(b) & \text{if } x = c, \\ \psi(x) & \text{if } x \neq b, c. \end{cases}$$

In other words, we swap the targets under ψ of b and c.

By construction, any edge e of J which meets b or c is mapped to an edge of F by ψ'. Since $\psi'(x) = \psi(x)$ when $x \neq b, c$, any bad edge of ψ' not meeting b or c is also a bad edge of ψ; thus ψ' has at least one fewer bad edge (namely ab) than ψ, which contradicts minimality of ψ. $\square$

Now we can give the proof of Theorem 3.

Proof of Theorem 3. Given a graph H and a natural number $k \geq 2$, set $\varepsilon = 1/8k^2|H|$ and $n_0 = (20k|H|)^{16k|H|} \geq (1/\varepsilon)(200k^2|H|/\varepsilon)^{4k|H|}$. Take any $n \geq n_0$, and let G be a two-coloured complete graph on $(\chi(H)-1)(n-1)+\sigma(H)$ vertices.

By Lemma 19, if G contains neither a red copy of P_n^k nor a blue copy of H, then we have a partition $V(G) = V_1 \cup \cdots \cup V_{\chi(H)-1} \cup L$ such that the following are true.

- $|L| \leq \varepsilon n$.
- $2n/3 \leq |V_i| < n$ for each i.
- For each i and $v \in V_i$, v has at most $\varepsilon|V_i|$ blue neighbours in V_i.
- For each i and j, no vertex in V_i has more than $\varepsilon|V_j|$ red neighbours in V_j.

For each i, let C_i be the set of vertices of L which send at least $6k\varepsilon n$ red edges to V_i.

Suppose that for some i we have $|V_i \cup C_i| \geq n$. An application of Lemma 20 (with F the red graph induced on $V_i \cup C_i$, $J = P_n^k$ and $\Delta = 2k$, noting that indeed $\varepsilon < 1/(4k^2+4)$) shows that there is a red copy of P_n^k in G, which is a contradiction. It follows that, for each i, we have $|V_i \cup C_i| \le n-1$; so

$$\left| V(G) \setminus \bigcup_{i=1}^{\chi(H)-1} (V_i \cup C_i) \right| \ge |V(G)| - (\chi(H)-1)(n-1) = \sigma(H) \ .$$

Thus there is a set S of $\sigma(H)$ vertices in L, each of which sends at least $|V_i| - 6k\varepsilon n$ blue edges to V_i for each i. Take a $\chi(H)$-colouring c of H in which the part with colour $\chi(H)$ has $\sigma(H)$ vertices. We construct a blue copy of H in G greedily as follows. Let T_1 be a set of $|c^{-1}(1)|$ vertices in V_1 each of which is blue-adjacent to every member of S; let T_2 be a set of $|c^{-1}(2)|$ vertices of V_2 each of which is blue-adjacent to every member of $S \cup T_1$, and so on. The number of vertices of V_i which are red-adjacent to some member of S is at most $\sigma(H)6k\varepsilon n$; the number which are red-adjacent to some previously chosen vertex of $T_1 \cup \cdots \cup T_{i-1}$ is at most $|H|\varepsilon|V_i|$.

Since $\sigma(H)6k\varepsilon n + |H|\varepsilon|V_i| \le (6k+1)|H|\varepsilon n < n/2 < |V_i| - |H|$, we never become stuck. We obtain a blue complete $\chi(H)$-partite graph which contains H. This completes the proof. □

The proof above is not the simplest way to obtain Theorem 3: it is possible to work directly with the structure of the auxiliary graph G^* constructed in the proof of Lemma 19. However, this proof lends itself to the generalisation required to prove Theorem 4.

4 Poor expanders are always-good

In this section we prove Theorem 4. We use the same general approach as in the previous section, but here we make use of the Szemerédi Regularity Lemma [40], together with a theorem of Böttcher, Schacht and Taraz [6], in order to replace the graph P_n^k in Theorem 3 with a general graph G of bounded maximum degree and fairly small bandwidth.

Given $\varepsilon > 0$, let G be an n-vertex graph. For U and V disjoint subsets of $V(G)$, let $e(U,V)$ denote the number of edges of G between U and V, and define the *density* $d(U,V)$ of the pair (U,V) as

$$d(U,V) = \frac{e(U,V)}{|U||V|} \ .$$

We call (U,V) an *ε-regular pair* if, for all pairs of subsets $U' \subset U$ and $V' \subset V$ with $|U'| \ge \varepsilon|U|$ and $|V'| \ge \varepsilon|V|$, we have $|d(U',V') - d(U,V)| < \varepsilon$.

Suppose we have a partition $V(G) = Z_0 \cup Z_1 \cup \cdots \cup Z_r$ satisfying the following properties.

- $|Z_0| \le \varepsilon n$.
- For each $1 \le i \le r$, there are at most εr sets Z_j such that (Z_i, Z_j) is not ε-regular.
- $|Z_1| = |Z_2| = \cdots = |Z_r|$.

Then we call this partition *ε-regular*. We call the partition classes *clusters* and we refer to Z_0 as the *exceptional cluster*. In his seminal work [40], Szemerédi proved that every sufficiently large graph has an ε-regular partition in which the number of clusters is bounded by a function of ε and is independent of the number of vertices. We shall use this result in the following form.

Theorem 21 (Regularity Lemma)**.** *For any $\varepsilon > 0$ and k_0, there exist K and n_0 such that, whenever $n \geq n_0$ and G is an n-vertex graph, G possesses an ε-regular partition with between k_0 and K clusters.*

When we have an ε-regular partition of a graph G, we associate with it a *cluster graph* $R(G)$ whose nodes are the clusters of the partition (excluding Z_0) and whose edges correspond to ε-regular pairs of clusters – possibly only those whose density is above some given *density threshold d*. One can easily prove that under very simple conditions, if $R(G)$ contains a fixed graph H, then G must also contain H as a subgraph. This is summarized in the following lemma (see, for instance, Diestel [19]).

Lemma 22. *For every $d > 0$, $\Delta \geq 1$, there exists $\varepsilon_{22} = \varepsilon_{22}(d, \Delta) \leq 1/2$ with the following property. Let G be an n-vertex graph and $R(G)$ be a cluster graph with $r \leq d^{\Delta} n/4$ clusters, $\varepsilon \leq \varepsilon_{22}$, and with density threshold d. Then, for every graph H with $\Delta(H) \leq \Delta$, if $R(G)$ contains H as a subgraph, then G also contains H.*

To handle bounded degree graphs with small bandwidth, we require the following theorem, essentially due to Böttcher, Schacht and Taraz [6].

