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Diamond Condition for Commuting Adjacency Matrices of Directed and Undirected Graphs

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***Abstract*—In the context of the stability analysis of interdependent networks through the eigenvalue evaluation of their adjacency matrices, we characterize algebraically and also geometrically necessary and sufficient conditions for the adjacency matrices of directed and undirected graphs to commute. We also discuss the problem of communicating the concepts, the theorems, and the results to a non-mathematical audience, and more generally across different disciplinary domains, as one of the fundamental challenges faced by the Internet Science community. Thus, the paper provides much more background, discussion, and detail than would normally be found in a purely mathematical publication, for which the proof of the diamond condition would require only a few lines. Graphical visualization, examples, discussion of important steps in the proof and of the diamond condition itself as it applies to graphs whose adjacency matrices commute are provided. The paper also discusses interdependent graphs and applies the results on commuting adjacency matrices to study when the interconnection matrix encoding links between two disjoint graphs commutes with the adjacency matrix of the disjoint union of the two graphs. Expected applications are in the design and analysis of interdependent networks.**

Keywords: Commuting adjacency matrices, algebraic graph theory, internet science.

I. Introduction

This paper is motivated by the wish to optimize the efficiency of the mathematical analysis of the stability of interdependent networks. The paper is also concerned with a wider and deeper question of fundamental importance to Internet Science. Namely, given the increasing reliance of every applied and theoretical aspect of the Internet on many and very different disciplines, how can the inevitable disciplinary language barriers that today's Internet scientists face be best addressed? For example, what takes place in the mind of a mathematician when solving a particular problem? And how can such insight be best communicated to a wider audience of applied scientists and engineers? What kind of formalism provides an optimal "user interface" for a given audience?

Some integration with a social science perspective helps in this regard. For example, whereas in computer science and in the hard sciences in general the term 'paradigm' usually means a deeper and more pervasive 'model', but still a model, in social science the dominant definition of paradigm is due to Thomas Kuhn (himself a physicist) as, paraphrasing, a body of theory, a community of practice, and a set of methodologies [1]. For a community of practice to exist, a shared language – or set of languages – is essential. Thus, the development of an Internet Science faces the problem of undoing the effect of the disciplinary Babel tower most of us have been inhabiting.

As discussed in [2], however, one must go beyond "translation" when attempting to communicate across disciplines, and take into account the different epistemologies relied upon in each discipline. Although the analytical tools to effect such a reflexive analysis of one's research are best drawn from social science, at this early stage of Internet Science development an explicit discussion of the epistemologies at play (e.g. integration with [3]) would make this paper too difficult to read. Therefore, we prefer to adopt a tutorial style where the problem at hand is approached from different mathematical points of view, drawing connections between them. Thus, the paper can also be seen as a reflexive account or "case study", in two different mathematical "languages" that tend to be used respectively by engineers and mathematicians, of how a particular mathematical fact, the "diamond condition" for commuting adjacency matrices, was hypothesized, formalized, and proven together with the development of visualization techniques. The two presentations are then related through geometrical visualization.

A. Background and Scope

This work grew out of P. Dini's Research Mobility visit to P. Van Mieghem's Network Architectures and Services group at TU Delft in September 2012, in the context of the EINS Network of Excellence.[1]

The proof of the diamond condition requires only a few lines. In this form, however, it can only be understood and appreciated by mathematicians, or by other scientists equipped with pencil, lots of paper, time and patience, and a strong applied mathematics background. This paper aims to make this important result as accessible and applicable as possible for a wide range of Internet scientists. Therefore, the paper follows a tutorial style that is organized into four main narratives:

- Section II: General facts and some elementary results about commuting matrices, written in mathematical language.

[1] http://www.internet-science.eu

- Section III: A "pedestrian" and "brute force" account of how the commutativity requirement for symmetric adjacency matrices can be expressed most simply in coordinate form, written in a language more common in applied mathematics contexts.
- Section IV: A detailed account of the mathematical facts leading up to the diamond condition, written almost completely in pure maths.
- Section V: A geometrical discussion and visualization of the equivalence of the two previous sections, written in a language more common to engineering contexts.

B. Mathematical Problem Statement

Given two separate undirected networks, for example an electric utility network and the network of computers that controls it, each can be characterized by its adjacency matrix. Call the two matrices A_1 and A_2, both symmetric. Now introduce some links between the networks. These links can be modelled with another matrix B_0, which is not necessarily square or symmetric. The two global matrices, therefore, are:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & B_0 \\ B_0^T & 0 \end{bmatrix} \tag{1}$$

which are shown here in block form. Note that even when B_0 is neither square nor symmetric B will be both. It turns out that whenever A and B commute the largest eigenvalue of the combined matrix is a linear combination of the largest eigenvalues of the individual matrices, as follows [4]:

$$\lambda_1(A + \alpha B) = \lambda_1(A) + \alpha\lambda_1(B), \tag{2}$$

where α is some real number. This is useful because it speeds up the evaluation of the eigenvalues of the connected network since they are linear combinations of the eigenvalues of A and B. Faster eigenvalue evaluation means more frequent monitoring of the stability of the combined network, enabling quicker intervention if something goes wrong in either network. Thus, the problem we are trying to solve is:

> *Given any two symmetric adjacency matrices A_1 and A_2, what are the constraints on the form B_0 should take so that A and B commute?*

II. Some General Discussion and Results

Before addressing the problem statement we define a few terms and review some known facts about matrices.

Definition 3 (Eigenspace). *An eigenspace of a matrix is the linear subspace consisting of all eigenvectors associated to a given eigenvalue. Its dimensionallity is equal to the multiplicity of the eigenvalue.*

Definition 4 (Simultaneous diagonalization). *If two matrices A and B are diagonalized by the same matrix U then they are simultaneously diagonalizable.*

Explanation: Since U diagonalizes both A and B we can write

$$\left.\begin{matrix} UAU^{-1} \\ UBU^{-1} \end{matrix}\right\}\text{Diagonal.} \tag{5}$$

Conjugation by any invertible U respects addition and multiplication of A and B:

$$UAU^{-1} + UBU^{-1} = U(A + B)U^{-1} \tag{6}$$
$$\begin{aligned}(UAU^{-1})(UBU^{-1}) &= UA(U^{-1}U)BU^{-1} \\ &= UABU^{-1} = U(AB)U^{-1}. \end{aligned} \tag{7}$$

So simultaneous diagonalizability, i.e. that $UA^{-1}U$ and UBU^{-1} are both diagonal, implies that $U(A + B)U^{-1}$ and $U(AB)U^{-1}$ are too. Therefore the eigenvalues of $A + B$ and AB are, respectively, sums and products of those of A and B.

