**Mireia Jofre-Bonet and Martin Pesendorfer**

# Optimal Sequential Auctions

## Working paper

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Mireia Jofre-Bonet† Martin Pesendorfer‡

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**Abstract**

Sequential sealed first-price and open ascending bid auctions are studied. We examine which auction rule achieves the low procurement cost. We show that the answer to this policy question depends on whether the items are complements or substitutes. With substitutes, the first-price auction is preferred, while with complementarities, the open ascending bid auction is preferred. We also illustrate the procurement cost minimizing auction and the auction rule preferred by the bidders. With substitutes, bidders prefer the open ascending bid auction, while with complements bidders prefer the first-price auction.

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†Department of Economics, City University, London, U.K. and LSHTM. Email: m.jofre-bonet@lse.ac.uk

‡Department of Economics, London School of Economics, London WC2A 2AE, U.K., CEPR and NBER. Email: m.pesendorfer@lse.ac.uk

An important result in the theoretical auction literature concerns the revenue equivalence of first-price and open ascending bid auctions, see Vickrey [1961], Myerson [1981] and Riley and Samuelson [1981]. A number of papers have studied the robustness of the equivalence result to departures from the basic assumptions including risk-aversion, see Riley and Samuelson [1981], Matthews [1983], Maskin and Riley [1984], budget constraints, see Che and Gale [1998], positively correlated valuations, see Milgrom and Weber [1982], and bidder asymmetry, see Maskin and Riley [2000a]. We relax the assumption of a single period auction and study sequential auctions. We take into account that winning an item may affect the winning bidder's values in the next auction.

A sequential auction game is a selling mechanism commonly used when a seller has a number of related items for sale. Typically, an individual item is allocated to a bidder at each round by means of either a sealed bid first-price or an open English auction. Usually the same auction format is used for early and late items, and there is no change in the auction format over time. As the auction proceeds sequentially, a bidder's valuation for an additional item may depend on the number of items acquired so far. Substitutes arise if the value of an additional item falls in the number of acquired items, while complements arise if the value increases in the number of acquired items. This paper explores the relationship between substitutes and complements, and the choice of auction format both from the bidders' and the auctioneer's point of view.

Substitutability is pervasive in a number of settings including sequential real-estate auctions, sequential eBay auctions for used durables, and livestock auctions. What these auctions have in common is that the incremental value of owning a second unit is lower than it was for the first. A private house buyer is interested in the purchase of a single house only, an eBay bidder may wish to buy a single durable good. Similarly, a farmer that wishes to purchase one bull for breeding will value a second one much less than the first[1]. Substitutability also arises in sequential procurement contracting when the technology exhibits decreasing returns to scale: The cost of the marginal contract is higher when the bidder is already committed to a previously won and uncompleted contract than when the bidder is uncommitted.

Complementarity arises when the value of an additional item increases with the number of items acquired so far: A complete cycle of paintings or a complete china placesetting may have a higher value than the sum of the individual item values. Complementarity may arise for procurement contracts when there are learning-by-doing effects or experience effects. Additionally, if an up-front investment is required to undertake a project, then this may induce complementarities: The first period winner has already sunk the investment so that she is more competitive in the second period auction.

Empirical studies documenting the importance of substitutes and complements in sequential auctions are abound: Substitutes are found in industries in which bidders' capacity is limited, as shown in recent papers such as Jofre-Bonet and Pesendorfer [2003] for sequential highway-paving procurement auctions; and List, Millimet and Price [2004] for sequential timber auctions. Jofre-Bonet and Pesendorfer [2003] devise an empirical technique to measure consistently the effect of substitutes and show that

[1] Bidder behavior at sequential cattle auctions is nicely described in Zulehner [2002].

the bid mark-up increase due to the existence of the substitution effect is substantial. Zulehner [2002] describes a negative correlation between the initial and the subsequent bids by the same bidder in sequential cattle auctions.

Wolfram [1998] documents that bids at sequential electricity auctions include a startup price and a no-load price, which enables bidders to indicate complementarities in electricity generation between adjacent time periods. Anton and Yao [1987] show that complementarities arise in sequential competition for defense contracts as the incumbent firm may achieve a higher experience level and thus a lower cost. Gandal [1997] documents complementarities in sequential cable television license auctions. Incumbency advantage in sequential procurement auctions for school milk contracts may arise due to sunk investments by diaries, see Pesendorfer [2000]. There are also complementarities between adjacent school milk contracts, see Marshall, Raiff, Richard and Schulenberg [2002].

Motivated by the empirical studies, we consider a buyer's procurement auction model in which bidders (sellers) have private information about their costs. We consider a two period procurement auction game in which every period a single contract is offered for sale. There are two bidders who become privately informed about their contract costs at the beginning of each period. We assume that the identity of the winner of the first auction is publicly observed before the second auction starts, and we consider situations in which winning the first contract may affect the distribution of the winning bidder's costs at the next auction. We shall say that the items are substitutes if at the second auction the first period winning bidder has on average a higher cost than a losing bidder, and the items are complements if instead the first period winning bidder has on average a lower cost than a losing bidder. The asymmetry in the second period arises endogenously as it depends on the first period's auction outcome. We study the payoff and procurement cost ranking of sealed first-price and open ascending bid or second-price auctions.

As most of the empirical auction evidence on substitutes and complements arises in procurement auctions, in this paper we state our results in terms of a buyer's procurement auction. An alternative model formulation exists for a seller's auction in which bidders have private information about their willingness to pay and the seller awards the item to the high bidder. This alternative model formulation has the same mathematical structure than the one we have chosen, and therefore, our subsequent results can be restated in terms of a seller's auction with the appropriate changes in place.

There is little prior work on the relationship between optimal sequential auctions and the substitutes or complements property of the items auctioned. Jeitschko and Wolfstetter [2002] consider a binary-valuations example and show that the English auction extracts (weakly) more rent than the first-price auction in a static asymmetric auction and also in a sequential auction with complements (or substitutes). Our model set-up differs from this paper as we consider the class of continuous valuation distributions. We obtain more general results, and some of them are different.[2] Most of the theoretical literature on sequential auctions has focused on the martingale property of sequential auction prices and deviations thereof, see Weber [1983]. Empirical

[2] If valuations are drawn from a continuous distribution function instead of a binary valuation distribution, then there is no clear preference for the English auction in static asymmetric auctions as is shown in Maskin and Riley [2000].

evidence on declining prices is documented in Ashenfelter [1989] for wine auctions. McAfee and Vincent [1993] explain declining prices with a model in which items are perfect substitutes, each bidder acquires at most one item, and bidders are risk averse. Carolyn Pitchik and Schotter [1988] study the effect of bidder budget-constraint on the second period auction outcome. Benoit and Krishna [2001] study whether it is better to sell the more valuable item first or second when bidders face budget constraints and information is complete. Branco [1998] shows that with complements auction prices decline.