Theorem 23 (Böttcher, Schacht and Taraz [6])**.** *For any $\mu, \gamma > 0$ and for any natural numbers χ and Δ there exists $\varepsilon_{23} > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_{23}$, there is a K_{23} such that, for all $K \geq K_{23}$, there exist $\beta > 0$ and n_{23} such that the following holds for all $n \geq n_{23}$ and $0 \leq \eta < 1$.*

Let F be an n-vertex graph, and $R(F)$ the cluster graph corresponding to an ε-regular partition of F with $r \leq K$ parts whose edges correspond to ε-regular pairs of density at least γ. Suppose that in $R(F)$ there is a copy of $P^{\chi-1}_{(\eta+\mu)r}$ with the further property that every χ-clique is contained in a $(\chi+1)$-clique of $R(F)$. Then whenever G is an ηn-vertex graph with maximum degree Δ, chromatic number χ and bandwidth βn, we have $G \subset F$.

To be specific, Lemma 8 of [6] provides a graph homomorphism with certain additional desirable properties from G to $R(F)$, in particular that no vertex of $R(F)$ is the target of 'too many' vertices of G. Since in our situation we seek to embed only ηn vertices into $(\eta+\mu)r$ clusters (and therefore we have at least $(\eta+\mu/2)n$ vertices in the union of the clusters of the $P^{\chi-1}_{(\eta+\mu)r}$ in $R(F)$ but only ηn in G, as opposed to the requirement in [6] for a spanning embedding), we may in particular presume that for each i the homomorphism allocates at most $(1-\mu/4)|Z_i|$ vertices of G to the cluster Z_i. One can then complete the embedding of G into F by using the Blow-up Lemma of Komlós, Sárközy and Szemerédi [31] (again following the method of [6], Proof of Theorem 2). It should be emphasized that the major source of difficulty in [6] is the requirement for a spanning embedding: obtaining an embedding covering a $(1-\mu)$-fraction of F (essentially our situation) is relatively trivial.

Whenever we use this, we will in fact find a copy of $P^{\chi}_{(\eta+\mu)r}$ in $R(F)$; this certainly contains a copy of $P^{\chi-1}_{(\eta+\mu)r}$ in which every χ-clique extends to a $(\chi+1)$-clique. For convenience, we presume the parameter n_{23} is chosen to be at least as large as required for Theorem 21 to provide an ε-regular partition.

In our proof of Theorem 4, we shall use Theorem 23 with $\eta = 1/(\chi(H)-1)$. Since it is required that $\eta < 1$, we need to treat separately the case when H is bipartite: we appeal to Theorem 7.

Our general strategy for proving Theorem 4 is very similar to that in the proof of Theorem 3. We will need to be able to apply a version of our stability result, Lemma 19, to cluster graphs. This means that we need the following variant of Lemma 19, where our two-coloured graphs are not complete, but rather have minimum degree $(1-\varepsilon)n$ for some $\varepsilon > 0$ (whose size we may choose as small as we desire).

Lemma 24 (Modified Lemma 19)**.** *Let H be a graph, k a natural number, and ε' a positive constant at most $1/2|H|^2$. There exists $\varepsilon_{24} < \varepsilon'$ such that for every positive $\varepsilon < \varepsilon_{24}$ there is an n_{24} for which the following holds. If $n \geq n_{24}$ and G is a two-coloured graph on $N \geq (\chi(H)-1)n - n/6$ vertices with $\Delta(\bar{G}) < \varepsilon n$, which contains neither a red copy of P^k_n nor a blue copy of H, then there is a partition $V(G) = V_1 \cup \cdots \cup V_{\chi(H)-1} \cup L$ with the following properties.*

- *$|L| \leq \varepsilon' n$.*
- *For each i, $2n/3 \leq |V_i| < n$.*
- *For each i and $v \in V_i$, v has at most $\varepsilon'|V_i|$ blue neighbours in V_i.*
- *For each i and j, no vertex in V_i has more than $\varepsilon'|V_j|$ red neighbours in V_j.*

Proof. (Sketch) This is an entirely straightforward modification of the proof of Lemma 19. There are two changes which must be made.

First, we can no longer use the Erdős-Szekeres bound $R(K_s, K_{|H|}) \leq \binom{|H|+s}{|H|}$ to find red s-cliques. It is easy to prove (albeit with slightly worsened bounds) that, if n is large enough, then any two-coloured graph with minimum degree $(1-\varepsilon)n$ contains either K_s or $K_{|H|}$.

Second, the auxiliary graph G^* must be defined slightly differently. Just as before, we say two cliques Q_i and Q_j are red-adjacent if the induced bipartite graph $G[Q_i, Q_j]$ contains a red $K_{2k,2k}$. If however Q_i and Q_j are not red-adjacent, then we have two possibilities. If there are at least $2\sqrt{\varepsilon}|Q_i||Q_j|$ non-edges of G in $G[Q_i, Q_j]$ then Q_i and Q_j are non-adjacent in G^*. If Q_i and Q_j are not red-adjacent, and the number of non-edges of G in $G[Q_i, Q_j]$ is less than $2\sqrt{\varepsilon}|Q_i||Q_j|$, then Q_i and Q_j are blue-adjacent. Observe that if there is a vertex Q_i with $\sqrt{\varepsilon}v(G^*)$ non-neighbours in G^*, then there are at least $2\sqrt{\varepsilon}|Q_i|(v(G^*)-1)|Q_i|$ nonedges in G^*

adjacent to Q_i, so a vertex $v \in Q_i$ of minimum degree in G has more than εn non-neighbours in G. This contradiction yields $\Delta(\bar{G^*}) < \sqrt{\varepsilon}v(G^*)$.

The rest of the proof goes through unchanged: note that since G^* is now not a complete graph, we use the full strength of Lemma 18 in this setting. $\square$

We are now ready to prove the main result of this section.

Proof of Theorem 4. Fix a graph H with $\chi(H) \geq 3$ and an upper bound Δ on the maximum degree of G. We need to show that, if $\beta > 0$ is sufficiently small, then for all sufficiently large n, every connected n-vertex graph G with $\Delta(G) \leq \Delta$ and $\mathrm{bw}(G) \leq \beta n$ satisfies $R(G, H) = (\chi(H) - 1)(n - 1) + \sigma(H)$.

We set $\gamma = \frac{1}{200\Delta^2|H|}$ and $\mu = \frac{1}{12\chi(H)^2}$, and choose ε' such that

$$\varepsilon' < \min\left\{\varepsilon_{22}(1/2, \Delta), \frac{1}{2|H|^2}, \frac{1}{\Delta^2+4}\right\} \text{ and } 4(\chi(H)+3)\varepsilon'+8\gamma < \min\left\{\frac{1}{\Delta^2+4}, \frac{1}{(6\Delta+2)|H|}\right\}$$

hold. We then choose ε such that $\varepsilon < \varepsilon'$, $\varepsilon < \varepsilon_{23}(\mu, \gamma, \chi(H), \Delta)$, and $\varepsilon < \varepsilon_{24}(H, \chi(H), \varepsilon')$.

