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A new approach to identifying generalized competing risks models with application to second-price auctions

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This paper proposes an approach to proving nonparametric identification for distributions of bidders' values in asymmetric second-price auctions. I consider the case when bidders have independent private values and the only available data pertain to the winner's identity and the transaction price. My proof of identification is constructive and is based on establishing the existence and uniqueness of a solution to the system of nonlinear differential equations that describes relationships between unknown distribution functions and observable functions. The proof is conducted in two logical steps. First, I prove the existence and uniqueness of a local solution. Then I describe a method that extends this local solution to the whole support.

This paper delivers other interesting results. I demonstrate how this approach can be applied to obtain identification in auctions with a stochastic number of bidders. Furthermore, I show that my results can be extended to generalized competing risks models.

Keywords. Second-price auctions, ascending auctions, asymmetric bidders, private values, nonparametric identification, competing risks, coherent systems.

JEL classification. C02, C14, C41, C65, D44.

1. Introduction

In auctions, researchers are often interested in learning models' economic primitives, particularly the joint distribution of bidders' values. Because this underlying distribution is not known a priori, it must be learned from the data. To obtain credible estimation results, a researcher must first study the identification question to determine whether the distribution of interest is identified or whether there are many distributions consistent with the data. The importance of this issue has generated many methodological papers on identification in auction models. This paper contributes to that literature.

This paper examines the nonparametric identification of the distributions of bidders' values in asymmetric second-price auctions. The identification analysis cannot be

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conducted without (a) imposing conditions on the joint distribution of bidders' signals and (b) specifying what data are available from the auctions' outcomes. This paper assumes that bidders have private values and that the only available data pertain to the winner's identity and the transaction price. Identification in this framework was first considered in Athey and Haile (2002).

It is well known that in second-price auctions within the private-values framework, a weakly dominant strategy for bidders entails submitting their true value.[1] This paper considers an equilibrium where bidders employ this strategy. In this case, even though the submitted bids directly reveal bidders' values, the joint distribution of these values cannot be identified nonparametrically because not all the bids are observed. This result is established in Athey and Haile (2002). The identification of the parameter of interest requires strengthening the model's assumptions. This paper shows that in our problem, it suffices to assume that bidders' values are independent. There are three main issues to address in obtaining this result. First, the distribution functions must be identified nonparametrically so as to avoid incorrect assumptions about their form. Second, there is a challenge posed by the asymmetry of the bidders participating in the auction. Finally, given that the transaction price is the value of the second-highest bid, the identification proof must be based on the second-order statistic.

One of the main contributions of this paper is to provide conditions on the observable data sufficient to guarantee point identification. Namely, I present conditions on the observables that are sufficient to show that the model can have at most one solution and, therefore, to ensure the identification of distribution functions. The main sufficient identification condition can be formulated in terms of the observables as well as in terms of the unobservables. It is interpretable and is weaker than identification conditions usually assumed in auctions.

This paper delivers another important result by presenting conditions on the observables that are necessary and sufficient for the existence of a solution to the model; thus, it is always known with certainty whether the model has a solution. Interestingly, these conditions for existence are a subset of the conditions sufficient to guarantee identification.

Another contribution of this paper is to prove that when there are only two types of bidders, identification always holds. This result is generalized for the case when there are only two types of bidders and the joint distribution of bidders' values is given by an Archimedean copula. I obtain a condition on the generating function of a copula that is sufficient for identification. This condition is satisfied for many classes of Archimedean copulas.

A methodological contribution of this paper is to suggest a new approach to proving identification in analyzed auction models. The idea behind this method is to establish the existence and uniqueness of a solution to a system of nonlinear differential equations that relate unknown underlying distribution functions to the observable data. This strategy includes two major steps. First, I show that the system has a unique solution on a subinterval of the support; this is what I call a local solution. Second, I demonstrate

[1] See, for example, Vickrey (1961) or Krishna (2002).

that this local solution can be extended to the whole support. This two-step approach is constructive and enables us to conduct a thorough qualitative analysis of the identification problem.

Furthermore, the techniques developed in this paper allow for generalizations of the auction setting. Using the case of three bidders, I outline the specifics of proving identification in second-price auctions in which the set of actual bidders is unknown and varies exogenously. Komarova (2009) shows that one can relax the support conditions and permit distributions to have different upper support points as well as holes in the support.

Within the private-values framework, second-price auctions are equivalent to ascending auctions. For proofs of identification in these two types of auctions, when the data indicate only the winner's identity and the winning price, researchers have referred to results in the statistical literature that examines identification in generalized competing risks models. Athey and Haile (2002) were first to observe that analyzed auctions can be considered a special case of these models.

In generalized competing risks models, an object that consists of different components fails as a result of the cumulative failure of several of its elements, and the only observed data pertain to the lifetime of the object and the set of components that had failed before the object's failure. Though the main identification result for these cases was obtained by Meilijson (1981), his proofs lack some essential details, most importantly, conditions on the observables or on the unknowns that guarantee identification. I show that my method, on the other hand, provides an exhaustive proof of identification in generalized models. For any of these models, I provide conditions on the observables and equivalent conditions on the unknowns that guarantee that the model cannot have more than one solution. I also explain why the existence of a solution cannot be proved in general and must be assumed. For a special class of generalized competing risks models, which encompasses our auction models, I present necessary and sufficient conditions for existence.

For a thorough overview of nonparametric identification in auctions, see Athey and Haile (2002, 2006, 2007) and references therein. These authors obtain numerous nonparametric identification results for various auctions settings. For the auction framework analyzed in this paper, Athey and Haile (2002) first explain why this framework is a special case of generalized competing risks models considered in Meilijson (1981), and then they refer to the Meilijson's result to obtain identification. Brendstrup and Paarsch (2006) deal specifically with asymmetric ascending auctions within the independent-private-values framework, considering both single-unit and multiunit settings. Their proof of identification repeats the proof by Athey and Haile, but provides a more detailed technical explanation of why the analyzed auction framework is, in fact, a special case of generalized competing risks models. Brendstrup and Paarsch also suggest some estimation methods and apply them to analyze fish auctions in Denmark. Banerji and Meenakshi (2004) and Meenakshi and Banerji (2005) also consider asymmetric ascending auctions within the independent-private-values framework by examining wheat markets in India. Similar to Athey and Haile (2002) and Brendstrup and Paarsch (2006), they cite Meilijson (1981) to obtain identification.

Another thread of the literature related to this paper applies the techniques of the theory of differential equations to identification problems. In auctions, examples of such papers are Campo, Perrigne, and Vuong (2003), Guerre, Perrigne, and Vuong (2009), Lebrun (1999), and Maskin and Riley (2003). Campo, Perrigne, and Vuong (2003) prove nonparametric identification for asymmetric first-price auctions with affiliated private values. Guerre, Perrigne, and Vuong (2009) address the nonparametric identification of utility functions for bidders in first-price auctions, specifically when the bidders are risk averse and have private values. Lebrun (1999) analyzes first-price auctions with independent private values and characterizes a Bayesian equilibrium as a solution to a system of nonlinear differential equations. He refers to standard results in the theory of differential equations to show that an equilibrium exists and that it is unique when the valuation distributions have a mass point at the lower support point. Maskin and Riley (2003) also analyze the uniqueness of an equilibrium in first-price auctions and prove it under a certain set of assumptions that includes an assumption about the positive atoms of the valuation distributions at the lower support point.

The rest of this paper is organized as follows. Section 2 reviews second-price auctions, outlines generalized competing risks models, and explains their connection to auctions. Section 3 states identification results for second-price auctions and considers identification in more general auction settings. Section 4 describes generalized competing risks models in detail and provides identification results for these models. Section 5 concludes. Proofs of propositions, lemmas, and theorems are collected in the Appendices.

2. Second-price auctions and generalized competing risks models

In this section, I first review second-price auctions. Next, I describe generalized competing risks models and show their connection to these auctions.

2.1 *Second-price auctions within the private-values framework*

A single object is up for sale and d buyers are bidding on it. The set of all bidders is known. Bids are submitted in sealed envelopes. The highest bidder wins and pays the value of the second-highest bid; thus, in these auctions, the second-highest bid is the winning price. Suppose that the bidders have private values and that they are aware of their value. It is known that in this setting, a weakly dominant strategy for bidders is to submit their true value—and this is an equilibrium that I consider later. In this paper, only the winner's identity and the winning price are observed in the auction outcomes.

It is worth mentioning that within the private-values framework, second-price auctions are equivalent to open ascending auctions. One form of ascending auctions is a "button auction," in which bidders hold down a button as the auctioneer raises the price. When the price gets too high for a bidder, she drops out by releasing the button. The auction ends when only one bidder remains. This person wins the object and pays the price at which the auction stopped.

2.2 *Generalized competing risks models*

Now I turn to a brief description of generalized competing risks models. Consider a machine that consists of several elements. A special case of these models is classical competing risks models. The classical models correspond to a situation in which a machine breaks down as soon as one of its components fails; the data available after the breakdown are the machine's lifetime and the element that caused the failure. One example of these models in economics is duration models. Also, the Roy model is isomorphic to classical competing risks. In the Roy model, a person chooses from a finite set of occupational alternatives to obtain the highest income and the outcomes of the choice (occupation and income) are observed. In biometrics, the death of an individual because of a particular disease when that person also faced several other diseases presents a classical competing risks model, based on a fundamental assumption that a single cause is behind every death.

Generalized competing risks models relax this assumption and consider cases in which a machine fails because of the cumulative failure of some of its elements rather than a single one. A *fatal set* for the machine is a subset of parts such that the failure of all the parts in the subset causes the failure of the machine; in other words, it is a set of the elements that failed before the machine broke down. In this paper, the machine's failure provides information only about the fatal set and the machine's lifetime. More details about generalized competing risks are given in Section 4.

2.3 *Second-price auctions as a special case of generalized competing risks models*

Athey and Haile (2002) were among the first investigators to notice the connection between second-price auctions and generalized competing risks models. To clarify the connection, I use the equivalence of second-price and ascending auctions within the private-values paradigm.

Consider a button auction, as described above, with d bidders. Notice that observing the identity of the winner is equivalent to observing the identities of the bidders who dropped out. Compare this auction framework to the following generalized competing risks model. Assume that a machine consists of d elements and works as long as at least two of its elements are functioning; in other words, the machine breaks once $d-1$ of its elements are dead. The set of these $d-1$ elements is fatal. Clearly, the breakdown of other $d-1$ components would also be fatal. A fatal set in this model is an analog of the set of bidders who dropped out, and the machine's lifetime is an analog of the winning price.

3. Identification in second-price auctions

In this section, I formulate identification results and present a mathematical description of the identification problem. Also, I discuss generalizations of the identification results. The proofs of the theorems, propositions, and lemmas of this section are collected in Appendix A.

3.1 *Statement of the identification problem*

Denote bidders' private values as X_i, $i = 1, \dots, d$. Assume that these values are independent and their distributions have densities on a common support $[t_0, T]$. This implies that distribution functions of bidders' values are absolutely continuous functions and $F_i(t_0) = 0$, $i = 1, \dots, d$. Point t_0 is not permitted to be $-\infty$, but it is allowed to have $T = +\infty$. Also assume that bidders' values at each auction are independent draws from the same joint distribution. We aim to learn this distribution from the available data. Note that in equilibrium, the bids' joint distribution coincides with the distribution of the bidders' private values. Therefore, if all the bids are observed, then the distribution of values can be clearly identified. If some of the bids are not observed, however, then neither the joint nor the marginal value distributions can be identified, as shown in Athey and Haile (2002). Given that our knowledge is often limited to the second-highest bid, I show that when the only available data pertain to the bid and the winner's identity, the marginal distributions of bidders' values can be identified if these values are independent.

Notation Throughout this paper, I use the following notations. A bid submitted by player i is denoted as b_i. Symbol M^{tr} represents the transpose of matrix M. The distribution function of X_i is denoted as F_i, $i = 1, \dots, d$. Function F_i is called positive (negative) if $F_i(t) > 0$ ($F_i(t) < 0$) for $t > t_0$. A vector-valued function $F = (F_1, \dots, F_d)^{\mathrm{tr}}$ on $[t_0, T]$ is called positive (negative) if each of its components F_i is a positive (negative) function. Function F is referred to as strictly increasing if each F_i is strictly increasing on $[t_0, T]$.

For simplicity, I first consider the case of three bidders and then generalize the results to any number of bidders. Because the winner's identity and the winning price are observed in an auction's outcome, then the probability of an event $\{\text{price} \leq t, i \text{ wins}\}$ is known for any $t \in [t_0, T]$ and any $i = 1, 2, 3$. So, for each bidder i, we observe the following subdistribution function G_i on $[t_0, T]$:

$$G_i(t) = \Pr(\text{price} \leq t, i \text{ wins}), \quad i = 1, 2, 3.$$

The comma in the definition of G_i stands for "and."

The identification problem is to determine whether there is only one collection of private-values distribution functions F_1, F_2, and F_3 that rationalize observable functions G_1, G_2, and G_3.

Identification results can be obtained under weaker support conditions. For instance, the distributions can be allowed to have different upper support points. Also, they can be allowed to have holes on (t_0, T), which means that F_i can have flat parts on this interval. It is essential, however, that each F_i is strictly increasing in a small neighborhood of t_0. A more detailed discussion of these extensions is given in the remark in the end of Section A.4.

I do not explicitly consider the case of observable heterogeneity, but all the results in this paper carry over when analyzed distributions are conditional on auction-specific observables.

3.2 *Necessary conditions on observables*

I start by describing the properties of observable functions G_i that follow from the model.

I say that *the model is not stated correctly* if at least one of the following conditions fails to hold: (i) bidders submit their true values; (ii) bidders have independent private values; (iii) bidders' values distributions have densities; (iv) bidders' values are distributed on $[t_0, T]$.

The next proposition indicates *necessary conditions on observable functions G_i* implied by the model.

Proposition 3.1. *If the model is stated correctly, then the following conditions hold.*
Necessary conditions (I):

(i) $G_i(t_0) = 0$, $i = 1, 2, 3$.

(ii) G_i *is absolutely continuous on* $[t_0, T]$, $i = 1, 2, 3$.

(iii) G_i *is strictly increasing on* $[t_0, T]$, $i = 1, 2, 3$.

Proof. By assumption, the distributions of private values X_i have densities on the common support $[t_0, T]$. This implies, in particular, that players submit bids equal to t_0 with probability 0. Also, t_0 is the lower support point for all distributions. These two facts give condition (i). Condition (ii) follows from the absolute continuity of the distributions of X_i. Condition (iii) is true because the support of each X_i is the connected interval $[t_0, T]$, without any holes in it. □

Even though these conditions are simple, it is worth indicating them because they are useful in the proof of identification. As we can see, all the properties of the private-values distributions, except for the assumption of independence and the boundary conditions $F_i(T) = 1$, $i = 1, 2, 3$, are used to establish Proposition 3.1. The independence assumption, combined with necessary conditions (I), gives the following result.

Proposition 3.2. *Suppose that the model is stated correctly. Let F be a solution to the model. Then*

$$\lim_{t\downarrow t_0} \frac{F_1}{\sqrt{\dfrac{G_2G_3}{G_1}}}(t) = 1, \qquad \lim_{t\downarrow t_0} \frac{F_2}{\sqrt{\dfrac{G_1G_3}{G_2}}}(t) = 1, \qquad \lim_{t\downarrow t_0} \frac{F_3}{\sqrt{\dfrac{G_1G_2}{G_3}}}(t) = 1. \tag{3.1}$$

Conditions (3.1) are formulated in terms of both observable and unobservable functions. They characterize a solution F to the model only in a neighborhood of t_0. To be more precise, they find the rate of convergence of unknown distribution functions F_i at t_0 in terms of observable functions G_i. These conditions are essential for proving identification.

The properties of G_i formulated in the next corollary also play an important role in establishing identification.

COROLLARY 3.3. *Suppose that the model is stated correctly. Then the following conditions hold.*

Necessary conditions (II):

$$\lim_{t\downarrow t_0}\frac{G_2G_3}{G_1}(t)=0,\qquad \lim_{t\downarrow t_0}\frac{G_1G_3}{G_2}(t)=0,\qquad \lim_{t\downarrow t_0}\frac{G_1G_2}{G_3}(t)=0. \tag{3.2}$$

The reasoning behind conditions (II) is that no matter how different the underlying distributions are, bidders' probabilities of winning do not have considerably different rates of convergence at t_0.

Now that I have presented necessary conditions on observables, I turn to describing the mathematical model of identification, and explain how necessary conditions (I) and (II) are employed in the identification proof.

3.3 *Mathematical model of the identification problem*

Assuming the independence of bidders' values, functions G_i can be expressed through F_i as follows. Let b_i, $i=1,2,3$, indicate the submitted bids. Then

$$\begin{aligned} G_1(t) &= \Pr\bigl(\max\{b_2,b_3\}<b_1, \max\{b_2,b_3\}\le t\bigr)\\ &= \Pr\bigl(\max\{X_2,X_3\}<X_1, \max\{X_2,X_3\}\le t\bigr)=\int_{t_0}^{t}(F_2F_3)'(1-F_1)\,ds. \end{aligned}$$

Functions G_2 and G_3 have similar expressions. Therefore, unknown distribution functions F_i are related to observable functions G_i by means of the system of integral-differential equations

$$\begin{aligned} G_1(t) &= \int_{t_0}^{t}(F_2F_3)'(1-F_1)\,ds,\\ G_2(t) &= \int_{t_0}^{t}(F_1F_3)'(1-F_2)\,ds,\\ G_3(t) &= \int_{t_0}^{t}(F_1F_2)'(1-F_3)\,ds. \end{aligned} \tag{3.3}$$

Notice that the left-hand and right-hand sides of the equations in (3.3) are absolutely continuous functions, allowing us to differentiate them and obtain the following system of differential equations almost everywhere (a.e.) on $[t_0, T]$:

$$\textit{Main system:}\qquad \begin{aligned} g_1 &= (F_2F_3)'(1-F_1),\\ g_2 &= (F_1F_3)'(1-F_2),\\ g_3 &= (F_1F_2)'(1-F_3), \end{aligned} \tag{DE}$$

where g_i stands for the a.e. derivative of G_i. I refer to system (*DE*) as the main system. Distribution functions F_i in this system must satisfy the initial conditions

$$\textit{Initial conditions:}\qquad F_i(t_0)=0,\quad i=1,2,3. \tag{IC}$$

I refer to problem (*DE*)–(*IC*) as the main problem. The definition below explains the meaning of a solution to (*DE*)–(*IC*).

Definition 3.1. Function $F = (F_1, F_2, F_3)^{\mathrm{tr}}$ is a solution to problem (*DE*)–(*IC*) on an interval $[t_0, t_0 + a]$, $t_0 + a \leq T$, if F_i, $i = 1, 2, 3$, are absolutely continuous on $[t_0, t_0 + a]$, satisfy equations (*DE*) a.e. on $[t_0, t_0 + a]$, and satisfy (*IC*).

The system of differential equations (*DE*) is a convenient tool because identifying functions F_i is equivalent to proving that problem (*DE*)–(*IC*) can have at most one positive solution F on $[t_0, T]$.

While proving uniqueness, the solutions are not restricted to be monotone. Therefore, if the unique solution recovered from the distributions of bids and winners' identities turns out to be *non*monotone, then it could be interpreted as evidence that the observed distributions of prices and winner's identities cannot be rationalized by the equilibrium in weakly dominant strategies in second-price or ascending auctions within the asymmetric independent private values (IPV) framework.

3.4 *Main results*

Proving identification does not require establishing the existence of a solution to (*DE*)–(*IC*) and only requires showing that (*DE*)–(*IC*) cannot have more than one solution. However, I start by presenting an existence result, because it gives conditions for the existence of a solution that are also used to obtain identification.

Theorem 3.4 (Existence of a Solution). *Let observable functions G_i satisfy conditions* (I) *and* (II). *Then problem* (*DE*)–(*IC*) *has a positive solution on* $[t_0, T]$.

Remember that all conditions on G_i required in this theorem are necessary conditions implied by the model. Therefore, conditions (I) and (II) are both necessary and sufficient conditions for the existence of a solution to the model. In particular, if even one of the conditions in (I) and (II) fails to hold, we can immediately conclude that the observable data cannot be rationalized by the equilibrium in weakly dominant strategies in second-price or ascending auctions within the asymmetric IPV framework.

The next theorem describes conditions on G_i that are sufficient to guarantee the identification of F_i.

Theorem 3.5 (Uniqueness of a Solution). *Let observable functions G_i satisfy conditions* (I) *and* (II), *and the following sufficient condition.*

Sufficient condition (III): *The function*

$$\left(\frac{g_1}{G_1} + \frac{g_2}{G_2} + \frac{g_3}{G_3}\right)\left(\sqrt{\frac{G_2 G_3}{G_1}} + \sqrt{\frac{G_1 G_3}{G_2}} + \sqrt{\frac{G_1 G_2}{G_3}}\right) \tag{3.4}$$

has a finite Lebesgue integral—that is, belongs to the class L^1—in a neighborhood of t_0. *Then problem* (*DE*)–(*IC*) *has a unique positive solution on* $[t_0, T]$.

The most important element in obtaining sufficient condition (III) is the result of Proposition 3.2. To acquire a better understanding of this condition, I write it in terms of distribution functions F_i.

REMARK 3.1. *Condition* (III) *is equivalent to the following condition*: *The function*

$$\left(\frac{F_1'}{F_1}+\frac{F_2'}{F_2}+\frac{F_3'}{F_3}\right)(F_1+F_2+F_3) \tag{3.5}$$

has a finite Lebesgue integral in a neighborhood of t_0.

A detailed explanation of this remark can be found in Section A.1 in Appendix A. Condition (3.5) is satisfied, for instance, if the ratio of any two distribution functions of values is bounded from above on $(t_0, T]$.[2] In particular, this is the case when the densities of the distributions of values are bounded above and below from zero on $[t_0, t_0+\eta]$ for a small $\eta > 0$.[3]

Now it is intuitive that the reasoning behind this condition is that the underlying distribution functions F_1, F_2, and F_3 are not too different around t_0 in a certain sense.

For instance, if the underlying distribution functions are $F_1 = t$, $F_2 = t^2$, and $F_3 = \exp(1-\frac{1}{t^2})$ on [0, 1], then the corresponding observable functions G_i do not satisfy condition (III). Figure 1 depicts such F_i. As we can see, around the lower boundary $t = 0$, the private value of bidder 3 first-order stochastically dominates the private values of bidders 1 and 2. Also, the third bidder's value distribution has a mass at 0 that is considerably smaller than the mass put at 0 by the value distributions for bidders 1 and 2. This means that bidders 1 and 2 win very rarely when the observed sale price is close to 0.

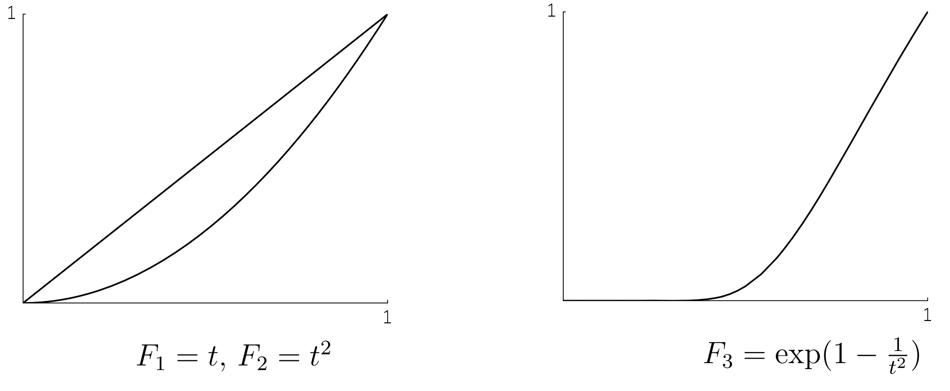


FIGURE 1. Underlying distribution functions.

