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Abstract

We give a general method of finding the optimal objective, and solution, values of a Mixed Integer Linear Programme over a Cone (MILPC) as a function of the coefficients (objective, matrix and right-hand side). In order to do this we first convert the matrix of constraint coefficients to a Normal Form (Modified Hermite Normal Form (MHNF)).Then we project out all the variables leaving an (attainable) bound on the optimal objective value. For (M)IPs, including MILPC, projection is more complex, than in the Linear programming (LP) case, yielding the optimal objective value as a *finite* disjunction of inequalities The method can also be interpreted as finding the 'minimal' strengthening of the constraints of the LP relaxation which yeilds an integer solution to the associated LP.

1 Introduction

For an LP the optimal solution can be obtained by solving the set of binding constraints as equations. These constraints give rise to a cone in the space of the structural variables. Our interest, here, is in solving the associated (Mixed) Integer Program over this cone. As is well known this MILPC does not generally solve the MILP associated with the original LP. This is because more constraints will be binding (in the sense of being non-redundant) than those for the original LP.

The solution of a (M)ILPC is an 'easy' problem in comparison with the general MILP problem. It is sometimes known as "the Group Problem" since it can be formulated as an optimisation problem over a finite Abelian Group. It is discussed by, Johnson [3] and in Nemhauser and Wolsey[4]. The convex hulls of the integer points within ILPCs are the Corner Polyhedra described by Gomory et al.[2]. Cuningham Green[1]also gives a method of successively enumerating all feasible solutions to an ILPC. Williams[7] describes how to calculate the *value* function of an MILPC by converting the constraint matrix into a succession of HNFs (Modified(M)HNF), leading to a double recursion. We use this MHNF here.

It is not suggested that the procedure described here is computationally viable for, other than, very small problems. However it is intended to give insight into the structure of the MILPC, and ultimately MILP problems.

We consider a MILP over a (dual feasible, pointed, LP) cone, where the LP solution is unique, in the following form. All coefficients are assumed to be integer.

$$\mathbf{M1} : Min\ z$$

$$-c_1x_1 - c_2x_{2-} ... - c_nx_n \geq -z$$

subject to:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n} \geq b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n} \geq b_2$$

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.

$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n \geq b_n$$

$$x_j \in \mathcal{Z} \quad j \in J_1, x_j \in \mathcal{R} \quad j \in J_2$$

By a succession of elementary integer column operations (represented by postmultiplication by a matrix E)and row interchanges (represented by premultiplication by a matrix T), we convert the matrix into a Modified Hermite Normal Form (MHNF) (see Williams[7]). This transformation is carried out by a succession of transformations of decreasing sub matrices into HNF. Williams[7] uses this to obtain the optimal Chvatal function for an IPC by means of a double recursion.The transformed matrix of coefficients, in MHNF, is:

$$A = \begin{pmatrix} -c_1^{'} \\ a_{11}^{'}, -a_{12}^{'}, ..., 0 \\ -a_{21}^{'}, \ a_{22}^{'}, -a_{23}^{'}, ..., 0 \\ . \\ . \\ . \\ -a_{n-1,1}^{'}, \ -a_{n-1,2}^{'}, ..., a_{n-1,n-1}^{'}, -a_{n-1,n}^{'} \\ -a_{n1}^{'}, \ -a_{n2}^{'}, ..., -a_{n,n-1}^{'}, a_{nn}^{'} \end{pmatrix}$$ where $c_1^{'}, a_{ii}^{'} > 0, a_{ij}^{'} \geq 0$ for $i \neq j$ In order to give this form a uniqueness it is convenient (but not necessary) to stipulate that $a_{ii}^{'} > a_{ij}^{'}$ for all $i \neq j, i < n$.

The model now takes the form:

$$\mathbf{M2} : Min \ z$$

subject to

$$-c_1^{'}x_1^{'} \geq -z$$

$$a_{11}^{'}x_1^{'} - a_{12}^{'}x_2^{'} \geq b_1^{'}$$

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$$-a'_{n-1,1}x'_1 - a'_{n-1,2}x'_2 - ... + a'_{n-1,n-1}x'_{n-1} - a'_{n-1,n}x'_n \geq b'_{n-1}$$

$$-a'_{n1}x'_1 - a'_{n2}x'_2 - ... - a'_{n-1,n-1}x'_{n-1} + a'_{nn}x'_n \geq b'_n$$

$$x'_j \in \mathcal{Z} \quad j \in J_1, x'_j \in \mathcal{R} \quad j \in J_2$$

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In order to eliminate a *real* variable between inequalities we make use of the following theorem given in Williams[8].

Theorem 1 $\exists x_j \ \{a_{ij}x_j \geq f_i \quad i \in I, \ -a_{kj}x_j \geq g_k \ k \in K\} \iff 0 \geq a_{kj}f_i + a_{ij}g_k \quad i \in I, k \in K$ *where* $a_{ij} > 0, i \in I \cup K, x_j \in \mathcal{R}$

Proof.

(i) $\Rightarrow$ *This is obtained by adding each inequality, in the form* $x_j \geq f_i/a_{ij}$ *to each inequality, in the form* $-x_j \geq g_k/a_{kj}$ *respectively to give* $f_i/a_{ij} \leq$ -g_k/a_{kj}, $i \in I, k \in K$ *ie* $0 \geq a_{kj}f_i + a_{ij}g_k \quad i \in I, k \in K$.