Let k_0 be such that $k_0 \geq 2/\varepsilon$, $k_0 > K_{23}(\mu, \gamma, \chi(H), \Delta, \varepsilon)$, and $k_0 > n_{24}(H, \chi(H), \varepsilon', \varepsilon)$. Let K and n_0 be constants such that the conclusion of Theorem 21 holds, with parameters ε and k_0, so in particular $K \geq k_0 > K_{23}(\mu, \gamma, \chi(H), \Delta, \varepsilon)$. Now let $\beta > 0$ and $n_{23} \geq n_0$ be constants such that the conclusion of Theorem 23 holds.

Take any $n \geq \max(n_{23}, K2^{\Delta+2})$, and set $\eta = 1/(\chi(H)-1) \leq 1/2$. Let G be any connected n-vertex graph with $\mathrm{bw}(G) \leq \beta n$ and $\Delta(G) \leq \Delta$, so $\chi(G) \leq \Delta + 1$.

Let F be a two-coloured complete graph on $(\chi(H) - 1)(n - 1) + \sigma(H)$ vertices. Our aim is to prove that F contains either a red copy of G or a blue copy of H.

By applying Theorem 21 to the red graph of F, we obtain an ε-regular partition, and hence a cluster graph $R(F)$ with some number r of vertices, $k_0 \leq r \leq K$. By moving the vertices of at most $\chi(H)$ clusters to the exceptional set Z_0, we may assume that $r = (\chi(H) - 1)m$ for some integer m. When (A, B) is a pair of clusters which is ε-regular, we have an edge AB in $R(F)$. This graph $R(F)$ is very nearly complete: each vertex has degree at least $(1 - \varepsilon)r$. When the density of red edges in (A, B) is more than γ, we colour AB red, otherwise we colour it blue. Notice that if AB is blue, then the density of blue edges in (A, B) is at least $1/2$.

If there is a blue copy of H in $R(F)$, then, by Lemma 22 with $d = 1/2$, F contains a blue copy of H, and we are done.

Also, if there is a red copy of $P^{\chi(H)}_{(\eta+\mu)r}$ in $R(F)$, then, by Theorem 23, there is a red copy of G in F, and again we are done.

Observe that $(\chi(H) - 1)(\eta + \mu)r - (\eta + \mu)r/6 < r$. Thus if $R(F)$ contains neither a red $P^{\chi(H)}_{(\eta+\mu)r}$ nor a blue H, then, by Lemma 19, we have a partition $V(R(F)) = W_1 \cup \cdots \cup W_{\chi(H)-1} \cup L'$ with the following properties.

- $|L'| \le \varepsilon'(\eta+\mu)r$.
- For each i, $2(\eta+\mu)r/3 \le |W_i| < (\eta+\mu)r$.
- For each i and $v \in W_i$, v has at most $\varepsilon'|W_i|$ blue neighbours in W_i.
- For each i and j, no vertex in W_i has more than $\varepsilon'|W_j|$ red neighbours in W_j.

Consider the cluster $A \in W_i$, and let $j \neq i$. Set

$$\delta = 2\left(4\varepsilon' + 2\varepsilon(\chi(H)-1) + 4(\gamma+\varepsilon)\right),$$

and let A' be the set of vertices in A which send more than $\delta n/2$ red edges to W_j. Suppose that $|A'| \ge \varepsilon|A|$.

In $R(F)$, A sends red edges to at most $\varepsilon'|W_j|$ clusters of W_j. To these clusters, A' sends at most $\varepsilon'|W_j| \cdot |A'|(N/r)$ red edges.

There are further at most εr clusters of W_j which are not adjacent in $R(F)$ to A, corresponding to non-ε-regular pairs in F. To these clusters, A' sends at most $\varepsilon r \cdot |A'|(N/r)$ red edges.

The remaining clusters of W_j are linked by blue edges in $R(F)$ to A. Hence, their red density is at most γ and they are ε-regular. Using ε-regularity, the total number of red edges from A' to these clusters is bounded by $|W_j| \cdot (\gamma+\varepsilon)|A'|(N/r)$.

Using that $|W_i| < (\eta+\mu)r = m + \mu r < 2m$ and $N/r = n/m + \sigma(H)/r < 2n/m$, we obtain

$$e_{\mathrm{red}}(A', \bigcup W_j) < (4\varepsilon' + 2\varepsilon(\chi(H)-1) + 4(\gamma+\varepsilon))|A'|n = \delta|A'|n/2.$$

On the other hand, it follows from the definition of A' that $e_{\mathrm{red}}(A, \bigcup W_j) > \delta n|A'|/2$, which is a contradiction. Hence, $|A'| < \varepsilon|A|$.

Since this holds for each cluster of W_i and every $j \neq i$, we can remove at most $(\chi(H)-2)\varepsilon|\bigcup W_i|$ vertices from $\bigcup W_i$ to obtain a set V_i of vertices of F which sends at most $\delta n/2$ red edges to $\bigcup W_j$ for every $j \neq i$.

Since $|V_i| > n/2$ for each i, we certainly have that for each $i \neq j$, every vertex in V_i has at most $\delta|V_j|$ red neighbours in V_j.

Given i, by an identical argument to that in the proof of Lemma 19, if there are more than $\delta^2|V_i|^2/6$ blue edges in V_i, then V_i contains a blue copy of $K_{|H|,|H|}$ which we can extend to a blue copy of H in F. It follows that we can remove at most $2\delta|V_i|/3$ vertices from V_i to obtain a set V_i' such that every vertex in V_i' has at most $\delta|V_i|/2$ blue neighbours in V_i. Thus, for each i, every vertex in V_i' has at least $(1-\delta)|V_i'|$ red neighbours in V_i'.

We can now complete the proof in an identical fashion to the proof of Theorem 3. We let the set L contain all those vertices of F which are in no set V_i'. We let C_i be the set of vertices in L which send at least $3\Delta\delta n$ edges to V_i'. By Lemma 20, if for some i we have $|V_i' \cup C_i| \ge n$, then we can find a red copy of G in F.

