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Optimal hitting time and perpetual option in a non-Lévy model: application to real options

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Optimal hitting time and perpetual option in a non-Lévy model: Application to Real options

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Abstract: We study the perpetual American option characteristics in the case where the underlying dynamics involve a Brownian motion and a point process with a stochastic intensity. No assumption on the distribution of the jump size is made and we work with an arbitrary positive or negative jump. After proving the existence of an optimal stopping time for the problem and characterizing it as the hitting time of an optimal boundary,we provide closed-form formulae for the option value, as well as for the Laplace transform of the optimal stopping time. These results are then applied to the analysis of a real option problem when considering the impact of a fundamental and brutal change in the investment project environment. The consequences of this impact, that can seriously modify, positively or negatively, the project's future cash flows and therefore the investment decision, are illustrated numerically via the study of some examples.
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1 Introduction

In this paper, we consider a stochastic process S with a dynamics involving a diffusion and a unique jump process. In such a framework, we aim at studying the following problem:

$$\operatorname*{ess\,sup}_{\tau \in \Upsilon_t} \mathbb{E}\left(e^{-\mu(\tau - t)} \left(S_\tau - 1\right)^+ \ / \ \mathcal{F}_t \right),$$

where Υ_t denotes the set of $\mathcal{F}_t-$stopping times taking values in $(t, +\infty[$ and μ is a positive real number. This issue comes from real options, and more precisely from the investor's problem to finance a project, when a (unique) sudden and drastic change can occur in the environment. No assumption is made on the impact of this shock. As a consequence, even if our study includes credit risk issues, it cannot be reduced to this type of problems. In particular, we can also consider positive consequences of the drastic change on the project.

Real options studies are usually written in a continuous framework for the underlying dynamics (see the seminal papers by Brennan and Schwartz. (1985), Mc Donald and Siegel (1986), Dixit and Pindyck (1993)). But the existence of crises and shocks on investment markets generates discontinuities. Some papers have been studying the impact of such instabilities on the investment decision process (see for instance Gerber and Shiu (1998), Mordecki (2002) or Barrieu and Bellamy (2004)).
The question of the impact of a brutal change in the project environment has not been studied so far, to our knowledge. Such a fundamental change in the regulation can seriously impact, positively or negatively, the project's future cash flows and therefore the investment decision. Many situations can lead to a major brutal change in the project environment: for instance, a currency devaluation, an increase in interest rates, the finalization of brand-new technologies or the entry of a new competing firm. Mathematically speaking, all these situations are similar and are equivalent to consider a single jump diffusion process.
This work aims at studying the impact of such a market perturbation on the decision taking. More precisely, the investor has a budget allocation problem. He has to determine whether the project is interesting and if so, to have some ideas of the optimal investment time as to allocate his capital among financial instruments with different maturities. Therefore, we need to specify the optimal benefit/cost ratio, the project investment value and the optimal time to enter the project for the investor when he considers the project. This problem is similar to the study of a perpetual American call option written on the project ratio with strike 1. However, since this paper aims at analyzing a real option problem, we do not adopt a risk neutral logic, and the calculations are made with respect to the prior probability measure, representative of the agent's beliefs. This does not induce any loss of generality and the results we provide can easily be extended to the standard framework, where the perpetual option is evaluated under an equivalent martingale measure.

In the second section of the paper, we introduce the framework of the study and present the model. The relationship between a real option problem and the pricing of a perpetual American option is also recalled. The main results on optimal stopping and perpetual options characteristics are given in Section 3. We obtain in this section closed form formulae without any assumption on the distribution of the jump size and with arbitrary positive or negative jump. Section 4 is devoted to the study of real option problems in a disrupted framework. We first illustrate the main results of this paper with different investment examples.Then we analyze the importance of the calibration of the model and study the consequences of a misspecification in a bearish environment. Finally, some proofs are delayed in the Appendix.

2 The model

2.1 Framework

In this paper, we consider a given economic agent having the opportunity to invest in a particular project. The stochastic framework is described by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ is the prior probability measure representing the beliefs and anticipations of the agent.
The project is characterized at any time $t \geq 0$ by its profit/cost ratio S_t. More precisely, this process satisfies

$$\begin{cases} dS_t = S_{t^-}\left(\alpha dt + \sigma dW_t + \Phi dD_t\right) \\ S_0 = s_0 \end{cases} \quad (1)$$

where:

- $(W_t; t \geq 0)$ is a $\mathbb{P}-$Brownian motion,
- The process $(D_t; t \geq 0)$ describes the potential shock of a drastic change in regulation on the project. It is defined as

$$D_t = \mathbf{1}_{t \geq T} \ ,$$

where T is a random variable of exponential law, with parameter λ. We denote by $(M_t; t \geq 0)$ the compensated martingale associated with $(D_t; t \geq 0)$:

$$M_t = D_t - \lambda \int_0^t (1 - D_s) ds \; . \tag{2}$$

- The jump size Φ is a random variable. More precisely, Φ measures the impact of the perturbation on the profit/cost ratio.

Therefore, the solution of the SDE (1) is:

$$S_t = s_0 \left(1 + \Phi\right)^{D_t} \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right]$$

We assume the Brownian motion and the random variable T to be independent. Moreover, the random variable Φ is taken independent of both the Brownian motion and the jump time process. Note that such an assumption is reasonable since the (regulation) shock is exogenous to the project we consider.

Assumptions: From now on, we make the following technical assumptions, ensuring that the problem we consider is well-defined and is non-degenerate:

$$\begin{cases} i) & -1 < \underline{\Phi} \leq \Phi < \overline{\Phi} \; ; \\ ii) & 0 < s_0 < 1 \; ; \\ iii) & \sigma > 0 \; ; \\ iv) & \mu > \max(0, \alpha, \alpha + \lambda \overline{\Phi}). \end{cases} \tag{H}$$

The first assumption $\Phi > -1$ implies that the ratio S remains strictly positive, therefore the investment opportunity is never worthless. Considering a bounded Φ ensures that the positive impact of a drastic change in the environment cannot be infinite. Therefore, the situation we study is neither catastrophic nor explosive ; Φ is also assumed to be not identically equal to 0, otherwise this study is reduced to the standard "Black-Scholes" framework.
The second assumption states that the project is interesting ($s_0 > 0$) but also that the problem is about a "true" decision since s_0 is (strictly) less than 1: delaying the project realization is only relevant in the case where the profits/costs ratio is less than one.
The third condition maintains a Brownian source of randomness in the process dynamics. It is indeed reasonable to assume that the ratio profits/costs randomly evolves even without a major modification of the project environment.
Finally, the fourth assertion of (H) is an integrability condition.

The available information structure is characterized by the filtration $(\mathcal{F}_t; t \geq 0)$. At each instant of time t, the agent knows what is the current value of the ratio profit/cost related to the project, and therefore knows $\sigma(S_s; 0 \leq s \leq t)$. This information includes the value of D_t: indeed, at each instant t, from the observation of the underlying process S, the agent knows whether the shock has already occurred or not. As a consequence, the filtration $\mathcal{F}_t$ is defined as:

$$\mathcal{F}_t = \sigma\left(S_s, D_s;\; 0 \leq s \leq t\right).$$

Note that the approach defined by the framework we consider differs from those of the models of the literature related to both real options and American options. In particular, real options studies are usually written in a continuous framework for the underlying dynamics (see the seminal papers by Brennan and Schwartz. (1985), Mc Donald and Siegel (1986), Dixit and Pindyck (1993)). But the existence of crises and shocks on investment markets generates discontinuities. Some papers have been studying the impact of such instabilities on the investment decision process (see for instance Gerber and Shiu (1998), Mordecki

(2002) or Barrieu and Bellamy (2004)). The question of the impact of a brutal change in the project environment however has not been studied so far, to our knowledge. Moreover, on the contrary to most of the existing literature on American options (see for instance Gerber and Shiu (1998), Mordecki (2002) or Chesney and Jeanblanc (2004), the underlying process S cannot be written in terms of a Lévy process X_t as $S_0 \exp(X_t)$. Quite recently, Muroi (2002) has considered a general framework involving a possible default for the underlying process of an American option. His approach is however different in the sense that it is focused on the numerical aspects and based upon partial differential equation methods. Moreover he does not provide closed-form characterizations of the American option.