(ii)$\Leftarrow$*Suppose* $0 \geq a_{kj}f_i + a_{ij}g_k$ *ie* $-a_{ij}g_k \geq a_{kj}f_i$. *This can expressed as* $-g_k/a_{kj} \geq f_i/a_{ij}$. *Let* $x_j = \max_i\{f_i/a_{ij}\}$ *(or* $\min_k\{-g_k/a_{kj}\}$*). Then* $a_{ij}x_j \geq f_i$ *and* $-a_{kj}x_j \geq g_k \ i \in I, k \in K$ ∎

Note that projecting out continuous variables from an LP polytope results in another polytope in a lower dimension.This is in contrast to the IP case where projection does not generally result in an IP in a lower dimension. In general it results in an optimization over a *finite* disjunction of polytopes in a lower dimension.

In the LP case, the original inequalities imply $f_i/a_{ij} \leq x_j \leq -g_k/a_{kj}$. Since x_j belongs to the continuum of $\mathcal{R}$ both (i) and (ii) of the above proof follow. However if $x_j \in \mathcal{Z}$ then (ii) does not follow. When integer variables are projected out the resultant inequalities must be strengthened beyond those for the LP case. One possible way of strenghthening is provided by the following theorem.

Theorem 2 $\exists x_j \ \{a_{ij}x_j \geq f_i \quad i \in I, \ -a_{kj}x_j \geq g_k \ k \in K\} \iff 0 \geq a_{kj}f_i + a_{ij}g_k + a_{kj}u_i, \ f_i + u_i \equiv 0(\bmod a_{ij}), \ u_i \in \{0, 1, 2, ..., a_{ij} - 1\}, \ i \in I, k \in K$,*where* $a_{ij}, a_{kj} > 0, i \in I \cup K, x_j \in \mathcal{Z}$

Proof.

(i) $\Rightarrow$*We can write the inequalities in the form* $a_{kj}f_i \leq a_{kj}a_{ij}x_j \leq -a_{ij}g_k$ *implying that a multiple of* $a_{kj}a_{ij}$ *lies between the left and rightmost terms. If we apply a non-negative 'correction term'* $a_{kj}u_i$

to the left side we have $a_{kj}f_i + a_{kj}u_i \leq a_{kj}a_{ij}x_j \leq -a_{ij}g_k$ *so long as* , $f_i + u_i \equiv 0(\bmod a_{ij})$.*This implies* $0 \geq a_{kj}f_i + a_{ij}g_k + a_{kj}u_i$.*Whatsmore there is no loss of generality in restricting* u_i *to the domain*$\{0, 1, 2, ..., a_{ij} - 1\}$.*Note that we could alternatively apply (different) correction terms to the right side.*

(ii)$\Leftarrow$*Suppose* $0 \geq a_{kj}f_i + a_{ij}g_k + a_{kj}u_i$ *and* $f_i + u_i \equiv 0(\bmod a_{ij})$ *where* $u_i \in \{0, 1, 2, ..., a_{ij} - 1\}$. *This can expressed as* $-g_k/a_{kj} \geq f_i/a_{ij} + u_i/a_{ij}$. *Let* $x_j = \max_i\{f_i/a_{ij} + u_i/a_{ij}\}$ *which is integral by virtue of the congruence. Then* $a_{ij}x_j \geq f_i$*and* $-a_{kj}x_j \geq g_k$, $i \in I, k \in K$. ■

When we project out an integer variable we, in general, produce congruence relations as well as inequalities. These must be taken account of in the elimination of subsequent variables. Before doing this it is convenient to eliminate the next variable, to be projected out, from all except one of the current set of congruence relations. This may be done by means of the Generalised Chinese Remainder Theorem (GCRT). This result is encapsulated in the following theorem.

Theorem 3 $ex \equiv d_l(\bmod m_l)$ $l \in L \iff ex \equiv \sum_l \lambda_l m_l' d_l(\bmod M)$, $0 \equiv d_l - d_s(\bmod \gcd(m_l, m_s))$ $l, s \in L$ *where* $M = \text{lcm}_l(m_l)$,$m_l m_l' = M$, $l \in L$ *and* $\sum_l \lambda_l m_l' = 1$

Proof. *(i)* $\Longrightarrow$ *The result that there exist* λ_l *such that* $\sum_l \lambda_l m_l' = \gcd_l(m_l') = 1$ *is well known and proved using the Euclidean Algorithm. We do not repeat the proof here. Multiplying each of the original congruences by* $\lambda_l m_l'$ *we obtain* $ex \equiv \sum_l \lambda_l m_l' d_l(\bmod M)$.*Subtracting the congruences in pairs we obtain* $0 \equiv d_l - d_s(\bmod \gcd(m_l, m_s))$.*(ii)*$\Longleftarrow$ *If* $0 \equiv d_l - d_s(\bmod \gcd(m_l, m_s))$ $l, s \in L$ *then* $\sum_l \lambda_l m_l' d_l \equiv d_s \sum_l \lambda_l m_l'(\bmod \gcd_l(\lambda_l M, m_s \sum_l \lambda_l m_l')$ $s \in L$.*Since* $ex \equiv \sum_l \lambda_l m_l' d_l(\bmod M)$ *and* $\sum_l \lambda_l m_l' = 1$ *this implies* $ex \equiv d_s(\bmod m_s)$ $s \in L$. ■

Having aggregated all the congruences, involving the variable to be eliminated, into one congruence (together with congruences involving the other variables) we are in a position to eliminate a variable between a set of inequalities and this congruence. However two cases need to be distinguished, depending on whether the new variable to be eliminated is integer or real. We consider the two cases in the following two theorems.