But if for each i we have $|V_i' \cup C_i| \le n-1$, then we can find a set S of $\sigma(H)$ vertices of L which each sends at least $(1-3\Delta\delta)n$ blue edges to each set V_i'. As in the proof of Theorem 3, we can now construct a blue copy of H in F greedily. This completes the proof. $\square$

5 Multi-colour problems

In this section, we prove Theorem 8, stating that if $\mathcal{G}$ is an always-good family of graphs, then $\mathcal{G}$ is also multicolour-always-good; that is, given $r \ge 2$ and graphs $H_1, \ldots, H_r$, there are integers W and Z (not depending on $\mathcal{G}$) such that, for all sufficiently large $G \in \mathcal{G}$, $R(G, H_1, H_2, \ldots, H_r) = W(|G|-1) + Z$.

Our first step is to give explicit definitions of the constants W and Z as the solutions to two further Ramsey-type problems involving $H_1, \ldots, H_r$.

The first is a variant of the standard Ramsey problem. We define the *homomorphism Ramsey number* $R_{\text{hom}}(H_1, \ldots, H_r)$ to be the smallest N such that, if F is any r-coloured complete graph on N vertices, there exists a colour i such that there is a graph homomorphism from H_i into the ith colour subgraph of F. Then we let

$$W = R_{\text{hom}}(H_1, \ldots, H_r) - 1.$$

It is clear that $R_{\text{hom}}(H_1, \ldots, H_r) \le R(H_1, \ldots, H_r)$, and sometimes this inequality is sharp – if all the graphs are complete graphs – but in general it is not; for example, $R_{\text{hom}}(C_3, C_5) = 5$ although $R(C_3, C_5) = 9$. It may be of independent interest to investigate the properties of R_{hom} further.

Given a graph G, a vertex x of G, and an integer $m \ge 1$, the *m-blow-up* of G at x is the graph obtained by replacing x with an independent set $\{x_1, \ldots, x_m\}$, each vertex of which has the same adjacencies as x has in G. The *m-blow-up* of the graph G is the graph obtained by blowing up by m at each vertex of G. It is clear that there is a homomorphism of H_i into the ith colour subgraph of G if and only if there is some m-blow-up of G whose ith colour subgraph contains H. Hence R_{hom} can be also defined in terms of blow-ups.

We define Z as the smallest natural number N with the following property. For any labelled graph F on $W + N$ vertices, with all edges incident to at least one of the first W vertices present, if F is r-coloured and F' is obtained from F as the m-blow-up of the first W vertices for some sufficiently large $m = m(H_1, \ldots, H_r)$, then there is some i such that H_i is contained within the ith colour subgraph of F'.

It is clear that, for any connected n-vertex graph G with $n \ge Z$, $R(G, H_1, \ldots, H_r) \ge W(n-1)+Z$. Indeed, let F be a labelled r-coloured graph on $W+Z-1$ vertices demonstrating that Z cannot be replaced by $Z-1$. We obtain an r-coloured complete graph F' on $W(n-1) + Z - 1$ vertices by taking the $(n-1)$-blow-up of the first W vertices of F and then replacing every non-edge with a red edge. Then certainly there is no red copy of G in F', and, by the definition of Z, for each $i \in [r]$, there is no copy of H_i in F'.

We now prove Theorem 8.

Proof. Given an always-good class of graphs $\mathcal{G}$, let $r \geq 2$ be any integer and $H_1, \ldots, H_r$ any collection of r graphs. Let W, Z, and $m = m(H_1, \ldots, H_r)$ be defined as above.

Suppose that ℓ is some integer sufficiently large that the following procedure succeeds.

Let F denote a complete $(W+1)$-partite graph $K_{\ell,\ell,\ldots,\ell,Z}$, with an r-colouring of its edges. Let $(U_1, V_1), \ldots, (U_{\binom{W}{2}}, V_{\binom{W}{2}})$ be an enumeration of all the pairs of parts of F, excluding the last part. Set Γ_0 equal to the set of vertices in these first W parts.

Now, for each $1 \leq i \leq \binom{W}{2}$ in turn, consider the complete bipartite r-coloured subgraph J_i of F induced by the pair $(U_i \cap \Gamma_{i-1}, V_i \cap \Gamma_{i-1})$. Let B_i be a maximum-size monochromatic complete balanced bipartite subgraph of J_i, and form Γ_i by deleting from Γ_{i-1} all vertices in $U_i \cap \Gamma_{i-1}$ and $V_i \cap \Gamma_{i-1}$ except those in B_i.

Observe that, at each step i, the r-coloured complete bipartite graph J_i must have one colour present with edge density at least $1/r$, and thus the Kövári-Sós-Turán Theorem (Theorem 16) provides a lower bound on the size of the monochromatic complete balanced bipartite subgraph B_i found at step i. In particular, by choosing ℓ sufficiently large, we may conclude that, at the end of the process, the set $U \cap \Gamma_{\binom{W}{2}}$ contains at least $r^Z m$ vertices for every part U of F, except the last one.

Now let F' be the r-coloured $(W+1)$-partite graph obtained from F by removing all vertices $\Gamma_0 - \Gamma_{\binom{W}{2}}$ – in other words, we take the r-coloured complete W-partite graph induced on $\Gamma_{\binom{W}{2}}$ and add back the last part of F. By construction, the edges between any two of the first W parts of F' form a monochromatic complete bipartite graph. Let F'' be obtained by deleting from the first W parts of F' a minimum set of vertices such that the edges from any vertex in the $(W+1)$-st part of F'' to any of the first W parts are monochromatic. By choice of ℓ, the first W parts of F each still contain at least m vertices. By the definition of W and Z, for some i, a copy of H_i of colour i is contained in F'', and, hence, in F.

Now, because $\mathcal{G}$ is always-good, in particular it is H-good for $H = K_{\ell,\ell,\ldots,\ell,Z}$. Note that $\chi(H) = W + 1$ and $\sigma(H) = Z$. Thus, whenever $G \in \mathcal{G}$ is sufficiently large, and F is any $\{\text{red}\} \cup [r]$-coloured complete graph on $W(|G| - 1) + Z$ vertices, either F contains a red copy of G, or F contains an r-coloured copy of $H = K_{\ell,\ell,\ldots,\ell,Z}$, and hence a copy of H_i of colour i for some $i \in [r]$. □

In the proof above, the required size of ℓ is a tower of height $O(W^2)$, and in turn W can be very large in comparison to the small graphs—for instance, if each of the small graphs is the clique K_s, then $W = 2^{\Omega(s)}$.

As an illustration of the use of Theorem 8, we show how to find the Ramsey numbers for a collection of odd cycles, provided they are suitably long.

Corollary 25. *For any odd integers $\ell_1, \ldots, \ell_r$, with $\ell_s > 2^s$ for each $1 \leq s \leq r$, and every sufficiently large n,*

$$R(C_n, C_{\ell_1}, \ldots, C_{\ell_r}) = 2^r(n-1) + 1.$$

Proof. Since the family of cycles is always-good by Theorem 3, and thus by Theorem 8 multicolour-always-good, we need only solve the two auxiliary Ramsey-type problems to find W and Z.