2.2 Perpetual option and investment problem

The investor has neither any obligation to undertake it nor any time constraint to take his decision. Therefore, the investment decision problem is often brought down to the study of a perpetual American option with the profit/cost ratio S as underlying and 1 as striking level (see for instance Mac Donald and Siegel (1986)). Solving the real option problem at any time t is equivalent to determine the value of the following quantity:

$$C_t = \operatorname*{ess\,sup}_{\tau \in \Upsilon_t} \mathbb{E}\left(e^{-\mu(\tau-t)} \left(S_\tau - 1\right)^+ \ / \ \mathcal{F}_t \right) \tag{3}$$

where Υ_t denotes the set of $\mathcal{F}_t$−stopping times taking values in $(t, +\infty[$ and where μ is the discount rate, which can be different from the instantaneous risk-free rate and represents the agent's preferences for the present, $\mathbb{E}$ is the expectation with respect to the prior probability measure $\mathbb{P}$.
The pair (D, S) being Markovian, we can rewrite C_t as follows

$$\begin{aligned} C_t &= \operatorname*{ess\,sup}_{\tau \in \Upsilon_t} \mathbb{E}\left(e^{-\mu(\tau-t)} \left(S_\tau - 1\right)^+ \ / \ \mathcal{F}_t \right) \\ &= \operatorname*{ess\,sup}_{\tau \in \Upsilon_t} \mathbb{E}\left(e^{-\mu(\tau-t)} \left(S_\tau - 1\right)^+ \ / \ S_t, D_t \right) \end{aligned}$$

or for all $(t, x, d) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \{0, 1\}$, adopting the natural notation $C(t, x, d)$:

$$C(t, x, d) = \operatorname*{ess\,sup}_{\tau \in \Upsilon_t} \mathbb{E}\left(e^{-\mu(\tau-t)} \left(S_\tau - 1\right)^+ \ / \ S_t = x, D_t = d \right) \tag{4}$$

Note that the computations are made with respect to the prior probability measure $\mathbb{P}$, corresponding to the agent's beliefs. This does not create any loss of generality. If the underlying asset of the investment option is traded on financial market, a risk-neutral valuation formula can be easily obtained by considering a risk-neutral probability measure, as underlined in the following remark:

Remark 1 *If a risk-neutral approach is adopted, then the only changes in the results come from both the modification of the discount rate μ in the instantaneous risk-free rate r, and a modification in the jump process intensity. More precisely, any equivalent martingale measure $\mathbb{P}^{(\psi,\gamma)}$ is defined in terms of a pair of processes (ψ, γ), satisfying the usual regularity conditions and being solution of*

$$\begin{cases} \alpha + \lambda\Phi\left(1 - D_t\right) - r + \sigma\psi_t + \lambda\Phi\gamma_t = 0 \\ \gamma_t > -1 \end{cases}$$

The Radon-Nikodym derivative $\left.\dfrac{d\mathbb{P}^{(\psi,\gamma)}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t^{(\psi,\gamma)}$ *is given by:* $dL_t^{(\psi,\gamma)} = L_t^{(\psi,\gamma)}\left(\psi_t dW_t + \gamma_t dM_t\right)$; $L_0^{(\psi,\gamma)} = 1$. *Therefore the intensity of $(D_t; t \geq 0)$ evaluated with respect to $\mathbb{P}^{(\psi,\gamma)}$ is equal to* $\lambda \int_0^t \left(1 + \gamma_s\right)\left(1 - D_s\right) ds$.

3 Main results on optimal stopping and perpetual options

This third section is devoted to the study of the perpetual call option, described in Subsection 2.2. We first prove the existence of an optimal stopping time in Equation (4)

$$C(t,x,d) = \text{ess}\sup_{\tau\in\Upsilon_t} \mathbb{E}\left(e^{-\mu(\tau-t)}\left(S_\tau-1\right)^+ \ / \ S_t = x, D_t = d\right)$$

and characterize it as the hitting time of an optimal frontier. Then, we look at the different characteristics of the perpetual option. More precisely, we provide closed-form formulae for the Laplace transform of the optimal stopping time and for the real option value.

3.1 Case $D_t = 1$

Two very different situations have to be studied separately, depending on whether the jump has already occurred or not (in other words, whether $d = 1$ or $d = 0$) at time t of the study. The arguments used in both cases are of a completely different nature.

We can first notice that in the situation where the jump has occurred before time t, the problem

$$C(t,x,1) = ess\sup_{\tau\in\Upsilon_t} \mathbb{E}\left(e^{-\mu(\tau-t)}\left(S_\tau-1\right)^+ \ / \ S_t = x, D_t = 1\right) \tag{5}$$

is similar to the standard one. As a consequence, the results are well known and therefore we simply state the results and sketch their proof. More precisely,

Proposition 1 *i) The optimal hitting time in Problem* (5) *exists and it is characterized as the first hitting time of a constant level* $L^{1,*}$*:*

$$\begin{aligned} C(t,x,1) &= \sup_{L\geq 1} \mathbb{E}\left(e^{-\mu(\tau_L-t)}\left(S_{\tau_L}-1\right)^+ \ / \ S_t = x, \ D_t = 1\right) \\ &= \mathbb{E}\left(e^{-\mu\left(\tau_{L^{1,*}}-t\right)}\left(S_{\tau_{L^{1,*}}}-1\right)^+ \ / \ S_t = x, \ D_t = 1\right) \end{aligned}$$

where τ_L *the first hitting time of* L

$$\tau_L = \inf\left\{t\geq 0 \ ; S_t \geq L\right\}.$$

ii) The optimal level $L^{1,*}$ *is given by:*

$$L^{1,*} = \frac{k_1}{k_1-1}$$

where k_1 *is the unique positive real such that* $\psi\left(k_1\right) = \mu$*, with* ψ*, the Lévy exponent of the Lévy process defined by* $X_s = \left(\alpha - \frac{1}{2}\sigma^2\right)s + \sigma W_s$*.*

i) The Laplace transform of the optimal time is:

$$\mathbb{E}\left(e^{-\mu\left(\tau_{L^{1,*}}-t\right)} \ / \ S_t = x, D_t = 1\right) = \left(\frac{x}{L^{1,*}}\right)^{\frac{L^{1,*}}{L^{1,*}-1}},$$

ii) The price of the real option is given by:

$$C(t,x,1) = \left(L^{1,*}-1\right)\left(\frac{L^{1,*}}{x}\right)^{-k_1}$$

where the optimal level $L^{1,*}$ *is given by Proposition 1.*

Proof.
Note that, on the set $\{t \geq T\}$, we necessarily have $\tau_L \geq T$ and as a consequence for any s, t such that $0 \leq T \leq t \leq s$:

$$S_s = x \exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right)(s-t) + \sigma\left(W_s - W_t\right)\right); S_t = x$$

Therefore, the proof of these results is similar to that of the standard Black-Scholes model (see for instance Merton (1992) section 8.8, or Karatzas et al.(1998) section 2.6) and we omit it.

Note that the formulae do not apparently depend on the jump size Φ. This comes from the fact that after the jump, the jump size does not play any role apart from the shift in the "initial condition", which becomes $s_0(1+\Phi)$ instead of s_0. This dependency is somehow hidden in the value x of the process at time t.

On the following , we only consider the case where $D_t = 0$, since this second situation where the jump has not already occurred at time t is not standard.

3.2 Existence and characterization of an optimal stopping time

In this section, we consider the question of the existence of an optimal stopping time for the problem (4). The following result holds true:

Theorem 2 *There exists an optimal stopping time for the problem* (4). *In other words, there exists* $\tau_t^* \in \Upsilon_t$ *such that*

$$C(t, x, d) = \mathbb{E}\left(e^{-\mu(\tau_t^* - t)}\left(S_{\tau_t^*} - 1\right)^+ \ / \ S_t = x, D_t = d\right)$$

Moreover, τ_t^* *is characterized as the first hitting time by the process* S *of a frontier* $L_t^{d,*}$*, fully known at time* t*. More precisely,*

$$\tau_t^* = \inf\left\{s \geq t; S_s \geq L_t^{d,*}\right\}$$

The problem is of a double nature: first we have to prove the existence of an optimal stopping time for:

$$C(t, x, 0) = \operatorname*{ess\ sup}_{\tau \in \Upsilon_t} \mathbb{E}\left(e^{-\mu(\tau - t)}\left(S_\tau - 1\right)^+ \ / \ S_t = x, D_t = 0\right) \tag{6}$$

and then we have to characterize it.