[2]If for any i and j, $\frac{F_i}{F_j} \le M$ on $(t_0, T]$, then the function in (3.5), which can be written as $\sum_{i=1}^3 \sum_{j=1}^3 \frac{F_i}{F_j} F_j'$, is bounded from above by $M \sum_{j=1}^3 F_j'$ on $(t_0, T]$. This implies that the function in (3.5) has a finite Lebesgue integral in a neighborhood of t_0.

[3]If for any i, $0 < M_1 \le F_i' \le M_2 < \infty$ on $[t_0, t_0+\eta]$, then for any i and j, $F_i' \le \frac{M_2}{M_1} F_j'$ on $[t_0, t_0+\eta]$ and, hence, $F_i \le \frac{M_2}{M_1} F_j$ on $[t_0, t_0+\eta]$, which implies that $\frac{F_i}{F_j} \le \frac{M_2}{M_1}$ on $(t_0, t_0+\eta]$. Clearly, $\frac{F_i}{F_j}$ is also bounded from above on $(t_0+\eta, T]$ since $F_i(t_0+\eta) > 0$, $i = 1, 2, 3$.

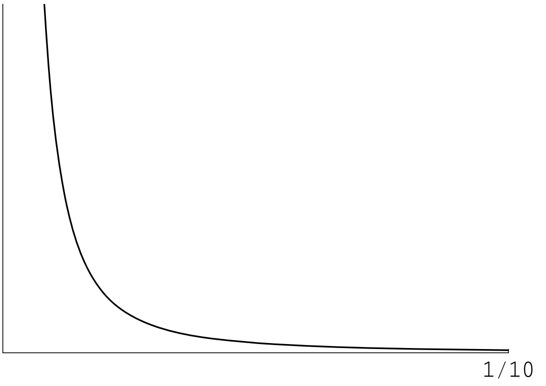


FIGURE 2. Function in (3.5) has an infinite Lebesgue integral in a neighborhood of 0.

Figure 2 shows the function in (3.5). The Lebesgue integral of this function in any small neighborhood of 0 is infinite due to the terms

$$\int_0^{\varepsilon} \frac{F_3'}{F_3}(F_1 + F_2) = \int_0^{\varepsilon} \frac{2}{t^3}\big(t + t^2\big) = \infty$$

for an arbitrary $\varepsilon > 0$. Thus, condition (III) is not satisfied. This sufficient condition captures the fact that the behavior of F_3 at $t = 0$ is unlikely to be identified because the bids from bidder 3 are almost never observed in a small neighborhood of 0. It also captures the fact that F_1 and F_2 are unlikely to be identified around $t = 0$ because the identities of bidders 1 and 2 are almost never observed there.

Condition (3.5) is satisfied if all F_i behave as power functions around t_0:

$$0 < \lim_{t \downarrow t_0} \frac{F_i(t)}{(t - t_0)^{\alpha_i}} < \infty$$

for some $\alpha_i > 0$, $i = 1, 2, 3$.

In identification results for the first-price auctions, it is usually assumed that the densities of all the distributions of the bidders' values are bounded away from zero and are finite on the support. For example, these conditions are imposed in Guerre, Perrigne, and Vuong (2009). Condition (III) is much weaker than these restrictions.

Suppose t_0 is not the lower support point of the distributions of private values, but is a binding reserve price in the intersection of the supports of bidders' private values. This reserve price is known and the bidders submit bids only when their private value is no less than t_0. Then (3.3) and (*DE*) have to be written in terms of truncated distribution functions

$$\overline{F}_i(t) = \frac{F_i(t) - F_i(t_0)}{1 - F_i(t_0)}, \quad t \geq t_0, i = 1, 2, 3,$$

and observable functions

$$\overline{G}_i(t) = \Pr(\text{price } \leq t, i \text{ wins}|\text{all bidders participate}), \quad t \geq t_0, i = 1, 2, 3,$$

assuming that the econometrician observes whether or not the event {all bidders participate} occurs, and observes the winning price and the winner's identity when this event takes place. Functions $\overline{F}_i$ and $\overline{G}_i$ satisfy conditions $\overline{F}_i(t_0) = 0$ and $\overline{G}_i(t_0) = 0$. The identification task becomes the task of establishing the uniqueness of $(\overline{F}_1, \overline{F}_2, \overline{F}_3)$, $t \geq t_0$, and all the conditions indicated in this section have to be verified for $\overline{F}_i$ and $\overline{G}_i$, $i = 1, 2, 3$. For more details, see Section 3.5.3.

This paper's identification proof comprises two major steps: establishing the local identification result and establishing the global identification result.

Namely, I first prove that problem (*DE*)–(*IC*) has only one positive solution F in a small neighborhood of t_0; this solution is what I call a local solution. Establishing the existence and uniqueness of a local solution is the most challenging part of the identification result because system (*DE*) has a singularity at t_0 due to $F_i(t_0) = 0$, $i = 1, 2, 3$. System (*DE*) can rewritten in a form that has the derivatives of F_i only on the left-hand side. In that form, the right-hand side is singular when functions F_i take zero values.

The local existence and uniqueness proofs use auxiliary functions $H_1 = F_2F_3$, $H_2 = F_1F_3$, and $H_3 = F_1F_2$. Because $F_1 = \sqrt{\frac{H_2H_3}{H_1}}$, $F_2 = \sqrt{\frac{H_1H_3}{H_1}}$, and $F_3 = \sqrt{\frac{H_1H_2}{H_3}}$, system (*DE*) can be rewritten equivalently in terms of functions H_i. The existence and uniqueness of a local solution are first established for this new system because it has a convenient form, even though it does not satisfy usual Cauchy–Lipschitz conditions, in particular, because of the initial restrictions $H_i(t_0) = 0$. More precisely, the existence of a local solution cannot be proven by using standard existence theorems in the theory of differential equations because (a) functions H_i must belong to a specific region defined by inequalities $0 \leq \frac{H_2H_3}{H_1} < 1$, $0 \leq \frac{H_1H_3}{H_1} < 1$, and $0 \leq \frac{H_1H_2}{H_3} < 1$; (b) the vector $(0, 0, 0)$ of the values of (H_1, H_2, H_3) at t_0 is not an interior point of this region. Instead, I use the so-called Tonelli approximation method, which chops an interval around t_0 into very small intervals and then exploits the form of the system for H_i to build special functions on these intervals step by step. These functions have an important property: when the lengths of the small intervals go to zero, the sequence of these functions has a subsequence that converges to a solution. Proving the uniqueness of a local solution is complicated by the fact that the right-hand side of the system for H_i does not satisfy a standard Lipschitz condition in H_i due to the initial conditions $H_i(t_0) = 0$, $i = 1, 2, 3$. I deal with this by establishing a differential inequality for functions H_i that implies uniqueness. The local existence and uniqueness results for H_i give the local existence and uniqueness results for F_i.

After proving the existence and the uniqueness of a local solution to (*DE*)–(*IC*), I show that it can be extended to a positive solution on the entire interval $[t_0, T]$, and that such an extension is unique.

To gain intuition, consider Figure 3. The picture on the left shows the local solution F found on some interval $[t_0, t_0 + c]$. The idea of constructing a global solution is to extend this solution F to the right at least to a small interval $(t_0 + c, t_0 + c_1]$, $c_1 > c$, in such a way that the extended solution solves (*DE*)–(*IC*) on $[t_0, t_0 + c_1]$. The picture on the right in Figure 3 shows this extended solution. Then this solution is extended even further to the right and so on. I show that if we continue this process in a certain way, then we will reach the upper support point T and, thus, find the solution on the whole support. An

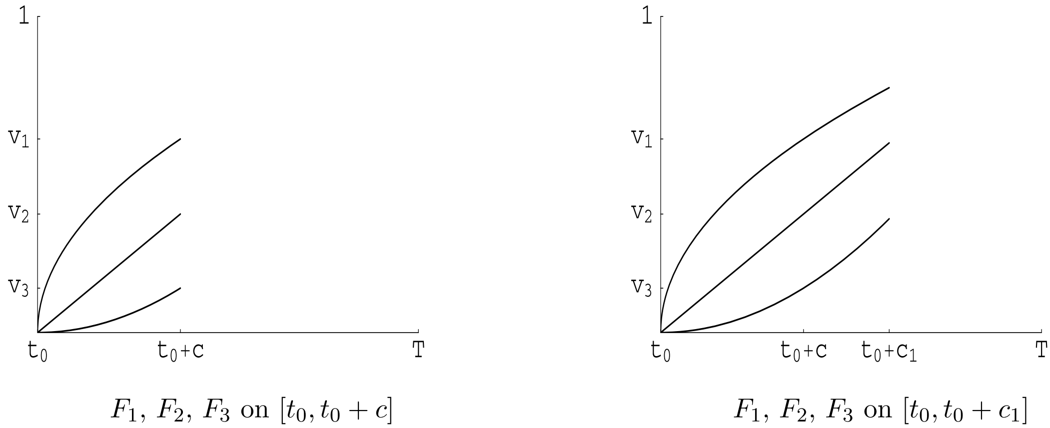


FIGURE 3. Solution to the main problem on $[t_0, t_0 + c]$ (left) and extended solution to the main problem on $[t_0, t_0 + c_1]$ (right).

analogous continuation argument for extending solutions to larger intervals is considered in Guerre, Perrigne, and Vuong (2009).

It is worth mentioning that in first-price auctions, Lebrun (1999) and Maskin and Riley (2003) avoid singularities at the lower support point by considering a reserve price and assuming that the values of underlying distributions at the reserve price are strictly positive. In this case, their systems of differential equations obey standard Cauchy–Lipschitz conditions and, thus, uniqueness is obtained in a straightforward way. In the framework of this paper, singularities remains present even if t_0 is a reserve price and $F_i(t_0) > 0$ for all i. As explained above, this happens because the system of differential equations has to be written in terms of truncated distributions functions $\frac{F_i(t)-F_i(t_0)}{1-F_i(t_0)}$, which take the value of 0 at t_0.

3.5 *Extensions*

This section discusses identification in (a) auctions with any number of bidders, (b) auctions with any number of bidders, but where there are only two types of bidders, and (c) auctions with a stochastic number of bidders.

3.5.1 *Any number of bidders* Here I show how the identification result for auctions with three bidders can be generalized to auctions with any number of bidders. I state main results and outline their proofs in Appendix A. The interpretations and intuitiveness of these results are similar to those in the case of three bidders.

The observable functions are

$$G_i(t) = \Pr(\text{price } \le t, i \text{ wins}) = \Pr\Big(\max_{j\neq i} b_j \le t, \max_{j\neq i} b_j < b_i\Big), \quad i = 1, \dots, d.$$

Propositions 3.6 and 3.7 below are the analogs of Propositions 3.1 and 3.2. Corollary 3.8 is analogous to Corollary 3.3.

Proposition 3.6. *If the model is stated correctly, then the following conditions hold.*

Necessary conditions (Id):

(i) $G_i(t_0) = 0$, $i = 1, \dots, d$.

(ii) G_i *are absolutely continuous on* $[t_0, T]$, $i = 1, \dots, d$.

(iii) G_i *are strictly increasing on* $[t_0, T]$, $i = 1, \dots, d$.

Proposition 3.7. *Suppose that the model is stated correctly. Let* F *be a solution to the model. Then*

$$\lim_{t \downarrow t_0} \frac{F_i}{\left(\frac{G_1 G_2 \cdots G_{i-1} G_{i+1} \cdots G_d}{G_i^{d-2}}\right)^{1/(d-1)}}(t) = 1, \quad i = 1, \dots, d.$$

Corollary 3.8. *Suppose that the model is stated correctly. Then the following conditions hold.*

Necessary conditions (IId):

$$\lim_{t \downarrow t_0} \frac{G_1 G_2 \cdots G_{i-1} G_{i+1} \cdots G_d}{G_i^{d-2}}(t) = 0, \quad i = 1, \dots, d.$$

The mathematical model of the identification problem is obtained in the following way. The definition of G_i and the independence of private values yield the following system of integral-differential equations that describes relationships between observable functions G_i and unknown distribution functions F_i:

$$G_i(t) = \int_{t_0}^{t} (F_1 \cdots F_{i-1} F_{i+1} \cdots F_d)'(1 - F_i)\, ds, \quad i = 1, \dots, d.$$

The differentiation of both sides of these equations gives us a system of differential equations

$$g_i = (F_1 \cdots F_{i-1} F_{i+1} \cdots F_d)'(1 - F_i), \quad i = 1, \dots, d. \tag{3.6}$$

Functions F_i in this system must satisfy initial conditions

$$F_i(t_0) = 0, \quad i = 1, \dots, d. \tag{3.7}$$

Theorem 3.9 below gives necessary and sufficient conditions for the existence of a solution to the model. Theorem 3.10 presents an identification result.

Theorem 3.9 (Existence of a Solution). *Let observable functions* G_i *satisfy conditions* (Id) *and* (IId). *Then problem* (3.6)–(3.7) *has a positive solution on* $[t_0, T]$.

Theorem 3.10 (Uniqueness of a Solution). *Let observable functions* G_i *satisfy conditions* (Id) *and* (IId), *and the following sufficient condition.*

Sufficient condition (IIId): *The function*

$$\sum_{i=1}^{d} \frac{g_i}{G_i} \cdot \sum_{i=1}^{d} \left(\frac{G_1 G_2 \cdots G_{i-1} G_{i+1} \cdots G_d}{G_i^{d-2}} \right)^{1/d-1}$$

has a finite Lebesgue integrable in a neighborhood of t_0. *Then problem* (3.6)–(3.7) *has a unique positive solution on* $[t_0, T]$.

The main identification condition (IIId) has an equivalent form in terms of the primitives of the model:

The function

$$\sum_{i=1}^{d} \frac{F_i'}{F_i} \cdot \sum_{i=1}^{d} F_i$$

has a finite Lebesgue integral in a neighborhood of t_0.

3.5.2 *Only two types of bidders: Independent values and special cases of dependent values*
Suppose that there are only two types of bidders. An econometrician observes the type of the winner but not the identity. Let d, the total number of bidders, and k, the number of bidders of type I, be known. Introduce F_{I} and F_{II} as

$$F_{\mathrm{I}}(t) = P(\text{value of type I bidder} \le t),$$
$$F_{\mathrm{II}}(t) = P(\text{value of type II bidder} \le t).$$

Then for each i, $i = 1, \ldots, d$,

$$\Pr(\text{price } \le t, i \text{ wins}) = \begin{cases} \displaystyle\int_{t_0}^{t} \left(F_{\mathrm{I}}^{k-1} F_{\mathrm{II}}^{d-k}\right)' (1 - F_{\mathrm{I}})\, ds, & \text{if bidder } i \text{ is of type I,} \\ \displaystyle\int_{t_0}^{t} \left(F_{\mathrm{I}}^{k} F_{\mathrm{II}}^{d-k-1}\right)' (1 - F_{\mathrm{II}})\, ds, & \text{if bidder } i \text{ is of type II.} \end{cases}$$

The following functions $\widetilde{G}_{\mathrm{I}}$ and $\widetilde{G}_{\mathrm{II}}$ are observed:

$$\begin{aligned} \widetilde{G}_{\mathrm{I}}(t) &= \Pr(\text{price } \le t, \text{a bidder of type I wins}), \\ \widetilde{G}_{\mathrm{II}}(t) &= \Pr(\text{price } \le t, \text{a bidder of type II wins}). \end{aligned} \tag{3.8}$$

Their relations to unobserved primitives $\tilde{F}_{\mathrm{I}}$ and $\tilde{F}_{\mathrm{II}}$ are

$$\begin{aligned} \widetilde{G}_{\mathrm{I}}(t) &= \sum_{i=1}^{d} 1(i \text{ is of type I}) \Pr(\text{price } \le t, i \text{ wins}) \\ &= k \int_{t_0}^{t} \left(F_{\mathrm{I}}^{k-1} F_{\mathrm{II}}^{d-k}\right)' (1 - F_{\mathrm{I}})\, ds, \end{aligned}$$

$$\begin{aligned}\widetilde{G}_{\mathrm{II}}(t) &= \sum_{i=1}^{d} 1(i \text{ is of type II}) \Pr(\text{ price } \le t, i \text{ wins}) \\ &= (d-k)\int_{t_0}^{t} \big(F_{\mathrm{I}}^{k} F_{\mathrm{II}}^{d-k-1}\big)'(1-F_{\mathrm{II}})\,ds.\end{aligned}$$

Proving the identification of F_{I} and F_{II} is equivalent to proving that the system of differential equations

$$\begin{aligned}\widetilde{G}_{\mathrm{I}}'/k &= \big(F_{\mathrm{I}}^{k-1} F_{\mathrm{II}}^{d-k}\big)'(1-F_{\mathrm{I}}), \\ \widetilde{G}_{\mathrm{II}}'/(d-k) &= \big(F_{\mathrm{I}}^{k} F_{\mathrm{II}}^{d-k-1}\big)'(1-F_{\mathrm{II}}),\end{aligned}$$

together with the initial conditions

$$F_{\mathrm{I}}(t_0)=0, \qquad F_{\mathrm{II}}(t_0)=0,$$

does not have more than one positive increasing solution $(F_{\mathrm{I}}, F_{\mathrm{II}})^{\mathrm{tr}}$.

The following theorem gives conditions on observable functions $\widetilde{G}_{\mathrm{I}}$ and $\widetilde{G}_{\mathrm{II}}$ that are both necessary and sufficient for the identification of F_{I} and F_{II}. It shows that in a situation with only two types, there is no need to verify any conditions similar to condition (IIId).

Theorem 3.11. *Suppose d and k are known, and the winner's type and the winning price are observed in the auction's outcomes. The following conditions on $\widetilde{G}_{\mathrm{I}}$ and $\widetilde{G}_{\mathrm{II}}$ are necessary and sufficient for the identification of F_{I} and F_{II}:*

(i) $\widetilde{G}_{\mathrm{I}}(t_0)=0$, $\widetilde{G}_{\mathrm{II}}(t_0)=0$.

(ii) *$\widetilde{G}_{\mathrm{I}}$ and $\widetilde{G}_{\mathrm{II}}$ are absolutely continuous on $[t_0, T]$.*

(iii) *$\widetilde{G}_{\mathrm{I}}$ and $\widetilde{G}_{\mathrm{II}}$ are strictly increasing on $[t_0, T]$.*

The result in Theorem 3.11 can be extended to the case when bidders' private values are dependent and their joint distribution is described by an Archimedean copula,

$$C(u_1, u_2, \dots, u_d) = \psi^{-1}\big(\psi(u_1)+\psi(u_2)+\cdots+\psi(u_d)\big),$$

where function ψ, the so-called generator, is defined on $(0, 1]$ and

$$\psi(1)=0, \qquad \lim_{x\to 0}\psi(x)=\infty, \qquad \psi'(x)<0, \qquad \psi''(x)>0.$$

In other words, the joint distribution of bidders' values has the following representation through the marginal distribution functions F_i:

$$F(t_1, \dots, t_d) = \psi^{-1}\big(\psi\big(F_1(t_1)\big)+\psi\big(F_2(t_2)\big)+\cdots+\psi\big(F_d(t_d)\big)\big).$$

As above, suppose that there are only two types of bidders and an econometrician knows the number of bidders of each type. The data pertain to the winning price and the winner's type but not the identity. Then functions $\widetilde{G}_{\mathrm{I}}$ and $\widetilde{G}_{\mathrm{II}}$, defined as in (3.8), are observable. The theorem below gives conditions on $\widetilde{G}_{\mathrm{I}}$, $\widetilde{G}_{\mathrm{II}}$, and ψ that are sufficient for the identification of F_{I} and F_{II}. I assume that the generator ψ of the copula function is known.

Theorem 3.12. *Suppose d and k are known, and the winner's type and the winning price are observed in the auction's outcomes. If $\widetilde{G}_{\mathrm{I}}$ and $\widetilde{G}_{\mathrm{II}}$ satisfy conditions* (i)–(iii) *in Theorem* 3.11 *and the function $\frac{\psi''(x)}{(\psi'(x))^2}$ is increasing, then F_{I} and F_{II} are identified.*

Archimedean copulas are often used in various applications due to their convenient form. This theorem can be applied, for instance, to Clayton, Gumbel, Frank, Joe, and Ali–Mikhail–Haq (AMH) copulas. Identification within the Archimedean family of copulas in a different auction framework is considered in Brendstrup and Paarsch (2007).

3.5.3 *Reserve price* Suppose t_0 is a binding reserve price in the intersection of the supports of bidders' private values.[4] A reserve price does not change bidders' behavior because it is still a weakly dominant strategy to bid one's value. Suppose that the reserve price t_0 is known to the bidders, and that a bidder does not submit a bid if her value is less than t_0. I assume that the set of potential bidders is known by an econometrician and does not change.[5] The econometrician observes whether or not the event {all bidders participate} occurs, and observes the winning price and the winner's identity when this event takes place. In addition, suppose that in any right-hand side neighborhood of t_0, densities F_i' are positive on sets that have positive Lebesgue measure.

Since only the second-highest bid is known and information about lower bids is not available, we have to consider the truncated distribution functions

$$\overline{F}_i(t) = \frac{F_i(t) - F_i(t_0)}{1 - F_i(t_0)}, \quad t \geq t_0, i = 1, \ldots, d,$$

and observable functions

$$\overline{G}_i(t) = \Pr(\text{price } \leq t, i \text{ wins}|\text{all bidders participate}), \quad t \geq t_0, i = 1, \ldots, d.$$

The identification task is to prove the uniqueness of $(\overline{F}_1, \ldots, \overline{F}_d)$.

[4] Now bidders can have different lower support points.

[5] Here are two examples of how the sets of potential bidders were determined in ascending auctions considered within the IPV framework. Athey, Levin, and Seira (2011) study federal auctions of timberland in California, in which the United States Forest Service sells logging contracts. They classify the bidders into two types: small firms that lack manufacturing capacity ("loggers") and larger firms with manufacturing capability ("mills"). To construct the number of potential logger bidders, Athey, Levin, and Seira count the number of distinct logging companies that entered an auction in the same geographic area in the prior year. They also do a similar count for mills. In fish auctions considered in Brendstrup and Paarsch (2006), the bidders are resale trade firms. After consulting with the auctioneer and after examining the raw data, the authors found that a total of seven potential bidders—two major and five minor—existed. Moreover, each of these bidders attended virtually every auction, so despite the presence of a reserve price, Brendstrup and Paarsch decided to ignore the issue of endogenous participation.