Simultaneous multi-unit auctions are studied by number of authors. Recent contributions to this literature include Milgrom [2000] and Ausubel [2004] who study simultaneous ascending bid auctions. When goods are complements, then selling the items in a bundle can increase seller's revenues, as shown in Palfrey [1983], Levin [1997] and Armstrong [2000]. These papers differ from our setting in that we do not consider simultaneous sales, but consider sequential auctions. In our setting, bidders can condition their behavior on past auction outcomes which are publicly observed.

The paper is organized as follows: The next section describes the two period model. We assume that the first period winner draws the second period cost from a distinct cost distribution than a losing bidder. Section II illustrates the bidding equilibrium in second-price and first-price auctions. Section III describes our main results. It compares the first-price and second-price equilibrium in terms of procurement cost and bidders' rent. Section IV illustrates the procurement cost minimizing auction rule. Section V concludes.

## I. Model

A two period game is considered.[3] Every period a single contract is offered for sale. There are two bidders which are denoted by $i = 1, 2$.[4] We sometimes refer to the bidder that won (lost) the first period auction as the "winner" ("loser"). A bid in the procurement auction indicates a price at which the bidder is willing to provide the project. The price may depend on the bidder's cost for the project, the perception about the cost of the other bidder, and on the rules of the auction game. We make the following assumptions on the bidders' costs and the auction game:

*Private cost*: Each bidder observes privately his/her own cost draw at the beginning of every period. The second period's cost draw is not known in the first period.[5] The assumption arises when time elapses between periods, or when the properties of the second contract become known at the beginning of the second period only.[6] The period cost draw is private information and not observed by other bidders or the auctioneer. The first period cost is drawn from the distribution function $F$. The distribution of the second period cost draw depends on the outcome of the first period

[3] The restriction to two periods simplifies the exposition, but is not needed. The subsequent analysis and results extend to a multi-period setting in which the substitutes or complements property arises between items sold in adjacent periods.

[4] The restriction to two bidders allows us to adopt equilibrium characterization and uniqueness results for asymmetric auctions, see Maskin and Riley [1996, 2000a, 2000b].

[5] A model in which second period cost draws are known in period one would be qualitatively similar, but would entail the additional feature that bidders update their beliefs about second period costs based on the observed first period bid.

[6] In highway paving contracts, the auctioneer reveals upcoming contracts a short period before the letting date only. Limited updating can also arise when the auctioneer does not reveal the first period bids but reveals the first period winner only. An example in which information updating is limited in this manner are sequential London Bus route auctions.

auction game. The winner draws from the distribution function $F_w$ and the loser from $F_l$. The distributions are continuous, differentiable, and have common interval support $S = \left(\underline{C}, \overline{C}\right) \subset \Re_+$. We denote with $F_i'(c)$ for $i = l, w$ the associated probability density function.

We assume that the cost distributions satisfy the (strict) monotone likelihood ratio property, see Milgrom [1981]. Based on this property, we define items as substitutes or complements using the two conditions below.

*Condition 1:* We shall say the items are *substitutes* if in the second period the winner is more likely to have a higher cost than a loser in the likelihood ratio sense,

$$(1) \qquad \frac{F_w'(c)}{F_w'(c')} > \frac{F_l'(c)}{F_l'(c')} \quad \text{for all } c, c' \in S \text{ with } c > c'.$$

The substitutes property (1) has the following intuitive implications on the cost distribution functions: (i) $F_l(c) > F_w(c)$ for all $c \in S$; (ii) $F_l(c)/F_l'(c) > F_w(c)/F_w'(c)$ for all $c \in S$; and (iii) $\left[1 - F_l(c)\right]/F_l'(c) < \left[1 - F_w(c)\right]/F_w'(c)$ for all $c \in S$. A proof of these properties is given in the appendix.

*Condition 2:* We shall say the items *complements* if in the second period the winner is more likely to have a lower second period cost than a loser in the likelihood ratio sense,

$$(2) \qquad \frac{F_w'(c)}{F_w'(c')} < \frac{F_l'(c)}{F_l'(c')} \quad \text{for all } c, c' \in S \text{ with } c > c'.$$

*Auction game:* We shall consider two distinct auction games in the period game: (i) a first-price sealed-bid auction in which the low bidder wins and pays his bid; and (ii) a second-price sealed-bid auction in which the low bidder wins and pays the bid of the other bidder, which under our assumptions is strategically equivalent to an open ascending price auction. We shall ignore ties, as the probability of a tie is zero with continuous probability distributions.

Bidders are risk neutral. They discount future payoffs with the common discount factor $\beta \in (0,1)$. Bidders' objective is to maximize the sum of first period and discounted second period payoffs.

A strategy in the first period specifies a bid as a function of the cost, $b_f(c)$. A strategy in the second period specifies a bid in the second period for the winning and losing bidder as a function of the period cost, $b_w(c), b_l(c)$. We omit the dependence of the second period strategy on the first period privately observed cost and publicly observed bids as these variables are not payoff relevant in the second period, and will not affect the outcome.

We are interested in symmetric *Perfect Bayesian Nash Equilibria,* PBNE.

*Definition*: A PBNE is a tuple $(b_f, b_w, b_l)$ such that (i) the strategies constitute a subgame perfect equilibrium; and (ii) the beliefs are consistent with Bayes rule.

The next section examines bidding behavior in standard auctions. We examine a second-price and a first-price auction. Then, we compare the auctions' outcomes and illustrate the auction rule that minimizes procurement costs.

## II. Standard Auctions

This section examines bidding behavior in standard auctions. We start with the second-price auction, and establish that there exists an efficient equilibrium. Then, we examine the first-price auction.

### II.A. Second-Price Auction

In a second-price procurement auction the low bidder wins. The price paid equals the opponent's bid and does not depend on the bidder's own bid.