Existence of an optimal stopping time For any $x \in R_*^+$, and for $D_t = 0$, we denote by $S_{\cdot}^{t,x,0}$ the trajectory of the process S such that $S_t^{t,x,0} = x$. Hence, for any $s \geq t$:

$$S_s^{t,x,0} = x\left(1+\Phi\right)^{D_s} \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)(s-t) + \sigma\left(W_s - W_t\right)\right] \tag{7}$$

The following result holds true:

Lemma 3 *The stopping time* τ_t^* *defined as*

$$\tau_t^* = \inf\left\{s \geq t \ / \ C\left(s, S_s^{t,x,0}, 0\right) = \left(S_s^{t,x,0} - 1\right)^+\right\}$$

is $\mathbb{P}$ *- almost surely finite.*

Proof. The constants $\underline{\Phi}$ and $\overline{\Phi}$ being defined in Assumptions (H), $S^{\underline{\Phi}}$ and $C^{\underline{\Phi}}$ are the process and the function defined as follows:

$$\text{for } s \geq t,\ S_s^{\underline{\Phi}} = x\left(1+\underline{\Phi}\right)\exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)(s-t) + \sigma\left(W_s - W_t\right)\right],$$

and

$$C^{\underline{\Phi}}(t,x) = \text{ess}\sup_{\tau\in\Upsilon_t} \mathbb{E}\left(e^{-\mu(\tau_t - t)}\left(S_{\tau_t}^{\underline{\Phi}} - 1\right)^+ \ / \ S_t^{\underline{\Phi}} = x\left(1+\underline{\Phi}\right)\right).$$

The process $S^{\overline{\Phi}}$ and the function $C^{\overline{\Phi}}$ are defined in a similar way.
Because the inequalities $-1 < \underline{\Phi} \leq \Phi < \overline{\Phi}$ hold true, we can write:

$$\begin{aligned}\tau_t^* &\leq \inf\left\{s\geq t \ / \ C^{\overline{\Phi}}(t,x) = \left(S_s^{t,x,0} - 1\right)^+\right\} \\ &\leq \inf\left\{s\geq t \ / \ C^{\overline{\Phi}}(t,x) = \left(S_s^{\underline{\Phi}} - 1\right)^+\right\} \\ &\leq \inf\left\{s\geq t \ / \ C^{\overline{\Phi}}(t,x) = k\left(S_s^{\overline{\Phi}} - \tfrac{1}{k}\right)^+\right\}\end{aligned}$$

where $k = \dfrac{1+\underline{\Phi}}{1+\overline{\Phi}}$. Now, fom Hypothesis ((H) assertion iv), we have

$$\inf\left\{s\geq t \ / \ C^{\overline{\Phi}}(t,x) = k\left(S_s^{\overline{\Phi}} - \frac{1}{k}\right)^+\right\} < \infty,\ \mathbb{P}.\ a.s.$$

■

Proposition 4 *There exists an optimal stopping time for Problem* (6)*. In other words, there exists* $\tau_t^* \in \Upsilon_t$ *such that*

$$C(t,x,0) = \mathbb{E}\left(e^{-\mu(\tau_t^* - t)}\left(S_{\tau_t^*} - 1\right)^+ \ / \ S_t = x, D_t = 0\right) \tag{8}$$

Proof. From equation (7), Assumption ((H) , i) and iv)), we deduce that the optimal stopping theory can apply (see for instance Shiryayev (1978) (Theorem 3.3) or El Karoui (1979)); therefore we conclude from Lemma 3 that the stopping time τ_t^* is optimal. ■

Characterization of the optimal stopping time In this second part of the proof, we want to characterize the optimal stopping time of Problem (6). More precisely, using the same type of arguments as in Jacka (1991), we get a characterization similar to the one obtained in the case where $D_t = 1$.
Let $\mathcal{Y}$ be the continuation region defined by:

$$\mathcal{Y} = \left\{(t,x) \ / \ C(t,x,0) > (x-1)^+\right\}$$

and $\mathcal{Y}_t$ denotes the section:

$$\mathcal{Y}_t = \{x \ / \ (t,x) \in \mathcal{Y}\}$$

We first gives the characterization of the section $\mathcal{Y}_t$ of the continuation region in terms of an optimal frontier. More precisely,

Proposition 5 *For any* $t > 0$*, there exists* $L_t^{0,*} > 1$ *such that*

$$\mathcal{Y}_t = \left[0, L_t^{0,*}\right[$$

Proof.

From Equation (8), the function $x \mapsto C(t,x,0)$ is continuous therefore it suffices to prove the following implication:

$$(x \in \mathcal{Y}_t \text{ and } 0 < y < x) \Longrightarrow (y \in \mathcal{Y}_t)$$

Let us assume that $x \in \mathcal{Y}_t$. As a consequence, $C(t,x,0) > (x-1)^+$. Let τ_t^x be defined by

$$\tau_t^x = \inf\left\{s \geq t \;/\; \left(s, S_s^{t,x,0}\right) \notin \mathcal{Y}\right\}$$

We have, on the one hand

$$C(t,x,0) \quad = \mathbb{E}\left(e^{-\mu(\tau_t^x - t)}\left(S_{\tau_t^x} - 1\right)^+ \;/\; S_t = x,\; D_t = 0\right) = \mathbb{E}\left(e^{-\mu(\tau_t^x - t)}\left(S_{\tau_t^x}^{t,x,0} - 1\right)^+\right)$$

and on the other hand, as τ_t^x is not optimal for $C(t,y,0)$

$$C(t,y,0) \geq \mathbb{E}\left(e^{-\mu(\tau_t^x - t)}\left(S_{\tau_t^x}^{t,x,0} - 1\right)^+\right)$$

Therefore

$$C(t,x,0) - C(t,y,0) \quad \leq \mathbb{E}\left(e^{-\mu(\tau_t^x - t)}\left[\left(S_{\tau_t^x}^{t,x,0} - 1\right)^+ - \left(S_{\tau_t^x}^{t,y,0} - 1\right)^+\right]\right)$$

The assumption $0 < y < x$ together with Equation (7) imply

$$\begin{aligned} C(t,x,0) - C(t,y,0) \quad &\leq \mathbb{E}\left(e^{-\mu(\tau_t^x - t)}\left(S_{\tau_t^x}^{t,x,0} - S_{\tau_t^x}^{t,y,0}\right)\right) \\ &\leq (x-y)\,\mathbb{E}\left(e^{(\alpha-\mu)(\tau_t^x - t) + \lambda\int_t^{\tau_t^x} \ln(1+\Phi)(1-D_s)ds}\,\mathcal{E}\left(\sigma W\right)_{\tau_t^y, t}\right) \end{aligned}$$

where $\mathcal{E}\left(\sigma W\right)_{\tau_t^y,t} = \exp\left(\sigma\left(W_{\tau_t^y} - W_t\right) - \frac{1}{2}\sigma^2\left(\tau_t^y - t\right)\right)$

Let $\widetilde{\mathbb{P}}$ be the probability measure, equivalent to $\mathbb{P}$, defined by:

$$\left.\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)$$

Then

$$C(t,x,0) - C(t,y,0) \leq (x-y)\,\widetilde{\mathbb{E}}\left(e^{(\alpha-\mu)(\tau_t^x - t) + \lambda\int_t^{\tau_t^x} \ln(1+\Phi)(1-D_s)ds}\right)$$