Unknown primitives $\overline{F}_i$ are related to observable functions $\overline{G}_i$ by means of the system of integral-differential equations

$$\overline{G}_i(t) = \int_{t_0}^{t} (\overline{F}_1 \cdots \overline{F}_{i-1}\overline{F}_{i+1} \cdots \overline{F}_d)'(1 - \overline{F}_i)\, ds, \quad i = 1, \ldots, d,$$

for $t \geq t_0$. The identification of $\overline{F}_i$ can be proven by applying the methods of Section 3.5.1 to the system of differential equations

$$\overline{G}_i'(t) = (\overline{F}_1 \cdots \overline{F}_{i-1}\overline{F}_{i+1} \cdots \overline{F}_d)'(1 - \overline{F}_i), \quad i = 1, \ldots, d,$$

considered together with the initial conditions

$$\overline{F}_i(t_0) = 0, \quad i = 1, \ldots, d.$$

All the conditions indicated in Section 3 have to be verified for $\overline{F}_i$ and $\overline{G}_i$, $i = 1, 2, 3$.

If, in addition, an econometrician always observes the set of actual bidders, then the values of $F_i(t_0)$ are identified from the data because

$$F_i(t_0) = P(i \text{ does not participate in auction}), \quad i = 1, \ldots, d.$$

The identification of $\overline{F}_i(t)$ for $t \geq t_0$ and the identification of $F_i(t_0)$ imply that distributions functions F_i are identified for $t \geq t_0$.

The main identification condition (III) in Section 3 did not allow the lower tails of value distributions to behave very differently. When t_0 is the reserve price, the behavior of the distributions in the lower tails does not matter because only the behavior in a right-hand side neighborhood of t_0 is important. In the example in Section 3.4, the value distributions are $F_1(t) = t$, $F_2(t) = t^2$, and $F_3(t) = \exp(1 - \frac{1}{t^2})$, $t \in [0, 1]$. As was shown, condition (III) is violated when $t_0 = 0$. Suppose t_0 is a reserve price that lies in $(0, 1)$. Then $\overline{F}_1(t) = \frac{t - t_0}{1 - t_0}$, $\overline{F}_2(t) = \frac{t^2 - t_0^2}{1 - t_0^2}$, and $\overline{F}_3(t) = \frac{\exp(1-1/t^2) - \exp(1-1/t_0^2)}{1 - \exp(1-1/t_0^2)}$ for $t \in [t_0, 1]$. The densities of the truncated distributions are equal to zero if $t \in [0, t_0)$. For $t \in (t_0, 1]$, the densities are equal to $\overline{F}_1'(t) = \frac{1}{1 - t_0}$, $\overline{F}_2'(t) = \frac{2t}{1 - t_0^2}$, and $\overline{F}_3'(t) = \frac{(2/t^3)\exp(1-1/t^2)}{1 - \exp(1-1/t_0^2)}$. Any of these densities is bounded from above and is bounded away from zero in a right-hand side neighborhood of t_0. Therefore, as follows from the result in footnote 3, condition (III) is satisfied for $\overline{F}_i(t)$, $i = 1, 2, 3$. To give an example of value distributions with different lower support points, for each $i = 1, 2, 3$, consider $F_i(t) = (t - t_{0i})^{\alpha_i}$, $\alpha_i > 0$, with the support $[t_{0i}, t_{0i} + 1]$. Suppose that $\max\{t_{01}, t_{02}, t_{03}\} < \min\{t_{01}, t_{02}, t_{03}\} + 1$ and that the reserve price t_0 lies in $(\max\{t_{01}, t_{02}, t_{03}\}, \min\{t_{01}, t_{02}, t_{03}\} + 1)$. Then condition (III) is satisfied for $\overline{F}_i(t)$, $i = 1, 2, 3$, because $\overline{F}_i'(t) = \alpha_i \frac{(t - t_{0i})^{\alpha_i - 1}}{1 - (t_0 - t_{0i})^{\alpha_i}}$, $i = 1, 2, 3$, is bounded from above and is bounded away from zero in a right-hand side neighborhood of t_0.

3.5.4 *Auctions with exogenous variation in the number of bidders* In this section, even though the set of potential buyers is observable to an econometrician and does not change, the set of actual participants is unobserved and varies exogenously. Denote the set of potential buyers as $\{1, \dots, d\}$. Formally, the participation is said to be *exogenous* if for all $A \subseteq \{1, \dots, d\}$ and for all $A' \subseteq A$,

$$\Pr\left(\bigcap_{i\in A'}(X_i \le t_i)\middle| A\right) = \Pr\left(\bigcap_{i\in A'}(X_i \le t_i)\right),$$

where $\Pr(\cdot|A)$ stands for the probability conditional on when A is the set of actual participants.

Athey and Haile (2002) note that such exogenous variation could arise when potential bidders face random shocks to the entry cost and these shocks are independent of bidders' private values, and bidders make decisions about entry before their values are realized. In this setting, bidders with favorable shocks enter the auction and then learn their values. If the seller does not set a reserve price, then all bidders who entered submit bids equal to their values. This creates exogenous variation in the set of actual participants. Another possible cause of exogenous variation is the bidders' use of mixed strategies in Bayesian Nash equilibrium in nonselective entry models, in which all the bidders observe the same constant entry cost and no bidder has private signals at the entry stage. Exogenous variation can also be created by a seller's restrictions on participation. McAfee and McMillan (1987) discuss the case of government-contract bidding where bidders are selected from a list of qualified bidders on a rotating basis.

Here I do not aim to present a complete general analysis of identification. Rather, I want to illustrate how the methods developed in this paper allow us to approach the identification problem.

Suppose that the number of bidders and their identities are determined by chance and the process through which bidders are selected is taken to be exogenous, embodied in the known to the econometrician probabilities p_A, $A \subseteq \{1, \dots, d\}$, where A is the set of actual bidders. The distributions of bidders' private values are assumed to have densities on a common support $[t_0, T]$. The observed data are the identity of the winner and the winning price.

To gain some insight while keeping the problem simple, I consider the case of three buyers.

The analysis below permits situations when some or all probabilities p_1, p_2, p_3 are strictly positive; that is, auctions with only one bidder as the actual participant may have positive probability. I suppose that the auction rule for such cases requires that the object is sold to the only participant at the price equal to t_0 and that the bidders are aware of this rule. In this case, if a bidder knows that she is the only actual participant, then she is indifferent about which amount from $[t_0, T]$ to bid. I do not make any assumptions about bidders' strategies in such situations. This does not affect the identification analysis because auctions with only one bidder are not helpful in identifying the distributions of bidders' values.

The only assumption imposed on probabilities p_A is the following:

If $p_{123} = 0$, *then all three values* p_{12}, p_{13}, p_{23} *are strictly positive.* (3.9)

This means that each bidder competes against any other bidder with a positive probability.

As before, the observed functions are $G_i(t) = \Pr(\text{price } \leq t, i \text{ wins})$, $i = 1, 2, 3$. Using the law of total probability, for $t \geq t_0$,

$$\begin{aligned} G_1(t) &= \Pr\bigl(\text{price } \leq t, 1 \text{ wins}|\{1\}\bigr) p_1 + \Pr\bigl(\text{price } \leq t, 1 \text{ wins}|\{1, 2\}\bigr) p_{12} \\ &\quad + \Pr\bigl(\text{price } \leq t, 1 \text{ wins}|\{1, 3\}\bigr) p_{13} + \Pr\bigl(\text{price } \leq t, 1 \text{ wins}|\{1, 2, 3\}\bigr) p_{123} \\ &= p_1 + \int_{t_0}^{t} \bigl(p_{12} F_2' + p_{13} F_3' + p_{123} (F_2 F_3)'\bigr)(1 - F_1)\, ds, \end{aligned}$$

because

$$\Pr\bigl(\text{price } \leq t, 1 \text{ wins}|\{1\}\bigr) = Pr\bigl(\text{price} = t_0, 1 \text{ wins}|\{1\}\bigr) = 1.$$

Similarly,

$$G_2(t) = p_2 + \int_{t_0}^{t} \bigl(p_{12} F_1' + p_{23} F_3' + p_{123} (F_1 F_3)'\bigr)(1 - F_2)\, ds,$$

$$G_3(t) = p_3 + \int_{t_0}^{t} \bigl(p_{13} F_1' + p_{23} F_2' + p_{123} (F_1 F_2)'\bigr)(1 - F_3)\, ds.$$

The differentiation of these equations a.e. on $[t_0, T]$ yields

$$\begin{aligned} g_1 &= \bigl(p_{12} F_2' + p_{13} F_3' + p_{123} (F_2 F_3)'\bigr)(1 - F_1), \\ g_2 &= \bigl(p_{12} F_1' + p_{23} F_3' + p_{123} (F_1 F_3)'\bigr)(1 - F_2), \\ g_3 &= \bigl(p_{13} F_1' + p_{23} F_2' + p_{123} (F_1 F_2)'\bigr)(1 - F_3). \end{aligned} \tag{3.10}$$

To prove identification, it has to be shown that system (3.10) with initial conditions

$$F(t_0) = 0, \quad i = 1, 2, 3, \tag{3.11}$$

does not have more than one positive solution on $[t_0, T]$.

My approach is to construct an auxiliary system by introducing new functions

$$\begin{aligned} H_1 &= p_{12} F_2 + p_{13} F_3 + p_{123} F_2 F_3, \\ H_2 &= p_{12} F_1 + p_{23} F_3 + p_{123} F_1 F_3, \\ H_3 &= p_{13} F_1 + p_{23} F_2 + p_{123} F_1 F_2. \end{aligned}$$

As shown in Section A.7 in Appendix A, assumption (3.9) guarantees that each function F_i, $i = 1, 2, 3$, has a unique representation in terms of H. Let $q_i(H)$ denote this representation. Then (3.10) can be written as the system of differential equations

$$H_i' = \frac{g_i}{1 - q_i(H)}, \quad i = 1, 2, 3.$$

The initial conditions on H_i are

$$\lim_{t \downarrow t_0} H_i(t) = 0, \quad i = 1, 2, 3.$$

The existence of a local solution to the auxiliary problem can be proven by applying techniques from Section A.3. First, I find necessary conditions on G_i. Assuming these conditions, I use the Tonelli approximations method to prove the local existence of a solution H to the auxiliary problem. Then I find a solution F to (3.10)–(3.11) from H by using formulas $F_i = q_i(H)$, $i = 1, 2, 3$. The extension techniques in Section A.4 would be used to show global identification.

A more detailed identification analysis of these auctions is given in Section A.7 in Appendix A.

4. Identification in generalized competing risks models

The main purpose of this section is to present conditions on observables sufficient to guarantee identification in generalized competing risks models.

In Section 2, I gave two examples of these models. First, I explained why we can consider second-price auctions to be a special case of these models. In the other example, I considered widely used classical competing risks models. I now proceed to a more detailed description of generalized competing risks models. For convenience, I use the terminology of reliability theory, which refers to these generalized models as coherent systems.[6] Essentially, a coherent system is a system that collapses because several of its elements fail.

Suppose that a machine with a coherent structure consists of d elements. Denote the elements' lifetimes as $X_1, \dots, X_d$ and denote the machine's lifetime as Z; the lifetime Z is a function of $X_1, \dots, X_d$. Conveniently, Z can be characterized by fatal sets. As defined in Section 2, a *fatal set* is a subset of parts such that the failure of all the parts in the subset causes the failure of the machine. Even more conveniently, Z can be described by the collection $I_1, \dots, I_m$ of *minimal fatal sets*, which are fatal sets that do not encompass other fatal sets. A machine is "alive" as long as in every I_j, $j = 1, \dots, m$, there is at least one part that is alive. The lifetime of the machine then can be expressed as

$$Z = \min_{j=1,\dots,m} \max_{i \in I_j} X_i.$$

The examples below clarify the structure of a coherent system. To guarantee that the probability of the simultaneous failure of several elements is 0, I suppose that the joint distribution of $X_1, \dots, X_d$ has a density. Also, X_i have the same support $[t_0, T]$.

Example 4.1. In a classical competing risks model with d risks, the collection of minimal fatal sets is $I_1 = \{1\}, \dots, I_d = \{d\}$, and the machine's lifetime is

$$Z = \min\{X_1, \dots, X_d\}.$$

Clearly, the number of minimal fatal sets coincides with the number of elements. Furthermore, there are no fatal sets other than sets I_i. Take, for instance, set $\{1, 2\}$. Although it is a superset of fatal sets $\{1\}$ and $\{2\}$, it is not fatal itself. Indeed, the death of these two elements could not cause the machine's failure, because the death of either of them would have led to failure earlier.

[6]The concept of a coherent system was introduced in Barlow and Proschan (1975).

Example 4.2. Consider a button auction with d bidders who have private values. In this case, the fatal sets are the sets of bidders who dropped out before the auction ended. The collection of minimal fatal sets is

$$I_i = \{1, \dots, i-1, i+1, \dots, d\}, \quad i = 1, \dots, d.$$

Here, element lifetimes X_i are bidders' private values and the lifetime Z is the winning price. Notice that the number of minimal fatal sets is the same as the number of bidders and there are no fatal sets besides I_i.

Example 4.3. Consider a machine with five parts. Let the collection of minimal fatal sets be $I_1 = \{1, 2, 3\}$, $I_2 = \{1, 2, 4\}$, $I_3 = \{1, 3, 4\}$, $I_4 = \{2, 3, 4\}$, $I_5 = \{1, 3, 5\}$, and $I_6 = \{2, 3, 5\}$. An example of a fatal set that is not a minimal fatal set is $\{1, 2, 3, 5\}$: It causes the failure of the machine when, for instance, the machine's elements break in the order of 5, 1, 2, and 3. Set $\{1, 2, 3, 4\}$, on the other hand, is not fatal, because all its three-element subsets are minimal fatal sets.

For coherent systems, the goal is to learn the marginal distributions of element lifetimes X_i from the joint distribution of observed "autopsy" data, which comprise the machine's lifetime Z and a *diagnostic set* D, which is the set of parts that have failed by time Z and which is revealed during the autopsy. This identification question is raised in Meilijson (1981). Meilijson claims that under certain restrictions on a coherent system's structure, the distributions of the components' lifetimes are identified if the lifetimes are independent. To formulate the identification result, he introduces an incidence matrix constructed in the following way. Given a collection of minimal fatal sets, the coherent system's incidence matrix is a matrix M such that $M(i, j) = 1$ if $j \in I_i$ and $M(i, j) = 0$ otherwise, $i = 1, \dots, m$, $j = 1, \dots, d$.

For example, in the three-bidder auctions considered in Example 4.2, the incidence matrix is

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

In classical competing risks models, on the other hand, the incidence matrix is the $d \times d$ identity matrix.

The main result of Meilijson (1981) says that if $X_1, X_2, \dots, X_d$ are nonatomic, independent, and possess the same essential infimum and supremum, and if the rank of M is d, then the joint distribution of Z and D uniquely determines the distribution of each X_j, $j = 1, \dots, d$.

The idea behind Meilijson's proof is (a) to use data only from those cases where set D is a minimal fatal set and (b) to obtain integral equations that relate the distribution functions of components' lifetimes to observable functions, and then apply to them a fixed point theorem for multidimensional functional spaces. Though Meilijson (1981) made important contributions, including the observation that only the data corresponding to minimal fatal sets can be considered and observation of the rank condition on the incidence matrix, the proofs lack some essential details. First, the author does not discuss necessary conditions on observable data besides mentioning them as a prospect for future research. As we have seen in the auction model, such conditions are crucial for obtaining the existence and uniqueness results. Second, he does not explore the existence of underlying distributions that rationalize the observables. A possible reason for this omission is the fact that in the majority of generalized competing risks models, existence cannot be proved and must be assumed, as I explain below. Nevertheless, I show that existence can be established for a special class of competing risks models and present conditions on observables that are necessary and sufficient for existence. Third, Meilijson's proof does not give conditions on observables that are sufficient to guarantee the uniqueness of underlying distributions consistent with the data. I provide these conditions for any generalized competing risks model. Finally, although the author mentions that the locally identified distribution functions can be extended to the whole support, he does not present a proof of this result. As in the auction, such a proof would require the identification result for the case in which all distribution functions have positive values at the initial point.

I suggest a new approach to identification in generalized competing risks models that offers a complete transparent proof of the identification result. I assume that the distributions of the components' lifetimes have densities, even though Meilijson (1981) obtains his result under the weaker assumption that the lifetimes' distributions functions are merely continuous. The idea behind my method is similar to the case of the auction; namely, I derive a system of nonlinear differential equations that relates the underlying distribution functions to observable functions, and then examine the existence and uniqueness issues for this system. I use the incidence matrix and assume the rank condition as in Meilijson (1981).

Now I turn to stating the main results for generalized competing risks models. An outline of Meilijson's method is given in Appendix B.

For every diagnostic set D, there is a corresponding observable function G_D:

$$G_D(t) = P(Z \leq t, D\text{—diagnostic set}).$$

Any diagnostic set is a fatal set. For any fatal set, all the minimal fatal sets it contains as subsets have a nonempty intersection. Because lifetimes X_i are independent,

$$G_D(t) = \int_{t_0}^{t} \left(\prod_{j \in C_D} F_j(s) \right)' \prod_{j \in D^c} \bigl(1 - F_j(s)\bigr) \prod_{j \in D \setminus C_D} F_j(s)\, ds, \tag{4.1}$$

where F_j is the distribution function of X_j, C_D is the intersection of all minimal fatal sets contained in D, and $D^c = \{1, \dots, d\} \setminus D$.

Let G_i be an observable function that corresponds to the minimal fatal set I_i, $i = 1, \dots, m$:

$$G_i(t) = \int_{t_0}^{t} \left(\prod_{j \in I_i} F_j(s) \right)' \prod_{j \in I_i^c} \bigl(1 - F_j(s)\bigr)\, ds, \quad i = 1, \dots, m. \tag{4.2}$$

System (4.2) of integral-differential equations is an analog of system (3.3). The differentiation of the equations in (4.2) yields the system of nonlinear differential equations

$$\left(\prod_{j\in I_i} F_j\right)' = \frac{g_i}{\prod_{j\in I_i^c}(1-F_j)}, \quad i=1,\dots,m. \tag{4.3}$$

I analyze this system together with initial conditions

$$F_i(t_0)=0, \quad i=1,\dots,d. \tag{4.4}$$

First, I consider the case in which the number of minimal fatal sets coincides with the number of the machine's components, that is, $m=d$. In this instance, M is a square matrix. Let k_{ij} stand for the (i,j) element of the inverse matrix M^{-1}.

In Theorem 4.1 below I formulate the existence result for problem (4.3)–(4.4) and describe conditions on G_i that guarantee it. Theorems 4.1 and 4.2 assume that lifetimes X_i, $i=1,\dots,d$, of the components are independent and their distributions have densities on a common support $[t_0,T]$.

THEOREM 4.1.[7] *Let $m=d$. Let functions G_i satisfy the following conditions*:

(i) $G_i(t_0)=0$, $i=1,\dots,d$.

(ii) G_i *are absolutely continuous on* $[t_0,T]$, $i=1,\dots,d$.

(iii) G_i *are strictly increasing on* $[t_0,T]$, $i=1,\dots,d$.

(iv) $\lim_{t\downarrow t_0}\prod_{j=1}^{d} G_j^{k_{ij}}(t)=0$, $i=1,\dots,d$.

Then problem (4.3)–(4.4) *has a solution F on* $[t_0,T]$.[8]

Notice that, from the model, conditions (i)–(iv) in this proposition are necessary on G_i. Indeed, (i)–(iii) follow directly from the definition of functions G_i. Given that conditions (i)–(iii) hold, condition (iv) can be obtained from (4.3). The interpretation of these conditions is similar to that of conditions (I) and (II) in the auction model.

An important difference between this case and the auction, however, is that even if problem (4.3)–(4.4) possesses a solution F and all F_i in this solution have the properties of distribution functions, the existence of a solution to the model is not guaranteed. Indeed, to satisfy the model, F must solve equation (4.1) for any diagnostic set D. System (4.3), however, accounts only for the minimal fatal sets. Therefore, after finding a solution to (4.3)–(4.4), we have to substitute it into (4.1) to verify that it solves this equation for any D. Because it is difficult (and perhaps impossible) to find conditions on functions G_D under which the model has a solution, it is common in reliability theory to assume existence. The only situation in which the conditions in Theorem 4.1 guarantee existence of a solution to the model is when $m=d$ and the only fatal sets in the model are minimal fatal sets. Notice that this is the case in the auction model analyzed in this paper.

[7]The proof of this theorem is available on request.

[8]I consider only positive solutions.

The next theorem provides conditions on G_i that are sufficient for the uniqueness of a solution to (4.3)–(4.4). The proof of this theorem is given in Appendix B.

THEOREM 4.2. *Let $m = d$. Suppose that all conditions on G_i in Theorem 4.1 are satisfied. Denote*

$$\Gamma_i(t) = g_i \sum_{l \in I_i^c} \sum_{h=1}^{d} |k_{lh}| \left(\prod_{j \neq h} G_j^{lj} \right) G_h^{k_{lh}-1}.$$

If for any $i = 1, \dots, d$, function

$$\Gamma_i \text{ has a finite Lebesgue integral in a small neighborhood of } t_0, \tag{4.5}$$

then problem (4.3)–(4.4) has a unique solution on $[t_0, T]$.

Because problem (4.3)–(4.4) has a unique solution, the model cannot have more than one solution. Therefore, the following corollary holds.

COROLLARY 4.3. *Let $m = d$. Suppose that all conditions in Theorem 4.2 are satisfied. Then a solution to the model, if it exists, is unique.*

When the number of minimal fatal sets exceeds d, that is, $m > d$, the existence of a solution to the model is always assumed. It is easy, however, to indicate conditions on observable functions that guarantee the uniqueness of a solution to the model when one exists. Consider any $d \times d$ full-rank submatrix of M. Without a loss of generality, suppose that this submatrix is formed by the first d rows in M. The subsystem of (4.3) that comprises the differential equations corresponding to the first d rows in M has only one solution if G_i satisfy the conditions in Theorem 4.2. Consequently, the model has at most one solution. We can find other sufficient conditions by choosing different submatrices of M.

The proofs of Theorems 4.1 and 4.2 use the same methods as those of Theorems 3.4 and 3.5. First, the existence and uniqueness of a solution are established locally and then globally.

5. CONCLUSION

This paper has provided methodological contributions by presenting a new way to prove identification in analyzed auction models. This approach, which employs the techniques of the theory of differential equations, is based on establishing the existence and uniqueness of a solution to the system of nonlinear differential equations that relates the underlying unknown distribution functions to the observable data. This method is constructive and provides new insight by looking at identification from a fresh perspective. Though it allows us to explore identification in more general auction settings, this approach is not limited to auctions only. As the paper has demonstrated, it can be applied to prove identification in a wide class of generalized competing risks models.

There are some issues that are worth exploring in future research. One of them would be to develop procedures for the estimation of the distribution functions of private values. One possible approach, which is discussed in Komarova (2009), is to use a sieve method based on the minimization of a certain sample objective function over a chosen sieve space. The identification result guarantees that the uniform probability limit of the sequence of such objective function has the unique arg min that coincides with the collection of true underlying value distributions. Such a uniqueness condition is usually required when proving consistency of extremum estimators. Alternatively, the Tonelli approach described in this paper can be used to construct the estimates of the auxiliary functions first and then use them to construct the estimates of the distribution functions of values. The identification result would guarantee that a sequence of Tonelli approximations constructed using consistent estimators of observable functions would converge to the unique solution to (*DE*)–(*IC*), that is, converge to the true value distributions. The sieve approach involves an optimization procedure and the choice of sieve spaces, whereas the Tonelli construction of an estimator involves integration and the choice of the length of intervals in the step-by-step procedure. Both methods rely on nonparametric estimators of the subdistribution functions of price that would be obtained from the observed data in a straightforward way. There are other methods that can be exploited. One method of interesting estimation issues that would come to light is the effect of irregularities near the boundary of the support on the rates of convergence of estimators in various norms.