In a one period model, the second-price auction has a dominant strategy equilibrium in which bidders submit a bid equal to their cost, $b = c$. In the dynamic two-period auction game, with positive discounting $\beta > 0$ and when the items are substitutes or complements, it is no longer an equilibrium to bid the cost. The reason is that winning confers an opportunity cost (benefit) at the next auction, which will influence optimal bidding and render bidding of the own cost unprofitable. An optimal bid choice will take into account both, the cost of the project and the opportunity cost. We shall begin with a discussion of the second period payoffs, then quantify the opportunity cost, and finally examine the first period bid choice.

The second-price auction has a dominant strategy equilibrium in the second period in which bidders submit a bid equal to their cost, $b = c$.[7] The dominant strategy equilibrium yields the efficient outcome. Ignoring the zero probability event of ties, the efficient allocation rule is given by:

$$q_w^e(c) = \begin{cases} 1 & \text{if } c_w < c_l; \\ 0 & \text{otherwise.} \end{cases}$$

and $q_l^e = 1 - q_w^e$. Let $Q_w^e, Q_l^e$ denote the interim efficient winning probabilities, $Q_w^e(c_w) = \int_{c_w}^{\overline{C}} q_w^e(c) F_l'(c_l) dc_l$ and $Q_l^e(c_l) = \int_{c_l}^{\overline{C}} q_l^e(c) F_w'(c_w) dc_w$, and let $\Pi_w^e$, $\Pi_l^e$ denote the ex ante expected period rent for the winner and loser associated with

[7] When the cost support is bounded, $\overline{C} < \infty$, then there exist also pooling equilibria, for example $b_w = 0$ and $b_l = \overline{C}$. The described pooling equilibrium involves weakly dominated strategies and it is not efficient. As customary, we shall ignore pooling equilibria and focus our analysis on the unique equilibrium surviving the iterated elminination of weakly dominated strategies.

the efficient allocation rule. Following Myerson [1981], the ex ante expected second period profits reduces to the expected virtual rent, $\Pi_l^e = \int_S F_l(c)\, Q_l^e(c)\, dc$ and $\Pi_w^e = \int_S F_w(c)\, Q_w^e(c)\, dc$. The expression is obtained by using the envelope theorem and integration by parts.

In the first period of the game, the period's gain plus the discounted expected second period payoff equals $b - c + \beta\Pi_w^e$ if the bidder wins, and it equals $\beta\Pi_l^e$ if the bidder loses. As the bidder is risk neutral, the rent increment between winning and losing, $[b - c + \beta(\Pi_w^e - \Pi_l^e)]$, determines the first period bid choice. The first term in the rent difference equals the usual expression of the bid minus the period cost. The second term, $\beta[\Pi_l^e - \Pi_w^e]$, denotes the opportunity cost (benefit) of winning, and enters as an additive constant when the bidder wins the item. As illustrated in the following proposition, the symmetric first period equilibrium bidding strategy will take the added constant into account.

**Proposition 1** *The symmetric first period equilibrium bid function in the second-price auction equals:*

$$b_f^{SP}(c) = c + \beta[\Pi_l^e - \Pi_w^e]. \tag{3}$$

The proof follows from standard arguments for second-price auctions by which bidders bid their cost and therefore there is no static mark-up component. The argument is based on the second-price auctions' property that the bid does not affect the price paid, and affects the winning probability only. The property implies that bidding below cost may result in a loss as then a contract may be won at a price below cost. Similarly, bidding above cost may also result in a loss as it hurts the chances of winning an item with a positive expected payoff.

The equilibrium bidding strategy in (3) has an intuitive explanation. With both, substitutes and complements, the opportunity cost (benefit) of winning equals the discounted payoff difference between losing and winning in the second period auction game, $\beta\left[\Pi_l^e - \Pi_w^e\right]$, and will be passed on to the auctioneer as an additive mark-up (mark-down) independent of the cost realization in the first period.

Observe though that the first period bid strategy, $b_f^{SP}(c) = c + \beta\left[\Pi_l^e - \Pi_w^e\right]$, is not a dominant strategy equilibrium as the value of the mark-up depends on the opponent's equilibrium bid strategy in the second period. If the opponent were to use a distinct bid function, say to bid half the second period cost only, then the optimal dynamic mark-up of the bidder would be altered and reduced.

Observe also that the PBNE in the second-price auction retains the efficiency property of the static second-price auction.

**Corollary 1** *The PBNE in the second-price auction is efficient.*

Corollary 1 follows from two properties of the equilibrium bid functions. These properties are: (i) that the mark-up is independent of the cost realization; and (ii) that the mark-up is identical for both bidders. These two properties imply that the low cost bidder will submit the low bid.

So far, we have characterized the bidding equilibrium in the second-price auction. Next, we consider the bidding equilibrium in the first-price auction.

### II.B. First-Price Auction

In a first-price auction, the low bidder wins and receives his bid. The period payoff of the winner is given by the bid minus the cost, while the loser receives zero.

In period 2 under both cases, substitutes and complements, there will be one efficient and one inefficient bidder. With substitutes, the winner will be less efficient in the second period in the sense of condition (1), while with complements, the loser will be less efficient in the second period in the sense of condition (2). The existence and the uniqueness of an equilibrium in first-price asymmetric auctions has been established by a number of authors, including Maskin and Riley [1996,2000], Athey [2001], and Jackson, Simon, Swinkels and Zame [2002]. We shall proceed by assuming that a unique equilibrium in the second period exists with equilibrium bid functions $(b_w, b_l)$.

The second period equilibrium allocation rule, ignoring the zero probability event of ties, is given by,

$$q_w^{FP}(c_w, c_l) = \begin{cases} 1 & \text{if } b_w(c_w) < b_l(c_l); \\ 0 & \text{otherwise.} \end{cases}$$

and $q_l^{FP} = 1 - q_w^{FP}$. It says that the bidder $i$, with $i = w, l$, wins the second contract when his bid is low. Let $Q_w^{FP}, Q_l^{FP}$ denote the interim expected winning probabilities, $Q_w^{FP}(c_w) = \int_{c_w}^{\overline{C}} q_w^{FP}(c_w, c_l) F_l'(c_l) dc_l$ and $Q_l^{FP}(c_l) = \int_{c_l}^{\overline{C}} q_l^{FP}(c_w, c_l) F_w'(c_w) dc_w$. Let $\Pi_l^{FP}$, $\Pi_w^{FP}$ denote the ex ante expected second period rent for the first period losing and winning bidder associated with the first-price allocation rule, $\Pi_l^{FP} = \int_S F_l(c) Q_l^{FP}(c) dc$ and $\Pi_w^{FP} = \int_S F_w(c) Q_w^{FP}(c) dc$.