Theorem 4 $\exists x_j$ $\{a_{ij}x_j \geq f_i \;\; i \in I, -a_{kj}x_j \geq g_k \;\; k \in K \;, ex_j \equiv d(\bmod m)\} \iff 0 \geq a_{kj}f_i + a_{ij}g_k + a_{kj}u_i, 0 \equiv d(\bmod \gcd(e, m)), f_i - \lambda_m a_{ij}d/\gcd(e, m) + u_i \equiv 0(\bmod a_{ij}m/\gcd(e, m))$ *where* $a_{ij}, a_{kj} > 0, i \in I \cup K, x_j \in \mathcal{Z}$, $\lambda_e m/\gcd(e, m) + \lambda_m e/\gcd(e, m) = 1$ *and* $u_i \in \{0, 1, 2, ..., a_{ij}m/\gcd(e, m) - 1\}$

Proof. *(i)* $\Rightarrow$*We can write the inequalities in the form* $a_{kj}ef_i \leq a_{ij}a_{kj}ex_j \leq -a_{ij}eg_k$. *From the congruence,*$a_{ij}a_{kj}ex_j \equiv a_{ij}a_{kj}d(\bmod\ a_{ij}a_{kj}m)$.*Let* $y = a_{ij}a_{kj}ex_j$.*Then* $y \equiv 0(\bmod a_{ij}a_{kj}e)$ *and* $y \equiv a_{ij}a_{kj}d(\bmod a_{ij}a_{kj}m)$. *Applying the GCRT gives* $0 \equiv d(\bmod \gcd(e, m))$ *and* $y \equiv \lambda_m a_{ij}a_{kj}ed/\gcd(e, m)$ $\bmod(a_{ij}a_{kj}\,\text{lcm}(e, m))$. *Therefore* $a_{kj}ef_i - \lambda_m a_{ij}a_{kj}ed/\gcd(e, m) \leq$*a multiple of* $a_{ij}a_{kj}\,\text{lcm}(e, m) \leq -a_{ij}eg_k - \lambda_m a_{ij}a_{kj}ed/\gcd(e, m)$. *Since* (e, m) *divides* d, *by the congruence, the leftmost expression, in the above inequality, is a multiple of* $a_{kj}e$. *Hence we can apply a non-negative 'correction term'* $a_{kj}eu_i$ *to the left side giving* $a_{kj}ef_i - \lambda_m a_{ij}a_{kj}ed/\gcd(e, m) + a_{kj}eu_i \equiv 0(\bmod a_{ij}a_{kj}\,\text{lcm}(e, m))$. *ie* $f_i - \lambda_m a_{ij}d/\gcd(e, m) + u_i \equiv 0(\bmod a_{ij}m/\gcd(e, m))$. u_i *can be restricted to the domain*$\{0, 1, 2, ..., a_{ij}m/\gcd(e, m) - 1\}$.*The resultant inequalities are* $0 \geq a_{kj}f_i + a_{ij}g_k + a_{kj}u_i$.*(ii)*$\Leftarrow$*Suppose* $0 \geq a_{kj}f_i + a_{ij}g_k + a_{kj}u_i, 0 \equiv$

$d(\bmod \gcd(e,m))$, $f_i - \lambda_m a_{ij} d/\gcd(e,m) + u_i \equiv 0(\bmod a_{ij}m/\gcd(e,m))$,where $a_{ij}, a_{kj} > 0, i \in I \cup K$, and $u_i \in \{0,1,2,...,a_{ij}m/\gcd(e,m) - 1\}$ The inequality can be expressed as $-a_{ij}g_k \geq a_{kj}f_i + a_{kj}u_i$.But $a_{kj}f_i + a_{kj}u_i \equiv \lambda_m a_{ij}a_{kj}d/\gcd(e,m)$ $(\bmod\ a_{ij}a_{kj}m/\gcd(e,m))$ by the second congruence. Let $a_{ij}a_{kj}x_j = \max_i\{a_{kj}f_i + a_{kj}u_i\}$ giving $x_j \in \mathcal{Z}$. Then $a_{ij}x_j \geq f_i$ and $-a_{kj}x_j \geq g_k, i \in I, k \in K$. Also $\exists i$ such that $a_{ij}x_j = f_i + u_i$. Combining this with the above congruence gives $x_j \equiv \lambda_m d/\gcd(e,m)$ $(\bmod m/\gcd(e,m))$ ie $x_j \equiv (1-\lambda_e m)d/\gcd(e,m)$ $(\bmod m/\gcd(e,m))$. This implies $ex_j \equiv d(\bmod m)\}$.
■

Theorem 5 *$\exists x_j$ $\{a_{ij}x_j \geq f_i \ \ i \in I,\ -a_{kj}x_j \geq g_k\ k \in K\ , ex_j \equiv d(\bmod m)\} \iff 0 \geq a_{kj}ef_i + a_{ij}eg_k + a_{kj}eu_i, ef_i - a_{ij}d + eu_i \equiv 0(\bmod a_{ij}m)\ i \in I, k \in K$*

where $a_{ij}, a_{kj} > 0, i \in I \cup K, x_j \in \mathcal{R}$ and $u_i \in [0,1,2,...,a_{ij}m/e)$.