From the assumption $(H), iv)$, $\widetilde{\mathbb{E}}\left(e^{(\alpha-\mu)(\tau_t^x - t) + \lambda\int_t^{\tau_t^x} \ln(1+\Phi)(1-D_s)ds}\right) \leq 1$ and

$$C(t,x,0) - C(t,y,0) \leq (x-y)$$

or

$$C(t,y,0) \quad \geq C(t,x,0) - (x-y) > (x-1)^+ - (x-y) > y-1$$

Let $\xi = \inf\left\{u \geq t \;/\; S_u^{t,y,0} \geq 2\right\}$, then

$$\begin{aligned} C(t,y,0) \quad &\geq \mathbb{E}\left(e^{-\mu(\xi - t)}\left(S_\xi^{t,y,0} - 1\right)^+\right) \\ &\geq \mathbb{E}\left(e^{-\mu(\xi - t)}\right) > 0 \end{aligned}$$

Therefore, for any $(t,y) \in \mathbb{R}^+ \times \mathbb{R}_*^+$, $C(t,y,0) > 0$. Finally, $C(t,y,0) > (y-1)^+$. Hence the result. ■

We can now characterize the optimal stopping time as the hitting time of a frontier:

Corollary 6 *The optimal stopping time τ_t^* of Problem* (6) *is characterized by:*

$$\tau_t^* = \inf\left\{s \geq t \ / \ S_s^{t,x,0} \geq L_t^{0,*}\right\}$$

where $L_t^{0,*}$ is defined in Proposition 5.

Proof. The proof is an immediate consequence of Proposition 5. ■

3.3 Perpetual option characteristics

We are now interested in finding the perpetual option characteristics, in particular its value. As previously, we only consider the situation where the jump has not occurred yet at time t. We now want to study the characteristics of the following perpetual option:

$$C(t,x,0) = \text{ess}\sup_{\tau\in\Upsilon_t} \mathbb{E}\left(e^{-\mu(\tau-t)}\left(S_\tau - 1\right)^+ \ / \ S_t = x, D_t = 0\right)$$

which can also be written, using Corollary 6 as

$$C(t,x,0) = \text{ess}\sup_{L_t^0\geq 1} \mathbb{E}\left(e^{-\mu\left(\tau_{L_t^0},-t\right)}\left(S_{\tau_{L_t^0}} - 1\right)^+ \ / \ S_t = x, D_t = 0\right)$$

For the sake of simplicity in the notation and computation but without loss of generality, we consider this problem at time $t=0$ (obviously, $D_0 = 0$). Therefore, we look at:

$$C(0,s_0,0) = \text{ess}\sup_{L^0\geq 1} \mathbb{E}\left(e^{-\mu\tau_{L^0}}\left(S_{\tau_{L^0}} - 1\right)^+ \ / \ S_0 = s_0\right) \tag{9}$$

We now present the main result of this paper, giving the perpetual option characteristics in a general framework where we do not know a priori the sign of the jump size Φ. The following two theorems provide these explicit formulae at time 0. In particular, the optimal frontier and a characterization of the optimal hitting time are now given in Theorem 7:

Theorem 7 *i) The optimal frontier of the problem* (9) *is given by*

$$L_\Phi^{0,*} = \frac{\eta(\Phi)}{\eta(\Phi) - 1 + \Gamma(\Phi)} \ ,$$

where

$$\begin{aligned}\eta(\Phi) &= \tfrac{2}{\sigma} f_0(\lambda, a_0) + \theta_0\sigma \\ &-\tfrac{2}{\sigma}\mathbb{E}\left(\mathbf{1}_{\Phi<0}\left(1+\Phi\right)^{\frac{\theta_0}{\sigma}}\right) f_0(\lambda,a_0) \\ &+\tfrac{2}{\sigma}\mathbb{E}\left[\mathbf{1}_{\Phi>0}\left[\tfrac{\lambda}{(\lambda+\mu)}\Delta f_{0,\Phi}(\lambda+\mu, a_0-\theta_0) - (1+\Phi)^{\frac{\theta_0}{\sigma}} f_\Phi(\lambda,a_0)\right]\right]\end{aligned}$$

$$\text{and} \quad \Gamma(\Phi) = \tfrac{2}{\sigma}\mathbb{E}\left[\mathbf{1}_{\Phi>0}\left(\tfrac{\lambda(1+\Phi)}{(\lambda+\mu-\alpha)}\Delta f_{0,\Phi}(\lambda+\mu-\alpha, v_0) - \tfrac{\lambda}{(\lambda+\mu)}\Delta f_{0,\Phi}(\lambda+\mu, a_0-\theta_0)\right)\right] \ .$$

ii) For every s_0 such that $0 < s_0 < L_\Phi^{0,*}$, the optimal hitting time is characterized by its Laplace transform:

$$\mathbb{E}(e^{-\mu\tau_{L_0^{0,*}(\Phi)}}) = \Theta(\Phi) + \left(\frac{s_0}{L_\Phi^{0,*}}\right)^{\frac{\theta_0}{\sigma}} \times (1 + K(\Phi))$$

where

$$\begin{aligned}K(\Phi) &= -F_0\left(s_0,\lambda,\ L_\Phi^{0,*},a_0\right) \\ &\quad +F_0(s_0,\lambda,\ L_\Phi^{0,*},a_0)\mathbb{E}\left[\mathbf{1}_{\Phi<0}\left(1+\Phi\right)^{\frac{\theta_0}{\sigma}}\right]+\mathbb{E}\left[\mathbf{1}_{\Phi>0}\left(1+\Phi\right)^{\frac{\theta_0}{\sigma}}F_\Phi(s_0,\lambda,\ L_\Phi^{0,*},a_0)\right] \\ \Theta(\Phi) &= -\tfrac{\lambda}{\lambda+\mu}\mathbb{E}\left[\mathbf{1}_{\Phi>0}\times\Delta F_{0,\Phi}(s_0,\lambda+\mu,\ L_\Phi^{0,*},a_0-\theta_0)\right]\ .\end{aligned}$$

where the various notations used are given in the Appendix in Subsection 6.1.

Proof. As to prove this result, we first consider the following two cases, where Φ is a constant denoted by φ, which is either positive (bullish environment) or negative (bearish environment). Finally, the general result is obtained by means of conditional expectation, as it is enough to consider the case where the jump size is a constant, from the independence of the random variable Φ and the filtration generated by the Brownian motion and the jump process.

1) Bullish environment

Assuming the random variable Φ to be a positive constant φ, we determine the optimal threshold on which the agent bases his decision.
We proceed in two steps to obtain $L_\Phi^{0,*}$ in a bullish framework (with $\varphi>0$):
Step 1) The functions F_φ and f_φ, defined by Equations (11) and (13) in Subsection 6.1, are linked by the relationship

$$\forall\lambda>0,\forall a\in\mathbb{R},\frac{\partial}{\partial L}F_\varphi(s_0,\lambda,L,a)\Big|_{s_0=L}=\frac{2}{\sigma L}f_\varphi(\lambda,a)\ .$$

Moreover, function F_φ satisfies $F_\varphi(L,\lambda,L,a)=0$.
Step 2) We denote by c the following function

$$c\left(s_0,L\right)=\mathbb{E}\left[e^{-\mu\tau_L}\left(S_{\tau_L}-1\right)\right]\ . \tag{10}$$

Then from Lemma 10 (see Appendix) we deduce:

$$\begin{aligned}\frac{\partial}{\partial L}c(s_0,L)\Big|_{s_0=L} &= \tfrac{2}{\sigma L}\left(1+\varphi\right)^{\frac{\theta_0}{\sigma}}\left(L-1\right)f_\varphi(\lambda,a_0) \\ &\quad +1-\tfrac{\theta_0}{\sigma L}(L-1)-\tfrac{2}{\sigma L}(L-1)f_0(\lambda,a_0) \\ &\quad -\tfrac{2\lambda(1+\varphi)}{\sigma(\lambda+\mu-\alpha)}\Delta f_{0,\varphi}(\lambda+\mu-\alpha,v_0) \\ &\quad +\tfrac{2\lambda}{\sigma L(\lambda+\mu)}\Delta f_{0,\varphi}(\lambda+\mu,a_0-\theta_0)\ .\end{aligned}$$

The unique solution of $\frac{\partial}{\partial L}c(s_0,L)\Big|_{s_0=L}=0$ is $L=L_\varphi^{0,*}=\frac{\eta^+(\varphi)}{\eta^+(\varphi)-1+\Gamma^+(\varphi)}$ where

$$\eta^+\left(\varphi\right)=\tfrac{2}{\sigma}f_0(\lambda,a_0)+\theta_0\sigma+\tfrac{2}{\sigma}\left[\tfrac{\lambda}{(\lambda+\mu)}\Delta f_{0,\varphi}(\lambda+\mu,a_0-\theta_0)-\left(1+\varphi\right)^{\frac{\theta_0}{\sigma}}f_\varphi(\lambda,a_0)\right]$$

and $\Gamma^+\left(\varphi\right)=\tfrac{2}{\sigma}\left[\tfrac{\lambda(1+\Phi)}{(\lambda+\mu-\alpha)}\Delta f_{0,\varphi}(\lambda+\mu-\alpha,v_0)-\tfrac{\lambda}{(\lambda+\mu)}\Delta f_{0,\varphi}(\lambda+\mu,a_0-\theta_0)\right]$.