The following Lemma describes properties of the equilibrium bid strategies and expected payoffs that will be essential in the subsequent arguments (the appendix contains formal proofs of lemmas, propositions and theorems).[8]

**Lemma 1** *For any $c$ in $S$:*

*(i) Under condition (1), when the contracts are substitutes, $b_w(c) < b_l(c)$, and $\Pi_w^e < \Pi_w^{FP} < \Pi_l^{FP} < \Pi_l^e$.*

*(ii) Under condition (2), when the contracts are complements, $b_w(c) > b_l(c)$, and $\Pi_w^e > \Pi_w^{FP} > \Pi_l^{FP} > \Pi_l^e$.*

The Lemma illustrates intuitive properties of asymmetric first-price auctions. The less efficient bidder bids more aggressively than the more efficient bidder. The reason is that the less efficient bidder expects tougher competition in the auction than the more efficient bidder. The strategic effect has the following implications: When contracts are substitutes, the first period winning bidder knows she is on average less efficient in the second period, and charges a smaller mark-up over costs than a losing bidder. The reduction in the mark-up implies that she wins more frequently in the second period than is efficient, i.e. winning despite of having a higher cost draw than the opponent. In turn, this implies that she makes an expected profit larger than is efficient. When contracts are complements, the first period winning bidder charges a higher mark-up over costs than a losing bidder, and thus she wins less frequently and makes less rent than is socially efficient.

Next, we consider the remaining element in the equilibrium construction: The first period bid strategy. The asymmetry in the second period will affect the first

[8] These properties have also been employed in the context of mergers, Waehrer [1999], and bidder collusion, Pesendorfer [2000].

period's bid choice as it introduces an opportunity cost (benefit) of winning. The optimal first period bid will take both, the opportunity cost (benefit) of winning and the period cost realization, into account. As the opportunity cost (benefit) does not depend on the period cost realization, it simply shifts the cost by a constant term, $\beta\left[\Pi_l^{FP}-\Pi_w^{FP}\right]$, and the equilibrium bid function takes the well known form.

**Proposition 2** *The first period equilibrium bid function in the first-price auction equals:*

$$b_f^{FP}(c)=c+\frac{\int_c^{\overline{C}}\left[1-F(x)\right]dx}{1-F(c)}+\beta\left[\Pi_l^{FP}-\Pi_w^{FP}\right]. \tag{4}$$

The proof follows from standard arguments for first-price auctions, see for example Proposition 2 in Riley and Samuelson [1981]. The equilibrium bid in equation (4) has an intuitive explanation. It equals the cost plus a mark-up. The mark-up has two components: (i) the static mark-up equal to the expected opponent's cost conditional on the opponent's cost exceeding the own cost, and (ii) a dynamic mark-up equal to the opportunity cost of winning.

There are two features of the equilibrium worth emphasizing: First, the sign of the dynamic mark-up under the first-price auction coincides with the sign of the opportunity cost under the efficient allocation rule. It is positive when the contracts are substitutes, and negative when the goods are complements.

Second, when contracts are substitutes, the dynamic mark-up is smaller under the first-price auction rule than under the efficient second-price allocation rule:

$$\left[\Pi_l^{FP}-\Pi_w^{FP}\right]<\left[\Pi_l^{e}-\Pi_w^{e}\right].$$

When contracts are complements, the dynamic mark-up is larger under the first-price auction rule than under the efficient second-price allocation rule:

$$\left[\Pi_l^{FP} - \Pi_w^{FP}\right] > \left[\Pi_l^e - \Pi_w^e\right].$$

These features follow from the payoff inequalities in Lemma 1 and are explained by the strategic bid shading in first-price auctions.

The features imply that when contracts are substitutes, the first period mark-up is higher with the efficient second period allocation than with the allocation of the first-price auction. On the other hand, when contracts are complements, the first period mark-up is lower with the efficient second period allocation than with the allocation of the first-price auction.

We shall see next that the first period mark-up ranking plays a central role in determining the bidder's rent and procurement cost ranking.

## III. Optimal Sequential Auction

This section describes our main results. We compare the outcome under the first-price auction and the second-price auction. Subsection III.A considers this issue from the perspective of the bidders and compares the bidders' rents. Subsection III.B compares the total procurement cost associated to the first-price and second-price auctions.

### III.A. Bidders' Rent

The equilibrium characterization in section II allows us to determine a bidder's rent under the first-price and second-price auction rule. By using the envelope theorem and integration by parts, the ex ante expected equilibrium game payoff in the first-price auction, $\Pi^{FP}$, and in the efficient second-price auction, $\Pi^{e}$, equal:

$$\Pi^j = \int_S F(c)\,[1 - F(c)]\,dc + \beta \Pi_l^j \qquad \text{for } j = FP, e. \tag{5}$$

Equation (5) consists of the usual expression for the bidder's information rent in the first period, and a modified expression in the second period that captures the expected payoff of a losing bidder. The modification arises as competition in the first period diminishes any expected payoff advantages (or disadvantages) of the winning bidder. This can be seen most clearly in the analysis in section II, where the first period bid passes any subsequent payoff losses (gains) of the winner on to the seller by adding the opportunity cost (benefit) of winning to the bid. Thus both, the winning and the losing bidder, expect to receive the losing bidder's second period rent only.

The following Theorem compares bidder's rent between the two auction formats.

**Theorem 1 (Payoff Ranking)**

*(i) Under condition (1), when the contracts are substitutes,* $\Pi^{FP} < \Pi^{e}$*.*

*(ii) Under condition (2), when the contracts are complements,* $\Pi^{FP} > \Pi^{e}$*.*

This Theorem establishes that bidders prefer the efficient second-price auction when the contracts are substitutes, while they prefer the first-price auction when contracts are complements. The result is already apparent in the differential bid shading behavior illustrated at the end of section II, as with substitutes (complements), bidders' first period bids and thus first period payoffs are lower (higher) with the first-price auction than in the efficient second-price auction.