Appendix A: Proofs of the results in Section 3

In the Appendix, I use the following notation. The notation $L^1[\tau, \xi]$ stands for the class of functions that have finite Lebesgue integrals on $[\tau, \xi]$. The Euclidean norm of vector $x = (x_1, \dots, x_d)$ is denoted as $\|x\|$ and $\|x\|_1$ stands for the norm of x, $\|x\|_1 = \sum_{i=1}^d |x_i|$. The right derivative of function v at point t is

$$D_R v(t) = \lim_{h \downarrow 0} \frac{v(t+h) - v(t)}{h}.$$

A.1 *Proofs of Proposition* 3.2, *Remark* 3.1, *and Corollary* 3.3

Proof of Proposition 3.2. It suffices to show that $\lim_{t \downarrow t_0} \frac{F_1}{\sqrt{G_2 G_3 / G_1}}(t) = 1$. Let $t_1 > t_0$ be very close to t_0 and let $0 < L < 1$ be such that $F_i(t) \le L$ for any $t \in (t_0, t_1)$, $i = 1, 2, 3$. Consider the first equation in system (3.3) and use it to obtain that

$$G_1(t_1) \ge \int_{t_0}^{t_1} (F_2 F_3)'(1 - L)\, ds = (1 - L) F_2(t_1) F_3(t_1),$$

$$G_1(t_1) \le F_2(t_1) F_3(t_1).$$

Similarly, using the other two equations in (3.3), obtain that

$$(1 - L) F_1(t_1) F_3(t_1) \le G_2(t_1) \le F_1(t_1) F_3(t_1),$$

$$(1 - L) F_1(t_1) F_2(t_1) \le G_3(t_1) \le F_1(t_1) F_2(t_1).$$

Because $F_1 = \sqrt{\frac{F_1F_2F_1F_3}{F_2F_3}}$, then

$$F_1(t_1) \le \frac{1}{1-L}\sqrt{\frac{G_2G_3}{G_1}}, \qquad F_1(t_1) \ge \sqrt{1-L}\sqrt{\frac{G_2G_3}{G_1}}.$$

Because $F_1(t_0) = 0$ and t_1 can be chosen arbitrarily close to t_0, then L can be arbitrarily close to 0. This implies that $\lim_{t\downarrow t_0} \frac{F_1}{\sqrt{G_2G_3/G_1}}(t) = 1$. □

Proof of Corollary 3.3. Conditions (3.2) follow from Proposition 3.2 and the fact that $\lim_{t\downarrow t_0} F_i(t) = 0$, $i = 1, 2, 3$. □

Proof of Remark 3.1. From (*DE*),

$$\lim_{t\downarrow t_0} \frac{g_1}{(F_2F_3)'}(t) = 1, \qquad \lim_{t\downarrow t_0} \frac{g_2}{(F_1F_3)'}(t) = 1, \qquad \lim_{t\downarrow t_0} \frac{g_3}{(F_1F_2)'}(t) = 1.$$

From (3.1),

$$\lim_{t\downarrow t_0} \frac{G_1}{F_2F_3}(t) = 1, \qquad \lim_{t\downarrow t_0} \frac{G_2}{F_1F_3}(t) = 1, \qquad \lim_{t\downarrow t_0} \frac{G_3}{F_1F_2}(t) = 1.$$

This implies, for instance, that

$$\lim_{t\downarrow t_0} \frac{g_1(t)/G_1(t)}{F_2'(t)/F_2(t) + F_3'(t)/F_3(t)} = 1.$$

To summarize, from (*DE*) and (3.1), one obtains that there are constants $L_1 > 0$ and $L_2 > 0$ such that

$$\begin{aligned}
&\left(\frac{F_1'}{F_1} + \frac{F_2'}{F_2} + \frac{F_3'}{F_3}\right)(F_1 + F_2 + F_3) \\
&\quad \ge L_1\left(\frac{g_1}{G_1} + \frac{g_2}{G_2} + \frac{g_3}{G_3}\right)\left(\sqrt{\frac{G_2G_3}{G_1}} + \sqrt{\frac{G_1G_3}{G_2}} + \sqrt{\frac{G_1G_2}{G_3}}\right), \\
&\left(\frac{F_1'}{F_1} + \frac{F_2'}{F_2} + \frac{F_3'}{F_3}\right)(F_1 + F_2 + F_3) \\
&\quad \le L_2\left(\frac{g_1}{G_1} + \frac{g_2}{G_2} + \frac{g_3}{G_3}\right)\left(\sqrt{\frac{G_2G_3}{G_1}} + \sqrt{\frac{G_1G_3}{G_2}} + \sqrt{\frac{G_1G_2}{G_3}}\right),
\end{aligned}$$

which implies the statement of this remark. □

A.2 *Strategy for proving identification*

Theorems 3.4 and 3.5 follow from the proofs in Sections A.3 and A.4.

As mentioned in Section 3.4, my strategy for proving identification consists of two logical steps: first establishing local identification and then establishing global identification.

It can be shown that (*DE*)–(*IC*) always has a negative local solution as well as a positive local solution.[9] Conditions for uniqueness in the theory of differential equations do not let us control the sign of solutions. Therefore, even though I am interested only in a positive solution and can neglect a negative one, sufficient conditions that guarantee uniqueness of a positive local solution cannot be derived from system (*DE*). To tackle this problem, I use auxiliary tools.

Auxiliary tools I transform (*DE*) into a new system by introducing auxiliary functions

$$H_1 = F_2 F_3, \qquad H_2 = F_1 F_3, \qquad H_3 = F_1 F_2.$$

Clearly, these functions are the distribution functions of $\max\{X_2, X_3\}$, $\max\{X_1, X_3\}$, and $\max\{X_1, X_2\}$, respectively. Functions F_i are expressed through H_i as $F_1^2 = \frac{H_2 H_3}{H_1}$, $F_2^2 = \frac{H_1 H_3}{H_2}$, and $F_3^2 = \frac{H_1 H_2}{H_3}$. Taking into account that F_i must be positive, I obtain

$$F_1 = \sqrt{\frac{H_2 H_3}{H_1}}, \qquad F_2 = \sqrt{\frac{H_1 H_3}{H_2}}, \qquad F_3 = \sqrt{\frac{H_1 H_2}{H_3}}. \tag{A.1}$$

Thus, for any point $t > t_0$, system (*DE*) can be written as

$$H_1' = \frac{g_1}{1 - \sqrt{\frac{H_2 H_3}{H_1}}},$$
$$H_2' = \frac{g_2}{1 - \sqrt{\frac{H_1 H_3}{H_2}}},$$
$$H_3' = \frac{g_3}{1 - \sqrt{\frac{H_1 H_2}{H_3}}}.$$

Note that initial conditions $H_i(t_0) = 0$ cannot be imposed because the right-hand sides of the equations in this system are undefined when H_i takes value 0. Instead, I can set conditions on the upper limit of H_i at t_0:

$$\lim_{t \downarrow t_0} H_i(t) = 0, \quad i = 1, 2, 3. \tag{IC_H}$$

The right-hand side of the last system is a vector-valued function that depends on t, H_1, H_2, and H_3. Denote it as

$$J(t, H) = \left(\frac{g_1(t)}{1 - \sqrt{\frac{H_2 H_3}{H_1}}}, \frac{g_2(t)}{1 - \sqrt{\frac{H_1 H_3}{H_2}}}, \frac{g_3(t)}{1 - \sqrt{\frac{H_1 H_2}{H_3}}} \right)^{\mathrm{tr}} \tag{A.2}$$

and rewrite the last system as

$$H'(t) = J\big(t, H(t)\big). \tag{DE_H}$$

[9] See Remark A.4 for further explanation.

I refer to (DE_H) as an auxiliary system and refer to problem (DE_H)–(IC_H) as an auxiliary problem.

Definition A.1. Function $H = (H_1, H_2, H_3)^{\mathrm{tr}}$ is a solution to (DE_H)–(IC_H) on an interval $(t_0, t_0 + a]$ if H_i are absolutely continuous on $(t_0, t_0 + a]$, satisfy (DE_H) a.e. on $(t_0, t_0 + a]$, and also satisfy (IC_H).

Proof road map Because formulas (A.1) account for the sign of F_i, we automatically consider positive solutions to (DE)–(IC). Thereafter, by a solution to (DE)–(IC), I will always mean a positive solution.

The local identification result is proved in steps. In the first step, I show that conditions (I) and (II) are sufficient to guarantee that problem (DE_H)–(IC_H), which is the auxiliary problem, has a local solution. In the second step, I use formulas (A.1) to find F_i from H_i and show that these F_i constitute a local solution to the main problem. Last, for the auxiliary problem, I establish that its local solution that was found in the first step is unique. This implies that for the main problem, its local solution that was found in the second step is the unique solution.

The global identification result is obtained from the local identification result by showing how the unique local solution to (DE)–(IC) can be extended to the unique solution on the whole support. The idea is to extend this local solution to small intervals progressively farther to the right until the upper support point T is reached.

The identification proof below employs techniques of the theory of differential equations. Descriptions of similar or related techniques can be found, for instance, in Tonelli (1928), Sansone (1948), Hartman (1964), Szarski (1965), Coddington and Levinson (1972), and Filippov (1988).

A.3 *Local identification*

Proving local identification is the most difficult part of the identification proof. I show that to establish the existence of a local solution, I only need conditions (I) and (II). To obtain local uniqueness, I use condition (III) as well as (I) and (II).

A.3.1 *Existence of a local solution* I start by finding an interval on which a local solution to the auxiliary problem (DE_H)–(IC_H) and a local solution to the main problem (DE)–(IC) exist. Then I prove local existence for (DE_H)–(IC_H) and use this result to establish local existence for (DE)–(IC).

Before moving on, I must introduce some notation and carry out preliminary technical work. First of all, I have to indicate the domain of function $J(t, H)$. Take into account formulas (A.1), which express F through H, and note that for the auxiliary problem, we want to prove not only that there is a local solution, but also that this solution is such that functions $\frac{H_2H_3}{H_1}$, $\frac{H_1H_3}{H_2}$, and $\frac{H_1H_2}{H_3}$ take values less than 1 and the following conditions hold:

$$\lim_{t\downarrow t_0} \frac{H_2H_3}{H_1}(t) = 0, \qquad \lim_{t\downarrow t_0} \frac{H_1H_3}{H_2}(t) = 0, \qquad \lim_{t\downarrow t_0} \frac{H_1H_2}{H_3}(t) = 0.$$

This accords with the fact that for function $J(t, H)$ to be well defined, the denominators in $J(t, H)$ must be separated from 0. To do this, choose any $\delta \in (0, 1)$ and allow H to take values only in the sets

$$\bar{H}_0(\delta) = (0, \infty)^3 \cap \{(h_1, h_2, h_3)^{\mathrm{tr}} : h_2 h_3 \leq \delta h_1, h_1 h_3 \leq \delta h_2, h_2 h_3 \leq \delta h_1\}.$$

Let $\bar{D}_0(\delta) = [t_0, T] \times \bar{H}_0(\delta)$ be the domain of $J(t, H)$ (a.e. with respect to t). As we can see, δ guarantees that the denominators in $J(t, H)$ are separated from 0 by the value $1 - \sqrt{\delta}$.

To determine an interval of existence for a local solution, I use conditions (II). Choose $\gamma > 0$ such that $\gamma/(1 - \sqrt{\delta})^2 \leq \delta$. Let $t_0 + a$, $a > 0$, be a point from $[t_0, T]$ such that

$$\forall \bigl(t \in [t_0, t_0 + a]\bigr) \quad \frac{G_2 G_3}{G_1}(t) \leq \gamma, \qquad \frac{G_1 G_3}{G_2}(t) \leq \gamma, \qquad \frac{G_1 G_2}{G_3}(t) \leq \gamma. \tag{A.3}$$

Conditions (II) guarantee that such $t_0 + a$ exists. Interval $[t_0, t_0 + a]$ is an interval on which a solution to problem (DE_H)–(IC_H) exists.

Auxiliary system with ε The right-hand side $J(t, H)$ of the auxiliary system (DE_H) has singularities in H when $H_1 = 0$ or $H_2 = 0$ or $H_3 = 0$. These singularities can be handled by using a very small $\varepsilon > 0$ and considering an auxiliary system with $\varepsilon > 0$,

$$H_1' = \frac{g_1}{1 - \sqrt{\dfrac{H_2 H_3}{H_1 + \varepsilon}}},$$

$$H_2' = \frac{g_2}{1 - \sqrt{\dfrac{H_1 H_3}{H_2 + \varepsilon}}},$$

$$H_3' = \frac{g_3}{1 - \sqrt{\dfrac{H_1 H_2}{H_3 + \varepsilon}}},$$

together with initial conditions

$$H_i(t_0) = 0, \quad i = 1, 2, 3. \tag{$IC_{H,\varepsilon}$}$$

Denote

$$J^{\varepsilon}(t, H) = \left(\frac{g_1(t)}{1 - \sqrt{\dfrac{H_2 H_3}{H_1 + \varepsilon}}}, \frac{g_2(t)}{1 - \sqrt{\dfrac{H_1 H_3}{H_2 + \varepsilon}}}, \frac{g_3(t)}{1 - \sqrt{\dfrac{H_1 H_2}{H_3 + \varepsilon}}}\right)^{\mathrm{tr}}$$

and rewrite the system with ε as

$$H'(t) = J^{\varepsilon}\bigl(t, H(t)\bigr). \tag{$DE_{H,\varepsilon}$}$$

The definition of a solution to ($DE_{H,\varepsilon}$)–($IC_{H,\varepsilon}$) is analogous to Definition A.1 and defines a solution on $[t_0, t_0 + a]$ instead of $(t_0, t_0 + a]$.

Introduce

$$\bar{H}(\delta) = [0,\infty)^3 \cap \{(h_1,h_2,h_3)^{\mathrm{tr}} : h_2h_3 \le \delta h_1,\, h_1h_3 \le \delta h_2,\, h_2h_3 \le \delta h_1\}$$

and let $\bar{D}(\delta) = [t_0, T] \times \bar{H}(\delta)$ be the domain of $J^\varepsilon(t,H)$ (a.e. with respect to t). The difference between $\bar{H}(\delta)$ and $\bar{H}_0(\delta)$ is that $\bar{H}(\delta)$ allows H_i to take value 0.

Lemma A.1. *Let observable functions G_i satisfy conditions* (I) *and* (II). *Let $J^\varepsilon(t,H)$ be defined on $\bar{D}(\delta)$. Then* ($DE_{H,\varepsilon}$)–($IC_{H,\varepsilon}$) *has a solution on* $[t_0, t_0+a]$.

Proof. To prove this result, I use a Tonelli approximation approach, which builds special approximations of a solution on very small intervals. These approximations have an important property; when the lengths of the intervals go to zero, the sequence of approximations has a subsequence converging to a solution to ($DE_{H,\varepsilon}$)–($IC_{H,\varepsilon}$).

Tonelli approximations are constructed in the following way. Consider, for example, intervals $[t_0, t_0+\frac{1}{k}], [t_0+\frac{1}{k}, t_0+\frac{2}{k}], \dots, [t_0+\frac{r}{k}, t_0+a]$, where $a \le \frac{r+1}{k}$ and k is very large. First, an approximation is built on $[t_0, t_0+\frac{1}{k}]$, then it is extended to interval $(t_0+\frac{1}{k}, t_0+\frac{2}{k}]$. Next, it is extended to $(t_0+\frac{2}{k}, t_0+\frac{3}{k}]$ and so on. This process is continued until the approximation is constructed on the whole interval $[t_0, t_0+a]$.

Now I turn to a description of the rule of constructing approximations. The integration of both sides in ($DE_{H,\varepsilon}$) yields $H(t) = \int_{t_0}^t J^\varepsilon(s,H)\,ds$. For a given k, denote a corresponding Tonelli approximation as $H^k = (H_1^k, H_2^k, H_3^k)$. Function H^k is defined according to the rule

$$H^k(t) = \int_{t_0}^t J^\varepsilon\left(s, H^k\left(s-\frac{1}{k}\right)\right) ds, \quad t \in [t_0, t_0+a]. \tag{A.4}$$

Choose a k that is large enough. To carry out the first step of constructing an approximation on $[t_0, t_0+\frac{1}{k}]$, let

$$H_i^k(t) = 0, \quad t \in [t_0-1, t_0], i = 1,2,3.$$

Let me show that formula (A.4) is meaningful. In the first step, it defines $H^k(t)$ for $t \in [t_0, t_0+\min\{\frac{1}{k}, a\}]$. Because $J^\varepsilon(s, H^k(s-\frac{1}{k})) = (g_1(s), g_2(s), g_3(s))^{\mathrm{tr}}$ for any $s \in [t_0, t_0+\min\{\frac{1}{k}, a\}]$ and $g_i \in L^1[t_0, t_0+a]$, then the integral on the right-hand side exists. For the next step to be well defined, I have to check that for $t \in [t_0, t_0+\min\{\frac{1}{k}, a\}]$, the values of the constructed function $H^k(t) = (H_1^k, H_2^k, H_3^k)^{\mathrm{tr}}$ belong to $\bar{H}(\delta)$. Indeed, $H_i^k(t) = G_i(t)$. Properties $\frac{H_2^k(t)H_3^k(t)}{H_1^k(t)} \le \delta$, $\frac{H_1^k(t)H_3^k(t)}{H_2^k(t)} \le \delta$, and $\frac{H_1^k(t)H_2^k(t)}{H_3^k(t)} \le \delta$ follow from (A.3) and the fact that $\gamma < \delta$. Therefore, $H^k(t) \in \bar{H}(\delta)$.

In the second step, formula (A.4) defines H^k on $[t_0+\frac{1}{k}, t_0+\min\{\frac{2}{k}, a\}]$. For $t \in [t_0+\frac{1}{k}, t_0+\min\{\frac{2}{k}, a\}]$, the Lebesgue integral on the right-hand side is finite because function $J^\varepsilon(s, H^k(s-\frac{1}{k}))\,ds$ is evidently measurable and bounded by a function that has a finite Lebesgue integral:

$$\left| J_i^\varepsilon\left(s, H^k\left(s-\frac{1}{k}\right)\right) ds\right| \le \frac{g_i(s)}{1-\sqrt{\delta}} \in L^1[t_0, t_0+a], \quad s \in \left[t_0, t_0+\min\left\{\frac{2}{k}, a\right\}\right].$$

Clearly, $H_i^k(t) > 0$. Because $H_2^k(t) \le \frac{G_2(t)}{1-\sqrt{\delta}}$, $H_3^k(t) \le \frac{G_3(t)}{1-\sqrt{\delta}}$, and $H_1^k(t) \ge G_1(t)$, then

$$\frac{H_2^k(t)H_3^k(t)}{H_1^k(t)} \le \frac{G_2(t)G_3(t)}{(1-\sqrt{\delta})^2 G_1(t)} \le \frac{\gamma}{(1-\sqrt{\delta})^2} \le \delta.$$

Likewise,

$$\frac{H_1^k(t)H_3^k(t)}{H_2^k(t)} \le \frac{G_1(t)G_3(t)}{(1-\sqrt{\delta})^2 G_2(t)} \le \frac{\gamma}{(1-\sqrt{\delta})^2} \le \delta,$$

$$\frac{H_1^k(t)H_2^k(t)}{H_3^k(t)} \le \frac{G_1(t)G_2(t)}{(1-\sqrt{\delta})^2 G_3(t)} \le \frac{\gamma}{(1-\sqrt{\delta})^2} \le \delta.$$

Therefore, $H^k(t) \in \bar{H}(\delta)$ for $t \in [t_0 + \frac{1}{k}, t_0 + \min\{\frac{2}{k}, a\}]$.

All subsequent steps are similar to the second step. By continuing to construct approximations in this manner, I can eventually define function H^k on the whole interval $[t_0, t_0 + a]$.

I take progressively smaller intervals and obtain a sequence of approximations $\{H^k\}$. Because for any k,

$$\begin{aligned} \|H^k(t)\|_1 &\le \frac{G_1(t) + G_2(t) + G_3(t)}{1-\sqrt{\delta}} \\ &\le \frac{G_1(t_0+a) + G_2(t_0+a) + G_3(t_0+a)}{1-\sqrt{\delta}}, \end{aligned} \tag{A.5}$$

functions H^k in this sequence are uniformly bounded. Moreover, sequence $\{H^k\}$ is equicontinuous, a property that is implied by inequality (A.6) and the absolute continuity of G_i on $[t_0, t_0 + a]$:

$$\|H^k(t) - H^k(\tau)\|_1 \le \frac{\|G(t) - G(\tau)\|_1}{1-\sqrt{\delta}}, \quad t, \tau \in [t_0, t_0 + a]. \tag{A.6}$$

According to the Arzela–Ascoli theorem, sequence $\{H^k\}$ is relatively compact in $C([t_0, t_0 + a], \bar{H})$, so it contains a subsequence $\{H^{k_m}\}$ such that for some function H^ε,

$$\sup_{t\in[t_0,t_0+a]} \|H^\varepsilon(t) - H^{k_m}(t)\|_1 \to 0$$

as $m \to \infty$. Because

$$J^\varepsilon\left(t, H^{k_m}\left(t - \frac{1}{k_m}\right)\right) \to J^\varepsilon\big(t, H^\varepsilon(t)\big) \quad \text{a.e. } [t_0, t_0 + a]$$

as $m \to \infty$, and a.e. on $[t_0, t_0 + a]$,

$$\left\| J^\varepsilon\left(t, H^{k_m}\left(t - \frac{1}{k_m}\right)\right)\right\|_1 \le \frac{g_1(t) + g_2(t) + g_3(t)}{1-\sqrt{\delta}} \in L^1[t_0, t_0 + a],$$

then according to the Lebesgue dominated convergence theorem, H solves

$$H^{\varepsilon}(t)=\int_{t_0}^{t} J^{\varepsilon}\bigl(s, H^{\varepsilon}(s)\bigr)\, ds, \quad t\in[t_0, t_0+a].$$

The last equation implies that H^{ε} is absolutely continuous and solves ($DE_{H,\varepsilon}$)–($IC_{H,\varepsilon}$) a.e. on $[t_0, t_0+a]$. □

Local existence for the auxiliary problem The next proposition formulates the local existence result for the auxiliary problem.

Proposition A.2. *Let observable functions G_i satisfy conditions* (I) *and* (II). *Let $J(t, H)$ be defined on $\bar{D}_0(\delta)$. Then* (DE_H)–(IC_H) *has a solution on* $(t_0, t_0+a]$.

Proof. Choose a sequence ε_m such that $\varepsilon_m \to 0$ as $m\to\infty$. For every ε_m, denote a solution constructed under Proposition A.1 for this ε_m as H^{ε_m}. As I proved, for every ε_m, function H^{ε_m} is absolutely continuous on $[t_0, t_0+a]$ and $H_i^{\varepsilon_m}(t)>0$, $t\in(t_0, t_0+a]$.