2) Bearish environment

In the case where the constant φ is non positive, the optimal frontier is given by

$$L_\varphi^{0,*}=\frac{\eta^-\left(\varphi\right)}{\eta^-\left(\varphi\right)-1}\ ,$$

where $\eta^-\left(\varphi\right)=\tfrac{2}{\sigma}\left[\left(1-\left(\left(1+\varphi\right)^{\frac{\theta_0}{\sigma}}\right)\right)f_0(\lambda,a_0)+\tfrac{\theta_0}{2}\right]$

The proof is similar that of the previous case and the result directly comes from Lemma 11 proved in Appendix.

Assertion *i*) is then a straightforward consequence of **1)** and **2)** and of the independence of the random variable Φ with respect to the Brownian motion and the jump process, whereas Assertion *ii*) comes from this independence, from Lemma 10, Lemma 11 and the value of the optimal threshold given in Assertion *i*). ■

The following Theorem gives the perpetual option value:

Theorem 8 *i) For every s_0 such that $0 < s_0 < L_\Phi^{0,*}$, the perpetual option value at time $t = 0$ is*

$$C(0, s_0, 0) \quad = J(\Phi) + (\ L_\Phi^{0,*} - 1) \left(\frac{s_0}{L_\Phi^{0,*}}\right)^{\frac{\theta_0}{\sigma}} (1 + K(\Phi)) \ ,$$

where K is defined in Theorem 7,

$$J(\Phi) \quad = -\frac{\lambda s_0}{\lambda+\mu-\alpha} \mathbb{E}\left[\mathbf{1}_{\Phi>0}(1+\Phi)\Delta F_{0,\Phi}(s_0, \lambda+\mu-\alpha,\ L_\Phi^{0,*}, v_0)\right] \ .$$

and the various notations used are those of Subsection 6.1.

Proof. The proof directly comes from Theorem 7, Lemma 10 and Lemma 11. ■

3.4 Comments

The model presented here, and especially the situation $D_t = 0$ is very different from the existing literature, as it corresponds to a non-Lévy model. The results are therefore different. Note however that if $\Phi \equiv 0$ $\mathbb{P}$ a.s., then all the formulae of this study are similar to those of the "Black-Scholes" model. The "interesting" situation is when D_t is different from 1. Otherwise, the formulae are similar to a standard "Black-Scholes"-type framework, where no possible jump of the underlying process is introduced (as it is equivalent to have a shift in the initial condition of our problem). Therefore, we focus on the non-standard results, when $D_t = 0$.

In this paper, Theorem 7 gives an explicit representation of the optimal boundary without any particular assumption either on the sign of the jump Φ or on its distribution. Whether Φ is positive or negative has a great impact on the structure of the different results since it shapes the optimal frontier differently. In fact, both situations are mathematically very different. The main reason is that when the jump is negative, the process S necessarily crosses the given frontier in a continuous way. However, this property does not hold any more if the jump is positive. Therefore, involved calculus and obtained results are dissimilar. Moreover, the results we obtain are strongly different to those of Lévy model.

When the jump is negative, even if, the optimal frontier is still of a standard shape:

$$L_\Phi^{0,*} = \frac{\eta^-(\Phi)}{\eta^-(\Phi) - 1} \ ,$$

both the Laplace transform and the value of the investment opportunity are different. The standard equations for the Laplace transform and the price are perturbed by a multiplicative factor K, this factor K being identical to the one in the bullish environment and we have

$$\begin{cases} \mathbb{E}(e^{-\mu\tau_{L_0^{0,*}(\Phi)}}) & = \left(\frac{s_0}{L_\Phi^{0,*}}\right)^{\frac{L_\Phi^{0,*}}{L_\Phi^{0,*}-1}} (1 + K(\Phi)) \\ C(0, s_0, 0) & = (\ L_\Phi^{0,*} - 1) \left(\frac{s_0}{L_\Phi^{0,*}}\right)^{\frac{L_\Phi^{0,*}}{L_\Phi^{0,*}-1}} (1 + K(\Phi)) \end{cases}$$

This multiplicative factor comes from the memory of the process used to model the profit and cost ratio. Obviously, in the limit situation where there is no possible jump ($\Phi \equiv 0$), the results coincide with those of the standard "Black-Scholes"-type framework as $K(0) = 0$.

When considering a positive jump, we obtain explicit formula for the three characteristics of the investment opportunity. The results we obtain are fundamentally different from those of Lévy models. In particular, the optimal frontier can not be written under the previous standard form (if we except the particular limit case where $\Phi = 0$). Indeed, an additional term $\Gamma^+(\Phi)$ is present in the denominator of the optimal frontier,

$$L_\Phi^{0,*} = \frac{\eta^+(\Phi)}{\eta^+(\Phi) - 1 + \Gamma^+(\Phi)} \ .$$

Both the Laplace transform and the investment value differ from the standard case by two different effects. There is first a common multiplicative factor K and also two additive factors J' (for the Laplace transform) and J (for the value of the investment):

$$\begin{cases} \mathbb{E}(e^{-\mu\tau_{L_0^{0,*}(\Phi)}}) & = \Theta(\Phi) + \left(\frac{s_0}{L_\Phi^{0,*}}\right)^{\frac{L_\Phi^{0,*}}{L_\Phi^{0,*}-1}} (1 + K(\Phi)) \ , \\ C(0, s_0, 0) & = J(\Phi) + (\ L_\Phi^{0,*} - 1) \left(\frac{s_0}{L_\Phi^{0,*}}\right)^{\frac{L_\Phi^{0,*}}{L_\Phi^{0,*}-1}} (1 + K(\Phi)) \ . \end{cases}$$

These differences can be explained as follows: as for the bearish case, the factor K comes from the memory of the process used to model the profit and cost ratio whereas the additive factors J' and J translate the fact that in this case the optimal boundary $L_\Phi^{0,*}$ is not necessarily crossed in a continuous way.

4 Application to a real option problem

In this section, we look more precisely at the investment problem (i.e. value of the investment project and optimal hitting time) when the investor has no particular knowledge about the type of impact a new regulation, a sudden shock may have on the project itself. More precisely, we do not know whether the sign of the jump will be positive or negative. This may be the case for investment projects related to new technologies when the prospects are uncertain. In this case, the related market witnesses a rapid evolution. New regulations can be adopted, modifying its structure. They can generate either a positive impact when improving the security and the transparency (bullish environment) or a negative impact when increasing its complexity (bearish environment). New competitors can also enter the market, attracted by these new prospects, and actual competitors can default, victims of the lack of experience and speculation. Finally, we may think of new techniques that can be patented either by the considered firm or by competitors.
In this section, we focus on the situation at time $t = 0$ without any loss of generality. In fact, if a new regulation is expected, or if some disruptive factor for the economic environment is likely to occur, one may think that the investor will delay his investment decision and wait for the actual occurrence of the shock to evaluate the impact on the investment project and to finally take his decision. This delay seems to be all the more important so since the potential impact (positive or negative) is unknown. In reality, however, the investor does not necessarily have the opportunity of delaying his decisions, and often has to decide as soon as the investment project appears (that is to say at time $t = 0$). This pressure may be due to several reasons. In the first place, the investor could face some legal constraints as for instance, in the search for concessions or licences (e.g. the universal mobile telecommunications systems (UMTS) where the investors have to take their decisions before a fixed deadline). The investor can also face some market and competition constraints, where the postponement of a decision can give to his competitors the opportunity of taking a stand and as a consequence, invalidate any possibility of investing in the strategic business field for the investor. So considering the study at time $t = 0$ appears to be relevant for the investment problem we analyze.