Equation (5) also shows that the payoff comparison across auction formats reduces to a comparison of the second period expected rent to the losing bidder. The result in Theorem 1 is then easily explained as with substitutes (complements), the second period losing bidder's rent is lowest under the first-price (efficient second-price) auction rule as is shown in Lemma 1 in section II. As described earlier, the intuition for this result lies in the fact that with substitutes (complements) the losing bidder bids less (more) aggressively than the winning bidder in the first-price auction resulting in a lower (higher) winning probability and thus a lower (higher) rent than that associated with the socially efficient outcome in the second-price auction.

Next, we consider the procurement cost.

### III.B. Procurement Cost

The total procurement cost of the first-price and the efficient second-price auction, $PC^{FP}$ and $PC^{e}$, equals the cost of the winning bidder plus the bidder's rent:

$$\begin{aligned} PC^{j} &= 2\int_{S} c\left[1-F(c)\right]F'(c)dc + \beta\int_{S}\int_{S}\left[c_l q_l^j\left(c_w,c_l\right)+c_w q_w^j\left(c_w,c_l\right)\right]F_l'(c_l)F_w'(c_w)dc_w dc_l \\ &\quad +2\int_{S} F(c)\left[1-F(c)\right]dc + 2\beta\int_{S} F_l(c)Q_l^j(c)dc \qquad \text{for } j=FP,e \end{aligned}$$

where the first period procurement cost can be attributed to the usual virtual cost of a bidder, $c+F/F'$. The second period procurement cost differs from the usual virtual cost expression. It equals the cost of the winning bidder, $c_l q_l^j + c_w q_w^j$, plus twice the second period rent of the losing bidder, $2\cdot F_l/F_l'$, for $j=FP,e$. The expression involving twice the losing bidder's rent arises as expected payoff advantages (disadvantages) of the winning bidder in the second period are passed on to the auctioneer with the first period bid choice, and both, winning and losing, bidders expect to receive the losing bidder's rent in the second period.

The procurement cost difference between the efficient second-price and the first-price auction arises in the second period only.[9] The magnitude of the difference amounts to:

$$
\begin{aligned}
PC^{e}-PC^{FP} &= \beta\int_{S}\int_{S}\left[c_{l}-c_{w}\right]\left[q_{l}^{e}\left(c_{w},c_{l}\right)-q_{l}^{FP}\left(c_{w},c_{l}\right)\right]F_{l}'(c_{l})F_{w}'(c_{w})dc_{w}dc_{l} \\
&\quad +2\beta\int_{S}\int_{S}F_{l}\left(c\right)\left[Q_{l}^{e}\left(c\right)-Q_{l}^{FP}\left(c\right)\right]dc. \qquad (6)
\end{aligned}
$$

The first term on the right hand side measures the efficiency loss of the first-price auction. This term is always negative. The second term on the right hand side reflects the difference in second period payoff of the losing bidder between the efficient allocation rule and the allocation rule of the first-price auction. Lemma 1 shows that with substitutes, the term is positive, while with complements it is negative.

The following Theorem states our central result.

**Theorem 2 (Procurement Cost Ranking)**

*(i) Under condition (1), when the contracts are substitutes,* $PC^{e}>PC^{FP}$.

*(ii) Under condition (2), when the contracts are complements,* $PC^{e}<PC^{FP}$.

Theorem 2 gives a clear policy recommendation: The efficient second-price auction is optimal when contracts are complements, while the first-price auction is optimal when contracts are substitutes. Observe also that the ranking in Theorem 2 is the reverse ranking of Theorem 1, which illustrated the bidders' rent ranking. The intuition is again based on the feature that bidders bid more (less) aggressively in the

[9] As bidders are ex-ante symmetric in period 1, the auction format does not matter as both the first-price and the second price auction select the efficient bidder as the winner.

first-price auction than in the efficient auction when contracts are substitutes (complements). This implies that when the items are substitutes (complements) a lower (higher) procurement cost under the first-price than under the efficient second-price auction rule.

Theorem 2 may seem surprising in light of a recent result in Maskin and Riley [2000a], which establishes that the procurement cost (or revenue) ranking between first-price and open auctions *is ambiguous* when bidders are asymmetric.[10] The ambiguity result in Maskin and Riley is obtained under the assumption of a single period auction game in which the bidders' asymmetry is taken as exogenously given. In our model, this scenario is equivalent to considering the second period auction game in isolation only. In contrast, Theorem 2 shows that when the asymmetries arise endogenously due to the first period auction outcome, then the ambiguity disappears and the total procurement cost, consisting of the first and the second periods' procurement cost, has a clear and *unambigous* ranking across auction formats.

In order to illuminate the ranking result in more detail, we shall use the techniques developed in Myerson [1981] to illustrate the procurement cost minimizing allocation rule. Doing so, will allow us to interpret the procurement cost ranking more intuitively. This is done in the next section.

## IV. Procurement Cost Minimization

[10] Maskin and Riley [2000a] show that there is a class of distribution functions such that the first-price auction is preferred. The class has the feature that asymmetries arise due to a shift (or stretch) in the distribution. They also show that there is a second class of distribution functions such that the open auction is preferred. The asymmetry in the second class is based on a shift of probability mass to the upper end point in the support.

We conclude the discussion with a brief illustration of the auction rule that minimizes the procurement cost. The illustration will enable us to interpret the procurement cost ranking of the first-price and efficient second-price auction more intuitively. We explore the commitment solution in which the auctioneer fixes the auction rule for periods one and two before the bidding starts, and we do not permit the auctioneer to modify the auction rule after period one.

The techniques developed in Myerson [1981] allow us to address this problem. We consider the set of incentive compatible auction rules that satisfy the voluntary participation constraints and incentive constraints in every period. Let $q_i^t(c_i, c_j)$ denote the probability that bidder $i$ receives the object in period $t$ when bidder $i$ announces cost $c_i$ and bidder $j$ announces cost $c_j$. Let $T_i^t$ denote the expected transfer payment of bidder $i$ (to the seller) when the bidder announces cost $c_i$ in period $t$ and let $Q_i^t$ denote the expected winning probability, $Q_i^t(c_i) = \int q_i^t(c_i, c_j) F_j'(c_j) dc_j$. The expected payoff of bidder $i$ in period $t$, for $t = 1, 2$ and $i = 1, 2$, when the bidder with cost $c_i$ reports cost $c_i'$ equals:

$$\begin{aligned}
\Pi_i^2(c_i, c_i') &= T_i^2(c_i') - c_i Q_i^2(c_i'). \\
\Pi_i^1(c_i, c_i') &= T_i^1(c_i') - c_i Q_i^1(c_i') + \beta Q_i^1(c_i') \int_S \Pi_w^2(x, x) F_w'(x) dx \\
&\quad + \beta \left[1 - Q_i^1(c_i')\right] \int_S \Pi_l^2(x, x) F_l'(x) dx.
\end{aligned}$$