Proposition 1. *If symmetric operators A and B (on a real or complex n-dimensional vector space V) commute, then they are simultaneously diagonalizable by an orthonormal basis.*[2]

Proof: Let A be a symmetric operator on a real or complex n-dimensional vector space V (e.g., one given by an $n \times n$ matrix). Then

$$A\vec{v} = \lambda\vec{v}, \tag{8}$$

where λ is an eigenvalue for some eigenvector $\vec{v} \in V$. Now pre-multiply both sides of (8) by B. Then, since A and B commute, we have:

$$\begin{aligned} BA\vec{v} &= \lambda B\vec{v} \\ A(B\vec{v}) &= \lambda(B\vec{v}), \end{aligned} \tag{9}$$

showing that $B\vec{v}$ is an eigenvector of A that belongs to the *same* λ-eigenspace of A as $\vec{v}$.[3] Let the eigenvalues of A be λ_i $(1 \leq i \leq k, k \leq n)$. Let $W_i \subseteq V$ denote the λ_i-eigenspace of A. Now V is the orthogonal direct sum $\bigoplus_{j=1}^{k} W_i$, of the W_i. Moreover, A acts on each W_i by stretching vectors in W_i by a factor λ_i:

$$W_i = \{\vec{w} : A\vec{w} = \lambda_i\vec{w}\}. \tag{10}$$

For each i, let $d_i = \text{Dim}(W_i) \leq n$. By (9), W_i is invariant under B, i.e., $B : W_i \to W_i$ for each i. Therefore, since B is a symmetric operator on V, it is also a symmetric operator on W_i.[4] Hence there exists an orthonormal basis $w_1^{(i)}, \ldots, w_{d_i}^{(i)}$ for W_i that diagonalizes B restricted to W_i.[5] This means that,

[2]Note: this does *not* mean that A and B will have the same (number of distinct) eigenvalues, nor that they will have the same eigenspaces. In general they will not. But all we need to prove is that there exists a basis of V that will diagonalize both operators.

[3]The dimension of an eigenspace equals the number of repeated eigenvalues associated with it. As a reminder of how to visualize an eigenspace, if $n = 3$ and the eigenspace is 2-dimensional, it is a plane embedded in $\mathbb{R}^3$.

[4]For a real matrix, being symmetric as an operator (for example, the inner product $\langle Ax, y\rangle = \langle x, Ay\rangle$ always holds) is equivalent to being symmetric as a matrix ($A = A^T$). Then, since A maps W_i to W_i and the inner product condition holds in V, it must hold in W_i too, so A restricted to W_i is a symmetric operator, which lets us diagonalize A restricted to W_i.

[5]Notice that we are free to choose such a basis since – given that A is a symmetric operator – each eigenspace is orthogonal to all the other eigenvectors. In the example of the plane embedded in $\mathbb{R}^3$, any two mutually perpendicular unit vectors lying in this plane form a valid basis for this W_i, and each will be scaled by λ_i regardless of the choice of their orientation in this plane. However, only *one* orientation of this basis system will diagonalize also d_i dimensions of B.

using this basis for W_i,

$$B\,\vec{w}_j^{(i)} = \mu_j^{(i)}\,\vec{w}_j^{(i)}, \tag{11}$$

where $\mu_j^{(i)}$ is an eigenvalue of B for $\vec{w}_j^{(i)} \in W_i$ $(1 \leq j \leq d_i)$.[6] Be that as it may, since each $w_j^{(i)}$ lies in W_i, we have by definition of W_i that

$$A\,w_j^{(i)} = \lambda_i w_j^{(i)}, \tag{12}$$

for $1 \leq i \leq k$ and $1 \leq j \leq d_i$. Now consider the orthonormal basis for V given by concatenating the bases for the various W_i $(1 \leq i \leq k)$:

$$w_1^{(1)}, \ldots, w_{d_1}^{(1)}, \ldots, w_1^{(k)}, \ldots, w_{d_k}^{(k)},$$

In this basis, the ith block of B that corresponds to the subspace W_i is diagonal by (11) and the block of A corresponding to W_i is diagonal by (12). Thus A and B are both diagonal in this orthonormal basis. $\square$

In the above basis, it is obvious that the eigenvalues of a linear combination of the commuting symmetric matrices A and B will be a linear combination of the eigenvalues of A and B. A similar statement clearly also holds for the eigenvalues of the product AB.

Proposition 2. *If the block diagonal matrix for the networks is*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \tag{13}$$

and for the interconnection graph one has

$$B = \begin{bmatrix} 0 & B_0 \\ B_0^T & 0 \end{bmatrix}, \tag{14}$$

then commutativity $AB = BA$ is equivalent to requiring

$$A_1 B_0 = B_0 A_2. \tag{15}$$

Proof: From

$$AB = \begin{bmatrix} 0 & A_1 B_0 \\ A_2 B_0^T & 0 \end{bmatrix} \quad \text{and} \tag{16}$$

$$BA = \begin{bmatrix} 0 & B_0 A_2 \\ B_0^T A_1 & 0 \end{bmatrix}, \tag{17}$$

for commutativity $A_1 B_0 = B_0 A_2$ and $A_2 B_0^T = B_0^T A_1$ must be true simultaneously. Conversely, assuming Eq. (15) is true, since A_1 and A_2 are both symmetric,

$$\begin{aligned} A_1^T B_0 &= B_0 A_2^T \\ [A_1^T B_0]^T &= [B_0 A_2^T]^T \\ B_0^T A_1 &= A_2 B_0^T, \end{aligned}$$

thus showing that the other condition is also true. $\square$

The spectrum of a graph is the set of eigenvalues (with multiplicities) of a matrix representation of the graph.

[6]In other words, B may have up to d_i distinct eigenvalues associated with subspace W_i, unlike A which only had 1 eigenvalue associated with W_i (by definition of W_i as an eigenspace of A).

Proposition 3. *If B_0 is invertible and A and B commute, then A_1 and A_2 have the same spectrum. Note that in this case A_1 and A_2 must be of the same size.*

Proof: By Prop. 2, $B_0 A_2 = A_1 B_0$. Hence,

$$\begin{aligned} B_0 A_2 B_0^{-1} &= A_1 B_0 B_0^{-1} \\ B_0 A_2 B_0^{-1} &= A_1, \end{aligned} \tag{18}$$

but conjugation does not change the spectrum of a matrix. $\square$

Unfortunately, although two isomorphic graphs have adjacency matrices with the same spectrum, the converse is not true: two adjacency matrices with the same spectrum need not correspond to two isomorphic graphs. Graphs that have the same spectrum but are not isomorphic are called *cospectral*. The issue of cospectral graphs is likely to play an important role in this problem.