In the following, we look at a real option problem in a disrupted framework, as previously described. We illustrate the results of section 3 and provide numerical applications. In such a context, one of the most important questions is certainly related to the calibration of the model. This is all the more important so since the shock induces a drastic decrease in the project cash flows. Therefore, we conclude this study by looking at the impact of a model misspecification in a bearish environment, i.e. when the impact of a shock is to be negative.

4.1 Illustration

We now illustrate the main results of this paper (Theorems 7 and 8) through numerical examples. More precisely, we consider an agent facing an investment problem at time 0, in a potentially disturbed environment. We focus especially on the sensitivity with respect to the jump size of both the Laplace transform of the hitting time and the investment value. The set of parameters we consider is:

$$\sigma = 0.2 \; ; \; \alpha = 0.05 \; ; \; \mu = 0.15.$$

In the following table we give some values of the optimal profit and cost ratio for different values of the jump intensity λ and the jump size φ.

Table 1: **Optimal profit-cost ratio**

$\varphi \setminus \lambda$	0.25	0.5	1	2
-0.99	1.407617	1.37467	1.364417	1.367871
-0.8	1.415605	1.382496	1.372175	1.375653
-0.6	1.443986	1.410464	1.399953	1.403498
-0.5	1.468846	1.435171	1.424556	1.428139
-0.4	1.503957	1.470402	1.459746	1.463347
-0.3	1.553697	1.520998	1.510504	1.514057
-0.2	1.625893	1.595907	1.586132	1.58945
-0.1	1.735855	1.713473	1.705999	1.708546
-0.05	1.814285	1.800001	1.795146	1.796805
0	*1.917891*	*1.917891*	*1.917891*	*1.917891*
0.05	2.061146	2.098451	2.13082	2.165905
0.1	2.259259	2.38787	2.545456	
0.15	2.530455	2.851521		
0.2	2.90661	3.639258		
0.25	3.446203			
0.3	4.267252			
0.35	5.645627			
0.4	8.406155			

The optimal ratio is monotonic with respect to the jump size φ**.** Moreover the optimal ratio is a non-increasing function of the jump intensity λ for negative values of φ, whereas it is a non-decreasing function of λ for positive values of the jump size. Note also that whatever the jump intensity is, this ratio tends to $\frac{\theta_0}{\theta_0 - \sigma}$ when the jump size tends to zero. This value corresponds to the optimal ratio when the model uncertainty is simply driven by a Brownian motion.

Table 2 below provides the variation of the Laplace transform $\mathbb{E}\left(e^{-\mu\tau_{L_\varphi^{0,*}}}\right)$ and those of the optimal investment value at time 0 with respect to the jump size φ for different values of λ**.**

Table 2: **Laplace Transform of the optimal time to invest**

φ	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$
-0.99	0.117852	0.066776	0.025591	0.00582
-0.9	0.118797	0.068404	0.027797	0.008327
-0.8	0.121859	0.073688	0.034962	0.016465
-0.7	0.127039	0.08268	0.047158	0.030297
-0.6	0.13414	0.09513	0.064061	0.049423
-0.5	0.142747	0.11047	0.084925	0.07296
-0.4	0.152147	0.127691	0.10844	0.099398
-0.3	0.161184	0.145118	0.132457	0.126331
-0.2	0.167994	0.159997	0.153506	0.149993
-0.1	0.169553	0.167775	0.165981	0.164525
-0.05	0.166918	0.166801	0.166368	0.165701
0	*0.160903*	*0.160903*	*0.160903*	*0.160903*
0.05	0.150273	0.146763	0.142857	0.138179
0.1	0.135036	0.123248	0.10860	
0.15	0.116245	0.093441		
0.2	0.095067	0.061498		
0.25	0.072806			
0,3	0.050901			
0,35	0.030943			
0.4	0.014657			

Table 3: **Investment value**

φ	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$
-0.99	0.048038	0.025019	0.009326	0.002141
-0.9	0.048642	0.025752	0.010179	0.003078
-0.8	0.050645	0.028185	0.013012	0.006185
-0.7	0.054217	0.032532	0.018065	0.011712
-0.6	0.059556	0.039048	0.025622	0.019942
-0.5	0.066926	0.048073	0.036055	0.031237
-0.4	0.076676	0.060066	0.049855	0.046056
-0.3	0.089247	0.075606	0.067620	0.064941
-0.2	0.105146	0.095343	0.089975	0.088414
-0.1	0.124766	0.119703	0.117183	0.116573
-0.05	0.135919	0.133441	0.132287	0.132032
0	*0.147691*	*0.147691*	*0.147691*	*0.147691*
0.05	0.159882	0.161456	0.161618	0.161111
0.1	0.171207	0.171558	0.167923	
0.15	0.179599	0.173483		
0.2	0.183058	0.16257		
0.25	0.179627			
0.3	0.167352			
0.35	0.144303			
0.4	0.10875			

Both tables provide highly dissymmetric values with respect to the sign of the jump. Neither the Laplace transform nor the value of the investment are monotonic with respect to φ. If the jump size tends to zero, then the Laplace transform tends to $\left(\frac{s_0(\theta_0-\sigma)}{\theta_0}\right)^{\frac{\theta_0}{\sigma}}$, and it coincides with the value of the Laplace transform of the optimal hitting time in a Brownian model. The same result holds for the investment value.

4.2 Misspecification impact in a bearish environment

Assume now that the agent who does not know the (negative) random jump size Φ, can however estimate it by its expectation $\mathbb{E}(\Phi)$. The question is then if he takes his investment decision by considering only the expected value, how will his decision be impacted, in particular his investment timing: will he invest too early or too late in the project? Note that the value of the investment project is obviously reduced since $C(0, s_0, 0)$ is determined by using an optimal frontier satisfying the supremum:

$$C(0, s_0, 0) = \text{ess}\sup_{L^0 \geq 1} \mathbb{E}\left(e^{-\mu \tau_{L^0}} \left(S_{\tau_{L^0}} - 1\right)^+ \ / \ S_0 = s_0\right)$$

Any other frontier (in particular the level obtained by considering the expected value of the jump size instead of the random variable) is sub-optimal, and the supremum is not reached for this particular value. Let us first focus on the optimal level of benefit and cost ratio when the agent considers the expected value as the estimate for the jump size. We establish that such a misspecification leads the investor to undervalue the optimal level of the benefit/cost ratio since:

Proposition 9 *Using previous notations, we have*

$$L_{\Phi}^{0,*} \geq L_{\mathbb{E}(\Phi)}^{0,*}.$$

Proof. Let g be the function defined as $g(\theta) = \frac{1}{2}\theta^2 + \frac{\theta}{\sigma}\left(\alpha - \frac{\sigma^2}{2}\right) - \mu$.
Both inequalities $0 < \theta_0$ and $0 \leq \alpha < \mu$ and the equalities $g(\theta_0) = 0$ and $g(\sigma) = \alpha - \mu$ imply $\frac{\theta_0}{\sigma} > 1$. Hence, by Jensen's inequality, we deduce

$$\mathbb{E}\left((1+\Phi)^{\frac{\theta_0}{\sigma}}\right) \geq (1 + \mathbb{E}(\Phi))^{\frac{\theta_0}{\sigma}} \ .$$

On the other hand, we have:

$$f_0(\lambda, a) \geq 0, \ \forall \lambda > 0, \ \forall a \in \mathbb{R} \ .$$

So, from Theorem 7, we may conclude:

$$L_{\Phi}^{0,*} \geq L_{\mathbb{E}(\Phi)}^{0,*} \ .$$

■

Therefore, assuming that the agent chooses his investment time according to $L_{\mathbb{E}(\Phi)}^{0,*}$, he will enter the project as soon as the benefit/cost ratio reaches this threshold (instead of using the optimal threshold $L_{\Phi}^{0,*}$). In that sense, it can be said that the model misspecification leads them to undertake the project too early. More precisely, the agent has some beliefs regarding the dynamics of the random benefit/cost ratio. He knows perfectly its dynamics when his information is perfect or has simply an estimation when his access to the information is limited. Given his beliefs, the agent is able to explicitly determine the optimal ratio and therefore his investment strategy using this ratio as the threshold to take his decision. He has no better way to decide when it is optimal to invest, as the knowledge of the dynamics of S simply gives his the Laplace transform of the optimal investment time and therefore simply a heuristic determination of an average investment time (see for instance Barrieu and Bellamy (2004)).