The incentive constraints take the form,

$$\text{(IC)} \quad \Pi_i^t(c_i, c_i) \geq \Pi_i^t(c_i, c_i') \text{ for all } c_i, c_i' \in S \text{ and for } i = 1, 2,\ t = 1, 2.$$

and the voluntary participation constraint take the form,

$$\text{(VP)} \quad \begin{aligned} \Pi_i^1(c_i, c_i) &\geq \int_S \Pi_l^2(x,x)F_l'(x)dx \quad \text{for all } c_i \in S \text{ and for } i = 1,2; \\ \Pi_i^2(c_i, c_i) &\geq 0 \qquad \text{for all } c_i \in S \text{ and for } i = w, l; \end{aligned}$$

where the participation payoff in the first period equals at least the expected payoff of a bidder that participates in the second period only, $\int_S \Pi_l^2(x,x)F_l'(x)dx$. The (VP) constraint assumes that a bidder that refrains from bidding in the first period cannot be prevented from participating in the second period auction. This formulation of the (VP) constraint comes closest to the (implicit) assumption in the sequential first-price and second-price auction, analyzed earlier, in which a bidder cannot be prevented from participating in the second period auction.[11]

The following Lemma states an expression for the procurement cost. We show in the appendix, by using the techniques developed in Myerson [1981], that this expression applies under (VP) and (IC).

**Lemma 2** *In any incentive compatible auction rule that satisfies (VP) and (IC), the functions $Q_i^t(c)$ for $i, t = 1, 2$, are monotone decreasing and the procurement cost*

[11] A weaker (VP) constraint arises if a non-participating bidder is banned from the second auction. With the weaker constraint, the first period reservation value becomes zero, $\Pi_i^1(c,c) \geq 0$ for all $c \in S$ and for $i = 1,2$, and the auctioneer can extract all the rent in the second period by charging bidders a fee in period one equal to the expected second period's rent and by using the efficient second-price auction in the second period. As the fee is collected in period one, before the second period private information is observed, it will not affect subsequent behavior and enable the auctioneer to collect all the (expected) rent.

*equals:*

$$
\begin{aligned}
PC &= \int_S \int_S \left[ \sum_{i=1}^{2} \left( c_i + \frac{F(c_i)}{F'(c_i)} \right) q_i^1(c_1, c_2) \right] F'(c_1) F'(c_2) \, dc_1 dc_2 \\
&\quad + \sum_{i=1,2} \left[ \Pi_i^1(\overline{C}, \overline{C}) - \beta \int F_l(x) Q_l^2(x) \, dx \right] \\
&\quad + \beta \int_S \int_S \left[ c_w q_w(c_w, c_l) + c_l q_l(c_w, c_l) + 2 \frac{F_l(c_l)}{F_l'(c_l)} q_l(c_w, c_l) \right] F_w'(c_w) F_l'(c_l) \, dc_w dc_l,
\end{aligned}
$$

*with* $\Pi_i^1(\overline{C}, \overline{C}) \geq \beta \int F_l(x) Q_l^2(x) \, dx$ *for* $i = 1, 2$.

The first term in the procurement cost accounts for the virtual cost in the first period; the second term reflects the voluntary participation constraint; and, the third term accounts for the second period virtual cost.

The optimal auction rule maximizes the above expression. Observe that the first expression is the usual procurement cost expression, which is maximized with a first-price or second-price auction. The second term reflects the voluntary participation constraint. The third expression differs as it takes the dynamic bidding effect into account. Pointwise minimization of the third expression yields the optimal rule (we ignore again the zero probability event of a tie).

**Proposition 3** *The procurement cost minimizing solution is a first-price (or second-price) auction followed by an auction with the following allocation rule:*

$$
q_w(c_w, c_l) = \begin{cases} 1 & \textit{if } c_w < c_l + 2\frac{F_l(c_l)}{F_l'(c_l)}; \\ 0 & \textit{otherwise.} \end{cases}
$$

*and* $q_l(c_w, c_l) = 1 - q_w(c_w, c_l)$.

The optimal second period allocation rule assigns an increased winning probability to the bidder who won the first period item under both, complements and substitutes. The amount of the increase relative to the efficient rule equals twice the virtual rent of the losing bidder. The optimal choice balances two opposing effects: On the one hand, an increase in the second period winning probability leads to an increase in second period rent differential between the winning and the losing bidder. In turn, the increased rent differential implies more aggressive bidding and thus induces the benefit of lower procurement cost in the first period. On the other hand, the increase in the second period winning probability comes at the cost of an increased inefficiency in the second period. At the optimum, the marginal benefit of the reduced first period procurement cost equals the marginal cost of the second period efficiency loss, and the usual marginal condition holds.

The result in Proposition 3 allows us to illustrate the ranking obtained in Theorem 2 intuitively. A graphical illustration is given in Figures 1 and 2. The Figures assume a uniform cost distribution for the losing bidder, $F_l(c) = c$, for $0 < c < 1$, and plot the losing bidder's costs, $c_l$, on the horizontal axis and the winning bidder's cost, $c_w$, on the vertical axis. Figure 1 assumes that the winning bidders' cost are drawn from the distribution function $F_w(c) = c^{3/2}$, which implies substitutes, and Figure 2 assumes the distribution function $F_w(c) = c^{1/4}$, which reflects complements. Line $I^*$ describes the optimal awarding rule characterized in Proposition 3 and given by the line $c_w = c_l + 2F_l(c_l)/F_l'(c_l)$. To the northwest of line $I^*$, the second period contract is awarded to the losing bidder, and to the southeast of line $I^*$, the second period contract is awarded to the winning bidder. Line $I^e$ describes the awarding rule under the efficient second-price auction, which coincides with the 45 degree line. To the northwest of line $I^e$, the second-price auction awards the second period contract to the losing bidder, while to the southeast of line $I^*$, it awards the second period contract to the winning bidder.