Proof. *(i) $\Rightarrow$We can write the inequalities in the form $ef_i/a_{ij} \leq ex_j \leq -eg_k/a_{kj}$ implying that $ef_i/a_{ij} - d \leq ex_j - d \leq -eg_k/a_{kj} - d$ ie a multiple of m lies between the left and rightmost expressions. We apply a non-negative 'correction term' to the left side. This correction term is from the continuum of the rationals, so may be scaled. To maintain correspondence with Theorem 4 it is convenient to denote it by eu_i/a_{ij} giving $ef_i/a_{ij} - d + eu_i/a_{ij} \equiv (\bmod m)$. ie $ef_i - a_{ij}d + eu_i \equiv (\bmod a_{ij}m)$. u_i can be restricted to the interval $[0,1,2,...,a_{ij}m/e)$.The resultant inequalities are $0 \geq a_{kj}ef_i + a_{ij}eg_k + a_{kj}eu_i$.(ii)$\Leftarrow$Suppose $0 \geq a_{kj}ef_i + a_{ij}eg_k + a_{kj}eu_i$ and $ef_i - a_{ij}d + eu_i \equiv 0(\bmod a_{ij}m)$ where $u_i \in [0,1,2,...,a_{ij}m/e)$. The inequalities expressed as $-eg_{kj}/a_{kj} \geq ef_i/a_{ij} + eu_i/a_{ij}$. Let $ex_j = \max_i\{ef_i/a_{ij} + eu_i/a_{ij}\}$ ie Then $a_{ij}x_j \geq f_i$ and $-a_{kj}x_j \geq g_k$ $, i \in I, k \in K$. Also $\exists i$ such that $ex_j = ef_i/a_{ij} + eu_i/a_{ij}$ ie $ef_i + eu_i = a_{ij}ex_j$.Combining this with the above congruence gives $ex_j \equiv d(\bmod m)$.* ■

Note that, in the MIP case, we generate inequalities corresponding to those in the LP case, but augmented by correction terms (defined over finite integer domains or finite intervals) which are subject to a series of congruence relations. Also note that there will be a number of alternate correction terms and representations of the congruence relations.

Note also that, in the MILP case, we generate inequalities corresponding to those in the LP case, but augmented by correction terms (defined over finite integer domains) which are subject to congruence relations. Also note that these correction terms are not the same as the surplus variables (* *am I absolutely sure this is not the case?**)

The full solution of a MILP by projection forms the subject of another paper. It is also discussed by Williams[6]. However projection of a MILPC is considerably simpler and forms the subject of this paper.

We will restrict ourselves to considering only models **M2** (equivalent to **M1**) where the cone is pointed, dual feasible and gives a unique optimal LP solution. When solving the LP relaxation of **M2** (equivalent to **M1**) the inequalities would be treated as equations and the system solved as such to give the minimum value of z. However when solving a MILPC (and a MILP) we need to introduce correction terms and congruences, as in the theorems above. Observe that in **M2** each variable has only one positive coefficient in the inequalites. Therefore this inequality is combined once with each of the other inequalities (so long as they have a non-zero coefficient). After each elimination there remains exactly one inequality with a positive coefficient for each variable. Each elimination of an integer variable, as the result of theorem 2, also produces one congruence relation and correction term, derived from the inequality with a positive coefficient, for the variable to be eliminated. After all x_j variables have been eliminated we have the result of the following theorem.

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Theorem 6 *The optimal solution to* **M2** *is given by*

M3: *Min z*

such that

$$\begin{vmatrix} a'_{11}, -a'_{12}, \;...., 0 \\ -a'_{21}, a'_{22}, -a'_{23}, ..., 0 \\ . \\ . \\ . \\ -a'_{n1}, -a'_{n2}, ..., a'_{n,n} \end{vmatrix} z \geq -c_1(-1)^n \begin{vmatrix} -a'_{12}, 0, \;...., 0, (b'_1 + u_1) \\ a'_{22}, -a'_{23}, 0, ..., 0, (b'_2 + u_2) \\ . \\ . \\ . \\ -a'_{n2}, -a'_{n3}, ..., a'_{nn}, (b'_n + u_n) \end{vmatrix} ....(1)$$

$$b'_1 + u_1 \equiv 0 \bmod (a'_{11,} a'_{12}) \;\; \textit{if } x'_1 \, \epsilon \, \mathcal{Z} \;\; \textit{where}$$

(p, q) *represents* $\gcd(p.q)$.