We first need to show that $R_{\mathrm{hom}}(C_{\ell_1}, \ldots, C_{\ell_r}) = 2^r + 1$. For the lower bound, it is a standard result that the edge-set of K_{2^r} can be partitioned into r colour classes so that no colour class contains an odd cycle. For such a colouring, there is no homomorphism from the odd cycle C_{ℓ_i} to the ith colour class, for any i.

For the upper bound, we proceed by induction on r. The result is trivial for $r = 1$, so we suppose $r > 1$. For any r-coloured K_{2^r+1}, either there is an odd cycle Q in colour r, in which case there is a homomorphism from C_{ℓ_r} to Q – here we use the assumption that $\ell_r \geq 2^r + 1$ – or the graph of edges coloured by r is bipartite, in which case one of its parts contains an $(r-1)$-coloured $K_{2^{r-1}+1}$, and the result follows by induction.

Secondly, we need to show that $Z = 1$. Any labelled graph F on $W + Z = R_{\mathrm{hom}}(C_{\ell_1}, \ldots, C_{\ell_r})$ vertices, with all edges incident to at least one of the first W vertices present, is complete. Hence, if F is r-coloured, then, for some i, there is a homomorphism from C_{ℓ_i} into the ith colour subgraph of F. Thus, F contains an odd cycle Q of length at most ℓ_i in colour i.

For $m = \max(\ell_1, \ldots, \ell_r)$, let F' be obtained from F as the m-blow-up of the first W vertices. Given an odd cycle Q in colour i, we have freedom to choose a homomorphism which maps only one vertex of C_{ℓ_i} to a chosen vertex of Q. By m-blowing-up the remaining vertices of Q, we obtain enough room to embed the remaining vertices of C_{ℓ_i}. $\square$

A well-known problem raised by Bondy and Erdős [4] is to determine the r-colour Ramsey number $R(C_n, \ldots, C_n)$, when n is odd. The lower bound $2^{r-1}(n-1) + 1$ is pointed out in that paper, and this is widely believed to give the correct value of the Ramsey number provided n is sufficiently large – our result above may be seen as giving a weak support for that conjecture. The conjecture was proved in the case $r = 3$ by Kohayakawa, Simonovits and Skokan [29]: for n odd and sufficiently large, $R(C_n, C_n, C_n) = 4n - 3$.

6 Powers of paths and cycles against themselves

Our purpose in this section is to give both upper and lower bounds on the Ramsey numbers $R(P_n^k, P_n^k)$ and $R(C_n^k, C_n^k)$, for fixed k and large n. In the next section, we will move on to consider general graphs with bounded maximum degree and limited bandwidth.

We start with a construction giving a lower bound better than the one from Burr's construction (Lemma 1). We begin by assuming that n is a multiple of $k+1$: for convenience we restate Theorem 10.

Theorem 10. *For $k \geq 2$,*

$$R(C_{(k+1)t}^k, C_{(k+1)t}^k),\ R(P_{(k+1)t}^k, P_{(k+1)t}^k) \geq t(k+1)^2 - 2k\ .$$

Note that $\chi(C^k_{(k+1)t}) = \chi(P^k_{(k+1)t}) = k+1$, while $\sigma(C^k_{(k+1)t}) = \sigma(P^k_{(k+1)t}) = t$, so Lemma 1 gives the lower bound $k[(k+1)t-1]+t$ on both Ramsey numbers, which is $kt-k$ below the value in Theorem 10.

Proof. We colour $K_{t(k+1)^2-2k-1}$ as follows. Partition $[t(k+1)^2-2k-1]$ into disjoint sets $A_1, \dots, A_k$ each on $kt-1$ vertices, $B_1, \dots, B_k$ each on $2t-1$ vertices, and C on $t-1$ vertices.

Now colour edges as follows. Within each set A_i we have only red edges. Within each set B_i we have only blue edges. Between two sets A_i and A_j, $i \neq j$, we have only blue edges; between B_i and B_j, $i \neq j$, only red edges.

For each i, we have only red edges between A_i and B_i, while between A_i and B_j for $i \neq j$ we have only blue edges. Finally, we take any colouring within C, and join all its vertices in blue to every A_i and in red to every B_i.

In the red graph, any copy of P^k_m, for any $m > k$, with one vertex in a set A_i must lie entirely within $A_i \cup B_i$, A_i is too small to contain k colour classes of $P^k_{(k+1)t}$, hence, B_i must contain two vertices from two distinct vertex classes, which is impossible because all its edges are blue. But if the sets A_i are not to be used, then an entire colour class of $P^k_{(k+1)t}$ would have to lie in C, which is again too small.

The argument showing that there is no copy of $P^k_{(k+1)t}$ in the blue graph is very similar. Suppose there is such a copy Q, and suppose first that it includes some vertex v of some B_i. The set T of the next k vertices on Q forms a blue clique adjacent to v, so there is at most one vertex of T in each of the A_j with $j \neq i$, and so at least one vertex of T in B_i. Thus Q lies within $B_i \cup \bigcup_{j \neq i} A_j$. Moreover, at most $k-1$ out of each set of $k+1$ consecutive vertices on Q are in the A_j, and so there are at least $2t$ vertices of Q in B_i, which is impossible. As before, if the sets B_i are not used for Q, then an entire colour class of Q would lie in C, which is again too small. $\square$

For $k > 2$, the construction above can be generalised. First we take an auxiliary red-blue-coloured graph J, which is a copy of $K_{k,k}$, with parts $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$, with the property that each a_i is incident with at least one blue edge and each b_i with at least one red edge. Each such J will give us a different construction of a two-coloured graph on $t(k+1)^2-2k-1$ vertices with no monochromatic $P^k_{(k+1)t}$, as follows. Take disjoint vertex sets $A_1, \dots, A_k, B_1, \dots, B_k, C$, with $|C| = t-1$. The set A_i has $(\ell_i+1)t-1$ vertices, where ℓ_i is the number of blue edges incident with a_i in J, and the set B_i has $(m_i+1)t-1$ vertices, where m_i is the number of red edges incident with b_i in J. The total number of vertices is always $t(k+1)^2-2k-1$.

As before, within each A_i we have red edges, and between different A_i we have blue edges, while within each B_i we have blue edges, and between different B_i we have red edges. The colouring inside C is arbitrary, and its vertices are joined in blue to every A_i and in red to every B_i. The edges between A_i and B_j all have the colour of the edge between a_i and b_j in J. The proof that such a two-coloured graph contains no monochromatic $P^k_{(k+1)t}$ is similar to that in Theorem 10.