Notice that the bounds in (A.5) and (A.6) do not depend on the value of ε; therefore,

$$\bigl\|H^{\varepsilon_m}(t)\bigr\|_1 \le \frac{\|G(t_0+a)\|_1}{1-\sqrt{\delta}}, \quad t\in[t_0, t_0+a],$$

and

$$\bigl\|H^{\varepsilon_m}(t)-H^{\varepsilon_m}(\tau)\bigr\|_1 \le \frac{\|G(t)-G(\tau)\|_1}{1-\sqrt{\delta}}, \quad t, \tau\in[t_0, t_0+a].$$

The last two inequalities and the Arzela–Ascoli theorem imply that sequence $\{H^{\varepsilon_m}\}$ is relatively compact in $C([t_0, t_0+a], \bar{H})$. Hence, it has a subsequence $H^{\varepsilon_{m_l}}$ such that for some function H,

$$\sup_{t\in[t_0, t_0+a]} \bigl\|H(t)-H^{\varepsilon_{m_l}}(t)\bigr\|_1 \to 0$$

as $l\to\infty$. Because

$$J^{\varepsilon}\bigl(t, H^{\varepsilon_{m_l}}(t)\bigr) \to J\bigl(t, H(t)\bigr) \quad \text{a.e. } [t_0, t_0+a]$$

as $l\to\infty$, and a.e. on $[t_0, t_0+a]$,

$$\bigl\|J^{\varepsilon_{m_l}}\bigl(t, H^{\varepsilon_{m_l}}(t)\bigr)\bigr\|_1 \le \frac{g_1(t)+g_2(t)+g_3(t)}{1-\sqrt{\delta}} \in L^1[t_0, t_0+a],$$

the Lebesgue dominated convergence theorem yields

$$H(t)=\int_{t_0}^{t} J\bigl(s, H(s)\bigr)\, ds, \quad t\in[t_0, t_0+a].$$

From the last equation, it can be concluded that H_i are absolutely continuous on $[t_0, t_0+a]$ and constitute a solution to (DE_H)–(IC_H) on $(t_0, t_0+a]$. □

It is remarkable that this existence result does not require any assumptions on observable G_i besides necessary conditions, which are satisfied in the model.

The proof of this proposition implies that if we take a solution H to (DE_H)–(IC_H) on $(t_0, t_0+a]$ and define the function for t_0 as $H(t_0)=(0,0,0)^{\mathrm{tr}}$, then this extended function is absolutely continuous on $[t_0, t_0+a]$ and clearly satisfies (DE_H)–(IC_H) a.e. on $[t_0, t_0+a]$. In other words, a solution H can be extended from $(t_0, t_0+a]$ to $[t_0, t_0+a]$.

The following explanation shows why I cannot use standard existence theorems to prove Proposition A.2. A general form of a system of differential equations is

$$x'(t)=v\bigl(t, x(t)\bigr),$$

where x and v are vector-valued functions. Let the initial condition be

$$x(t_0)=x_0.$$

In our problem, x is function H and $v(t,x)$ is $J(t,H)$.[10] Existence theorems are usually proved for the situation in which the domain of v is $[t_0-h, t_0+h]\times B(x_0)$ or $[t_0, t_0+h]\times B(x_0)$, where $B(x_0)$ is an open ball with the center in x_0.[11] This property implies, for example, that x_0 is an interior point in the domain of v with respect to x. Existence theorems are also proven for some more general cases, but all require, at the very least, x_0 to be an interior point in the domain of v with respect to x, and this domain must satisfy certain properties. Because of the specificity of sets $\bar{H}_0(\delta)$ and $\bar{H}(\delta)$, and the fact that the point of the initial conditions $(0,0,0)^{\mathrm{tr}}$ is on the border of these sets, I cannot apply any of those results. The method of Tonelli approximation allows me to take into account the specificity of $\bar{H}_0(\delta)$ and $\bar{H}(\delta)$ by verifying at each step that the values of the constructed Tonelli function belong to the domain $\bar{H}(\delta)$.

Local existence for the main problem Now that I have established the local existence result for the auxiliary problem (DE_H)–(IC_H), I can turn to proving that the main problem (DE)–(IC) has a local solution. This result is easy to obtain if we recall how H and F are related in formulas (A.1).

Theorem A.3. *Let observable functions G_i satisfy conditions* (I) *and* (II). *Then* (DE)–(IC) *has a solution on* $[t_0, t_0+a]$.

Proof. Let H be a solution to (DE_H)–(IC_H) on $(t_0, t_0+a]$. For $t>t_0$, define F_i according to formulas (A.1), and let $F_i(t_0)=0$, $i=1,2,3$. It follows from (DE_H) that $1\le \frac{H_i(t)}{G_i(t)}\le \frac{1}{1-\sqrt{\delta}}$ for $t\in(t_0, t_0+a]$. Then

$$F_1(t)=\sqrt{\frac{H_2H_3}{H_1}}(t)\le \frac{1}{1-\sqrt{\delta}}\sqrt{\frac{G_2G_3}{G_1}}(t), \quad t\in(t_0, t_0+a],$$

[10] Even though initial conditions (IC_H) characterize the limit at t_0 rather than the value at t_0, this does not matter because, as I mentioned above, solution H can be extended from $(t_0, t_0+a]$ to $[t_0, t_0+a]$.

[11] For systems with discontinuous right-hand sides, this result is illustrated in Filippov (1988).

which implies that F_1 is continuous at t_0 because

$$0 \le \lim_{t\downarrow t_0} F_1(t) \le \frac{1}{1-\sqrt{\delta}} \lim_{t\downarrow t_0} \sqrt{\frac{G_2 G_3}{G_1}}(t) = 0.$$

Continuity of F_2 and F_3 at t_0 is established in a similar way. Because functions F_i are absolutely continuous on $[t_0+\Delta, t_0+a]$ for any $\Delta \in (0, a)$ and are continuous at point t_0, they are absolutely continuous on $[t_0, t_0+a]$. It is evident that F_i solve equations (*DE*) a.e. on $[t_0, t_0+a]$. □

Observe that because $J(t, H)$ is defined on $\bar{D}_0(\delta)$ and, therefore, a solution H to (DE_H)–(IC_H) takes values only in $\bar{H}_0(\delta)$, the values of the corresponding functions F_i belong to $[0, \sqrt{\delta}]$ only. The goal, however, is to identify F_i for all values in $[0, 1]$. This will be possible because δ can be arbitrarily close to 1.

Remark A.4. The last thing about the local existence that is worth mentioning concerns the comment made in Section A.2 about the existence of a negative function F that satisfies (*DE*) a.e. in a neighborhood of t_0 and also satisfies (*IC*). Note that functions F_i are expressed through H_i as $F_1^2 = \frac{H_2H_3}{H_1}$, $F_2^2 = \frac{H_1H_3}{H_2}$, and $F_3^2 = \frac{H_1H_2}{H_3}$, as follows from the definition of functions H_i. Taking into account that F_i are positive, I obtained (A.1) and substituted these formulas into (*DE*) to obtain the auxiliary system (DE_H). However, if I were looking for negative solutions, I would substitute formulas

$$F_1 = -\sqrt{\frac{H_2H_3}{H_1}}, \qquad F_2 = -\sqrt{\frac{H_1H_3}{H_2}}, \qquad F_3 = -\sqrt{\frac{H_1H_2}{H_3}}$$

into (*DE*) and obtain a different form of the auxiliary system:

$$\begin{aligned} H_1' &= \frac{g_1}{1+\sqrt{\frac{H_2H_3}{H_1}}}, \\ H_2' &= \frac{g_2}{1+\sqrt{\frac{H_1H_3}{H_2}}}, \\ H_3' &= \frac{g_3}{1+\sqrt{\frac{H_1H_2}{H_3}}}. \end{aligned} \tag{A.7}$$

Using the techniques of this section, it can be shown that (A.7) with initial conditions (IC_H) has a local solution H. This implies there is a negative function F that solves (*DE*) a.e. in a neighborhood of t_0.

A.3.2 *Uniqueness of a local solution* The next step in the proof of local identification is to show that (*DE*)–(*IC*) has only one local solution. Local existence was proved without imposing any assumptions on G_i besides necessary conditions (I) and (II). To establish local uniqueness, I will assume that condition (III) is also satisfied. In fact, condition (III) is the most important condition for proving uniqueness.

I start by stating the local uniqueness result. It relies mostly on conditions (3.1), which find the rate of convergence of F_i at t_0 in terms of observable functions G_i.

THEOREM A.5. *Let observable functions G_i satisfy conditions* (I), (II), *and* (III). *Then* (DE)–(IC) *has only one solution in a neighborhood of* t_0.

The idea of the proof of this theorem is to take two local solutions to problem (DE)–(IC) and show that they coincide on their common interval of existence.

Suppose that F and $\tilde{F}$ are two local solutions to (DE)–(IC) with a common interval of existence $[t_0, t_0 + c]$, $c > 0$. Let H_i and $\tilde{H}_i$ be corresponding auxiliary functions:

$$
\begin{aligned}
H_1 &= F_2F_3, \qquad & H_2 &= F_1F_3, \qquad & H_3 &= F_1F_2, \\
\tilde{H}_1 &= \tilde{F}_2\tilde{F}_3, \qquad & \tilde{H}_2 &= \tilde{F}_1\tilde{F}_3, \qquad & \tilde{H}_3 &= \tilde{F}_1\tilde{F}_2.
\end{aligned}
$$

Clearly, if functions H and $\tilde{H}$ are identical, then F and $\tilde{F}$ coincide.

The lemma below is key to proving that functions H and $\tilde{H}$ are identical.

LEMMA A.6. *Functions H and $\tilde{H}$ satisfy the inequality, a.e. on* $[t_0, t_0 + c]$,

$$
\|H'(t) - \tilde{H}'(t)\|_1 \le \Gamma_0(t)\|H(t) - \tilde{H}(t)\|_1, \tag{A.8}
$$

where

$$
\Gamma_0(t) = C\left(\frac{g_1}{G_1}(t) + \frac{g_2}{G_2}(t) + \frac{g_3}{G_3}(t)\right)\left(\sqrt{\frac{G_2G_3}{G_1}(t)} + \sqrt{\frac{G_1G_3}{G_2}(t)} + \sqrt{\frac{G_1G_2}{G_3}(t)}\right)
$$

and $C > 0$ is some constant.

PROOF. From (DE_H), we obtain

$$
H_i' - \tilde{H}_i' = \frac{g_i(F_i - \tilde{F}_i)}{(1 - F_i)(1 - \tilde{F}_i)}, \qquad i = 1, 2, 3. \tag{A.9}
$$

From equalities

$$
\begin{aligned}
H_1 - \tilde{H}_1 &= F_2(F_3 - \tilde{F}_3) + \tilde{F}_3(F_2 - \tilde{F}_2), \\
H_2 - \tilde{H}_2 &= F_1(F_3 - \tilde{F}_3) + \tilde{F}_3(F_1 - \tilde{F}_1), \\
H_3 - \tilde{H}_3 &= F_1(F_2 - \tilde{F}_2) + \tilde{F}_2(F_1 - \tilde{F}_1),
\end{aligned}
$$

we find that on $(t_0, t_0+c]$,

$$
\begin{aligned}
F_1-\tilde{F}_1 &= -\frac{F_1}{\tilde{F}_3(F_2+\tilde{F}_2)}(H_1-\tilde{H}_1)+\frac{F_2}{\tilde{F}_3(F_2+\tilde{F}_2)}(H_2-\tilde{H}_2) \\
&\quad +\frac{1}{F_2+\tilde{F}_2}(H_3-\tilde{H}_3), \\
F_2-\tilde{F}_2 &= \frac{\tilde{F}_2}{\tilde{F}_3(F_2+\tilde{F}_2)}(H_1-\tilde{H}_1)-\frac{F_2\tilde{F}_2}{F_1\tilde{F}_3(F_2+\tilde{F}_2)}(H_2-\tilde{H}_2) \\
&\quad +\frac{F_2}{F_1(F_2+\tilde{F}_2)}(H_3-\tilde{H}_3), \\
F_3-\tilde{F}_3 &= \frac{1}{F_2+\tilde{F}_2}(H_1-\tilde{H}_1)+\frac{\tilde{F}_2}{(F_2+\tilde{F}_2)F_1}(H_2-\tilde{H}_2) \\
&\quad -\frac{\tilde{F}_3}{(F_2+\tilde{F}_2)F_1}(H_3-\tilde{H}_3).
\end{aligned}
\tag{A.10}
$$

According to (3.1), there exist constants $C_1>0$, $C_2>0$ such that on $(t_0, t_0+c]$,

$$
\begin{aligned}
&C_1 \le \frac{F_1}{\sqrt{\frac{G_2G_3}{G_1}}} \le C_2, \qquad C_1 \le \frac{F_2}{\sqrt{\frac{G_1G_3}{G_2}}} \le C_2, \qquad C_1 \le \frac{F_3}{\sqrt{\frac{G_1G_2}{G_3}}} \le C_2, \\
&C_1 \le \frac{\tilde{F}_1}{\sqrt{\frac{G_2G_3}{G_1}}} \le C_2, \qquad C_1 \le \frac{\tilde{F}_2}{\sqrt{\frac{G_1G_3}{G_2}}} \le C_2, \qquad C_1 \le \frac{\tilde{F}_3}{\sqrt{\frac{G_1G_2}{G_3}}} \le C_2
\end{aligned}
$$

(t_0+c can be taken close enough to t_0). Then on $(t_0, t_0+c]$,

$$
\begin{aligned}
|F_1-\tilde{F}_1| &\le K\frac{1}{G_1}\sqrt{\frac{G_2G_3}{G_1}}|H_1-\tilde{H}_1|+K\sqrt{\frac{G_3}{G_1G_2}}|H_2-\tilde{H}_2| \\
&\quad +K\sqrt{\frac{G_2}{G_1G_3}}|H_3-\tilde{H}_3|, \\
|F_2-\tilde{F}_2| &\le K\sqrt{\frac{G_3}{G_1G_2}}|H_1-\tilde{H}_1|+K\frac{1}{G_2}\sqrt{\frac{G_1G_3}{G_2}}|H_2-\tilde{H}_2| \\
&\quad +K\sqrt{\frac{G_1}{G_2G_3}}|H_3-\tilde{H}_3|, \\
|F_3-\tilde{F}_3| &\le K\sqrt{\frac{G_2}{G_1G_3}}|H_1-\tilde{H}_1|+K\sqrt{\frac{G_1}{G_2G_3}}|H_2-\tilde{H}_2| \\
&\quad +K\frac{1}{G_3}\sqrt{\frac{G_1G_2}{G_3}}|H_3-\tilde{H}_3|,
\end{aligned}
\tag{A.11}
$$

where $K > 0$ is a constant expressed in terms of C_1 and C_2. Let $L > 0$ be a constant that bounds F_i and $\tilde{F}_i$ from above on $[t_0, t_0 + c]$. Denote $C = \frac{K}{(1-L)^2}$. Inequalities (A.11) and equations (A.9) imply that, a.e. on $[t_0, t_0 + c]$,

$$\|H' - \tilde{H}'\|_1 \le C\left(\frac{g_1}{G_1} + \frac{g_2}{G_2} + \frac{g_3}{G_3}\right)\left(\sqrt{\frac{G_2 G_3}{G_1}} + \sqrt{\frac{G_1 G_3}{G_2}} + \sqrt{\frac{G_1 G_2}{G_3}}\right)\|H - \tilde{H}\|_1.$$

□

Establishing inequality (A.8) is the most challenging part of proving local uniqueness.

Notice that because H and $\tilde{H}$ solve the auxiliary problem (DE_H)–(IC_H), then, a.e. on $(t_0, t_0 + c]$,

$$H'(t) = J\big(t, H(t)\big),$$
$$\tilde{H}'(t) = J\big(t, \tilde{H}(t)\big).$$

Therefore, inequality (A.8) can be rewritten as

$$\big\|J\big(t, H(t)\big) - J\big(t, \tilde{H}(t)\big)\big\|_1 \le \Gamma_0(t)\big\|H(t) - \tilde{H}(t)\big\|_1.$$

This last inequality is a generalized local Lipschitz condition for function $J(t, H)$ with respect to variable H. It holds only for the values of functions H and $\tilde{H}$ at the same point t, but not for any two arbitrary values of variable H.

The following two lemmas prove that inequality (A.8) together with condition (III) yield that H and $\tilde{H}$ are identical functions and, therefore, prove Theorem A.5.

Lemma A.7. *Let $z : [\tau, \xi] \to \Re^n$ be an absolutely continuous function. Then $\|z\|_1$ has the right derivative $D_R\|z\|_1$ a.e. on $[\tau, \xi]$ and*

$$D_R\big\|z(t)\big\|_1 \le \big\|z'(t)\big\|_1 \quad \textit{a.e. on } [\tau, \xi].$$

Proof. Hartman (1964) proves a similar lemma for smooth functions for the maxnorm and the Euclidean norm. First, for any fixed i, consider function $|z_i|$. Since z_i is absolutely continuous, $|z_i|$ is absolutely continuous too. Then $D_R|z_i(t)|$ exists a.e. on $[\tau, \xi]$.

Let $t \in [\tau, \xi]$ be a point in which z_i has a derivative. Use the definition of the right derivative,

$$D_R\big|z_i(t)\big| = \lim_{h \to +0} \frac{|z_i(t + h)| - |z_i(t)|}{h},$$

to conclude that $D_R|z_i(t)| = z_i'(t)$ if $z_i(t) > 0$ and $D_R|z_i(t)| = -z_i'(t)$ if $z_i(t) < 0$. Indeed, if $z_i(t) > 0$, then $z_i(t + h) > 0$ for small enough h, and $D_R|z_i(t)| = z_i'(t)$. In a similar way, we consider the case $z_i(t) < 0$. If $z_i(t) = 0$, then

$$D_R\big|z_i(t)\big| = \lim_{h \to 0+} \frac{|z_i(t + h)|}{h} = \left|\lim_{h \to 0+} \frac{z_i(t + h)}{h}\right| = \big|z_i'(t)\big|.$$

In all three cases, $D_R|z_i(t)| \le |z_i'(t)|$.

Function $\|z\|_1$ is the sum of absolutely continuous functions and, hence, is absolutely continuous. Then, a.e. on $[\tau, \xi]$,

$$D_R\|z(t)\|_1 = D_R\left(\sum_{i=1}^{n}|z_i(t)|\right) = \sum_{i=1}^{n} D_R|z_i(t)| \le \sum_{i=1}^{n}|z_i'(t)| = \|z'(t)\|_1. \qquad \square$$

Lemma A.8. *Let function $v:[\tau, \xi] \to \Re$ be absolutely continuous. Suppose that $v(\tau) = 0$ and, a.e. on $[\tau, \xi]$,*

$$D_R v(t) \le \Gamma(t)v(t), \quad \textit{where } \Gamma \in L^1[\tau, \xi].$$

Then

$$v(t) \le 0, \quad t \in [\tau, \xi].$$

Proof. Results similar to the one in this lemma have been obtained by researchers on a more general level. However, it is easier to prove this lemma directly than to show how it follows from more general results.

Function $\phi(t) = v(t)e^{-\int_\tau^t \Gamma(s)\,ds}$ is absolutely continuous as the product of two absolutely continuous function and

$$D_R\phi(t) = D_R\big(v(t)\big)e^{-\int_\tau^t \Gamma(s)\,ds} - \Gamma(t)v(t)e^{-\int_\tau^t \Gamma(s)\,ds} \le 0 \quad \text{a.e. } [\tau, \xi].$$

Szarski (1965) uses Zygmund's lemma to show that if ϕ is absolutely continuous and $D_R\phi(t) \le 0$ a.e. on $[\tau, \xi]$, then ϕ is nonincreasing on $[\tau, \xi]$. Since $\phi(\tau) = 0$, then $\phi(t) \le 0$ on $[\tau, \xi]$ and, hence, $v(t) \le 0$ on $[\tau, \xi]$. $\square$

Let me explain in more detail how these two lemmas imply that functions H and $\tilde{H}$ coincide on $[t_0, t_0 + c]$. Consider $[\tau, \xi] = [t_0, t_0 + c]$. In the first lemma, take $z(t) = H(t) - \tilde{H}(t)$ and use inequality (A.8) to obtain

$$D_R\|H(t) - \tilde{H}(t)\|_1 \le \Gamma_0(t)\|H(t) - \tilde{H}(t)\|_1.$$

In the second lemma, let $v(t) = \|H(t) - \tilde{H}(t)\|_1$ and $\Gamma(t) = \Gamma_0(t)$. Because condition (III) holds, then according to this lemma, $\|H(t) - \tilde{H}(t)\|_1 \le 0$, $t \in [t_0, t_0 + c]$. This means that $\|H(t) - \tilde{H}(t)\|_1 = 0$, $t \in [t_0, t_0 + c]$, or, in other words, functions H and $\tilde{H}$ coincide on $[t_0, t_0 + c]$. In turn, this implies that functions F and $\tilde{F}$ coincide on $[t_0, t_0 + c]$ too.

To summarize, I have shown that, given conditions (I), (II), and (III) on observable functions G_i, problem (*DE*)–(*IC*) has the unique solution F in a neighborhood of t_0. As mentioned in Section 3, this solution is assumed to be monotone.

A.4 *Global identification*

Now I establish that the local solution to (*DE*)–(*IC*) can be extended to a solution on the entire interval $[t_0, T]$ and that such an extension is unique.

Consider Figure 3 and the local solution F on $[t_0, t_0+c]$ depicted on the left in this figure. Notice that all functions F_i take positive values at t_0+c and these values are known. Denote them as $v_i = F_i(t_0+c)$, $v_i > 0$. To extend the local solution to the right, I need to solve system (DE) in a right-hand side neighborhood of t_0+c given that functions F_i in a solution to this system take values v_i at t_0+c. Clearly, results of Theorems A.3 and A.5 cannot be used for this problem because the methods in these theorems were developed for the situation when all initial values of F_i are 0. Therefore, to carry out the extension process, I first need to prove the local existence and uniqueness result for the case when all the initial values of F_i are positive.

A.4.1 *Positive initial values* Let $t_1 \in (t_0, T)$ and let functions F_i satisfy initial conditions

$$F_i(t_1) = v_i, \quad i = 1, 2, 3, \tag{A.12}$$

where v_i are known, $0 < v_i < 1$. Notice that the values of $G_i(t_1)$ are known.

I first consider the auxiliary system (DE_H). The initial conditions on functions H_i are obviously

$$H_1(t_1) = v_2 v_3, \qquad H_2(t_1) = v_1 v_3, \qquad H_3(t_1) = v_1 v_2. \tag{A.13}$$

Proposition A.9. *Let observable functions G_i satisfy conditions* (I). *Then* (DE_H)–(A.13) *has a solution in a right-hand neighborhood of t_1.*

Proof. The proof uses the Tonelli approximations approach. It is similar to the proof of Lemma A.1 and differs from it in technical details.