$$\begin{vmatrix} \lambda_1, \mu_1, 0, \\ a'_{11}, -a'_{12}, (b'_1 + u_1) \\ -a'_{21}, a'_{22}, (b'_2 + u_2) \end{vmatrix} \equiv 0 \bmod (\begin{vmatrix} a'_{11,}, -a'_{12} \\ -a'_{21}, a'_{22} \end{vmatrix}, a'_{23}(a'_{11}, a'_{12})) \;\; \textit{if } x'_2 \, \epsilon \, \mathcal{Z}$$

$$\textit{where } \mu_1 a'_{11} + \lambda_1 a'_{12} = (a'_{11}, a'_{12})$$

$$\begin{vmatrix} 0, \lambda_2, \mu_2, 0 \\ a'_{11}, -a'_{12}, 0, (b'_1 + u_1) \\ -a'_{21}, a'_{22}, -a'_{23}, (b'_2 + u_2) \\ -a'_{31}, -a'_{32}, a'_{33}, (b'_3 + u_3) \end{vmatrix} \equiv 0 \bmod (\begin{vmatrix} a'_{11,}, -a'_{12,} 0 \\ -a'_{21}, a'_{22}, -a'_{23} \\ -a'_{31}, -a_{32}, a_{33} \end{vmatrix}, a'_{34} \begin{vmatrix} a'_{11,}, -a'_{12} \\ -a'_{21}, a'_{22} \end{vmatrix}) \textit{if } x'_3 \, \epsilon \, \mathcal{Z}$$

$$\textit{where } \mu_2 \begin{vmatrix} a'_{11,}, -a'_{12} \\ -a'_{21}, a'_{22} \end{vmatrix} + \lambda_2 a'_{11} a'_{23} = (\begin{vmatrix} a'_{11,}, -a'_{12} \\ -a'_{21}, a''_{22} \end{vmatrix}, a'_{11} a'_{23})$$

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$$\begin{vmatrix} 0, ..., 0, \lambda_{n-1}, \mu_{n-1}, 0 \\ a'_{11}, -a'_{12}, 0, ..., 0, (b'_1 + u_1) \\ -a'_{21}, a'_{22}, -a'_{23}, ..., 0, (b'_2 + u_2) \\ . \\ . \\ . \\ -a'_{n1}, -a'_{n2}, ..., -a'_{nn-1}, (b'_n + u_n) \end{vmatrix} \equiv 0 \bmod (\begin{vmatrix} a'_{11,}, -a'_{12,} 0, ..., 0 \\ -a'_{21}, a'_{22}, 0, ..., 0 \\ -a'_{31}, -a'_{32}, 0, ..., 0 \\ . \\ . \\ . \\ -a'_{n1}, -a'_{n2}, ..., -a'_{nn-1}, a'_{nn} \end{vmatrix}, a'_{n-1,n} \begin{vmatrix} a'_{11,}, -a'_{12,} 0, ..., 0 \\ -a'_{21}, a'_{22}, 0, ..., 0 \\ -a'_{31}, -a'_{32}, 0, ..., 0 \\ . \\ . \\ . \\ -a'_{n-1,1}, -a'_{n-1,2}, ..., a'_{n-1n-1} \end{vmatrix})$$

$$u_i \,\epsilon\, \{0,1,...,\begin{vmatrix} a'_{11},,-a'_{12,}0,...,0 \\ -a'_{21},a'_{22},0,...,0 \\ -a'_{31},-a'_{32},0,...,0 \\ . \\ . \\ . \\ -a'_{i1},-a'_{i2},...,-a'_{i,n-1},a'_{ii} \end{vmatrix} / \begin{vmatrix} a'_{11},,-a'_{12,}0,...,0 \\ -a'_{21},a'_{22},0,...,0 \\ -a'_{31},-a'_{32},0,...,0 \\ . \\ . \\ . \\ -a'_{i-1,1},-a'_{i-1,2},...,-a'_{i-1,n-1},a'_{i-1,i-1} \end{vmatrix} -$$

$1\}$ *if* $x'_i \,\epsilon\, \mathcal{Z}$. $u_i = 0$ *if* $x_i \in \mathcal{R}$

Note that whereas **M2** *(and* **M1***) had an infinite number of integer solution we are now, in* **M3**, *looking for the optimum over a finite number of values of the* u_i.

To prove this theorem we successively eliminate the variables using theorems 2, to 5. .For convenience we repeat **M2**.

$$\mathbf{M2} : Min\ z$$

subject to

$$-c'_1 x'_1 \geq -z$$

$$a'_{11}x'_1 - a'_{12}x'_2 \geq b'_1$$

$$. \\ . \\ .$$

$$-a'_{n-1,1}x'_1 - a'_{n-1,2}x'_2 - ... + a'_{n-1,n-1}x'_{n-1} - a'_{n-1,n}x'_n \geq b'_{n-1}$$

$$-a'_{n1}x'_1 - a'_{n2}x'_2 - ... - a'_{n-1,n-1}x'_{n-1} + a'_{nn}x'_n \geq b'_n$$

$$x'_j \in \mathcal{Z} \quad j \in J_1, x'_j \in \mathcal{R} \quad j \in J_2$$

We eliminate x'_1 using theorem 2. This results in:

$$\mathbf{M2'(n-1)} : Min\ z$$

subject to

$$-c'_1 a'_{12} x'_2 \geq -a'_{11} z + c'_1 (b''_1 + u_1)$$

$$\begin{vmatrix} a'_{11},,-a'_{12} \\ -a'_{21},a'_{22} \end{vmatrix} x'_2 - a''_{11} a'_{23} x'_3 \geq \begin{vmatrix} a'_{11},,(b'_1+u_1) \\ -a'_{21},b'_2 \end{vmatrix}$$

$$\begin{vmatrix} a'_{11}, -a'_{12} \\ -a'_{31}, -a'_{32} \end{vmatrix} x'_2 + a'_{11}a'_{33}x'_3 - a'_{11}a'_{34}x'_3 \geq \begin{vmatrix} a'_{11}, (b'_1 + u_1) \\ -a'_{31}, b'_2 \end{vmatrix}$$

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$$\begin{vmatrix} a'_{11}, -a'_{12} \\ -a'_{n+1,1}, -a'_{n+1,2} \end{vmatrix} x'_2 - a'_{11}a'_{n+1,3}x'_3 - ... + a'_{11}a'_{n+1,n+1}x'_{n+1} \geq \begin{vmatrix} a'_{11}, (b'_1 + u_1) \\ -a'_{n+1,1}, b'_2 \end{vmatrix}$$

$$a'_{12}x'_2 \equiv -(b'_1 + u_1) \bmod(a'_{11})$$

$$u_1 \epsilon \{1, 2, ..., a'_{11} - 1\}$$

$$x'_j \in \mathcal{Z} \quad j \in J_1, x'_j \in \mathcal{R} \quad j \in J_2$$

Note that the determinants $\begin{vmatrix} a'_{11}, -a'_{12} \\ -a'_{i1}, -a'_{i2} \end{vmatrix}$ for $i \geq 3$ are all ≤ 0. Hence $\mathbf{M2}'(\mathbf{n})$, with variables $x'_2, ..., x'_{n+1}$,takes the same form as **M2**, where, in the inequalities, exactly one of the coefficients of x'_2 has a positive coefficient. However x'_2 is also included in a congruence. Using theorem 4 we associate the next correction term with this inequality. It can be seen that this correction term must be a multiple of a'_{11}. Therefore the correction term can be taken as $a'_{11}u_2$.The result of eliminating x'_2 is then:

$$\mathbf{M2}'(\mathbf{n-2}) : Min\ z$$

subject to

$$-c'_1a'_{12}a'_{23}x'_3 \geq -\begin{vmatrix} a'_{11}, -a'_{12} \\ -a'_{21}, a'_{22} \end{vmatrix} z - c_1 \begin{vmatrix} -a'_{12}, (b'_1 + u_1) \\ a'_{22}, (b'_2 + u_2) \end{vmatrix}$$

$$\begin{vmatrix} a'_{11}, -a'_{12}, 0 \\ -a'_{21}, a'_{22}, -a'_{23} \\ -a'_{31}, -a'_{32}, a'_{33} \end{vmatrix} x'_3 - a'_{34} \begin{vmatrix} a'_{11}, -a'_{12} \\ -a'_{21}, a'_{22} \end{vmatrix} x'_4 \geq \begin{vmatrix} a'_{11}, -a'_{12}, (b'_1 + u_1) \\ -a'_{21}, a'_{22}, (b'_2 + u_2) \\ -a'_{31}, -a'_{32}, b'_3 \end{vmatrix}$$

$$\begin{vmatrix} a'_{11}, -a'_{12}, 0 \\ -a'_{21}, a'_{22}, -a'_{23} \\ -a'_{41}, -a'_{42}, -a'_{43} \end{vmatrix} x'_3 + a'_{44} \begin{vmatrix} a'_{11}, -a'_{12} \\ -a'_{21}, a'_{22} \end{vmatrix} x'_4 - a'_{55} \begin{vmatrix} a'_{11}, -a'_{12} \\ -a'_{21}, a'_{22} \end{vmatrix} x'_5 \geq \begin{vmatrix} a'_{11}, -a'_{12}, (b'_1 + u_1) \\ -a'_{21}, a'_{22}, (b'_2 + u_2) \\ -a'_{41}, -a'_{42}, b'_3 \end{vmatrix}$$

.
.
.

$$\begin{vmatrix} a'_{11}, -a'_{12}, 0 \\ -a'_{21}, a'_{22}, -a'_{23} \\ -a'_{n1}, -a'_{n2}, -a'_{n3} \end{vmatrix} x'_3 - a'_{n+1,3} \begin{vmatrix} a'_{11}, -a'_{12} \\ -a'_{21}, a'_{22} \end{vmatrix} x'_3 - ... + a'_{n+1,n+1} \begin{vmatrix} a'_{11}, -a'_{12} \\ -a'_{21}, a'_{22} \end{vmatrix} x'_{n+1} \geq \begin{vmatrix} a'_{11}, -a'_{12}, (b'_1 + u_1) \\ -a'_{21}, a'_{22}, (b'_2 + u_2) \\ -a'_{n1}, -a'_{n2}, b'_3 \end{vmatrix}$$