[Figures 1 and 2 about here]

Figures 1 and 2 also illustrate the outcome under the first-price auction. In both cases, the asymmetric first-price equilibrium can be calculated numerically[12] and the resulting optimal first-price allocation rule is described by line $I^{FP}$. To the northwest of line $I^{FP}$ the first-price auction awards the second period contract to the losing bidder, while to the southeast of line $I^{FP}$ the first-price auction awards the second period contract to the winning bidder. We are now in a position to compare all three auction rules, and highlight the key features of the comparison:

The efficient line $I^e$ is to the right of the procurement cost minimizing line $I^*$. The reason is that the winning bidder receives the item less frequently under the efficient second-price auction than under the procurement cost minimizing rule.

The first-price awarding rule $I^{FP}$ lies entirely either to the left or to the right of the efficient rule $I^e$. With substitutes, the first-price awarding rule $I^{FP}$ is to the left, while with complements it is to the right of $I^e$.

Now, consider the case of substitutes, as illustrated in Figure 1. With substitutes, the first-price awarding rule $I^{FP}$ is to the left of the efficient rule $I^e$, as it assigns the

[12] Marshall, Meurer, Richard and Stromquist [1994] describe numerical methods to calculate the asymmetric first-price auction equillibrium $b_l, b_w$. The boundary is then the set of points, such that $b_l(c_l) = b_w(c_w)$.

item to the winning bidder more frequent than is socially efficient. As a result, the first-price cut-off rule is closer to the optimal rule $I^*$ than the efficient rule. We can conclude that the first-price auction dominates the efficient second-price auction.

Finally, consider the case of complements, as illustrated in Figure 2. With complements, the first-price rule $I^{FP}$ lies to the right of the efficient rule $I^e$, and is thus further away from the procurement cost minimizing rule $I^*$ than the efficient rule. So, in this case, we can conclude that the first-price auction is dominated by the second-price auction in terms of efficiency and also in terms of reduced procurement costs.

## V. Conclusions

In this paper, we have examined optimal sequential auctions when items are complements or substitutes in the sense that an item's value increases or decreases with the number of items acquired already. We have found that the existence of complementarity or substitutability between sequentially auctioned items has consequences on the procurement costs associated with different auction rules. Our analysis has definite policy recommendations for an auctioneer that wants to minimize procurement costs:

(i) If the items are substitutes, then it is optimal to use a sealed first-price auction rather than an open ascending price auction (or sealed second-price auction).

(ii) If the items are complements, then it is optimal to use an open ascending price auction (or sealed second-price auction) rather than a sealed first-price auction.

The explanation is intuitive: Enhancing the winning probability of the first round winner in the second period leads to increased competition in the first period, and thus to lower procurement costs. With substitutes, the first-price auction correctly favors the first period winning bidder yielding lower procurement costs than the socially efficient second-price auction, which does not favor any bidder. In contrast, when items are complements, the first-price auction incorrectly favors the first period losing bidder resulting in an increased procurement cost vis-a-vis the efficient second-price auction.

It is tempting to try to explain observed auction rules and relate them to our results on the complementarity and the substitutability of the items for sale: Casual empiricism suggests that job contract bidding for governmental institutions tends to be conducted in a sealed bid format, while fine art, antiques, wine, and livestock are mostly conducted openly.

The empirical evidence from procurement auctions for highway paving jobs and forest timber sales described earlier confirms the existence of substitutability between the items. To the extent that these procurement jobs do have a technology with decreasing returns to scale, the auctioneers' chosen first-price sealed auction format is adequate in order to minimize procurement costs.

The empirical evidence on art auctions is largely anecdotal and there is no conclusive evidence on complementarities.[13] Yet, the purpose behind the purchase may be indicative of complementarities or substitutabilities between the items from the bidder's point of view. Thus, knowing the purpose motivating most of bidders' bids may be important for the auctioneer in order minimize procurement costs: When facing a bidders that are mostly trying to complete a collection, complementarities may exist

[13] See Ashenfelter and Graddy [2004] for a survey on art auctions.

and an open ascending auction should be preferred. When faced with bidders that desire to acquire at most one item each, the auctioneer should anticipate the existence of substitutes and a sealed first-price auction should be chosen. For example, some of the empirical evidence on livestock auctions suggests the existence of substitutes and, according to our findings, it may be beneficial to the auctioneer to switch to a sealed bid format in those instances.

Although there is some evidence that the cost minimizing auction format is chosen in a number of settings, a more throughout empirical investigation of the auctioneer's choice of auction format is required to answer this question in more detail and to understand its implications in each case.

# Appendix

**Properties implied by condition (1):**

*(i)* $F_l(c) > F_w(c)$ *for all* $c \in S$*;*

*(ii)* $F_l(c)/F_l'(c) > F_w(c)/F_w'(c)$ *for all* $c \in S$*; and*

*(iii)* $[1 - F_l(c)]/F_l'(c) < [1 - F_w(c)]/F_w'(c)$ *for all* $c \in S$*.*

**Proof.** Condition (1), the monotone likelihood ratio property, implies

$$\text{(A1)} \qquad F_l'(c')F_w'(c) > F_l'(c)F_w'(c') \quad \text{for all } c, c' \in S \text{ with } c > c'.$$

Integrating both sides of the inequality over $c'$ from the lower endpoint of the support $S$ to $c$, yields

$$\text{(A2)} \qquad \frac{F_w'(c)}{F_l'(c)} > \frac{F_w(c)}{F_l(c)} \quad \text{for all } c \in S,$$

which implies property (ii).

Next, integrate both sides of (A1) over $c$ from $c'$ to the upper endpoint of the support $S$, yields

$$\text{(A3)} \qquad \frac{1 - F_w(c')}{1 - F_l(c')} > \frac{F_w'(c')}{F_l'(c')} \quad \text{for all } c' \in S,$$

which implies property (iii).

Combining (A3) and (A2) gives

$$\frac{1 - F_w(c)}{1 - F_l(c)} > \frac{F_w(c)}{F_l(c)} \quad \text{for all } c \in S,$$

which implies property (i). ■

**Proof of Lemma 1.** We consider the case of substitutes (1). The case of complements follows by mimicking the steps in the argument with permuted bidder identity.