5 Conclusion

In this paper, We provide closed-form formulae for the different characteristics of a perpetual option in a framework where the uncertainty is brought by a Brownian motion and a point process with stochastic intensity. An application of these results to a real option problem is then presented. More precisely, assuming that a unique shock occurs in the context of an investment project, we studied the investment problem when the project's environment can be subject to a brutal and fundamental change (for instance, in the regulation or in its competitors' strengths and numbers). Mathematically speaking, all these situations are similar and are equivalent to consider a single jump diffusion process. The aim has been to study the impact of such a market perturbation on the decision. More precisely, assuming that the investor has a budget allocation problem at time 0, he has to determine whether the project is interesting and if so, to have some ideas of the optimal investment time as to allocate his capital among financial instruments with different maturities. In this framework, the investor determines his strategy at time 0 by characterizing the optimal profit/cost ratio, and then has some anticipations about the optimal time to invest in the project by looking at the Laplace transform at time 0 of the first hitting time of this optimal ratio. We have obtained closed-form formulae for the optimal benefit/cost ratio, the investment value and the optimal time to enter the project, without any assumption on the distribution of the jump size and with arbitrary positive or negative jump.

The framework of the investment problem studied in this paper can be extended. For example, it is also possible to consider an investment environment that could be potentially by different shocks. In this case, the profit/cost ratio is then written on the following form

$$dS_t = S_{t^-}\left(\alpha dt + \sigma dW_t + \sum_{i=1}^{i=k} \Phi_i dD_{it}\right) ,$$

where $D_{it} = 1_{t \geq T_i}$ and $T_1, T_2 - T_3, ..., T_k - T_{k-1}$ is an increasing sequence of independent random variables with exponential law, the parameter of $T_i - T_{i-1}$ being λ_i . Such an extension seems rather natural as it leads to a framework allowing for different impacts of different natures on the investment field. In this sense, the uncertainty prevailing in investment problems could be represented.

6 Appendix

6.1 Notation

We give a series of various notations which are used throughout the paper. In spite of being rather heavy, these notations are as simplified as possible and allow us to give explicit formulae for the different quantities characterizing the investment project.

i) Let $\mathcal{N}$ be the cumulative distribution of the Gaussian law :

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt .$$

ii) For $s_0 > 0$, $\nu > 0$, $L \geq 1$, $\varphi \geq 0$ and a in $\mathbb{R}$, let $F_\varphi(\nu, L, a)$ be defined as

$$\begin{aligned} F_\varphi(s_0, \nu, L, a) &= \int_0^{+\infty} \nu e^{-\nu t} \mathcal{N}\left(\tfrac{1}{\sigma\sqrt{t}} \ln(\tfrac{L}{s_0(1+\varphi)}) - a\sqrt{t}\right) dt \\ &- \left(\frac{L}{s_0}\right)^{\frac{2a}{\sigma}} \int_0^{+\infty} \nu e^{-\nu t} \mathcal{N}\left(-\tfrac{1}{\sigma\sqrt{t}} \ln(\tfrac{L(1+\varphi)}{s_0}) - a\sqrt{t}\right) dt \end{aligned} \tag{11}$$

and $\Delta F_{0,\varphi}(s_0, \nu, L, a)$ is defined by

$$\Delta F_{0,\varphi}(s_0, \nu, L, a) = F_\varphi(s_0, \nu, L, a) - F_0(s_0, \nu, L, a) . \tag{12}$$

iii) For $\nu > 0, \varphi \geq 0$ and a in $\mathbb{R}$, let $f_\varphi(\nu, a)$ be defined as

$$\begin{aligned} f_\varphi(\nu, a) &= \int_0^{+\infty} \nu e^{-\nu t} \frac{1}{\sqrt{2\pi t}} \exp(-\tfrac{1}{2}(\tfrac{1}{\sigma\sqrt{t}} \ln(1+\varphi) + a\sqrt{t})^2) dt \\ &-a \int_0^{+\infty} \nu e^{-\nu t} \mathcal{N}\left(-\tfrac{1}{\sigma\sqrt{t}} \ln(1+\varphi) - a\sqrt{t}\right) dt \end{aligned} \tag{13}$$

and $\Delta f_{0,\varphi}(\nu,a)$ id defined by

$$\Delta f_{0,\varphi}(\nu,a) = f_\varphi(\nu,a) - f_0(\nu,a)\ . \tag{14}$$

iv) We denote by a_0, θ_0 and v_0 the following parameters

$$a_0 = \frac{1}{\sigma}\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2\mu\sigma^2}\ ; \tag{15}$$

$$\theta_0 = \frac{1}{\sigma}\left(\frac{\sigma^2}{2}-\alpha+\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2\mu\sigma^2}\right)\ ; \tag{16}$$

$$v_0 = \frac{\alpha+\frac{\sigma^2}{2}}{\sigma}\ . \tag{17}$$

Lemma 10 *Assuming that the jump size is a positive constant φ, the following equalities hold:*

$$i)\ \mathbb{E}\left(e^{-\mu\tau_L}\times \mathbf{1}_{\tau_L=T}\right) = -\left(\frac{\lambda}{\lambda+\mu}\right)\times\left(\Delta F_{0,\varphi}(s_0,\lambda+\mu,L,a_0-\theta_0)\right)$$

$$ii)\ \mathbb{E}\left(e^{-\mu\tau_L}\mathbf{1}_{\tau_L>T}\right) = \left(\frac{(1+\varphi)\,s_0}{L}\right)^{\frac{\theta_0}{\sigma}}\times F_\varphi(s_0,\lambda,L,a_0)$$

$$iii)\ \mathbb{E}(e^{-\mu\tau_L}\mathbf{1}_{\tau_L<T}) = \left(\frac{s_0}{L}\right)^{\frac{\theta_0}{\sigma}}\times\left[1-F_0(s_0,\lambda,L,a_0)\right]$$

and

$$\begin{aligned} iv)\ c(s_0,L) &= \mathbb{E}\left[e^{-\mu\tau_L}\left(S_{\tau_L}-1\right)\right] \\ &= (L-1)\left(\tfrac{s_0}{L}\right)^{\frac{\theta_0}{\sigma}}\left[1-F_0(s_0,\lambda,L,a_0)\right]+(L-1)\left[\left(\tfrac{(1+\varphi)s_0}{L}\right)^{\frac{\theta_0}{\sigma}}F_\varphi(s_0,\lambda,L,a_0)\right] \\ &-\left[\tfrac{\lambda s_0(1+\varphi)}{\lambda+\mu-\alpha}\Delta F_{0,\varphi}(s_0,\lambda+\mu-\alpha,L,v_0)\right]+\tfrac{\lambda}{\lambda+\mu}\left[\Delta F_{0,\varphi}(s_0,\lambda+\mu,L,a_0-\theta_0)\right] \end{aligned}$$