$$(b'_1 + u_1) \equiv 0 \bmod (a'_{11}, a'_{12})$$
$$u_1 \epsilon \{1, 2, ..., a'_{11} - 1\}$$

$$(a'_{21}(a'_{11}, a'_{12}) + \lambda_1 \begin{vmatrix} a'_{11}, -a'_{12} \\ -a'_{21}, a'_{22} \end{vmatrix})(b'_1 + u_1) + a'_{11}(a'_{11}, a'_{12})(b'_2 + u_2) + a'_{11}a'_{23}(a'_{11}, a'_{12})x'_3 \equiv 0 \bmod (a'_{11} \begin{vmatrix} a'_{11}, -a'_{12} \\ -a'_{21}, a'_{22} \end{vmatrix})$$

$$u_2 \epsilon \{1, 2, ..., \begin{vmatrix} a''_{11}, -a'_{12} \\ -a'_{21}, a'_{22} \end{vmatrix} / (a'_{11}, a'_{12}) - 1\}$$

$$\text{where } \mu_1 a'_{11} + \lambda_1 a'_{12} = (a'_{11}, a'_{12})$$

The second congruence can be simplified to:

$$\begin{vmatrix} \lambda_1, \mu_1, 0, \\ a'_{11}, -a'_{12}, (b'_1 + u_1) \\ -a'_{21}, a'_{22}, -a'_{23}, (b'_2 + u_2) \end{vmatrix} + a'_{23}(a'_{11}, a'_{12})x'_3 \equiv 0 \bmod (\begin{vmatrix} a'_{11}, -a'_{12} \\ -a'_{21}, a'_{22} \end{vmatrix})$$

$$x'_j \in \mathcal{Z} \quad j \in J_1, x'_j \in \mathcal{R} \quad j \in J_2$$

The model now takes the same form as $\mathbf{M2}'(\mathbf{n-1})$ (by renaming variables and coefficients). The proof follows by induction on n.

It is worth pointing out that the inequality in **M3** could also be written using the untransformed coefficient matrix given in, **M1**, taking account of row interchanges ie:

$$\begin{vmatrix} a_{11}, a_{12}, ..., a_{1n} \\ a_{21}, a_{22}, ..., a_{2n}. \\ . \\ . \\ . \\ a_{n1}, a_{n2}, ..., a_{nn} \end{vmatrix} z \geq \begin{vmatrix} -c_1, -c_2, ..., -c_n, 0 \\ a_{11}, ..., a_{1n}, (b_1 + u''_1) \\ a_{21}, ..., a_{2n}, (b_2 + u''_2) \\ . \\ . \\ . \\ a_{n1}, ..., a_{nn}, (b_n + u''_n) \end{vmatrix}$$

where u''_i is the correction term associated with the row of the transformed matrix corresponding to the original row with RHS coefficient b_i.

Once the optimal values of the u_i have been found from **M3** the optimal solution (in terms of the original variables) is given by:

$$x_j = \begin{vmatrix} a_{11}, a_{12}, ..., a_{1,,j-1}, b_1 + u_1, a_{1,,j+1}, ..., a_{1,n} \\ a_{21}, a_{22}, ..., a_{2,j-1}, b_2 + u_2, a_{2,,j+1}, ..., a_{2,n} \\ . \\ . \\ . \\ a_{n1}, a_{n2}..., a_{n,j-1}, b_2 + u_n, a_{n,,j+1}, ..., a_{n,n} \end{vmatrix} / \begin{vmatrix} a_{11}, a_{12}, ..., a_{1,n} \\ a_{21}, a_{22}, ..., a_{2,n} \\ . \\ . \\ . \\ a_{n1}, a_{n2}, ..., a_{n,n} \end{vmatrix} \quad j \;\epsilon\; J$$

2 A Numerical Example

We consider the model

$$
\begin{aligned}
& \quad Minimize - x_1 + x_2 + x_3 \\
8x_1 - 12x_2 - 2x_3 &\geq 3 \\
4x_1 - x_2 - 2x_3 &\geq 2 \\
-12x_1 + 4x_2 + 6x_3 &\geq 13
\end{aligned}
$$

In MHNF **M3** becomes

$$
\begin{aligned}
-x_1^{'} &\geq -z \\
6x_1^{'} - 2x_2^{'} &\geq 13 \\
-2x_1^{'} + x_2^{'} - x_3^{'} &\geq 2 \\
-2x_1^{'} - 10x_2^{'} + 36x_3^{'} &\geq 3
\end{aligned}
$$

Where $A = \begin{bmatrix} 1,-1,-1 \\ 8,-12,-2 \\ 4,-1,-2 \\ -12,4,6 \end{bmatrix}$ $E = \begin{bmatrix} 0,0,1 \\ 0,1,-3 \\ 1,-1,4 \end{bmatrix}$ $T = \begin{bmatrix} 0,0,1 \\ 0,1,0 \\ 1,0,0 \end{bmatrix}$ $T\,AE = A^{/} = \begin{bmatrix} -1,0,0 \\ 6,-2,0 \\ -2,1,-1 \\ -2,-10,36 \end{bmatrix}$
$T \begin{bmatrix} 3 \\ 2 \\ 13 \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \\ 3 \end{bmatrix}$