Let me first specify the domain of the right-hand side $J(t, H)$ of the auxiliary system (DE_H) and find a solution's interval of existence. Let $\Delta > 0$ be any number such that $\Delta < \min\{1 - v_1, 1 - v_2, 1 - v_3\}$. Define set

$$\begin{aligned}\bar{H}(\Delta) = [0, \infty)^3 \cap \big\{&(h_1, h_2, h_3)^{\mathrm{tr}} : h_2 h_3 \le (v_1 + \Delta)^2 h_1, \\ &h_1 h_3 \le (v_2 + \Delta)^2 h_2, h_2 h_3 \le (v_3 + \Delta)^2 h_1\big\}.\end{aligned}$$

Let the domain of $J(t, H)$ be $\bar{D}(\Delta) = [t_1, T] \times \bar{H}$. For a given Δ, I can always choose a $\gamma > 0$ small enough so that

$$(1+\gamma)^2 v_1^2 \le (v_1 + \Delta)^2, \qquad (1+\gamma)^2 v_2^2 \le (v_2 + \Delta)^2, \qquad (1+\gamma)^2 v_3^2 \le (v_3 + \Delta)^2.$$

Because $\lim_{t \downarrow t_1} G_i(t) = G_i(t_1)$, there exists a point $t_1 + a_1$, $a_1 > 0$, from $[t_1, T]$ such that

$$\begin{aligned}&G_1(t_1 + a_1) - G_1(t_1) \le \gamma v_2 v_3 (1 - v_1 - \Delta), \\ &G_2(t_1 + a_1) - G_2(t_1) \le \gamma v_1 v_3 (1 - v_2 - \Delta), \\ &G_3(t_1 + a_1) - G_3(t_1) \le \gamma v_1 v_2 (1 - v_3 - \Delta).\end{aligned}$$

Interval $[t_1, t_1 + a_1]$ is an interval on which a local solution exists.

Now I construct Tonelli approximations. For any natural number k, let

$$H_1^k(t) = v_2 v_3, \qquad H_2^k(t) = v_1 v_3, \qquad H_3^k(t) = v_1 v_2$$

for $t \in [t_1 - 1, t_1]$. Denote $v_0 = (v_2 v_3, v_1 v_3, v_1 v_2)^{\mathrm{tr}}$ and let v_0^i be i's coordinate of v_0, $i = 1, 2, 3$. Define function

$$H^k(t) = v_0 + \int_{t_1}^{t} J\left(s, H^k\left(s - \frac{1}{k}\right)\right) ds, \quad t \in [t_1, t_1 + a_1]. \tag{A.14}$$

This formula is meaningful. In the first step it defines H on $[t_1, t_1 + \min\{\frac{1}{k}, a_1\}]$. For t from this interval, the Lebesgue integral on the right-hand side is finite because the integrand is bounded from above by functions from $L^1[t_1, t_1 + a_1]$:

$$\left| J_i\left(s, H^k\left(s - \frac{1}{k}\right)\right)\right| \le \frac{g_i(s)}{1 - v_i}, \quad s \in \left[t_1, t_1 + \min\left\{\frac{1}{k}, a_1\right\}\right].$$

Evidently, for $t \in [t_1, t_1 + \min\{\frac{1}{k}, a_1\}]$,

$$\begin{aligned} H_1^k(t) &= v_2 v_3 + \frac{G_1(t) - G_1(t_1)}{1 - v_1}, \\ H_2^k(t) &= v_1 v_3 + \frac{G_2(t) - G_2(t_1)}{1 - v_2}, \\ H_3^k(t) &= v_1 v_2 + \frac{G_3(t) - G_3(t_1)}{1 - v_3}. \end{aligned}$$

Let me show that $H^k(t) \in \bar{H}$ for $t \in [t_1, t_1 + \min\{\frac{1}{k}, a_1\}]$. Consider, for instance, $\frac{H_2^k H_3^k}{H_1^k}$. Because

$$\begin{aligned} H_2^k(t) &\le v_1 v_3 + \frac{G_2(t_1 + a_1) - G_2(t_1)}{1 - v_2} \le v_1 v_3 + \frac{\gamma v_1 v_3 (1 - v_2 - \Delta)}{1 - v_2} \le (1 + \gamma) v_1 v_3, \\ H_3^k(t) &\le (1 + \gamma) v_1 v_2, \\ H_1^k(t) &\ge v_2 v_3, \end{aligned}$$

then

$$\frac{H_2^k(t) H_3^k(t)}{H_1^k(t)} \le (1 + \gamma)^2 v_1^2 \le (v_1 + \Delta)^2.$$

Likewise,

$$\frac{H_1^k(t) H_3^k(t)}{H_2^k(t)} \le (v_2 + \Delta)^2, \qquad \frac{H_1^k(t) H_2^k(t)}{H_3^k(t)} \le (v_3 + \Delta)^2.$$

In the second step, formula (A.14) defines H on $[t_1 + \frac{1}{k}, t_1 + \min\{\frac{2}{k}, a_1\}]$. For t from this interval, the Lebesgue integral on the right-hand side is finite because

$$\left| J_i\left(s, H^k\left(s - \frac{1}{k}\right)\right)\right| \le \frac{g_i(s)}{1 - v_i - \Delta} \in L^1[t_1, t_1 + a_1], \quad s \in \left[t_1, t_1 + \min\left\{\frac{2}{k}, a_1\right\}\right].$$

Note that $H^k(t) \in \bar{H}$ for $t \in [t_1 + \frac{1}{k}, t_1 + \min\{\frac{2}{k}, a_1\}]$. Indeed,

$$\begin{aligned}
H_2^k(t) &\le v_1 v_3 + \frac{G_2(t_1 + a_1) - G_2(t_1)}{1 - v_2 - \Delta} \le v_1 v_3 + \gamma v_1 v_3 = (1+\gamma) v_1 v_3, \\
H_3^k(t) &\le (1+\gamma) v_1 v_2, \\
H_1^k(t) &\ge v_2 v_3.
\end{aligned}$$

Therefore,

$$\frac{H_2^k(t) H_3^k(t)}{H_1^k(t)} \le (1+\gamma)^2 v_1^2 \le (v_1 + \Delta)^2.$$

In a similar way, I can show that for $t \in [t_1 + \frac{1}{k}, t_1 + \min\{\frac{2}{k}, a_1\}]$,

$$\frac{H_1^k(t) H_3^k(t)}{H_2^k(t)} \le (v_2 + \Delta)^2, \qquad \frac{H_1^k(t) H_2^k(t)}{H_3^k(t)} \le (v_3 + \Delta)^2.$$

This process continues and defines function H^k on the whole interval $[t_1, t_1 + a_1]$.

Now let me obtain the properties of sequence $\{H^k\}$. Inequality

$$\|H^k(t)\|_1 \le (1+\gamma)(v_2 v_3 + v_1 v_3 + v_1 v_2)$$

for all $t \in [t_1, t_1 + a_1]$ implies that sequence $\{H^k\}$ is uniformly bounded.

Because for any $t, \tau \in [t_1, t_1 + a_1]$,

$$\begin{aligned}
\|H^k(t) - H^k(\tau)\|_1 &\le \frac{|G_1(t) - G_1(\tau)|}{1 - v_1 - \Delta} + \frac{|G_2(t) - G_2(\tau)|}{1 - v_2 - \Delta} + \frac{|G_3(t) - G_3(\tau)|}{1 - v_3 - \Delta} \\
&\le \frac{\|G(t) - G(\tau)\|_1}{1 - \max\{v_1 + \Delta, v_2 + \Delta, v_3 + \Delta\}},
\end{aligned}$$

and G_i are absolutely continuous on $[t_1, t_1 + a_1]$, then sequence $\{H^k\}$ is equicontinuous. According to the Arzela–Ascoli theorem, $\{H^k\}$ is relatively compact in $C([t_1, t_1 + a_1], \bar{H})$. Hence, it contains a subsequence H^{k_m} such that for some function H,

$$\sup_{[t_1, t_1 + a_1]} \|H(t) - H^{k_m}(t)\|_1 \to 0 \quad \text{as } m \to \infty.$$

Because

$$J\left(t, H^{k_m}\left(t - \frac{1}{k_m}\right)\right) \to J\big(t, H(t)\big) \quad \text{a.e. on } [t_1, t_1 + a_1]$$

and a.e. on $[t_1, t_1 + a_1]$,

$$\left| J\left(t, H^{k_m}\left(t - \frac{1}{k_m}\right)\right)\right| \le \frac{g_1(t) + g_2(t) + g_3(t)}{1 - \max\{v_1 + \Delta, v_2 + \Delta, v_3 + \Delta\}} \in L^1[t_1, t_1 + a],$$

then by the Lebesgue dominated convergence theorem, $H(t)$ solves

$$H(t) = v_0 + \int_{t_1}^{t} J\bigl(s, H(s)\bigr)\, ds, \quad t \in [t_1, t_1 + a_1],$$

which implies that H_i are absolutely continuous and solve (DE_H)–(A.13) on $[t_1, t_1 + a_1]$. □

The existence result of Proposition A.2 also required G_i to satisfy conditions (II). Note that because the values of the underlying distribution functions F_i at t_1 are separated from 0, then the result of Proposition A.9 does not require any conditions on the behavior of G_i around t_1.

The next theorem establishes the local existence and uniqueness result for problem (DE)–(A.12). It is noteworthy that conditions $F_i(t_1) > 0$ guarantee the uniqueness result without any additional conditions on functions G_i.

Theorem A.10. *Let observable functions G_i satisfy conditions* (I). *Then* (DE)–(A.12) *has only one solution in a right-hand neighborhood of t_1.*

Proof. According to Proposition A.9, problem (DE_H)–(A.13) has a solution H on $[t_1, t_1 + a_1]$, $a_1 > 0$. Use this solution to find functions

$$F_1 = \sqrt{\frac{H_2 H_3}{H_1}}, \qquad F_2 = \sqrt{\frac{H_1 H_3}{H_2}}, \qquad F_3 = \sqrt{\frac{H_1 H_2}{H_3}}.$$

Clearly, $F = (F_1, F_2, F_3)^{\mathrm{tr}}$ is absolutely continuous and solves (DE)–(A.12) on $[t_1, t_1 + a_1]$.

The uniqueness proof is based on obtaining a generalized local Lipschitz condition (A.8). Let F and $\tilde{F}$ be two local solutions of (DE)–(A.12). Without a loss of generality, assume that $[t_1, t_1 + a_1]$ is their common interval of existence. Let H and $\tilde{H}$ be their corresponding auxiliary functions:

$$\begin{aligned} H_1 &= F_2 F_3, & H_2 &= F_1 F_3, & H_3 &= F_1 F_2, \\ \tilde{H}_1 &= \tilde{F}_2 \tilde{F}_3, & \tilde{H}_2 &= \tilde{F}_1 \tilde{F}_3, & \tilde{H}_3 &= \tilde{F}_1 \tilde{F}_2. \end{aligned}$$

Functions H and $\tilde{H}$ solve the auxiliary system (DE_H) a.e. on $[t_1, t_1 + a_1]$.

The proof of the uniqueness part of this theorem is much easier than the proof for problem (DE)–(IC). Indeed, for (DE)–(IC), the difficulty of proving uniqueness stemmed from the fact that all F_i had values 0 at t_0. Now all $F_i(t_1)$ are positive. Use (A.10) and the fact that F_i are separated from 0 in a neighborhood of t_1 (without a loss of generality, a_1 is small enough) to obtain

$$|F_i - \tilde{F}_i| \le K \|H - \tilde{H}\|_1$$

on $[t_1, t_1 + a_1]$ for some constant K. Exploit (A.9) and establish that for some constant C,

$$\bigl\|H'(t) - \tilde{H}'(t)\bigr\|_1 \le C\bigl(g_1(t) + g_2(t) + g_3(t)\bigr)\bigl\|H(t) - \tilde{H}(t)\bigr\|_1$$

a.e. on $[t_1, t_1 + a_1]$. Because $g_i \in L^1[t_1, t_1 + a_1]$, then Lemmas A.7 and A.8 imply that H and $\tilde{H}$ coincide on $[t_1, t_1 + a_1]$. Hence, F and $\tilde{F}$ coincide on this interval too. □

A.4.2 *Extension of the local solution to the whole support* Now I turn to the final element of the identification proof. I demonstrate how the unique local solution to (*DE*)–(*IC*) can be uniquely extended to a solution on the whole support. Throughout this section, I assume that functions F_i obtained from H_i are strictly monotone, that is, the ratios $\frac{H_2H_3}{H_1}$, $\frac{H_1H_3}{H_2}$, and $\frac{H_1H_2}{H_3}$ are strictly increasing.

To begin, recall that in the proof of the existence result in Section A.3.1, function $J(t,H)$ was defined on $\bar{D}_0(\delta)$ and the values of function H were restricted to set $\bar{H}(\delta)$ for a chosen $0<\delta<1$:

$$\bar{H}_0(\delta)=(0,\infty)^3\cap\{(h_1,h_2,h_3)^{\mathrm{tr}}:h_2h_3\le\delta h_1,h_1h_3\le\delta h_2,h_2h_3\le\delta h_1\}.$$

Because the local solution to the auxiliary problem takes values only in this set, the functions F_i in the corresponding local solution to the main problem (*DE$_H$*)–(*IC$_H$*) take values in $[0,\sqrt{\delta}]$ only. However, we also want to identify F_i when these functions take values above $\sqrt{\delta}$. Notice that $\delta<1$ could be chosen arbitrarily close to 1 and this will allow the extension of the local solution to the whole support.

Fix δ, $0<\delta<1$, and let the domain of $J(t,H)$ be $\bar{D}_0(\delta)=[t_0,T]\times\bar{H}_0(\delta)$ (a.e. with respect to t). Theorem A.5 proved that given conditions (I), (II), and (III), system (*DE$_H$*) with initial conditions (*IC$_H$*) has the unique solution $H=(H_1,H_2,H_3)$ on some interval $[t_0,t_0+c]$. Denote $t_1=t_0+c$ and calculate

$$x_{i1}=H_i(t_1),\quad i=1,2,3.$$

Because H_i are strictly increasing functions, then $x_{i1}>0$. Note that $H(t_1)\in\bar{H}_0(\delta)$. If $H(t_1)$ is an interior point in $\bar{H}_0(\delta)$, that is, if

$$\frac{x_{11}x_{21}}{x_{31}}<\delta,\qquad\frac{x_{11}x_{31}}{x_{21}}<\delta,\qquad\frac{x_{21}x_{31}}{x_{11}}<\delta,$$

then $(t_1,H(t_1))$ is an interior point of $\bar{D}_0(\delta)$ and, therefore, $J(t,H)$ is defined in a neighborhood of this point. This means that the auxiliary system (*DE$_H$*), considered for $t\ge t_1$, with initial conditions

$$H_i(t_1)=x_{i1},\quad i=1,2,3,$$

is a well defined problem. In light of the results of Proposition A.9 and Theorem A.10, this problem has a unique solution H on some interval $[t_1,t_1+\mu]$, $\mu>0$. Thus, I can uniquely extend the local solution found on $[t_0,t_1]$ to a solution on the interval $[t_0,t_1+\mu]$. Note that the value of $H(t_1+\mu)$ belongs to $\bar{H}_0(\delta)$. If this value is in the interior of set $\bar{H}_0(\delta)$, I can extend the solution even farther to the right and continue this process until I reach a point in which the value of function H becomes located on the border of set $\bar{H}_0(\delta)$. This point determines the solution's right maximal interval of existence for the given value of δ.

Definition A.2. An interval $[t_0,\xi]$ is the maximal interval of existence of solution H to (*DE$_H$*)–(*IC$_H$*) if there does not exist an extension of H over an interval $[t_0,\xi+\eta]$ such that $\eta>0$ and H remains a solution to (*DE$_H$*)–(*IC$_H$*).

In the case that I am currently considering, the solution's maximal interval of existence is determined by the value of δ that was chosen to define set $\bar{H}_0(\delta)$. The proposition below yields an explicit formula for this interval.

Proposition A.11. *Let function $J(t, H)$ be defined on $\bar{D}_0(\delta)$. Assume that all conditions on G_i that guarantee existence and uniqueness of a local solution to (DE_H)–(IC_H) are satisfied. The maximal interval of existence of solution H to (DE_H)–(IC_H) is $[t_0, T_\delta]$, where T_δ is such that*

$$\max\left\{\frac{H_2(T_\delta)H_3(T_\delta)}{H_1(T_\delta)}, \frac{H_1(T_\delta)H_3(T_\delta)}{H_2(T_\delta)}, \frac{H_1(T_\delta)H_2(T_\delta)}{H_3(T_\delta)}\right\} = \delta.$$

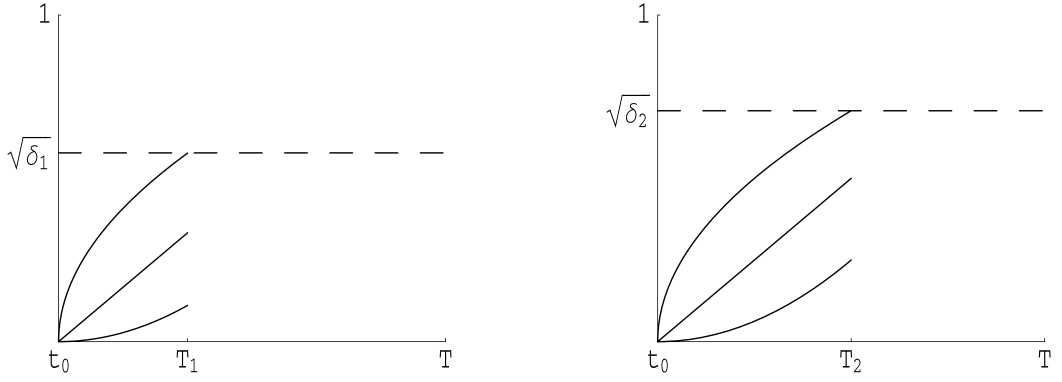
This proposition follows from the discussion above and, therefore, it is left without a proof.

Proposition A.11 implies that for the given δ, $[t_0, T_\delta]$ is the maximal interval of existence of a corresponding solution F to problem (DE)–(IC). Also, the values of functions F_i on $[t_0, T_\delta]$ belong to $[0, \sqrt{\delta}]$ and, for point T_δ,

$$\max\{F_1(T_\delta), F_2(T_\delta), F_3(T_\delta)\} = \sqrt{\delta}.$$

Figure 4 depicts maximal intervals of existence of a solution F for values δ_1 and δ_2, where $\delta_2 > \delta_1$. Maximal interval $[t_0, T_{\delta_1}]$ corresponds to δ_1 and maximal interval $[t_0, T_{\delta_2}]$ corresponds to δ_2. Because functions F_i are strictly increasing, then $T_{\delta_2} > T_{\delta_1}$. Intuitively, if δ approaches 1, then the maximal interval of existence approaches support $[t_0, T]$. The theorem below establishes this fact.

Theorem A.12. *Consider a strictly increasing sequence δ_n, $n \geq 1$, such that $\delta_n < 1$ and $\delta_n \to 1$ as $n \to \infty$. Assume that all conditions on G_i that guarantee the existence and uniqueness of a local solution to problem (DE_H)–(IC_H) are satisfied. Let $[t_0, T_{\delta_n}]$ be the*



F_1, F_2, F_3 on $[t_0, T_{\delta_1}]$ F_1, F_2, F_3 on $[t_0, T_{\delta_2}]$

Figure 4. Maximal intervals of existence of a solution to the main problem: $[t_0, T_{\delta_1}]$ corresponds to δ_1 (left) and $[t_0, T_{\delta_2}]$ corresponds to δ_2, where $\delta_2 > \delta_1$ (right).

maximal interval of existence for the solution to (DE_H)–(IC_H) *when* $J(t, H)$ *is defined on* $\bar{D}_0(\delta_n)$. *Then* T_{δ_n} *is determined from*

$$\max\left\{\frac{H_2(T_{\delta_n})H_3(T_{\delta_n})}{H_1(T_{\delta_n})}, \frac{H_1(T_{\delta_n})H_3(T_{\delta_n})}{H_2(T_{\delta_n})}, \frac{H_1(T_{\delta_n})H_2(T_{\delta_n})}{H_3(T_{\delta_n})}\right\} = \delta_n \tag{A.15}$$

and T_{δ_n} *is a strictly increasing sequence. If*

$$F_i(T) = 1, \quad i = 1, 2, 3, \tag{A.16}$$

then $T_{\delta_n} \to T$ *as* $n \to \infty$.

Proof. Proposition A.11 clearly implies (A.15). Because functions $\frac{H_2H_3}{H_1}$, $\frac{H_1H_3}{H_2}$, and $\frac{H_1H_2}{H_3}$, and sequence δ_n are strictly increasing, (A.15) implies that sequence T_{δ_n} is strictly increasing. Because T_{δ_n} increases and is bounded from above by T, it converges to some point $\bar{T} \le T$. If $\bar{T} < T$, then we get a contradiction with the condition $\delta_n \to 1$ and conditions (A.16). Thus, $\bar{T} = T$. □

Taking into account that $F_1^2 = \frac{H_2H_3}{H_1}$, $F_2^2 = \frac{H_1H_3}{H_2}$, and $F_3^2 = \frac{H_1H_2}{H_3}$, we can see that Theorem A.12 guarantees that by choosing δ arbitrarily close to 1, we will identify F_i on the whole support $[t_0, T]$. This completes the proof of identification.

Remark Here I briefly discuss what happens when distributions have different upper support points or holes in the support.

Let τ_i denote the upper support point of the distribution of values for bidder i, $i = 1, 2, 3$. Without a loss of generality, $\tau_1 \le \tau_2 \le \tau_3$. The identification of F_i, $i = 1, 2, 3$, on $[t_0, \tau_1]$ can be shown by using techniques from Sections A.2–A.4. It holds that $F_1(\tau_1) = 1$. If $\tau_1 = \tau_2 < \tau_3$, then $F_2(\tau_1) = 1$ and $F_3(\tau_1) < 1$, and F_3 is not identified on $(\tau_2, \tau_3]$ since there are no prices observed in this interval. If $\tau_1 < \tau_2$, then $F_2(\tau_1) < 1$ and $F_3(\tau_1) < 1$. The values of $F_2(\tau_1)$ and $F_3(\tau_1)$ are known and strictly positive, and they are initial conditions for the system

$$F_2' = \frac{g_3(1 - F_2)}{1 - G_2 - G_3}, \qquad F_3' = \frac{g_2(1 - F_3)}{1 - G_2 - G_3}$$

considered for $t \in [\tau_1, \tau_2]$. This system relates observables and unobservables for bidders 2 and 3 on $[\tau_1, \tau_2]$. The identification of F_2 and F_3 on $[\tau_1, \tau_2]$ can be shown by using techniques from Section A.4. Then $F_2(\tau_2) = 1$. If $\tau_2 = \tau_3$, then $F_3(\tau_2) = 1$ and, thus, all distribution functions are fully identified. If $\tau_2 < \tau_3$, then $F_3(\tau_2) < 1$, and F_3 is not identified on $(\tau_2, \tau_3]$ since there are no prices observed in this interval. To summarize, F_i, $i = 1, 2, 3$, are identified from t_0 and up to the second-highest upper support point. For more details, see Komarova (2009).

As for the holes in the support, suppose that each F_i is strictly increasing in a small neighborhood of t_0, but can have flat parts on (t_0, T). Because observable functions G_i

are strictly increasing in a neighborhood of t_0, the local identification result can be established as in Theorems A.3 and A.5. The proof of Theorem A.10 for positive initial values and the rest of the global extension techniques require F_i to be increasing, but not necessarily *strictly* increasing, outside of a small neighborhood of t_0. Thus, F_i can have flat parts on (t_0, T).

A.5 *Auctions with any number of bidders*

Proofs of Propositions 3.6 and 3.7 and Corollary 3.8 are similar to those of Propositions 3.1 and 3.2 and Corollary 3.3.