We look at a few more simple cases of solutions to the commutative interconnection problem. We can always take $B_0 = 0$, the $m \times n$ zero matrix, for a trivial example (i.e. no coupling). If the two graphs are isomorphic then we can always take $B_0 = I$ (or a permutation matrix that gives the mapping of the isomorphism on nodes). As a fourth and distinct case,

Proposition 4. *Let A_1 be an $m \times m$ matrix and A_2 $n \times n$. Using an $m \times n$ matrix of all 1s for B_0 works to yield $AB = BA$, with A and B as defined in* (13) *and* (14)*, if and only if all the row sums of A_1 are equal to all the column sums of A_2. In particular this always works if A_1 and A_2 are adjacency matrices of regular graphs of the same degree.*

Proof: Let

$$A_1 = \left.\begin{bmatrix} \overbrace{\begin{matrix} 0 & a_{12} & \cdots & a_{1m} \\ a_{21} & 0 & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & 0 \end{matrix}}^{m} \end{bmatrix}\right\} m$$

$$B_0 = \left.\begin{bmatrix} \overbrace{\begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{matrix}}^{n} \end{bmatrix}\right\} m$$

$$A_2 = \left.\begin{bmatrix} \overbrace{\begin{matrix} 0 & a^{12} & \cdots & a^{1n} \\ a^{21} & 0 & \cdots & a^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a^{n1} & a^{n2} & \cdots & 0 \end{matrix}}^{n} \end{bmatrix}\right\} n$$

Then,

n

$$A_1 B_0 = \left.\left[\overbrace{\begin{matrix} \sum_{j=1}^m a_{1j} & \cdots & \sum_{j=1}^m a_{1j} \\ \sum_{j=1}^m a_{2j} & \cdots & \sum_{j=1}^m a_{2j} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^m a_{mj} & \cdots & \sum_{j=1}^m a_{mj} \end{matrix}}\right]\right\} m \tag{19}$$

$$B_0 A_2 = \left.\left[\overbrace{\begin{matrix} \sum_{i=1}^n a^{i1} & \cdots & \sum_{i=1}^n a^{in} \\ \sum_{i=1}^n a^{i1} & \cdots & \sum_{i=1}^n a^{in} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n a^{i1} & \cdots & \sum_{i=1}^n a^{in} \end{matrix}}^{n}\right]\right\} m \tag{20}$$

Both (19) and (20) are $m \times n$ matrices and they are equal as long as the sums of the rows of A_1 equal the sums of the columns of A_2. In particular, this condition is clearly satisfied in the case of regular graphs of the same degree. $\square$

III. Lie Bracket and the Coordinate Form of the Commutativity Condition for Undirected Graphs

To investigate which matrices commute we can use a Lie algebra approach [5]. Let F be a field. A Lie algebra over F is a vector space L together with a bilinear map called the Lie bracket:

$$L \times L \to L, \qquad (x, y) \to [x, y], \qquad x, y \in L \tag{21}$$

such that

$$[x, x] = 0 \qquad \forall x \in L \tag{22}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \qquad \forall x, y, z \in L \tag{23}$$

Thus the Lie bracket returns a vector, which can be bracketed recursively with another vector as shown, for example, by the Jacobi identity (23). In $\mathbb{R}^3$, the familiar vector or cross-product satisfies the Lie bracket axioms. Bilinearity implies that

$$\begin{aligned} 0 = [x + y, x + y] &= [x, x] + [x, y] + [y, x] + [y, y] \\ &= [x, y] + [y, x], \end{aligned} \tag{24}$$

from which we find that

$$[x, y] = -[y, x] = 0, \qquad \forall x, y \in L. \tag{25}$$

Now let V be an n-dimensional vector space over F. Let $gl(V)$ be the set of all linear maps $V \to V$, so it can be regarded as a set of matrices. This is itself an n^2-dimensional vector space[7] over F with canonical basis E_{ij}, where E_{ij} is a unit vector since it is an $(n \times n)$ matrix all of whose entries are 0 except for a single 1, for one value of $1 \le i \le n$ and one value of $1 \le j \le n$, which equals 1. It becomes a Lie algebra if we define

$$[A, B] = AB - BA, \qquad \forall A, B \in gl(V), \tag{26}$$

with the usual matrix product.

[7] The axioms of a vector space can indeed be satisfied by a set whose elements are matrices.

Although Erdmann and Wildon (for example) develop the theory of abstract Lie algebras fairly extensively, in this paper we only need to rely on the definition of the Lie bracket. In particular, for the problem at hand we start by asking what conditions $(n \times n)$ symmetric adjacency matrices A and B need to satisfy in order to commute, i.e.,

$$[A, B] = 0. \tag{27}$$

We will later investigate how any such conditions might relate to the interdependent graphs and in particular to the form of the B_0 connecting matrix. We may rewrite A and B as follows:

$$A = \sum_{ij} a_{ij} E_{ij}, \; a_{ij} = a_{ji} \in \{0, 1\} \text{ and } 1 \le i, j \le n; \tag{28}$$

$$B = \sum_{ij} b_{ij} E_{ij}, \; b_{ij} = b_{ji} \in \{0, 1\} \text{ and } 1 \le i, j \le n, \tag{29}$$

The summation limits on all the indices are always from 1 to n unless underwise stated. Substituting into (27),

$$\begin{aligned} &\left[\sum_i \sum_j a_{ij} E_{ij}, \sum_k \sum_l b_{kl} E_{kl}\right] \\ &\qquad = \sum_i \sum_j \sum_k \sum_l a_{ij} b_{kl} [E_{ij}, E_{kl}] = 0 \end{aligned} \tag{30}$$

An example of this straightforward calculation for $n = 2$ is provided in [6]. The Lie brackets of the unit matrices (basis vectors) follow these easily verifiable rules:

$$\text{Case 1:} \qquad j = k, i = l, \qquad [E_{ij}, E_{kl}] = E_{il} - E_{kj} \tag{31}$$

$$\text{Case 2:} \qquad j = k, i \ne l, \qquad [E_{ij}, E_{kl}] = E_{il} \tag{32}$$

$$\text{Case 3:} \qquad j \ne k, i = l, \qquad [E_{ij}, E_{kl}] = -E_{kj} \tag{33}$$

$$\text{Case 4:} \qquad j \ne k, i \ne l, \qquad [E_{ij}, E_{kl}] = 0. \tag{34}$$

The four cases can be written more compactly using the Kronecker delta δ_{ij}, which takes values zero and one with $\delta_{ij} = 1$ if and only if $i = j$ (see also [7]):

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj} \tag{35}$$

We now expand $[A, B]$ as in (30), according to which of these cases applies to (i, j, k, l). Full details for the example $n = 3$ are provided in [6]. In the following, we discuss the general case, showing the specialization to $n = 3$ only for the final step of each case.