When $k+1$ does not divide n, still Burr's construction can be improved upon. For powers of paths, the adjustments required are small, so we concentrate on powers of cycles.

Theorem 26. *For $k \geq 2$ and any $1 \leq r \leq k$,*

$$R(C^k_{(k+1)t+r}, C^k_{(k+1)t+r}) \geq (k+1)[(k+2)t + 2r - 2] + r \ .$$

The lower bound on $R(C^k_{(k+1)t+r}, C^k_{(k+1)t+r})$ coming from Lemma 1 is $(k+1)[(k+1)t + r - 1] + r$.

Proof. We colour $K_{(k+1)((k+2)t+2r-2)+r-1}$ as follows. Partition $[(k+1)((k+2)t+2r-2)+r-1]$ into disjoint sets $A_1, \dots, A_{k+1}$ each on $kt + r - 1$ vertices, $B_1, \dots, B_{k+1}$ each on $2t + r - 1$ vertices, and C on $r - 1$ vertices.

Now we colour edges as in the proof of the previous theorem. The proof that such a two-coloured graph contains no monochromatic $C^k_{(k+1)t+r}$ follows the proof of Theorem 10 and we omit it here. □

We believe that the constructions above are, at least asymptotically, optimal.

The rest of this section is devoted to proving the upper bounds on $R(P^k_n, P^k_n)$ and $R(C^k_n, C^k_n)$ given in Theorem 11. Our first step is to prove an upper bound on $R(P_n, P^k_n)$, for which we need the following three results.

First, we recall the Erdős-Gallai extremal theorem for cycles [22].

Theorem 27 (Erdős-Gallai [22])**.** *Let G be a graph on n vertices, and c an integer, $3 \leq c \leq n$. Then either G contains a cycle of length at least c or*

$$e(G) < (c-1)(n-1)/2 + 1 \ .$$

Second, we need a result on maximum cycles in graphs. The lemma below is simple, and convenient for our purposes: a much stronger result has recently been proved by Kohayakawa, Simonovits and Skokan [30]

Lemma 28. *Given a graph G containing vertex disjoint cycles C_t and $C_{t'}$, if G contains no cycle of length greater than t, then the bipartite graph $G[V(C_t), V(C_{t'})]$ contains no copy of $K_{s,s}$, where $s = \left\lceil \frac{t}{t'} \right\rceil + 2$.*

Proof. Suppose not, and let G, C_t, $C_{t'}$ form a counterexample. Now G contains a copy of the bipartite graph $K_{s,s}$ whose parts are in $V(C_t)$ and $V(C_{t'})$, so in particular there are two vertices of this complete bipartite graph in C_t which are joined in C_t by a path P of length at least $\frac{s-1}{s}t$, and two more in $C_{t'}$ joined by a path P' in $C_{t'}$ of length at least $\frac{s-1}{s}t'$. The vertices in $V(P) \cup V(P')$ form a cycle of length at least $\frac{s-1}{s}(t + t') > t$, which is a contradiction. □

Third, a standard greedy method allows us to find a copy of P^k_n in a very dense graph on only slightly more than n vertices.

Lemma 29. *Let k and n be natural numbers, and ε a real number, satisfying $0 < \varepsilon \leq (k+3)^{-1}$ and $n > 3\varepsilon^{-2}$. If H is any graph on at least $n + (k+2)\varepsilon n$ vertices, such that the complement $\overline{H}$ contains no cycle of length at least $\varepsilon^2 n$, then H contains a copy of P_n^k.*

Proof. By Theorem 27, $\overline{H}$ has at most $(\varepsilon^2 n - 1)(|H| - 1)/2 + 1 < \varepsilon^2 |H| n/2$ edges. If $\overline{H}$ had more than $n + k\varepsilon n$ vertices of degree greater than εn, then it would have at least $(n + k\varepsilon n)\frac{\varepsilon n}{2}$ edges, which is a contradiction. So at least $n + k\varepsilon n$ vertices of $\overline{H}$ have degree less than εn. Let H' be the subgraph of H induced by these vertices. Then H' has at least $n + k\varepsilon n$ vertices, the neighbourhood of any set of k vertices of $\overline{H'}$ contains at most $k\varepsilon n$ vertices, and so, in H', every set of k vertices has at least n common neighbours. We can embed P_n^k into H' by a simple greedy procedure: we choose any vertex to be the first vertex of the path, any neighbour to be the second vertex of the path, and so on. At each embedding step, we only need to find a vertex which is adjacent to all of the last k vertices embedded, and which has not yet been used in the embedding. Such a vertex is guaranteed to exist since any k vertices of H' have at least n common neighbours, and we only need to embed a total of n vertices. $\square$

Now we can prove our upper bound on the Ramsey number of a path versus a power of a path.

Lemma 30. *For any natural number k,*

$$R(P_n, P_n^k) \leq \left(k + 1 + \frac{1}{k+1}\right) n + o(n) \ .$$

Note that this upper bound is significantly larger than the lower bound $R(P_n, P_n^k) \geq k(n-1) + \sigma(P_n^k) \sim \left(k + \frac{1}{k+1}\right) n$. We conjecture that the lower bound is correct. An improvement in this upper bound would improve the upper bound in Theorem 11 by a corresponding amount, but this is not the source of the factor of 2 between our lower and upper bounds.

Proof. We show that, for any $0 < \varepsilon \leq (k+3)^{-1}$, the Ramsey number $R(P_n, P_n^k)$ is bounded above by

$$\left(k + 1 + \frac{1}{k+1} + (k+3)\varepsilon\right) n \tag{1}$$

for

$$n > \left(16(2k+1)\varepsilon^{-8}\right)^{4\varepsilon^{-2}} \ . \tag{2}$$

Accordingly, we assume that n is indeed greater than $\left(16(2k+1)\varepsilon^{-8}\right)^{4\varepsilon^{-2}}$. Let G be a two-edge-coloured complete graph on $\left(k + 1 + \frac{1}{k+1} + (k+3)\varepsilon\right) n$ vertices which contains no red P_n. We choose successively vertex-disjoint maximum-length red cycles in G. Let V_1 be the vertex set of the longest red cycle of G, V_2 the vertex set of the longest red cycle of $G - V_1$, and so on.

Since $P_n \subset C_n$, we have $n - 1 \geq |V_1| \geq |V_2| \geq \cdots$. Let r be the greatest index such that $|V_r| \geq \varepsilon^2 n$, and let $W = V(G) - \bigcup_{i=1}^{r} V_i$. Since the sets V_i are disjoint, we have $r \leq (k + 1 + \frac{1}{k+1} + (k+3)\varepsilon)\varepsilon^{-2} < \varepsilon^{-3}$, independently of n.