Applying **M3** we have:

$$8z \geq 488 + 26u_3 + 72u_2 + 2u_3$$

subject to:

$$
\begin{aligned}
u_1 &\equiv 1(\mathrm{mod}\,2) \\
36u_1 + 3u_3 &\equiv 0(\mathrm{mod}\,46) \\
& \quad u_1 \epsilon \{0,1\}, u_2 \epsilon \{0\}, u_3 \epsilon \{0,1,3,4\}
\end{aligned}
$$

The optimal solution is $u_1 = 1,\ u_2 = 0,\ u_3 = 3$ resulting in $z = 65, x_1 = 56, x_2 = 20, x_3 = 101$.(The optimal solution to the LP relaxation results from setting $u_1 = 0,\ u_2 = 0,\ u_3 = 0$, giving $z = 61$).

3 A Geometrical Interpretation

Initially the model **M3** is viewed in (x_1, x_2) space. Correction terms u_1 are applied to give possible strengthenings to the first constraint as illustrated in figure 1 by the sublattice of feint lines.

Then, when x_1 is projected out, we view the model in (x_2, x_3) space in figure 2. Projection has implied a congruence involving x_2 and u_1 ,represented by the sublattice of horizontal feint lines.An extra correction term u_2 is introduced to give possible strengthenings to the new first constraint. These strengthenings give rise to the sublattice represented by the slanted feint lines. The intersection of the two sublattices, created to date, is represented (in (x_2, x_3) space) by the bold dots. In this way the lattice of integer solutions is bult up following successive projections of the model into lower dimensions.

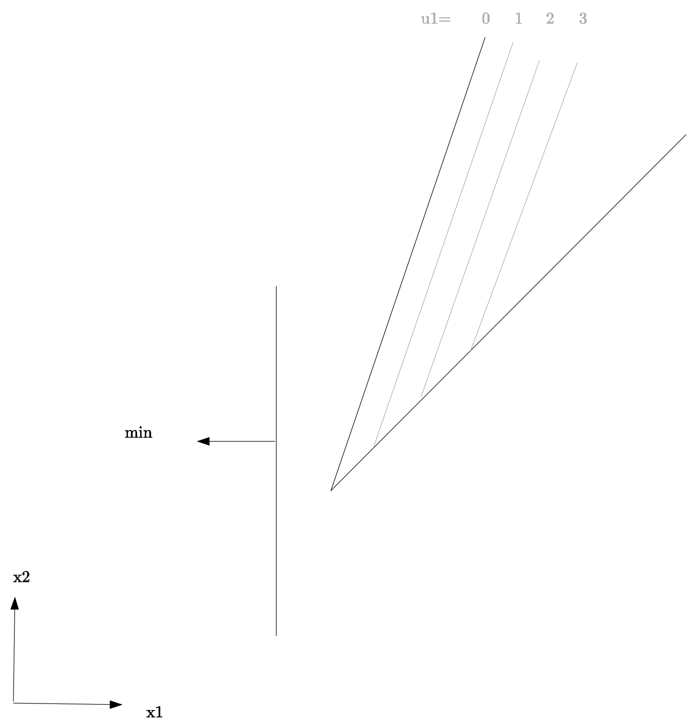

Figure 1

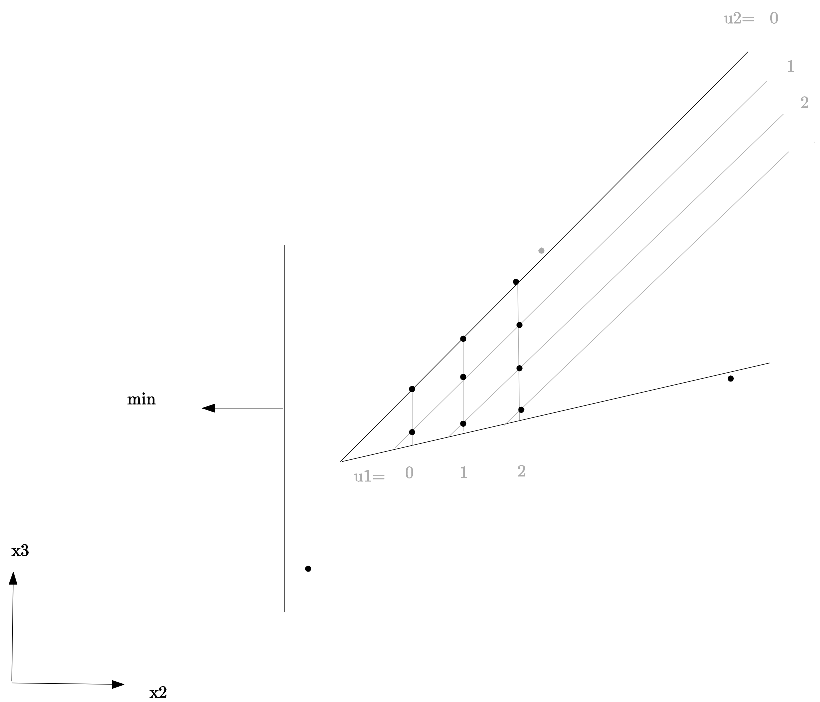

Figure 2

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