Case 1: $j = k, i = l$

$$\sum_i \sum_{j \ne i} a_{ij} b_{ji} (E_{ii} - E_{jj}) = 0, \tag{36}$$

since A and B are both symmetric. E.g. for $n = 3$,

$$\begin{aligned} (36) = \; & a_{12} b_{21} (E_{11} - E_{22}) + a_{13} b_{31} (E_{11} - E_{33}) + \\ & a_{21} b_{12} (E_{22} - E_{11}) \; + a_{23} b_{32} (E_{22} - E_{33}) + \\ & a_{31} b_{13} (E_{33} - E_{11}) + a_{32} b_{23} (E_{33} - E_{22}) = 0, \end{aligned} \tag{37}$$

Case 2: $j = k, i \ne l$

$$\sum_i \sum_j \sum_{l \ne i} a_{ij} b_{jl} E_{il} = \sum_i \sum_{l \ne i} \left(\sum_j a_{ij} b_{jl}\right) E_{il}. \tag{38}$$

For $n = 3$, taking advantage of the fact that A and B are symmetric adjacency matrices (i.e. with zero diagonal), the above simplifies to:

$$\sum_i \sum_{l \neq i} \left(\sum_j a_{ij} b_{jl} \right) E_{il}$$
$$= a_{13}b_{32}E_{12} + a_{12}b_{23}E_{13} + a_{23}b_{31}E_{21} + a_{21}b_{13}E_{23} + a_{32}b_{21}E_{31} + a_{31}b_{12}E_{32} \quad (39)$$

We will pair this expression with the analogous result for Case 3, which is calculated next.

Case 3: $j \neq k, i = l$

$$\sum_i \sum_j \sum_{k \neq j} a_{ij} b_{ki} (-E_{kj}) = -\sum_j \sum_{k \neq j} \left(\sum_i a_{ij} b_{ki} \right) E_{kj} \quad (40)$$

For zero diagonals and $n = 3$,

$$-\sum_j \sum_{k \neq j} \left(\sum_i a_{ij} b_{ki} \right) E_{kj}$$
$$= -a_{31}b_{23}E_{21} - a_{21}b_{32}E_{31} - a_{32}b_{13}E_{12} - a_{12}b_{31}E_{32} - a_{23}b_{12}E_{13} - a_{13}b_{21}E_{23} \quad (41)$$

Adding the two non-zero results for the 3×3 example,

$$\begin{aligned}
&\sum_i \sum_j \sum_k \sum_l a_{ij} b_{kl} [E_{ij}, E_{kl}] \\
&= a_{13}b_{23}(E_{12} - E_{21}) + a_{12}b_{23}(E_{13} - E_{31}) + \\
&\quad a_{23}b_{13}(E_{21} - E_{12}) + a_{12}b_{13}(E_{23} - E_{32}) + \\
&\quad a_{23}b_{12}(E_{31} - E_{13}) + a_{13}b_{12}(E_{32} - E_{23}) \\
&= (a_{13}b_{23} - a_{23}b_{13})(E_{12} - E_{21}) + \\
&\quad (a_{12}b_{23} - a_{23}b_{12})(E_{13} - E_{31}) + \\
&\quad (a_{12}b_{13} - a_{13}b_{12})(E_{23} - E_{32}) \quad (42)
\end{aligned}$$

Setting this equal to zero, therefore, is the condition for two 3×3 symmetric adjacency matrices A and B to commute. The generalization of this expression to $n \times n$ matrices requires a bit of work which, as before, is shown in detail in [6]. What we have shown so far is that the Lie bracket of two arbitrary symmetric matrices A and B is given by the sum of Eqs. (38) and (40). Expansion of these two sums, simplification due to symmetry and zero diagonals, and rearrangement eventually leads to the general commutativity condition:

$$\begin{aligned}
&\sum_i \sum_j \sum_k \sum_l a_{ij} b_{kl} [E_{ij}, E_{kl}] \\
&= \sum_i \sum_{l \neq i} \left(\sum_j a_{ij} b_{jl} \right) E_{il} - \sum_j \sum_{k \neq j} \left(\sum_i a_{ij} b_{ki} \right) E_{kj} \\
&= \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} \sum_{\substack{j=1 \\ j \neq i,k}}^{n} (a_{ij}b_{kj} - a_{kj}b_{ij})(E_{ik} - E_{ki}) = 0 \quad (43)
\end{aligned}$$

The next section provides a more abstract, more elegant, and more efficient derivation and proof of this condition, along with the analogous one for directed graphs. The much greater insight afforded by this more "mathematical" work then leads naturally to a simple geometrical interpretation for commutativity and to constructive tests that are easily codable for both directed and undirected graphs for any n. The strategy is to work with the simplest possible "unit graphs", i.e. single directed or undirected edges, and to generalize to any digraph or graph by writing it as a linear combination of these with coefficients in $\{0, 1\}$. The geometrical condition for undirected graphs is of course equivalent to the result obtained above.

IV. Diamond Condition for Commuting Directed and Undirected Graphs

A *directed graph* or *digraph* $\Gamma = (V, E)$ is a set V of vertices together with a set of edges $E \subseteq V \times V$. It is called a *graph*, or *undirected graph*, if E is a symmetric relation, i.e. $E = E^{-1} = \{(v_2, v_1) : (v_1, v_2) \in E\}$. If Γ is an undirected graph, we say "$\{v_1, v_2\}$ is an edge of Γ" if either (hence both) of (v_1, v_2) or (v_2, v_1) are edges of Γ. In this paper we shall assume Γ has no self-loops, i.e. $(v, v) \notin E$ for any $v \in V$. Moreover, if we write (i, j) is an edge of Γ, this shall be taken to assume $i \neq j$.[8]

For finite graphs, $|V| = n$ is a natural number and it is convenient to take $V = \{1, \ldots, n\}$. We then denote by E_{ij} the $n \times n$ matrix having zeroes in all positions, except for a 1 in row i, column j. The notation δ_{ij} is the Kronecker delta taking values zero and one with $\delta_{ij} = 1$ if and only if $i = j$.