Proof.
i) We have

$$\mathbb{E}\left(e^{-\mu\tau_L}\times\mathbf{1}_{\tau_L=T}\right)=\frac{\lambda}{\lambda+\mu}\mathbb{E}^{\psi_1}\left(\mathbf{1}_{\tau_L=T}\right)\ ,$$

where $\psi_1=\frac{\mu}{\lambda}$ and where $\mathbb{P}^{\psi_1}$ is the $\mathbb{P}-$equivalent probability measure, specified by its Radon-Nikodym derivative

$$\left.\frac{d\mathbb{P}^{\psi_1}}{d\mathbb{P}}\right|_{\mathcal{F}_t}=\mathcal{E}\left(\psi_1 M\right)_t=\exp\left(\ln\left(1+\psi_1\right)D_t-\lambda\psi_1\int_0^t\left(1-D_s\right)ds\right)$$

and $\mathbb{E}^{\psi_1}$ is the $\mathbb{P}^{\psi_1}$- expectation. Hence we get

$$\begin{aligned}\mathbb{E}^{\psi_1}\left(\mathbf{1}_{\tau_L=T}\right) &= \int_0^{+\infty}(\lambda+\mu)e^{-(\lambda+\mu)t}\left[\mathcal{N}\left(\frac{y-u_0t}{\sqrt{t}}\right)-e^{2u_0y}\mathcal{N}\left(\frac{-y-u_0t}{\sqrt{t}}\right)\right]dt \\ &-\int_0^{+\infty}(\lambda+\mu)e^{-(\lambda+\mu)t}\left[\mathcal{N}\left(\frac{x-u_0t}{\sqrt{t}}\right)-e^{2u_0y}\mathcal{N}\left(\frac{x-2y-u_0t}{\sqrt{t}}\right)\right]dt\ .\end{aligned}$$

where $Y_t^{u_0}=u_0t+W_t$, $M_t^{u_0}=\sup\left(Y_s^{u_0},\ 0\le t\le s\right)$ with $u_0=\frac{\alpha-\frac{\sigma^2}{2}}{\sigma}$, $y=\frac{1}{\sigma}\ln(\frac{L}{s_0})$ and $x=\frac{1}{\sigma}\ln(\frac{L}{s_0(1+\varphi)})$.
and we finally have

$$
\begin{aligned}
\mathbb{E}\left(e^{-\mu\tau_L}\times \mathbf{1}_{\tau_L=T}\right) &= \left(\frac{\lambda}{\lambda+\mu}\right)\left(F_0(s_0,\lambda+\mu,L,u_0)-F_\varphi(s_0,\lambda+\mu,L,u_0)\ \right)\\
&= -\frac{\lambda}{\lambda+\mu}\left(\Delta F_{0,\varphi}(s_0,\lambda+\mu,L,a_0-\theta_0)\right)
\end{aligned}
$$

ii) We have

$$
\mathbb{E}\left(e^{-\mu\tau_L}\mathbf{1}_{\tau_L>T}\right) = \left(\frac{(1+\varphi)\,s_0}{L}\right)^{\frac{\theta_0}{\sigma}}\times \mathbb{E}^{\theta_0}\left(\mathbf{1}_{\tau_L>T}\right)\ ,
$$

where $\mathbb{P}^{\theta_0}$ is the $\mathbb{P}-$equivalent probability measure, such that $\left.\frac{d\mathbb{P}^{\theta_0}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}\left(\theta_0 W\right)_t = \exp\left(\theta_0 W_t - \frac{1}{2}\theta_0^2 t\right)$ and where $\mathbb{E}^{\theta_0}$ is the $\mathbb{P}^{\theta_0}$- expectation. We can write

$$
\mathbb{E}\left(e^{-\mu\tau_L}\mathbf{1}_{\tau_L>T}\right) = \left(\frac{(1+\varphi)\,s_0}{L}\right)^{\frac{\theta_0}{\sigma}}\times \mathbb{P}^{\theta_0}\left(M_T^{a_0,\theta_0}<y, Y_T^{a_0,\theta_0}<x\right)
$$

with

$$
\begin{aligned}
&Y_t^{a_0,\theta_0} = a_0 t + W_t - \theta_0 t, \quad && M_t^{a_0,\theta_0} = \sup\left(Y_s^{a_0,\theta_0},\ 0\le t\le s\right)\\
&y = \tfrac{1}{\sigma}\ln\left(\tfrac{L}{s_0}\right), && x = \tfrac{1}{\sigma}\ln\left(\tfrac{L}{s_0(1+\varphi)}\right).
\end{aligned}
$$

Therefore we obtain

$$
\mathbb{E}\left(e^{-\mu\tau_L}\mathbf{1}_{\tau_L>T}\right) = \left(\frac{(1+\varphi)\,s_0}{L}\right)^{\frac{\theta_0}{\sigma}}\times F_\varphi(s_0,\lambda,L,a_0)\ .
$$

iii) The third assertion directly comes from

$$
\mathbb{E}(e^{-\mu\tau_L}\mathbf{1}_{\tau_L<T}) = \left(\tfrac{s_0}{L}\right)^{\frac{\theta_0}{\sigma}}\times \mathbb{E}^{\theta_0}\left(\mathbf{1}_{\tau_L<T}\right) = \left(\tfrac{s_0}{L}\right)^{\frac{\theta_0}{\sigma}}\left(1-F_0(s_0,\lambda,L,a_0)\right)\ .
$$

iv) We now consider $c\,(s_0,L) = \mathbb{E}\left[e^{-\mu\tau_L}\left(S_{\tau_L}-1\right)\right]$.
We have

$$
\mathbb{E}\left(S_{\tau_L}e^{-\mu\tau_L}\times \mathbf{1}_{\tau_L=T}\right) = \frac{\lambda s_0(1+\varphi)}{\lambda+\mu-\alpha}\mathbb{E}^{\psi_2}\left(\mathbf{1}_{\tau_L=T}\right)\ ,
$$

where $\psi_2 = \frac{\mu-\alpha}{\lambda}$, and $\mathbb{P}^{\psi_2}$ is the probability measure defined by $\left.\frac{d\mathbb{P}^{\psi_2}}{d\mathbb{P}^{\sigma}}\right|_{\mathcal{F}_t} = \mathcal{E}\left(\sigma W\right)_t\times\mathcal{E}\left(\psi_2 M\right)_t$.
Then we get

$$
\begin{aligned}
\mathbb{E}\left(S_{\tau_L}e^{-\mu\tau_L}\times \mathbf{1}_{\tau_L=T}\right) &= \frac{s_0(1+\varphi)\lambda}{\lambda+\mu-\alpha}F_0(s_0,\lambda+\mu-\alpha,L,v_0)\\
&\quad -\frac{s_0(1+\varphi)\lambda}{\lambda+\mu-\alpha}F_\varphi(s_0,\lambda+\mu-\alpha,L,v_0)\ ,
\end{aligned}
$$

where v_0 is defined by (17). We get the conclusion from i), ii), iii), from the equality

$$
\begin{aligned}
c(s_0,L) &= (L-1)\mathbb{E}\left(e^{-\mu\tau_L}\times \mathbf{1}_{\tau_L<T}\right) + (L-1)\mathbb{E}\left(e^{-\mu\tau_L}\times \mathbf{1}_{\tau_L>T}\right)\\
&\quad +\mathbb{E}\left(e^{-\mu\tau_L}\left(S_{\tau_L}-1\right)\times \mathbf{1}_{\tau_L=T}\right),
\end{aligned}
$$

■

Lemma 11 *Let L be a frontier such that $L \geq 1$ and $L > s_0$. For any φ in $]-1,0[$, we have*

$$\mathbb{E}(e^{-\mu\tau_L}\mathbf{1}_{\tau_L>T}) = \left(\frac{(1+\varphi)\,s_0}{L}\right)^{\frac{\theta_0}{\sigma}} \times F_0(s_0,\lambda,L,a_0)\ ,$$

where a_0, θ_0 and $F_0(s_0,\lambda,L,a)$ are defined by (15), (16) and (11) respectively

Proof. For any φ in $]-1,0[$, we can write

$$\mathbb{E}\left(e^{-\mu\tau_L}\mathbf{1}_{\tau_L>T}\right) = \left(\tfrac{(1+\varphi)s_0}{L}\right)^{\frac{\theta_0}{\sigma}} \times \mathbb{E}^{\theta_0}\left(\mathbf{1}_{\tau_L>T}\right)\ .$$

and in this case we have

$$\mathbb{E}^{\theta_0}\left(\mathbf{1}_{\tau_L>T}\right) = F_0(s_0,\lambda,L,a_0)\ .$$

■

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