Proof of Theorem 3.9. I can use the same approach as in the case of three bidders. System (3.6) can be rewritten in a convenient form by introducing d auxiliary functions $H_1, H_2, \dots, H_d$ that stand for the distribution functions of $\max\{X_2, X_3, \dots, X_d\}$, $\max\{X_1, X_3, \dots, X_d\}, \dots, \max\{X_1, X_2, \dots, X_{d-1}\}$, respectively:

$$H_1 = F_2 F_3 \cdots F_d, \qquad H_2 = F_1 F_3 \cdots F_d, \qquad \dots, \qquad H_d = F_1 F_2 \cdots F_{d-1}.$$

For $t > t_0$, functions F_i can be expressed through H_i as

$$F_1 = \left(\frac{H_2 H_3 \cdots H_d}{H_1^{d-2}}\right)^{1/(d-1)}, \qquad \dots, \qquad F_d = \left(\frac{H_1 H_2 \cdots H_{d-1}}{H_d^{d-2}}\right)^{1/(d-1)}. \tag{A.17}$$

Therefore, (3.6) can be rewritten as

$$H_i' = \frac{g_i}{1 - \left(\frac{H_1 \cdots H_{i-1} H_{i+1} \cdots H_d}{H_i^{d-2}}\right)^{1/(d-1)}}, \qquad i = 1, \dots, d. \tag{A.18}$$

This system, together with initial conditions

$$\lim_{t \downarrow t_0} H_i(t) = 0, \qquad i = 1, \dots, d \tag{A.19}$$

constitutes an auxiliary problem. To deal with discontinuities in H on the right-hand side in (A.18), I introduce a very small number $\varepsilon > 0$ and obtain an auxiliary system with ε:

$$H_i' = \frac{g_i}{1 - \left(\frac{H_1 \cdots H_{i-1} H_{i+1} \cdots H_d}{H_i^{d-2} + \varepsilon}\right)^{1/(d-1)}}, \qquad i = 1, \dots, d.$$

As in the case of three bidders, first I can establish local existence for the auxiliary system with ε. Then I can show the existence of a local solution to the auxiliary problem (A.18)–(A.19) by letting $\varepsilon \to 0$. After that, I can use formulas (A.17), which express F through H, to prove that the main problem (3.6)–(3.7) has a local solution.[12] $\square$

Proof of Theorem 3.10. The existence part of this theorem follows from Theorem 3.9. To prove the uniqueness part, let F and $\tilde{F}$ be two solutions to (3.6)–(3.7) with a common interval of existence $[t_0, t_0 + c]$, $c > 0$. Let

$$H_i = F_1 \cdots F_{i-1} F_{i+1} \cdots F_d, \qquad \tilde{H}_i = \tilde{F}_1 \cdots \tilde{F}_{i-1} \tilde{F}_{i+1} \cdots \tilde{F}_d, \quad i = 1, \dots, d.$$

The idea is to derive an inequality similar to (A.8). Use (A.17) and (A.18) to obtain that, a.e. on $[t_0, t_0 + c]$,

$$H_i' - \tilde{H}_i' = \frac{g_i(F_i - \tilde{F}_i)}{(1 - F_i)(1 - \tilde{F}_i)}. \tag{A.20}$$

The definitions of H and $\tilde{H}$ allow me to express $H - \tilde{H}$ through $F - \tilde{F}$ as

$$H - \tilde{H} = B(F, \tilde{F})(F - \tilde{F}),$$

where a $d \times d$ matrix $B(F, \tilde{F})$ depends on F and $\tilde{F}$ in the manner

$$B(F, \tilde{F}) = \begin{pmatrix} 0 & F_3 F_4 \cdots F_d & \tilde{F}_2 F_4 \cdots F_d & \tilde{F}_2 \tilde{F}_3 F_5 \cdots F_d & \cdots & \tilde{F}_2 \tilde{F}_3 \cdots \tilde{F}_{d-1} \\ F_3 F_4 \cdots F_d & 0 & \tilde{F}_1 F_4 \cdots F_d & \tilde{F}_1 \tilde{F}_3 F_5 \cdots F_d & \cdots & \tilde{F}_1 \tilde{F}_3 \cdots \tilde{F}_{d-1} \\ \cdots & \cdots & \cdots & \cdots & & \\ F_2 F_3 \cdots F_{d-1} & \tilde{F}_1 F_3 \cdots F_{d-1} & \tilde{F}_1 \tilde{F}_2 F_4 \cdots F_d & \tilde{F}_1 \tilde{F}_2 \tilde{F}_3 \cdots F_d & \cdots & 0 \end{pmatrix}.$$

The result of Proposition 3.7 implies that $\lim_{t \downarrow t_0} \frac{F_i}{\tilde{F}_i}(t) = 1$. Therefore, for a t close enough to t_0 (without a loss of generality, I can assume that $t_0 + c$ is close enough to t_0), matrix $B(F, \tilde{F})$ can be written as

$$B(F, \tilde{F}) = \big(I + M_{o(1)}(F, \tilde{F})\big) B_0(F),$$

where I is the $d \times d$ identity matrix, $M_{o(1)}(F, \tilde{F})$ is a $d \times d$ matrix such that each of its elements is $o(1)$ as $t \to t_0$, and $B_0(F) = B(F, F)$:

$$B_0(F) = \begin{pmatrix} 0 & F_3 F_4 \cdots F_d & F_2 F_4 \cdots F_d & F_2 F_3 F_5 \cdots F_d & \cdots & F_2 F_3 \dots F_{d-1} \\ F_3 F_4 \cdots F_d & 0 & F_1 F_4 \cdots F_d & F_1 F_3 F_5 \cdots F_d & \cdots & F_1 F_3 \cdots F_{d-1} \\ \cdots & \cdots & \cdots & \cdots & & \\ F_2 F_3 \cdots F_{d-1} & F_1 F_3 \cdots F_{d-1} & F_1 F_2 F_4 \cdots F_d & F_1 F_2 F_3 \cdots F_d & \cdots & 0 \end{pmatrix}.$$

[12] A detailed proof of Theorem 3.9 is available on request.

Matrix $B_0(F)$ is symmetric and invertible at any point $t \neq t_0$. The inverse matrix is

$$B_0^{-1}(F) = \frac{1}{(d-1)F_1F_2\cdots F_d} \times \begin{pmatrix} -(d-2)F_1^2 & F_1F_2 & F_1F_3 & F_1F_4 & \cdots & F_1F_d \\ F_1F_2 & -(d-2)F_2^2 & F_2F_3 & F_2F_4 & \cdots & F_2F_d \\ \cdots & \cdots & \cdots & \cdots & & \\ F_1F_d & F_2F_d & F_3F_d & F_4F_d & \cdots & -(d-2)F_d^2 \end{pmatrix}.$$

Thus, $F - \tilde{F}$ can be expressed through $H - \tilde{H}$ as

$$F - \tilde{F} = B_0^{-1}(F)\big(I + M_{o(1)}(F, \tilde{F})\big)^{-1}(H - \tilde{H}). \tag{A.21}$$

The next step is to bound on $[t_0, t_0 + c]$ the absolute values of the elements in $B_0^{-1}(F)$ by observable functions. This is achieved by using the result of Proposition 3.7. Take, for instance, the element $B_0^{-1}(F)_{11}$ in the first row and the first column:

$$\begin{aligned} \big|B_0^{-1}(F)_{11}\big| &= \left|\frac{(d-2)F_1}{(d-1)F_2\cdots F_d}\right| \\ &\leq K_{11} \frac{\left(\dfrac{G_2\cdots G_d}{G_1^{d-2}}\right)^{1/(d-1)}}{\displaystyle\prod_{i=2}^{d}\left(\dfrac{G_1\cdots G_{i-1}G_{i+1}\cdots G_d}{G_i^{d-2}}\right)^{1/(d-1)}} = \frac{K_{11}}{G_1}\left(\frac{G_2\cdots G_d}{G_1^{d-2}}\right)^{1/(d-1)} \end{aligned}$$

for some constant K_{11}. Consider another cell in $B_0^{-1}(F)$, for example, the element $B_0^{-1}(F)_{12}$ in the first row and the second column:

$$\begin{aligned} \big|B_0^{-1}(F)_{12}\big| &= \left|\frac{1}{(d-1)F_3\cdots F_d}\right| \\ &\leq K_{12}\prod_{i=3}^{d}\left(\frac{G_1\cdots G_{i-1}G_{i+1}\cdots G_d}{G_i^{d-2}}\right)^{-1/(d-1)} \\ &= \frac{K_{12}}{G_1}\left(\frac{G_1G_3\cdots G_d}{G_2^{d-2}}\right)^{1/(d-1)} \end{aligned}$$

for some constant K_{12}. For the other elements, bounds are found in a similar way. Then equations (A.20) and (A.21) yield that, a.e. on $[t_0, t_0 + c]$,

$$\big\|H' - \tilde{H}'\big\|_1 \leq C\sum_{i=1}^{d}\left(\frac{G_1G_2\cdots G_{i-1}G_{i+1}\cdots G_d}{G_i^{d-2}}\right)^{1/(d-1)} \cdot \sum_{i=1}^{d}\frac{g_i}{G_i}\|H - \tilde{H}\|_1$$

for some constant C. The last inequality and Lemmas A.7 and A.8 imply that H and $\tilde{H}$ coincide on $[t_0, t_0 + c]$ and, hence, F and $\tilde{F}$ coincide on $[t_0, t_0 + c]$. $\square$

A.6 *Auctions with two types of bidders*

Proof of Theorem 3.11. Note that for each i, $G_i(t)=\widetilde{G}_{\mathrm{I}}(t)/k$ if bidder i is of type I and $G_i(t)=\widetilde{G}_{\mathrm{II}}(t)/(d-k)$ if bidder i is of type II. To establish necessity, use Proposition 3.6, which implies that conditions (i)–(iii) of this theorem are satisfied for $\widetilde{G}_{\mathrm{I}}/k$ and $\widetilde{G}_{\mathrm{II}}/(d-k)$. Clearly, then conditions (i)–(iii) hold for $\widetilde{G}_{\mathrm{I}}$ and $\widetilde{G}_{\mathrm{II}}$ too. Sufficiency follows from Theorem 3.12 by considering $\psi(x)=-\ln x$, $x\in(0,1]$. □

Proof of Theorem 3.12. Without a loss of generality, suppose that bidders $1,\dots,k$ are of type I and bidders $k+1,\dots,d$ are of type II. Introduce functions

$$\begin{aligned}\Sigma_{\mathrm{I}}(t)&=C\big(1,\underbrace{F_{\mathrm{I}}(t),\dots,F_{\mathrm{I}}(t)}_{k-1},\underbrace{F_{\mathrm{II}}(t),\dots,F_{\mathrm{II}}(t)}_{d-k}\big)\\&=\psi^{-1}\big((k-1)\psi\big(F_{\mathrm{I}}(t)\big)+(d-k)\psi\big(F_{\mathrm{II}}(t)\big)\big),\\ \Sigma_{\mathrm{II}}(t)&=C\big(\underbrace{F_{\mathrm{I}}(t),\dots,F_{\mathrm{I}}(t)}_{k},\underbrace{F_{\mathrm{II}}(t),\dots,F_{\mathrm{II}}(t)}_{d-k-1},1\big)\\&=\psi^{-1}\big(k\psi\big(F_{\mathrm{I}}(t)\big)+(d-k-1)\psi\big(F_{\mathrm{II}}(t)\big)\big).\end{aligned}$$

Function Σ_{I} is the distribution function of $\max\{X_2,\dots,X_k,X_{k+1},\dots,X_d\}$. As can be seen, this maximum does not contain the private value of the first bidder of type I. Due to the exchangeability property of the joint distribution of private values, Σ_{I} is also the distribution function of $\max\{X_1,\dots,X_{j-1},X_{j+1},\dots,X_{k+1},\dots,X_d\}$ for any $j=2,\dots,k$. Thus, we can consider any such maximum that excludes the private value of one of the bidders of type I.

Function Σ_{II} is the distribution function of $\max\{X_1,\dots,X_k,X_{k+1},\dots,X_{d-1}\}$. Due to exchangeability, it is also the distribution function of $\max\{X_1,\dots,X_k,\dots,X_{j-1},X_{j+1},\dots,X_d\}$ for any $j=k+1,\dots,d-1$.

Suppose bidder i is of type I. Then

$$\begin{aligned}G_i(t)&=\Pr(\text{price }\le t, i\text{ wins})\\&=P\Big(\max_{j\neq i}X_j<X_i,\max_{j\neq i}X_j\le t\Big)\\&=P\Big(\max_{j\neq i}X_j\le t,X_i>t\Big)+P\Big(\max_{j\neq i}X_j<X_i,X_i\le t\Big)\\&=P\Big(\max_{j\neq i}X_j\le t\Big)-P\Big(\max_{j\neq i}X_j\le t,X_i\le t\Big)+P\Big(\max_{j\neq i}X_j<X_i,X_i\le t\Big)\\&=\Sigma_{\mathrm{I}}(t)-\psi^{-1}\big(k\psi\big(F_{\mathrm{I}}(t)\big)+(d-k)\psi\big(F_{\mathrm{II}}(t)\big)\big)+P\Big(\max_{j\neq i}X_j<X_i,X_i\le t\Big).\end{aligned}$$

Let $Q_i(s_1,s_2)$ denote the value of the distribution function of $(\max_{j\neq i}X_j,X_i)$ at (s_1,s_2):

$$\begin{aligned}Q_i(s_1,s_2)&=P\Big(\max_{j\neq i}X_j\le s_1,X_i\le s_2\Big)\\&=\psi^{-1}\big((k-1)\psi\big(F_{\mathrm{I}}(s_1)\big)+(d-k)\psi\big(F_{\mathrm{II}}(s_1)\big)+\psi\big(F_{\mathrm{I}}(s_2)\big)\big).\end{aligned}$$

It can be shown that the joint density of $(\max_{j\neq i} X_j, X_i)$ is $\frac{\partial^2 Q_i}{\partial s_1 \partial s_2}(s_1, s_2) = \frac{\partial^2 Q_i}{\partial s_2 \partial s_1}(s_1, s_2)$. Then

$$P\Big(\max_{j\neq i} X_j < X_i, X_i \le t\Big) = \int_{t_0}^{t} \frac{\partial Q_i}{\partial s_2}(s, s)\, ds$$
$$= \int_{t_0}^{t} \frac{\psi'(F_{\mathrm{I}}(s))F'_{\mathrm{I}}(s)}{\psi'(\psi^{-1}(k\psi(F_{\mathrm{I}}(s)) + (d-k)\psi(F_{\mathrm{II}}(s))))}\, ds.$$

Now use the fact that $\widetilde{G}_{\mathrm{I}}(t) = kG_i(t)$ to obtain

$$\frac{\widetilde{G}_{\mathrm{I}}(t)}{k} = \Sigma_{\mathrm{I}}(t) - \psi^{-1}\big(k\psi\big(F_{\mathrm{I}}(t)\big) + (d-k)\psi\big(F_{\mathrm{II}}(t)\big)\big)$$
$$+ \int_{t_0}^{t} \frac{\psi'(F_{\mathrm{I}}(s))F'_{\mathrm{I}}(s)}{\psi'(\psi^{-1}(k\psi(F_{\mathrm{I}}(s)) + (d-k)\psi(F_{\mathrm{II}}(s))))}\, ds.$$

Similarly, considering $G_i(t)$ when bidder i is of type II, obtain that

$$\frac{\widetilde{G}_{\mathrm{II}}(t)}{d-k} = \Sigma_{\mathrm{II}}(t) - \psi^{-1}\big(k\psi\big(F_{\mathrm{I}}(t)\big) + (d-k)\psi\big(F_{\mathrm{II}}(t)\big)\big)$$
$$+ \int_{t_0}^{t} \frac{\psi'(F_{\mathrm{II}}(s))F'_{\mathrm{II}}(s)}{\psi'(\psi^{-1}(k\psi(F_{\mathrm{I}}(s)) + (d-k)\psi(F_{\mathrm{II}}(s))))}\, ds.$$

Denote $G_{\mathrm{I}}(t) = \widetilde{G}_{\mathrm{I}}(t)/k$ and $G_{\mathrm{II}}(t) = \widetilde{G}_{\mathrm{II}}(t)/(d-k)$. Differentiate the equation for $\widetilde{G}_{\mathrm{I}}$ to obtain

$$G'_{\mathrm{I}}(t) = \Sigma'_{\mathrm{I}}(t) - \frac{k\psi'(F_{\mathrm{I}}(t))F'_{\mathrm{I}}(t) + (d-k)\psi'(F_{\mathrm{II}}(t))F'_{\mathrm{II}}(t)}{\psi'(\psi^{-1}(k\psi(F_{\mathrm{I}}(t)) + (d-k)\psi(F_{\mathrm{II}}(t))))}$$
$$+ \frac{\psi'(F_{\mathrm{I}}(t))F'_{\mathrm{I}}(t)}{\psi'(\psi^{-1}(k\psi(F_{\mathrm{I}}(t)) + (d-k)\psi(F_{\mathrm{II}}(t))))}$$
$$= \Sigma'_{\mathrm{I}}(t) - \frac{(k-1)\psi'(F_{\mathrm{I}}(t))F'_{\mathrm{I}}(t) + (d-k)\psi'(F_{\mathrm{II}}(t))F'_{\mathrm{II}}(t)}{\psi'(\psi^{-1}(k\psi(F_{\mathrm{I}}(t)) + (d-k)\psi(F_{\mathrm{II}}(t))))}$$
$$= \Sigma'_{\mathrm{I}}(t) - \frac{(k-1)\psi'(F_{\mathrm{I}}(t))F'_{\mathrm{I}}(t) + (d-k)\psi'(F_{\mathrm{II}}(t))F'_{\mathrm{II}}(t)}{\psi'\Big(\psi^{-1}\Big(\dfrac{k}{d-1}\psi(\Sigma_{\mathrm{I}}(t)) + \dfrac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}}(t))\Big)\Big)}$$
$$= \Sigma'_{\mathrm{I}}(t) - \frac{\psi'(\Sigma_{\mathrm{I}}(t))\Sigma'_{\mathrm{I}}(t)}{\psi'\Big(\psi^{-1}\Big(\dfrac{k}{d-1}\psi(\Sigma_{\mathrm{I}}(t)) + \dfrac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}}(t))\Big)\Big)}.$$

In a similar way, obtain

$$G'_{\mathrm{II}}(t) = \Sigma'_{\mathrm{II}}(t) - \frac{\psi'(\Sigma_{\mathrm{II}}(t))\Sigma'_{\mathrm{II}}(t)}{\psi'\Big(\psi^{-1}\Big(\dfrac{k}{d-1}\psi(\Sigma_{\mathrm{I}}(t)) + \dfrac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}}(t))\Big)\Big)}.$$

Thus, the system of differential equations for identifying Σ_{I} and Σ_{II} is

$$\begin{aligned}\Sigma_{\mathrm{I}}' &= \frac{G_{\mathrm{I}}'}{1-\dfrac{\psi'(\Sigma_{\mathrm{I}})}{\psi'\left(\psi^{-1}\left(\dfrac{k}{d-1}\psi(\Sigma_{\mathrm{I}})+\dfrac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}})\right)\right)}},\\ \Sigma_{\mathrm{II}}' &= \frac{G_{\mathrm{II}}'}{1-\dfrac{\psi'(\Sigma_{\mathrm{II}})}{\psi'\left(\psi^{-1}\left(\dfrac{k}{d-1}\psi(\Sigma_{\mathrm{I}})+\dfrac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}})\right)\right)}}.\end{aligned}\tag{A.22}$$

This system is analyzed together with initial conditions

$$\Sigma_{\mathrm{I}}(t_0)=\Sigma_{\mathrm{II}}(t_0)=0.\tag{A.23}$$

It is enough to show that problem (A.22)–(A.23) cannot have more than one solution in a neighborhood of t_0 and, thus, cannot have more than one solution on the whole support. This will also imply that F_{I} and F_{II} are identified because F_{I} and F_{II} are uniquely determined by Σ_{I} and Σ_{II} as

$$\begin{aligned}F_{\mathrm{I}} &= \psi^{-1}\left(\frac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}})-\frac{d-k-1}{d-1}\psi(\Sigma_{\mathrm{I}})\right),\\ F_{\mathrm{II}} &= \psi^{-1}\left(\frac{k}{d-1}\psi(\Sigma_{\mathrm{I}})-\frac{k-1}{d-1}\psi(\Sigma_{\mathrm{II}})\right).\end{aligned}$$

System (A.22) implies that for any point from the support,

$$\begin{aligned}&k\Sigma_{\mathrm{I}}+(d-k)\Sigma_{\mathrm{II}}-(d-1)\psi^{-1}\left(\frac{k}{d-1}\psi(\Sigma_{\mathrm{I}})+\frac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}})\right)\\ &\quad=kG_{\mathrm{I}}+(d-k)G_{\mathrm{II}}.\end{aligned}$$

Suppose that problem (A.22)–(A.23) has two solutions $(\Sigma_{\mathrm{I}},\Sigma_{\mathrm{II}})$ and $(\tilde{\Sigma}_{\mathrm{I}},\tilde{\Sigma}_{\mathrm{II}})$ with a common interval of existence $[t_0,t_0+c]$. Let us show that for any $t\in[t_0,t_0+c]$, $\Sigma_{\mathrm{I}}(t)\geq\tilde{\Sigma}_{\mathrm{I}}(t)$ if and only if $\Sigma_{\mathrm{II}}(t)\leq\tilde{\Sigma}_{\mathrm{II}}(t)$. Fix $t\in(t_0,t_0+c]$. From the equation

$$\begin{aligned}&k\Sigma_{\mathrm{I}}+(d-k)\Sigma_{\mathrm{II}}-(d-1)\psi^{-1}\left(\frac{k}{d-1}\psi(\Sigma_{\mathrm{I}})+\frac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}})\right)\\ &\quad=k\tilde{\Sigma}_{\mathrm{I}}+(d-k)\tilde{\Sigma}_{\mathrm{II}}-(d-1)\psi^{-1}\left(\frac{k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{I}})+\frac{d-k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{II}})\right),\end{aligned}$$

obtain that

$$\begin{aligned}&k\left(1-\frac{\psi'(\Sigma_{\mathrm{I}}^*)}{\psi'\left(\psi^{-1}\left(\dfrac{k}{d-1}\psi(\Sigma_{\mathrm{I}}^*)+\dfrac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}})\right)\right)}\right)(\Sigma_{\mathrm{I}}-\tilde{\Sigma}_{\mathrm{I}})\\ &\quad=(d-k)\left(1-\frac{\psi'(\Sigma_{\mathrm{II}}^*)}{\psi'\left(\psi^{-1}\left(\dfrac{k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{I}})+\dfrac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}}^*)\right)\right)}\right)(\tilde{\Sigma}_{\mathrm{II}}-\Sigma_{\mathrm{II}}),\end{aligned}\tag{A.24}$$

where $\Sigma_{\mathrm{I}}^* = \alpha\Sigma_{\mathrm{I}} + (1-\alpha)\tilde{\Sigma}_{\mathrm{I}}$ for some $\alpha = \alpha(\Sigma_{\mathrm{I}}(t), \tilde{\Sigma}_{\mathrm{I}}(t), \Sigma_{\mathrm{II}}(t)) \in [0,1]$ and $\Sigma_{\mathrm{II}}^* = \beta\Sigma_{\mathrm{II}} + (1-\beta)\tilde{\Sigma}_{\mathrm{II}}$ for some $\beta = \beta(\tilde{\Sigma}_{\mathrm{I}}(t), \Sigma_{\mathrm{II}}(t), \tilde{\Sigma}_{\mathrm{II}}(t)) \in [0,1]$. Note that for $t < T$,

$$\frac{\psi'(\Sigma_{\mathrm{I}})}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\Sigma_{\mathrm{I}}) + \frac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}})\right)\right)} < 1,$$

$$\frac{\psi'(\tilde{\Sigma}_{\mathrm{I}})}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{I}}) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{II}})\right)\right)} < 1.$$