The *adjacency matrix* of a graph or digraph Γ is defined as

$$A(\Gamma) = \sum_{(i,j) \in E}^{n} E_{ij}. \quad (44)$$

For $n \times n$ square matrices A and B, their *Lie bracket* is $[A, B] = AB - BA$. A and B commute if and only if $[A, B] = 0$.

A. Commuting Digraphs

We study when two digraphs have commuting adjacency matrices. In this case we say that the (di)graphs commute. Note we can do this even if the (di)graphs do not have the same number of nodes: If Γ has vertices V and Γ' has vertices V', possibly $V \cap V' \neq \emptyset$, we enumerate $V \cup V' = \{v_1, \ldots, v_n\}$, and consider, without loss of generality, each of the (di)graphs as having edges connecting nodes amongst the v_i's that belong to them. NB: Whether or not graphs commute does depend on whether and how their nodes are identified, e.g. they always commute if their sets of nodes are disjoint!

Observation 1. $E_{ij}E_{kl} = \delta_{jk}E_{il}$

Thus,

Lemma 5 (Lie Bracket of Directed Edges)**.**

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}. \quad (45)$$

[8]Note that parentheses in (i, j) imply that order matters, whereas curly brackets $\{i, j\}$, which are more generally used to denote a set, imply that either ordering of the indices is equivalent. Thus, parentheses are used for the edges of digraphs and curly brackets for the edges of bidirectional graphs.

Corollary 6. *Distinct E_{ij} and E_{kl} commute unless (and only unless) the directed edges (i,j) and (k,l) are abutting (i.e. $j=k$ or $i=l$, or both).*

Corollary 7. *E_{ij} does not commute with E_{ji}, for any choice of $i,j \in V \cup V'$ $(i \neq j)$.*

Corollary 8. *Disjoint edges always commute. That is, E_{ij} commutes with E_{kl} if the four vertices i, j, k, and l are pairwise distinct.*

Observation 2.

$$[A,B] = \sum_{i}^{n}\sum_{j}^{n}\sum_{k}^{n}\sum_{l}^{n} a_{ij}b_{kl}\,[E_{ij},E_{kl}]. \tag{46}$$

Proposition 9 (Simple Necessary Directed Quadrilateral Condition for Commuting Digraphs)**.** *Let $[A,B]=0$ for adjacency matrices of directed graphs. For each vertex k where an edge (i,k) of Γ meets an edge (k,j) of Γ', there exists a vertex k' such that edge $(i,k') \in \Gamma'$ and edge $(k',j) \in \Gamma$.*

Proof: Abutting edges account for the only way to generate nonzero coefficients in front of E_{ij} and E_{ji} in the expansion of the Lie bracket by Lemma 5. In the expansion of the Lie bracket of A and B (Observation 2), for every k with an edge (i,k) in Γ and an edge (k,j) in Γ', we have

$$[E_{ik},E_{kj}] = \delta_{kk}E_{ij} - \delta_{ji}E_{kk} = E_{ij}. \tag{47}$$

For every such k, we also have, since (k',j) is in Γ and (i,k') in Γ', the summand

$$[E_{k'j},E_{ik'}] = \delta_{ji}E_{k'k'} - \delta_{k'k'}E_{ij} = -E_{ij}, \tag{48}$$

which cancels the former. However, there can be no cancellation if there is no k' corresponding to k. □

Visual Interpretation for Digraphs. Each pair of such summands in the proof corresponds to a quadrilateral whose directed edges give two paths from i to j with edges coming alternatingly from the two graphs. Along one of the paths the Γ edge is first and along the other the Γ' edge is first. (Note that $k=k'$ is possible if the two digraphs share a two-step path from i to j. Also $i=j$ can occur.) Moreover, it is easy to give necessary and sufficient conditions for directed graphs to commute:

Theorem 10 (**Diamond Condition for Commuting Digraphs**)**.** *If Γ and Γ' are directed graphs, with adjacency matrices A and B respectively, then A and B commute if and only if the diamond condition holds: For all nodes i and j, the number of two-step paths from i to j consisting of an edge of Γ followed by an edge of Γ' is equal to the number of two-step paths from i to j consisting of an edge of Γ' followed by an edge of Γ.*

Proof: Since we are multiplying adjacency matrices, the first number mentioned gives the (i,j)-entry of AB, the second number gives (i,j)-entry of BA. Hence $[A,B]=0$ in its (i,j)-entry if and only if these numbers are equal. □

B. Commuting Undirected Graphs

We now characterize when undirected graphs have commuting adjacency matrices. Notation: Let $E_{\{i,j\}} = E_{ij} + E_{ji}$. Obviously $E_{\{i,j\}} = E_{\{j,i\}}$. If Γ is an undirected graph, then A is a symmetric matrix. Clearly in this case

$$A(\Gamma) = \sum_{\{i,j\}\in E}^{n} E_{\{i,j\}}. \tag{49}$$

Observation 3. *For undirected graphs with adjacency matrices A and B,*

$$[A,B] = \sum_{\substack{i<j\\k<l}}^{n} a_{ij}b_{kl}\,[E_{\{i,j\}},E_{\{k,l\}}]. \tag{50}$$

Lemma 11 (Lie Bracket of Undirected Edges)**.**

$$\begin{aligned}[E_{\{i,j\}},E_{\{k,l\}}] = \delta_{jk}(E_{il}-E_{li}) + \delta_{jl}(E_{ik}-E_{ki}) +\\ \delta_{ik}(E_{jl}-E_{lj}) + \delta_{il}(E_{jk}-E_{kj}).\end{aligned} \tag{51}$$

Moreover, at most one of the summands is nonzero. The bracket is zero if and only if $|\{i,j,k,l\}| \neq 3$. Thus an edge commutes with another unless (and only unless) they share a single vertex.