If $|W| \geq n + (k+2)\varepsilon n$, then the graph of blue edges in W satisfies the conditions of Lemma 29, so G contains a blue copy of P_n^k. Therefore we will assume $|W| < n + (k+2)\varepsilon n$.

Let $s = \left\lceil \frac{n}{\varepsilon^2 n} \right\rceil + 2 < 2\varepsilon^{-2}$. By Lemma 28, for any $1 \leq i < j \leq r$, there is no red copy of $K_{s,s}$ in G whose parts are in V_i and V_j respectively. We wish to use this together with Lemma 17 to find a blue copy of P_n^k (which has maximum degree $2k$). We will use the fact that P_n^k is a subgraph of the complete $(k+1)$-partite graph with parts of size $\left\lceil \frac{n}{k+1} \right\rceil$. Observe that no part V_i has size greater than n, and the union of all the parts has size at least $\left(k + \frac{1}{k+1} + \varepsilon\right) n$.

Now choose ℓ_1 to be the smallest index such that

$$\sum_{i=1}^{\ell_1} \left(|V_i| - 4s^2 n^{\frac{2s-1}{2s}}(2k+1)\right) \geq \left\lceil \frac{n}{k+1} \right\rceil .$$

Since $4s^2 n^{\frac{2s-1}{2s}}(2k+1)r < \varepsilon n$ (here we use (2)), by (1) this is possible and, furthermore, $\sum_{i=1}^{\ell_1} |V_i| < n$ (in fact, this sum can exceed $2\left\lceil \frac{n}{k+1} \right\rceil + \varepsilon n$ only when $\ell_1 = 1$).

For each $2 \leq j \leq k$ in succession, let ℓ_j be the smallest index such that

$$\sum_{i=\ell_{j-1}+1}^{\ell_j} \left(|V_i| - 4s^2 n^{\frac{2s-1}{2s}}(2k+1)\right) \geq \left\lceil \frac{n}{k+1} \right\rceil .$$

Again, this is possible because $\sum_{i=1}^{\ell_{j-1}} |V_i| < (j-1)n$ and $4s^2 n^{\frac{2s-1}{2s}}(2k+1)r < \varepsilon n$, and we also have $\sum_{i=\ell_{j-1}+1}^{\ell_j} |V_i| < n$.

We apply Lemma 17 to the parts $V_1, \ldots, V_r$ of G. Let $V_1', \ldots, V_r'$ be the parts of G' as in the lemma; since for each $1 \leq i < j \leq r$ the sets V_i and V_j are blue-adjacent, the parts V_i' and V_j' span a complete bipartite graph. Let $W_1 = \bigcup_{i=1}^{\ell_1} V_i'$, $W_j = \bigcup_{i=\ell_{j-1}+1}^{\ell_j} V_i'$ for each $2 \leq j \leq k$, and $W_{k+1} = \bigcup_{i=\ell_k+1}^{r} V_i'$. Since $|W_1|, \ldots, |W_k| < n$ and (1) holds, we are guaranteed to find that $|W_{k+1}| \geq \lceil \frac{n}{k+1} \rceil$. The W_j form the parts of a complete $(k+1)$-partite subgraph of G', so that P_n^k can be embedded into G'. By Lemma 17, G contains a blue copy of P_n^k. $\square$

It is now straightforward to prove our desired bounds on $R(P_n^k, P_n^k)$ and $R(C_n^k, C_n^k)$.

Proof of Theorem 11. Given $\varepsilon > 0$, let $s = s(n)$ be any sufficiently slowly growing function of n and n_0 be any sufficiently large integer. Suppose $n > n_0$.

Suppose first that we seek either a monochromatic P_n^k, or a monochromatic C_n^k where $k+1$ divides n. Let G be a two-coloured complete N-vertex graph, where

$$N = \left(2k + 2 + \frac{2}{k+1}\right) n + \varepsilon n \ .$$

We partition $V(G)$ into a collection $\mathcal{R}$ of red s-cliques, $\mathcal{B}$ of blue s-cliques, and a leftover set of at most 2^{2s} vertices. Without loss of generality, we assume that $|\mathcal{R}| \geq |\mathcal{B}|$.

We call two cliques R_i and $R_j \in \mathcal{R}$ red-adjacent when $G[R_i, R_j]$ contains a copy of $K_{4k,4k}$, and blue-adjacent otherwise. This defines the two-coloured complete graph G^* on $\mathcal{R}$. For $t = n/s + \varepsilon n/4s(k+2)$, we have

$$|G^*| \geq \frac{N - 2^{2s}}{2s} \geq \frac{N - \varepsilon n/2}{2s} \geq \left(k + 1 + \frac{1}{k+1}\right) t + \frac{\varepsilon t}{8}.$$

If we find a red copy of P_t in G^*, then we immediately find a red copy both of P_n^k and of C_n^k in G, as $st \geq n$.

But by Lemma 30, if we do not have in G^* a red copy of P_t, then we do have a blue copy of P_t^k; by Lemma 17, we find in G a copy of P_n^k and, provided that $k+1$ divides n, also of C_n^k, as required.

If we seek a monochromatic copy of C_n^k and $k+1$ does not divide n, then observe that (provided $n > (k+1)^2$) we have $\chi(C_n^k) = k+2$. We use the same strategy, now applying Lemma 30 to find either a red P_t or blue P_t^{k+1} in the (by assumption larger) graph G^*, to obtain the desired result. $\square$

We note that the primary reason why the upper bound we obtain is larger than the conjectured value by approximately a factor of 2 is that, in this proof, we simply throw away the minority colour cliques.

7 Ramsey numbers of poor expanders

In this section we prove Theorem 12. We prove this theorem by combining Theorem 23 with a variation of Theorem 11.

Lemma 31. *Given sufficiently small $\varepsilon > 0$ and integer k, there exists n_0 such that the following holds. Let $n \geq n_0$ and G be any three-edge-colouring of the complete graph $K_{(2k+3)n}$, with edges coloured either 'red', 'blue', or 'bad', such that no more than εn bad edges meet any single vertex. Then G contains either a red or a blue copy of P_n^k.*

The proof of this lemma is a straightforward modification of the proof of Theorem 11, in much the same way as Lemma 24 is a straightforward modification of Lemma 18. As there, we must replace the Erdős-Szekeres bound with an easy modification to find red and blue cliques, and, as there, we must permit our auxiliary graph G^* to contain some non-edges (but not too many at any vertex). It is straightforward to check that the remainder of the proof is insensitive to this change; we omit the details.

We are now in a position to complete the proof of Theorem 12.