Because $\frac{\tilde{\Sigma}_{\mathrm{I}}}{\Sigma_{\mathrm{I}}} \to 1$, $\frac{\tilde{\Sigma}_{\mathrm{II}}}{\Sigma_{\mathrm{II}}} \to 1$ as $t \downarrow t_0$, then for t close enough to t_0 (c is chosen to be small enough),

$$\frac{\psi'(\Sigma_{\mathrm{I}}^*)}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\Sigma_{\mathrm{I}}^*) + \frac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}})\right)\right)} < 1,$$

$$\frac{\psi'(\Sigma_{\mathrm{II}}^*)}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{I}}) + \frac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}}^*)\right)\right)} < 1.$$

Therefore, (A.24) gives that $\Sigma_{\mathrm{I}}(t) \geq \tilde{\Sigma}_{\mathrm{I}}(t)$ if and only if $\Sigma_{\mathrm{II}}(t) \leq \tilde{\Sigma}_{\mathrm{II}}(t)$. The next step is to show that this fact and the fact that the function $\frac{\psi''(x)}{(\psi'(x))^2}$ is increasing imply that

$$(\Sigma_{\mathrm{I}}' - \tilde{\Sigma}_{\mathrm{I}}')(\Sigma_{\mathrm{I}} - \tilde{\Sigma}_{\mathrm{I}}) \leq 0, \qquad (\Sigma_{\mathrm{II}}' - \tilde{\Sigma}_{\mathrm{II}}')(\Sigma_{\mathrm{II}} - \tilde{\Sigma}_{\mathrm{II}}) \leq 0 \quad \text{a.e. } [t_0, t_0 + c].$$

Suppose that for a given point $t \in (t_0, t_0 + c]$, at which the derivatives Σ_{I}' and $\tilde{\Sigma}_{\mathrm{I}}'$ exist, it holds that $\Sigma_{\mathrm{I}} \geq \tilde{\Sigma}_{\mathrm{I}}$. Let us prove that $\Sigma_{\mathrm{I}}' - \tilde{\Sigma}_{\mathrm{I}}' \leq 0$. From (A.22), obtain that

$$\begin{aligned}\Sigma_{\mathrm{I}}' - \tilde{\Sigma}_{\mathrm{I}}' = \frac{G_{\mathrm{I}}'}{W_{\mathrm{I}}}\Bigg(&\frac{\psi'(\Sigma_{\mathrm{I}})}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\Sigma_{\mathrm{I}}) + \frac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}})\right)\right)} \\ &- \frac{\psi'(\tilde{\Sigma}_{\mathrm{I}})}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{I}}) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{II}})\right)\right)}\Bigg),\end{aligned}$$

where

$$\begin{aligned}W_{\mathrm{I}} = &\left(1 - \frac{\psi'(\Sigma_{\mathrm{I}})}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\Sigma_{\mathrm{I}}) + \frac{d-k}{d-1}\psi(\Sigma_{\mathrm{II}})\right)\right)}\right) \\ &\times \left(1 - \frac{\psi'(\tilde{\Sigma}_{\mathrm{I}})}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{I}}) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{II}})\right)\right)}\right).\end{aligned}$$

Because $\Sigma_{\mathrm{II}} \le \tilde{\Sigma}_{\mathrm{II}}$, then

$$\Sigma_{\mathrm{I}}' - \tilde{\Sigma}_{\mathrm{I}}' \le \frac{g_{\mathrm{I}}}{W_{\mathrm{I}}}\left(\frac{\psi'(\Sigma_{\mathrm{I}})}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\Sigma_{\mathrm{I}}) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{II}})\right)\right)} - \frac{\psi'(\tilde{\Sigma}_{\mathrm{I}})}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{I}}) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{II}})\right)\right)}\right).$$

Now we want to show that the difference in the parentheses is nonpositive.

Because $\psi'(\tilde{\Sigma}_{\mathrm{I}})/(\psi'(\psi^{-1}(\frac{k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{I}}) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{II}})))) < 1$, then $\tilde{\Sigma}_{\mathrm{I}} > \psi^{-1}(\frac{k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{I}}) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{II}}))$ and, therefore, $\Sigma_{\mathrm{I}} > \psi^{-1}(\frac{k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{I}}) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{\mathrm{II}}))$. Thus, if we show that the function

$$\frac{\psi'(y_1)}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2)\right)\right)}$$

is decreasing in y_1 when y_1 and y_2 are close to 0 and $y_1 > \psi^{-1}(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2))$, then we will establish that $\Sigma_{\mathrm{I}}' - \tilde{\Sigma}_{\mathrm{I}}' \le 0$. The derivative of this function with respect to y_1 is

$$\frac{\psi''(y_1)}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2)\right)\right)} - \frac{k(\psi'(y_1))^2\psi''\left(\psi^{-1}\left(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2)\right)\right)}{(d-1)\left(\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2)\right)\right)\right)^3}.$$

For this derivative to be nonpositive, it is sufficient that

$$\frac{\psi''(y_1)}{(\psi'(y_1))^2} \ge \frac{\psi''\left(\psi^{-1}\left(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2)\right)\right)}{\left(\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2)\right)\right)\right)^2}.$$

The last inequality holds because of the assumption that $\frac{\psi''(x)}{(\psi'(x))^2}$ is increasing and the condition $y_1 > \psi^{-1}(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2))$.

To summarize, we have established that $(\Sigma_{\mathrm{I}}' - \tilde{\Sigma}_{\mathrm{I}}', \Sigma_{\mathrm{I}} - \tilde{\Sigma}_{\mathrm{I}}) \le 0$ a.e. on $[t_0, t_0 + c]$, that is,

$$\frac{d}{dt}(\Sigma_{\mathrm{I}} - \tilde{\Sigma}_{\mathrm{I}})^2 \le 0 \quad \text{a.e. } [t_0, t_0 + c].$$

This inequality and (A.23) imply that Σ_{I} and $\tilde{\Sigma}_{\mathrm{I}}$ coincide in a neighborhood of t_0. In a similar way, it can be shown that

$$\frac{d}{dt}(\Sigma_{\mathrm{II}} - \tilde{\Sigma}_{\mathrm{II}})^2 \leq 0 \quad \text{a.e. } [t_0, t_0 + c],$$

and, therefore, Σ_{II} and $\tilde{\Sigma}_{\mathrm{II}}$ coincide in a neighborhood of t_0. □

A.7 *Auctions with exogenous variation in the number of bidders*

Similar to the main case, identification follows from several conditions on G_i, $i = 1, 2, 3$. The conditions implied by the model are the following: (i) $G_i(t_0) = p_i$, $i = 1, 2, 3$; (ii) G_i is absolutely continuous on $[t_0, T]$, $i = 1, 2, 3$; (iii) G_i is strictly increasing on $[t_0, T]$, $i = 1, 2, 3$. Depending on the values of p_A, $A \subseteq \{1, 2, 3\}$, some additional restrictions on G_i may be required.

An important part of proving identification is to demonstrate that under assumption (3.9), each F_i, $i = 1, 2, 3$, has a unique representation through H. I consider two cases: one with $p_{123} > 0$ and the other with $p_{123} = 0$.

Case $p_{123} > 0$ Rewrite functions H_i, $i = 1, 2, 3$, as

$$H_1 = p_{123}\left(F_2 + \frac{p_{13}}{p_{123}}\right)\left(F_3 + \frac{p_{12}}{p_{123}}\right) - \frac{p_{12}p_{13}}{p_{123}},$$

$$H_2 = p_{123}\left(F_1 + \frac{p_{23}}{p_{123}}\right)\left(F_3 + \frac{p_{12}}{p_{123}}\right) - \frac{p_{12}p_{23}}{p_{123}},$$

$$H_3 = p_{123}\left(F_1 + \frac{p_{23}}{p_{123}}\right)\left(F_2 + \frac{p_{13}}{p_{123}}\right) - \frac{p_{13}p_{23}}{p_{123}}.$$

Taking into account that F_i are positive for $t > t_0$, derive the formulas

$$F_1 = -\frac{p_{23}}{p_{123}} + \frac{1}{p_{123}}\sqrt{\frac{(p_{123}H_2 + p_{12}p_{23})(p_{123}H_3 + p_{13}p_{23})}{p_{123}H_1 + p_{12}p_{13}}},$$

$$F_2 = -\frac{p_{13}}{p_{123}} + \frac{1}{p_{123}}\sqrt{\frac{(p_{123}H_1 + p_{12}p_{13})(p_{123}H_3 + p_{13}p_{23})}{p_{123}H_2 + p_{12}p_{23}}},$$

$$F_3 = -\frac{p_{12}}{p_{123}} + \frac{1}{p_{123}}\sqrt{\frac{(p_{123}H_1 + p_{12}p_{13})(p_{123}H_2 + p_{12}p_{23})}{p_{123}H_3 + p_{13}p_{23}}}.$$

The expressions on the right-hand sides of these equations are $q_1(H)$, $q_2(H)$, and $q_3(H)$, respectively. If all three values p_{12}, p_{13}, and p_{23} are strictly positive, it can shown that conditions (i)–(iii) above are the only conditions required for identification. Otherwise, the proof of uniqueness requires stronger conditions in the spirit of condition (III) in Theorem 3.5. Notice that the situation of $p_{12} = p_{13} = p_{23} = 0$ is almost identical to the paper's main case.

Case $p_{123}=0$ Assumption 3.9 implies that $p_{12}>0$, $p_{13}>0$, $p_{23}>0$. Because

$$H_1 = p_{12}F_2 + p_{13}F_3,$$
$$H_2 = p_{12}F_1 + p_{23}F_3,$$
$$H_3 = p_{13}F_1 + p_{23}F_2,$$

F_i are expressed through H_i as

$$F_1 = -\frac{p_{23}}{2p_{13}p_{12}}H_1 + \frac{1}{2p_{12}}H_2 + \frac{1}{2p_{13}}H_3,$$
$$F_2 = \frac{1}{2p_{12}}H_1 - \frac{p_{13}}{2p_{12}p_{23}}H_2 + \frac{1}{2p_{23}}H_3,$$
$$F_3 = \frac{1}{2p_{13}}H_1 + \frac{1}{2p_{23}}H_2 - \frac{p_{12}}{2p_{13}p_{23}}H_3.$$

The expressions on the right-hand sides of these equations are $q_1(H)$, $q_2(H)$, and $q_3(H)$, respectively. It is easy to show that conditions (i)–(iii) above are sufficient to guarantee identification. As we can see, in both cases F_i are uniquely expressed in terms of H_i.

Appendix B: Identification in generalized competing risks models

First, I outline Meilijson's approach. From (4.3), Meilijson obtains a system of integral equations that do not contain the derivatives of F_j,

$$F(t) = \exp\left\{\tilde{T}\log\int_{t_0}^{t}\exp\{-\bar{M}\log\bigl(1-F(s)\bigr)\,dG(s)\}\right\},$$

where matrix $\bar{M}$ is such that $\bar{M}(i,j) = 1 - M(i,j)$ and $\tilde{T} = (M^{\mathrm{tr}}M)^{-1}M^{\mathrm{tr}}$. He suggests applying to these equations a fixed point theorem for multidimensional functional spaces. As I mentioned, however, his proofs miss important parts.

I now turn to describing my method. The rank condition implies that $m \geq d$, that is, there are at least as many minimal fatal sets as the number of the elements in a coherent system. First, I consider the case of $m = d$ and assume that the rank condition for the incidence matrix M holds, that is, M is invertible. Introduce auxiliary functions

$$H_i = \prod_{j\in I_i} F_j, \quad i = 1,\dots,d,$$

and denote $H = (H_1,\dots,H_d)^{\mathrm{tr}}$. The rank condition guarantees that functions F_i, $i = 1,\dots,d$, taking into account that they are positive, are uniquely expressed through functions H_i, $i = 1,\dots,d$, via multiplication, division, and taking a rational root. Indeed,

$$\log H_i = \sum_{j\in I_i}\log F_j, \quad i = 1,\dots,d.$$

These equations can be rewritten as $\log H = M \log F$; therefore, $F = \exp\{M^{-1} \log H\}$, that is,

$$F_i = \prod_{j=1}^{d} H_j^{k_{ij}}, \quad i = 1, \dots, d. \tag{B.1}$$

Similar to the auction problem, I obtain an auxiliary system of differential equations by rewriting (4.3) in terms of H:

$$H_i' = \frac{g_i}{\prod_{j\in I_i^c}\left(1 - \prod_{l=1}^{d} H_l^{k_{jl}}\right)}, \quad i = 1, \dots, d. \tag{B.2}$$

Functions H_i satisfy initial conditions

$$\lim_{t\downarrow t_0} H_i(t) = 0, \quad i = 1, \dots, d. \tag{B.3}$$

As with the auction, the existence and uniqueness theorems, Theorems 4.1 and 4.2, can be proved in steps. First, the results are obtained locally and then globally.

The existence of a local solution to (4.3)–(4.4) can be proved in the following way. First, to avoid discontinuities in H, I can modify the auxiliary system (B.2) by introducing a very small number ε when necessary. Using Tonelli approximations, I can establish the existence of a local solution for the auxiliary system with ε. After that, I can take the limit as $\varepsilon \to 0$ and show the existence of a local solution for (B.2)–(B.3). Then I can use formulas (B.1) to obtain the existence of a local solution to problem (4.3)–(4.4). To establish local uniqueness, I obtain a generalized local Lipschitz condition on H_i.

Finally, I can show that the unique local solution can be extended to the whole support and that such an extension is unique. Again, the monotonicity of F_i in this solution has to be assumed.

Below I prove the local uniqueness part of Theorem 4.2.

Proof of Theorem 4.2. Let F and $\tilde{F}$ be two local solutions to (4.3)–(4.4) with a common interval of existence $[t_0, t_0 + c]$. Let H and $\tilde{H}$ be the corresponding auxiliary functions. Then H and $\tilde{H}$ solve auxiliary system (B.2) a.e. on $(t_0, t_0 + c]$. Denote the right-hand side of (B.2) as

$$J(t, H) = \left(\frac{g_1(t)}{\prod_{j\in I_1^c}\left(1 - \prod_{l=1}^{d} H_l^{k_{jl}}\right)}, \dots, \frac{g_d(t)}{\prod_{j\in I_d^c}\left(1 - \prod_{l=1}^{d} H_l^{k_{jl}}\right)}\right).$$

The plan is to derive a generalized local Lipschitz condition on H_i and then use Lemmas A.7 and A.8 to establish that H and $\tilde{H}$ coincide. This will imply that F and $\tilde{F}$ coincide. Consider $H_i - \tilde{H}_i$ for any i and let $|I_i^c|$ be the number of elements in I_i^c. Then, a.e. on

$[t_0, t_0 + c]$,

$$\begin{aligned}
\left|H_i' - \tilde{H}_i'\right| &= \frac{g_i}{\prod_{j \in I_i^c}(1 - F_j)} - \frac{g_i}{\prod_{j \in I_i^c}(1 - \tilde{F}_j)} \\
&= \frac{g_i}{\prod_{j \in I_i^c}(1 - F_j) \prod_{j \in I_i^c}(1 - \tilde{F}_j)} \left(\prod_{j \in I_i^c}(1 - F_j) - \prod_{j \in I_i^c}\left(1 - F_j + (F_j - \tilde{F}_j)\right) \right) \\
&\leq \frac{g_i}{\prod_{j \in I_i^c}(1 - F_j) \prod_{j \in I_i^c}(1 - \tilde{F}_j)} 2^{|I_i^c|-1} \sum_{j \in I_i^c} |F_j - \tilde{F}_j| \leq C_i g_i \sum_{j \in I_i^c} |F_j - \tilde{F}_j|
\end{aligned}$$

for some constant C_i. Differences $|F_j - \tilde{F}_j|$ can be bounded from above by expressions of $|H_j - \tilde{H}_j|$. According to (B.1), for $t > t_0$,

$$F_j - \tilde{F}_j = \prod_{l=1}^{d} H_l^{k_{jl}} - \prod_{l=1}^{d} \tilde{H}_l^{k_{jl}};$$

therefore,

$$F_j - \tilde{F}_j = \sum_{h=1}^{d} \prod_{l<h} H_l^{k_{jl}} \prod_{m>h} \tilde{H}_m^{k_{jm}} \left(H_h^{k_{jh}} - \tilde{H}_h^{k_{jh}}\right).$$

For $x_1, x_2 > 0$, by the mean value theorem,

$$x_l^{\alpha} - x_2^{\alpha} = \alpha\left(\theta x_1 + (1 - \theta)x_2\right)^{\alpha - 1}(x_1 - x_2),$$

where $\theta = \theta(x_1, x_2) \in [0, 1]$. If $\alpha \geq 1$, then

$$\left|x_l^{\alpha} - x_2^{\alpha}\right| \leq \alpha\left(\max\{x_1, x_2\}\right)^{\alpha - 1}|x_1 - x_2|.$$

If $\alpha < 1$, then

$$\left|x_l^{\alpha} - x_2^{\alpha}\right| \leq |\alpha|\left(\min\{x_1, x_2\}\right)^{\alpha - 1}|x_1 - x_2|.$$

Because $H_h(t), \tilde{H}_h(t) > 0$ for $t > t_0$, then for $t > t_0$,

$$\left|H_h^{k_{jh}}(t) - \tilde{H}_h^{k_{jh}}(t)\right| \leq W_{jh}(t)\left|H_h(t) - \tilde{H}_h(t)\right|,$$

where

$$W_{jh}(t) = \left(1(k_{jh} \geq 1) \max\left\{H_h(t), \tilde{H}_h(t)\right\} + 1(k_{jh} < 1) \min\left\{H_h(t), \tilde{H}_h(t)\right\}\right)^{k_{jh}-1}.$$

Because $\lim_{t \downarrow t_0} \frac{H_h(t)}{G_h(t)} = 1$ and $\lim_{t \downarrow t_0} \frac{\tilde{H}_h(t)}{G_h(t)} = 1$, then $\lim_{t \downarrow t_0} \frac{W_{jh}(t)}{G_h^{k_{jh}-1}(t)} = 1$. Hence, for $t > t_0$,

$$|F_j - \tilde{F}_j| \leq L_j \sum_{h=1}^{d} \left(\prod_{l \neq h} G_l^{k_{jl}} \right) G_h^{k_{jh}-1} |k_{jh}||H_h - \tilde{H}_h|$$

for some constants $L_j > 0$. Thus, a.e. on $[t_0, t_0 + c]$,

$$\left|H_i'(t) - \tilde{H}_i'(t)\right| \le D_i g_i \sum_{j \in I_i^c} \sum_{h=1}^{d} \left(\prod_{l \ne h} G_l^{k_{jl}}(t)\right) G_h^{k_{jh}-1}(t) |k_{jh}| \left|H_h(t) - \tilde{H}_h(t)\right|$$

for some constants $D_i > 0$ and, consequently,

$$\left\|H'(t) - \tilde{H}'(t)\right\|_1 \le C\bigl(\Gamma_1(t) + \cdots + \Gamma_d(t)\bigr)\left\|H(t) - \tilde{H}(t)\right\|_1$$

for some constant $C > 0$. This inequality and Lemmas A.7 and A.8 imply that $H(t) = \tilde{H}(t)$, $t \in [t_0, t_0 + c]$. □

References

Athey, S. and P. A. Haile (2002), "Identification of standard auction models." *Econometrica*, 70 (6), 2107–2140. [270, 271, 273, 274, 287]

Athey, S. and P. Haile (2006), "Empirical models of auctions." Unpublished manuscript. [271]

Athey, S. and P. Haile (2007), "Nonparametric approaches to auctions." In *The Handbook of Econometrics*, Vol. 6A (J. J. Heckman and E. L. Leamer, eds.), North-Holland, Amsterdam. [271]

Athey, S., J. Levin, and E. Seira (2011), "Comparing open and sealed bid auctions: Theory and evidence from timber auctions." *Quarterly Journal of Economics*, 126 (1), 207–257. [285]

Banerji, A. and J. V. Meenakshi (2004), "Buyer collusion and efficiency of government intervention in wheat markets in northern India: An asymmetric structural auctions analysis." *American Journal of Agricultural Economics*, 86 (1), 236–253. [271]

Barlow, R. E. and F. Proschan (1975), *Statistical Theory of Reliability and Life Testing: Probability Models*. Holt, Rinehart & Winston, New York. [289]

Brendstrup, B. and H. J. Paarsch (2006), "Identification and estimation in sequential, asymmetric, English auctions." *Journal of Econometrics*, 134, 69–94. [271, 285]

Brendstrup, B. and H. J. Paarsch (2007), "Semiparametric identification and estimation in multi-unit, English auctions." *Journal of Econometrics*, 141, 84–108. [285]

Campo, S., I. Perrigne, and Q. Vuong (2003), "Asymmetry in first-price auctions with affiliated private values." *Journal of Applied Econometrics*, 18, 179–207. [272]

Coddington, E. A. and N. Levinson (1972), *Theory of Ordinary Differential Equations*. Tata McGraw-Hill Publishing Company, New Delhi. [297]

Filippov, A. F. (1988), *Differential Equations With Discontinuous Righthand Sides*. Kluwer Academic, Dordrecht. [297, 302]

Guerre, E., I. Perrigne, and Q. Vuong (2009), "Nonparametric identification of risk aversion in first-price auctions under exclusion restrictions." *Econometrica*, 77 (4), 1193–1227. [272, 279, 281]

Hartman, P. (1964), *Ordinary Differential Equations*. Wiley, New York. [297, 306]

Komarova, T. (2009), "Nonparametric identification in asymmetric second-price auctions: A new approach." Working Paper CWP31/09, CEMMAP. [271, 294, 314]

Krishna, V. (2002), *Auction Theory*. Academic Press, San Diego. [270]

Lebrun, B. (1999), "First-price auctions and the asymmetric N bidders case." *International Economic Review*, 40, 125–142. [272, 281]

Maskin, E. and J. Riley (2003), "Uniqueness of equilibrium in sealed high-bid auctions." *Games and Economic Behavior*, 45 (2), 395–409. [272, 281]

McAfee, R. P. and J. McMillan (1987), "Auctions with a stochastic number of bidders." *Journal of Economic Theory*, 43, 1–19. [287]

Meenakshi, J. V. and A. Banerji (2005), "The unsupportable support price: An analysis of collusion and government intervention in paddy auction markets in north India." *Journal of Development Economics*, 76, 377–403. [271]

Meilijson, I. (1981), "Estimation of the lifetime distribution of the parts from the autopsy statistics of the machine." *Journal of Applied Probability*, 18, 829–838. [271, 290, 291]

Sansone, G. (1948), *Equazioni Differenziali Nel Campo Reale*. Zanichelli, Bologna. [297]

Szarski, J. (1965), *Differential Inequalities*. Polish Scientific Publishers, Warszawa. [297, 307]

Tonelli, L. (1928), "Sulle equazioni funzionali del tipo di Volterra." *Bulletin of the Calcutta Mathematical Society*, 20, 31–48. [297]

Vickrey, W. (1961), "Counterspeculation, auctions, and competitive sealed tenders." *Journal of Finance*, 16 (1), 8–37. [270]

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