Proof: Using the distributivity of the Lie bracket over sums and the formula for the Lie bracket of directed edges, we see that

$$\begin{aligned}
&[E_{\{i,j\}},E_{\{k,l\}}] && (52)\\
&\quad = [E_{ij}+E_{ji}, E_{kl}+E_{lk}]\\
&\quad = [E_{ij}, E_{kl}+E_{lk}] + [E_{ji}, E_{kl}+E_{lk}]\\
&\quad = [E_{ij},E_{kl}] + [E_{ij},E_{lk}] + [E_{ji},E_{kl}] + [E_{ji},E_{lk}]\\
&\quad = \delta_{jk}E_{il} - \delta_{li}E_{kj} + \delta_{jl}E_{ik} - \delta_{ki}E_{lj} +\\
&\qquad \delta_{ik}E_{jl} - \delta_{lj}E_{ki} + \delta_{il}E_{jk} - \delta_{kj}E_{li} && (53)
\end{aligned}$$

Collecting terms multiplied by the same δ's now yields the result. If all of i,j,k and l are distinct then this is zero since all the δ's are zero. If there are only two distinct vertices, it follows that $\{i,j\} = \{k,l\}$; then this Lie bracket is the bracket of a matrix with itself and hence zero. In the case of 3 distinct vertices, we have two undirected edges sharing exactly one vertex, so only the δ corresponding to the unique shared vertex is nonzero. □

Corollary 12. *$E_{\{i,j\}}$ commutes with $E_{\{k,l\}}$ if and only if edges $\{i,j\}$ and $\{k,l\}$ are (1) identical or (2) share no vertex.*

NB: As can be seen from Lemma 11, $[A,B]$ need not be symmetric even if both A and B are symmetric.

Proposition 13 (Simple Necessary Quadrilateral Condition for Commuting Undirected Graphs)**.** *Let $[A,B]=0$ for adjacency matrices of undirected graphs. For each vertex k where an edge $\{i,k\}$ of Γ meets an edge $\{k,j\}$ of Γ' $(i \neq j)$, there is* *a vertex k' so that $\{j, k'\}$ is an edge in Γ and $\{k', i\}$ is an edge in Γ'.*[9]

Proof: In the expansion of $[A, B]$ in Observation 3 by Lemma 11, E_{ij} can occur only with a $+1$ coefficient due to summands of the form $[E_{\{i,k\}}, E_{\{k,j\}}]$ $(k \in V \cap V')$ and only with a -1 coefficient due to summands of the form $[E_{\{j,k'\}}, E_{\{k',i\}}]$ $(k' \in V \cap V')$. Hence the number of such summands of each type must be equal. □

Visual Interpretation. Each pair of such summands in the proof corresponds to a quadrilateral comprised of alternating edges from Γ and Γ'. Note that $k = k'$ is possible if the two edges where the graphs meet occur in both graphs.

Similarly, but more simply than in the directed case, it is easy to give necessary and sufficient conditions for directed graphs to commute:

Theorem 14 (Diamond Condition for Commuting Undirected Graphs). *If Γ and Γ' are undirected graphs, with adjacency matrices A and B respectively, then A and B commute if and only if, for all nodes i and j, the number of two-step paths from i to j consisting of an edge of Γ followed by an edge of Γ' is equal to the number of two-step paths from j to i consisting of an edge of Γ followed by an edge of Γ'.*

Proof: This follows from Theorem 10 by noting that the second number in Theorem 10 is equal, for undirected graphs, to the number of two-step paths from j to i consisting of an edge of Γ followed by an edge of Γ'. □

Note that edges in the Diamond Condition comprise (possibly degenerate) quadrilaterals with edges belonging alternatingly to Γ and Γ'.

C. Examples of Commutative Interconnections

We will now develop a geometrical visualization through the corresponding graphs. We recall the motivational problem of connecting two undirected networks in such a way that their adjacency matrices commute, Eq. (1) in Section I-B.

1) Connecting to n-Cycles: We examine the case where the two networks are both simple cycles with n nodes ($n \geq 3$). In this case $A_1 = A_2$ is the $n \times n$ matrix

$$A_1 = A_2 = \sum_{i=1}^{n} E_{\{i,i+1\}}, \tag{54}$$

where the indices are taken modulo n, so $n + 1 \equiv 1$. The two n-cycles can be visualized as straight-line networks with wrap-around (so-called periodic boundary conditions), where the last node connects back to the first. The two cycles are visualized in red as shown in Figure 1 with an example set of blue interconnections given by $B_0 = I$, the identity matrix.

[9]We call an edge *red* if it belongs to Γ and *blue* if it belongs to Γ'. With that, although one can formulate this proposition in terms of coterminous red-blue and blue-read paths from i to j as in Proposition 9, due to undirectedness we can formulate a quadrilateral condition in terms of a closed red-blue-red-blue loop around the perimeter of the quadrilateral i, k, j, k', which is easier to check visually (see next section). An edge is *purple* if it is both red and blue.

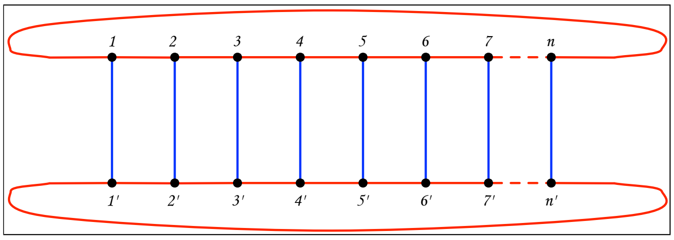

Fig. 1. Two disjoint n-cycle graphs (red) with interconnections (blue), according to the identity matrix I

We already know we can take B_0 to be the zero, identity or all-ones matrices. Regard the above (disconnected, two-component) red graph as Γ. We are interested in finding interconnection graph Γ' with nodes $\{1, 2, \ldots, n, 1', 2', \ldots, n'\}$ and edges of the form $\{i, j'\}$, with i a node of the first cycle and j' a node of the other one.

Blue edges connect each i to i', yielding a graph for which the diamond condition (Theorem 14) obviously holds (Figure 1). We now consider how the resulting graph can be extended minimally with more interconnection edges, while still preserving commutativity. If we want to add $\{1, 2'\}$ to the interconnection edges, the Diamond Condition (Theorem 14) says that we must complete to a quadrilateral the edges $\{1, 2\}$ and $\{1, 2'\}$. We can do this with $\{2, 3'\}$, which together with $\{2', 3'\}$ gives a quadrilateral.[10] The addition of the new edge thus requires another edge. Continuing in this way, we can add all $\{i, (i + 1)'\}$ to Γ'. This yields a regular graph of degree 4 consisting of the two cycles and the interconnections, see Figure 2. By Theorem 14, Γ and Γ' commute, giving us a new example. Here B_0 has entry $b_{ij} = 1$ if and only if $i = j$ or $j \equiv i + 1 \mod n$.