Proof of Theorem 12. Given Δ, let $\mu = 1/(30(\Delta+1)^2)$ and $\gamma = 1/2$. For k, $2 \leq k \leq \Delta+1$, let $\beta_{23} = \beta_{23}(k), \varepsilon_{23} > 0$, and $n_{23} = n_{23}(k)$ be constants such that whenever $n \geq n_{23}$, Theorem 23 permits the embedding into a $(2k+4)n$-vertex graph F (possessing a suitable ε_{23}-regular partition) of any n-vertex graph G with $\Delta(G) \leq \Delta$, $\chi(G) = k$, and $\mathrm{bw}(G) \leq \beta_{23}n$.

We set $\beta = \min\{\beta_{23}(k), 2 \leq k \leq \Delta+1\}$ and $n_0 = \max\{n_{23}(k), 2 \leq k \leq \Delta+1\}$. Let G be any n-vertex graph with $\Delta(G) \leq \Delta$ and $\mathrm{bw}(G) \leq \beta n$. Set $k = \chi(G) \leq \Delta+1$, and let F be any complete 2-coloured graph on $(2k+4)n$ vertices.

By Theorem 21, F possesses an ε_{23}-regular partition. Let $R(F)$ be the corresponding $(2k+3)m$-vertex cluster graph, with edges coloured 'red' when they correspond to ε_{23}-regular pairs whose density of red edges is at least $\frac{1}{2}$, 'blue' when they correspond to ε_{23}-regular pairs whose density of red edges is less than $\frac{1}{2}$, and 'bad' otherwise.

By Lemma 31 applied to $R(F)$, $R(F)$ contains either a red or a blue copy of P_m^k. By symmetry we may presume that it is a red copy.

Observe that $\frac{1}{2k+4} + \mu \leq \frac{1}{2k+3}$. It follows that we may set $\eta = \frac{1}{2k+4}$ and apply Theorem 23 to the graph formed by the red edges of F, with the ε_{23}-regular partition given, to find a copy of G; this is a red copy of G in F, completing the proof. □

We made no effort to optimise the constants implicit in either Lemma 31 or Theorem 12. It seems very likely that given any $\varepsilon > 0$ there is $\delta > 0$ such that the following is true for sufficiently large n. If F is any two-coloured complete graph on $R(P_n^k, P_n^k) + \varepsilon n$ vertices, then even after deleting δn edges meeting each vertex of F, there remains either a red or a blue copy of P_n^k in F. It would then follow that given $\varepsilon > 0$, if G is any n-vertex graph with maximum degree Δ, chromatic number k and bandwidth βn, where β is sufficiently small and n sufficiently large, then $R(G, G) \leq R(P_n^k, P_n^k) + \varepsilon n$.

It seems likely that $R(G, G) \leq R(P_n^{k-1}, P_n^{k-1}) + \varepsilon n$ is true. However to prove this (at least by the methods used here) one would need to be able to find in the Szemerédi cluster graph not only a monochromatic $(k-1)$st power of a path of sufficient length, but also a structure (for example an appropriately positioned $(k+1)$-clique in the same colour) permitting redistribution of vertices between colour classes.

8 Open Problems

We collect here some open problems related to our work, including some that we have mentioned in the paper.

First, we wonder whether there is scope for some improvement in Theorem 4: can we weaken the hypothesis that the bandwidth be sublinear?

Problem 32. *Is there, for any $d \geq 3$, a constant $\varepsilon_d > 0$ such that the class $\mathcal{G}_{d,\varepsilon_d n}$ of graphs G with maximum degree d and bandwidth at most $\varepsilon_d |G|$ is always-good?*

Another possibility for weakening the hypotheses of Theorem 4 is to replace the bound on the maximum degree by a bound on the degeneracy of G.

Conjecture 33. *For each fixed d, and each function $\beta(n) = o(n)$, the class $\mathcal{G}'_{d,\beta}$ of graphs G with degeneracy at most d and bandwidth at most $\beta(|G|)$ is always-good.*

We discussed in the introduction the need for n to be quite large in terms of $|H|$ in order for $R(P_n^k, H)$ to be as small as $(\chi(H) - 1)(n - 1) + \sigma(H)$, for $k \geq 2$. However there seems to be no such barrier for $k = 1$.

Conjecture 34. *For every graph H, $R(P_n, H) = R(C_n, H) = (\chi(H) - 1)(n - 1) + \sigma(H)$ whenever $n \geq \chi(H)|H|$.*

We believe that, for many graphs H, even just $n \geq |H|$ suffices. One such an example is the $R(C_n, K_\ell)$ with $n \geq \ell$ case of this conjecture, which is an old question of Erdős, Faudree, Rousseau and Schelp [21]. Even in this case the best result is that the formula holds for $n \geq 4\ell + 2$, due to Nikiforov [33].

We believe that our lower bound on $R(P_n^k, P_n^k)$ is in fact the correct value for this Ramsey number. We state this conjecture for n a multiple of $k + 1$, for convenience, but we believe that our construction in Section 6 is optimal for all sufficiently large values of n.

Conjecture 35. *For $k \geq 2$, and n a sufficiently large multiple of $k + 1$, we have*

$$R(P_n^k, P_n^k) = (k + 1)n - 2k.$$

We believe that the same result is also true for C_n^k.

A proof of the above conjecture would give some improvement in the bound in Theorem 12. As mentioned in the introduction, we expect the following to be true.

Conjecture 36. *For each $\Delta \geq 1$, there exist n_0, β and C such that, whenever $n \geq n_0$ and H is an n-vertex graph with maximum degree at most Δ and bandwidth at most βn, we have $R(H, H) \leq (\chi(H) + C)n$.*

As discussed at the end of the previous section, we may be able to take C to be arbitrarily small.

The graph P_n^3 is easily seen to be planar for every n; by Theorem 10 we have $R(P_n^3, P_n^3) \geq 4n-6$ when 4 divides n. We know of no planar graphs with larger Ramsey number bar a few small graphs ($R(K_4, K_4) = 18$, $R(K_5 - e, K_5 - e) = 22$, see [35]), but we have not made any serious efforts to discover such. Chen and Schelp proved [13] that there exists an absolute constant C such that $R(H, H) \leq Cn$ for every n-vertex planar graph H. The best value known to us for C is obtained by combining a theorem of Graham, Rödl and Ruciński [27] (essentially Theorem 9) with the Kierstead-Trotter bound [28] that all planar graphs are 10-arrangeable, which yields $C \approx 10^{200}$. By Corollary 15 we can reduce C to 12 for bounded degree planar graphs. We offer the following conjecture.

Conjecture 37. *For every sufficiently large n and every planar graph H on n vertices, we have $R(H, H) \leq 12n$.*

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