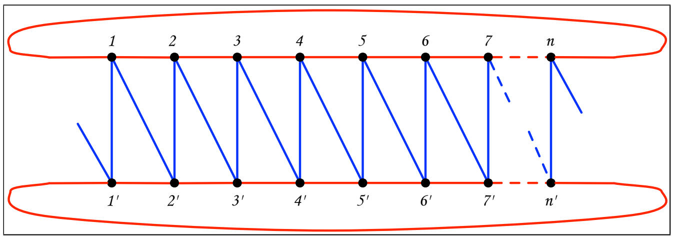

Fig. 2. Example of commuting B_0 interconnection graph with two n-cycle graphs

2) More Examples Connecting Two n-Cycles: Although we constructed the above example using the quadrilateral condition, we can now easily see many similar graphs that will also work (we also get an independent verification of the commutativity of A and the interconnection matrix B just constructed):

Define D_k to be the $n \times n$ matrix with entries $d_{ij}^{(k)} = 1$ if $j = i + k \mod n$ and zero otherwise. D_0 is the identity matrix I and $D_0 + D_1$ is the interconnection matrix B_0 we just constructed above (Figure 2).

[10]Also $\{1, 2'\}$ (blue), $\{2', 3'\}$ (red), $\{3', n\}$ (blue), $\{n, 1\}$ (red) would work, which can be continued to a different solution.

Observation 4. *The graph Γ constructed from the two n-cycles with interconnection matrix D_k is isomorphic to the graph with the two-cycles connected by links $\{i, i'\}$. Note that the latter are encoded by the identity matrix $D_0 = I$. Due to the isomorphism, the diamond condition for the graph with interconnection matrix based on D_k holds since it holds for the graph with interconnection matrix based on $I = D_0$.*

It follows from Proposition 2 that $A_1 D_k = D_k A_1$. Hence we have a commuting interconnection by setting $B_0 = D_k$ for any $1 \le k \le n$. Notice that the interconnections encoded by D_k have no edges in common with those encoded by D_m unless $k = m$. It follows that any sum of distinct D_i's is a zero-one matrix.

Let $S \subseteq \{1, 2, \ldots, n\}$. Let Γ' be the interconnection graph with matrix

$$B_S = \sum_{k \in S} \left[\begin{array}{cc} 0 & D_k \\ D_k^T & 0 \end{array} \right]. \tag{55}$$

Then B_S commutes with A, the matrix of two disjoint n-cycles, since A_1 commutes with each D_k. These yield 2^n examples of commuting interconnection graphs, including our example constructed in this section (taking $S = \{0, 1\}$), the identity ($S = \{0\}$), the zero matrix ($S = \emptyset$), and J, the all-ones matrix for $S = \{0, 1, 2, \ldots, n\}$. Note that each element of S adds one edge at every node to the graph. Thus we have proved the following:

Proposition 15 (Connecting Two n-Cycles via Commutative Interconnection)**.** *Let A be the adjaceny matrix of two disjoint n-cycles ($n \ge 3$) as given above. For any subset S of $\{1, \ldots, n\}$, the interconnection matrix B_S commutes with A. Moreover, including the interconnection links gives a regular graph of degree $|S| + 2$.*

3) Graphs on which Groups Act: We can generalize the results just presented.

Let Δ be any n-node graph with an automorphism group G, which acts[11] *regularly* on Δ: that is, for all $\pi, \pi' \in G$, if $\pi(v) = \pi'(v)$ for some v in the graph then $\pi = \pi'$. Take two disjoint copies of Δ with nodes $\{1, \ldots, n\}$ and $\{1', \ldots, n'\}$, respectively. We can create a commutative interconnection for fixed $\pi \in G$: connect each i in Δ to $\pi(i)'$ in the disjoint copy. For $\pi = e$, the identity automorphism, this is clearly a commutative interconnection. For $\pi \ne e$, the graph with the interconnections constructed according to π is clearly isomorphic to the one with $\pi = e$ since the nodes i' with a prime are simply relabelled by $(\pi(i))'$. Define D_π to be the $n \times n$ matrix of zeros and ones with entries $d_{ij}^{(\pi)} = 1$ if and only if $j = \pi(i)$. If D_π and $D_{\pi'}$ are both 1 at position (i, j), it follows that $j = \pi(i) = \pi'(i)$, whence by regularity $\pi = \pi'$. Interconnecting using the identity matrix yields a graph satisfying the diamond condition, hence so does interconnecting using any $\pi \in G$. And, like before, due to the isomorphism, it follows that $A_1 D_\pi = D_\pi A_1$, where A_1 is the adjacency matrix of Δ.

[11] *G acts on* $\Delta = (V, E)$ means each $\pi \in G$ permutes V, and for all $v_1, v_2 \in V$, $\{v_1, v_2\} \in E$ iff $\{\pi(v_1), \pi(v_2)\} \in E$.

Let B_π be the symmetric interconnection matrix based on D_π. $AB_\pi = B_\pi A$, where A is the adjacency block diagonal matrix based on $A_1 = A_2$. Then for any subset $S \subseteq G$,

$$B_S = \sum_{\pi \in S} B_\pi \tag{56}$$

commutes with the adjacency matrix A of the two disjoint copies of Δ. This is clear since A commutes with each B_π.

V. Visualization, Discussion and Conclusion

Figure 3 shows four examples of commuting 6-node graphs that demonstrate the Diamond Condition (Theorem 14). The purple edges indicate overlap between the two graphs Γ and Γ', and the examples were chosen to show the Diamond Condition for different levels of overlap.

Table I shows a visualization of Lemma 5, as the Lie bracket of the four possible combinations of single-edge directed graphs with one vertex in common. Table II, on the other hand, shows a visualization of Lemma 11. The insights encapsulated in these two lemmas and displayed in these tables were the kernel of the proof, in the following sense. Our initial work focused on reviewing background material and rederiving known results, as shown in Section II. The second phase of the work sought to express the commutativity condition through the Lie bracket for A and B matrices of arbitrary size, as shown in Section III. It was at this point that C. L. Nehaniv started to think about when "elemental" graphs of single edges commute, leading to the development presented in Section IV. The two tables then capture all the possible component interactions for the directed and undirected cases, from which the corresponding Diamond Conditions follow naturally for graphs of any size by linear superposition, matching the "brute force" condition (43).

Acknowledgment

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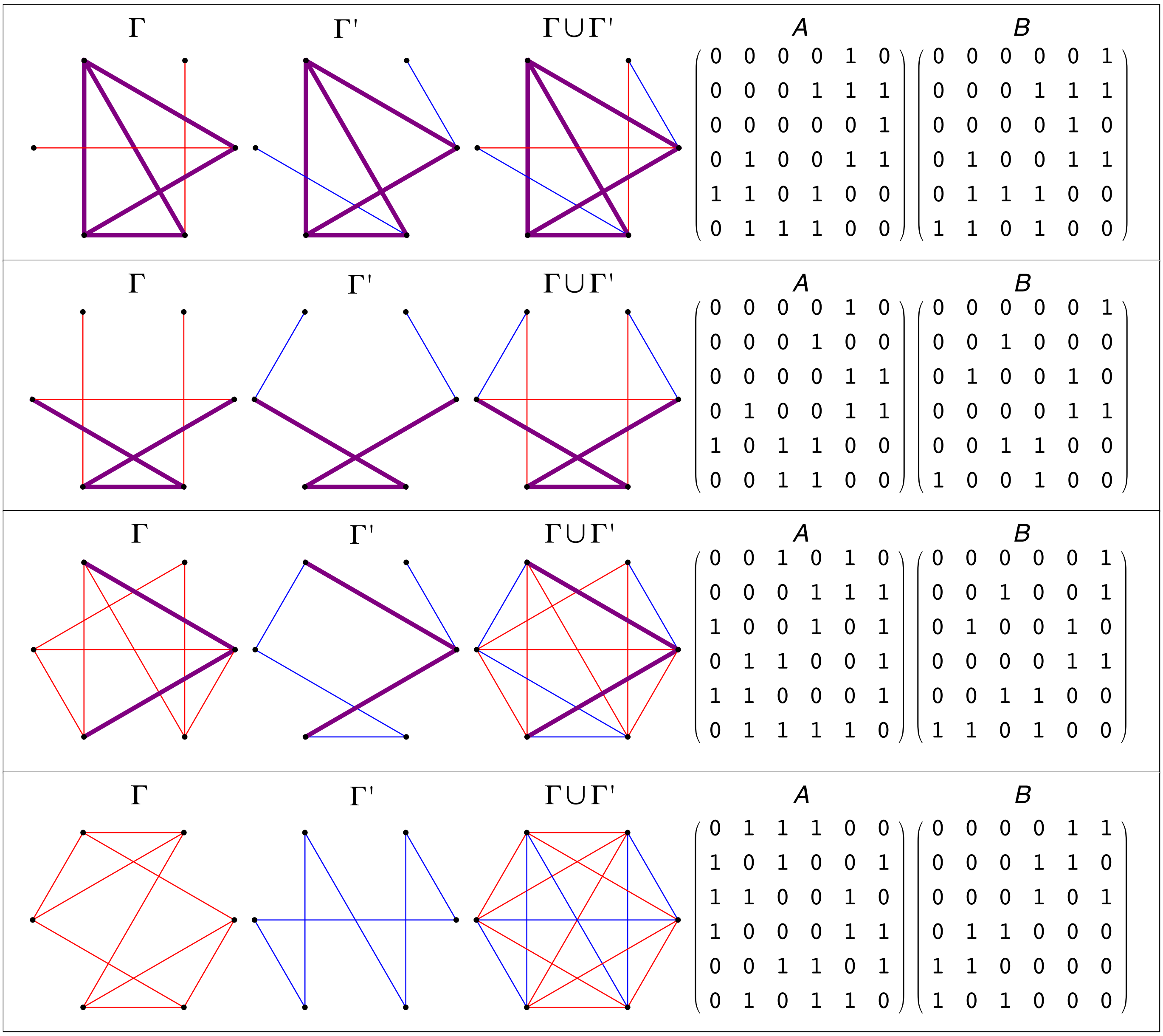

Fig. 3. **Example of Diamond Condition for undirected graphs (Theorem 14) and $n = 6$, verifiable in the $\Gamma \cup \Gamma'$ graph. Purple edges indicate overlap between Γ and Γ'. From the top: large overlap, medium overlap, small overlap, no overlap.**

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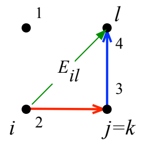
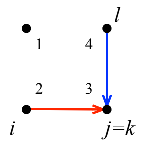
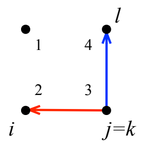
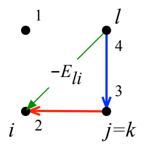
Bracket of Edges	Adjacency Matrix A of Γ	Adjacency Matrix B of Γ'	Lie Bracket	Kronecker δ Notation	Output
	$\begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{bmatrix}$	$[E_{ij}, E_{kl}] = \begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix} - \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$	$\delta_{jk}E_{il} - \cancel{\delta_{li}}E_{kj}$	E_{il}
	$\begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{bmatrix}$	$[E_{ij}, E_{lk}] = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} - \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$	$\cancel{\delta_{jl}}E_{ik} - \cancel{\delta_{ki}}E_{lj}$	0
	$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{bmatrix}$	$[E_{ji}, E_{kl}] = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} - \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$	$\cancel{\delta_{ik}}E_{jl} - \cancel{\delta_{lj}}E_{ki}$	0
	$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{bmatrix}$	$[E_{ji}, E_{lk}] = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} - \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{bmatrix}$	$\cancel{\delta_{il}}E_{jk} - \delta_{kj}E_{li}$	$-E_{li}$

TABLE I

Visualization of Lemma 5 for $n = 4$ example of single-edge digraphs with one common vertex

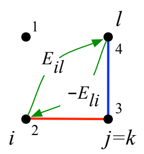
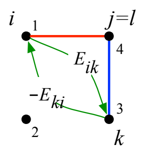
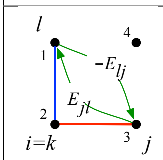
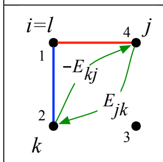
Bracket of Edges	Adjacency Matrix A of Γ	Adjacency Matrix B of Γ'	Lie Bracket $[E_{\{i,j\}}, E_{\{k,l\}}]$	Example-Specific Output	General Expression
	$\begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix} - \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{bmatrix}$	$E_{24} - E_{42}$	$\delta_{jk}(E_{il} - E_{li})$
	$\begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} - \begin{bmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix}$	$E_{13} - E_{31}$	$\delta_{jl}(E_{ik} - E_{ki})$
	$\begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix} - \begin{bmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$	$E_{31} - E_{13}$	$\delta_{ik}(E_{jl} - E_{lj})$
	$\begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{bmatrix} - \begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix}$	$E_{42} - E_{24}$	$\delta_{il}(E_{jk} - E_{kj})$

TABLE II

Visualization of Lemma 11 for $n = 4$ example of single-edge undirected graphs with one common vertex