First, we prove that $b_l(c) > b_w(c)$. Let $\phi_i(b)$ denote the inverse of the bid function for $i = w, l$. Theorem 1 in Lebrun [1999] establishes that bid functions are strictly increasing in costs and Theorem 2 in Lebrun [1999] establishes the existence of equilibrium with common bid support. Thus, the inverse of the bid function exists and is strictly increasing. We can write the payoff of bidder $w$ as

$$\max_b [b - c] [1 - F_l(\phi_l(b))],$$

and it's associated necessary first order condition implies

$$\text{(A4)} \qquad \frac{1}{b - \phi_w(b)} = \frac{F_l'(\phi_l(b)) \phi_l'(b)}{1 - F_l(\phi_l(b))}$$

Similarly, bidder $l$'s payoff is given by

$$\max_b [b - c] [1 - F_w(\phi_w(b))],$$

and, the associated necessary first order condition implies

$$\text{(A5)} \qquad \frac{1}{b - \phi_l(b)} = \frac{F_w'(\phi_w(b)) \phi_w'(b)}{1 - F_w(\phi_w(b))}.$$

Consider a point $c \in S$ such that $b_w(c) = b_l(c)$. The first order condition and condition (1) imply that $b_l'(c) > b_w'(c)$. As the support of bids is identical, this implies $b_l(c) > b_w(c)$.

Second, we show that $Q_w^e(c) < Q_w^{FP}(c)$ and $Q_l^{FP}(c) < Q_l^e(c)$: From the first part above, we can deduce that the inverse bid functions satisfy, $\phi_l(b) < \phi_w(b)$

for all $b$ contained in the interior of the support of bids. The winning probability $Q_l^{FP}(c) = 1 - F_w(\phi_w(b_l(c)))$ is less than the efficient probability $Q_l^e(c) = 1 - F_w(c)$ as $c < \phi_w(b_l(c))$, and $Q_w^{FP}(c) = 1 - F_l(\phi_l(b_w(c)))$ exceeds the efficient probability $Q_w^e(c) = 1 - F_l(c)$.

Third, we establish the payoff inequalities $\Pi_w^e < \Pi_w^{FP} < \Pi_l^{FP} < \Pi_l^e$. The interim expected equilibrium payoff of a bidder of type $i = l, w$ with cost $c$ equals $\Pi_i^{FP}(c) = (b_i(c) - c)\, Q_i^{FP}(b_i(c))$. Now, the envelope theorem implies that

$$\frac{d}{dc}\Pi_i^{FP}(c) = Q_i^{FP}(b_i(c)).$$

As $\Pi_i^{FP}\left(\overline{C}\right) = 0$, the interim expected payoff can be written as $\Pi_i^{FP}(c) = \int_c^{\overline{C}} Q_i^{FP}(x)dx$ and, by definition, $\Pi_i^e(c) = \int_c^{\overline{C}} Q_i^e(x)dx$. Now, the payoff inequalities $\Pi_w^e(c) < \Pi_w^{FP}(c)$ and $\Pi_l^{FP}(c) < \Pi_l^e(c)$ follow from the probability inequalities in the second part, $Q_w^e(c) < Q_w^{FP}(c)$ and $Q_l^{FP}(c) < Q_l^e(c)$. As the payoff payoff inequalities hold for all $c \in S$, they hold also ex ante, before costs are observed, which establishes the claim.

The final inequality that we need to establish is $\Pi_w^{FP} < \Pi_l^{FP}$: Let $G_i(b)$ denote the probability that a bid $b$ wins the auction for bidder $i = l, w$. We begin by showing that $G_w(b) < G_l(b)$ and then establish the inequality on profits. Let $\underline{b}$ denote the lower endpoint of the support of bids. As $\phi_l(b) < \phi_w(b)$ for all $b$ contained in the interior of the support of bids, conditions (A4) and (A5) imply that

$$F_w'(\phi_w(b))\,\phi_w'(b)\,/\,[1 - F_w(\phi_w(b))] < F_l'(\phi_l(b))\,\phi_l'(b)\,/\,[1 - F_l(\phi_l(b))]\,.$$

This can be written as $-(d/db)\ln[1 - F_w(\phi_w(b))] < -(d/db)\ln[1 - F_l(\phi_l(b))]$. Since $[1 - F_w(\phi_w(\underline{b}))] = [1 - F_l(\phi_l(\underline{b}))]$, this implies $\ln[1 - F_w(\phi_w(b))] > \ln[1 - F_l(\phi_l(b))]$, or equivalently, $G_l(b) = 1 - F_w(\phi_w(b)) > 1 - F_l(\phi_l(b)) = G_w(b)$. Now,

$$\begin{aligned}
\Pi_w^{FP}(c) &= [b_w(c) - c]\, G_w(b_w(c)) \\
&< [b_w(c) - c]\, G_l(b_w(c)) \\
&\leq [b_l(c) - c]\, G_l(b_l(c)) \\
&= \Pi_l^{FP}(c),
\end{aligned}$$

which establishes that $\Pi_w^{FP}(c) < \Pi_l^{FP}(c)$ for all $c \in S$.

Now, the ex ante payoff difference can be written as:

$$\begin{aligned}
\Pi_w^{FP} - \Pi_l^{FP} &= \int_S \Pi_w^{FP}(c)\, F_w'(c)dc - \int_S \Pi_l^{FP}(c)\, F_l'(c)dc \\
&< \int_S \Pi_l^{FP}(c)\, [F_w'(c) - F_l'(c)]\, dc \\
&= \int_S \left(-\frac{\partial \Pi_l^{FP}(x)}{\partial c}\right) [F_w(x) - F_l(x)]\, dx \\
&\leq 0
\end{aligned}$$

The first inequality uses $\Pi_w^{FP}(c) < \Pi_l^{FP}(c)$ for all $c \in S$. The second equality follows from integration by parts. The final inequality is based on two observations: First, the term in square brackets is negative from property (i) of condition (1). Second, the term in round brackets is positive as $-\frac{\partial \Pi_l^{FP}(c)}{\partial c} = -Q_l^{FP}(c) < 0$. This completes the proof. ■

**Proof of Theorem 1.** By using equation (5), the difference between the second-price game payoff and the first-price game payoff, equals,

$$\Pi^e - \Pi^{FP} = \beta \left[\Pi_l^e - \Pi_l^{FP}\right].$$

By Lemma 1, the right hand side is positive under condition (1), and negative under condition (2). ■

**Proof of Theorem 2.** The efficient winning probabilities are given by $Q_l^e(c) = 1 - F_w(c)$ and $Q_w^e(c) = 1 - F_l(c)$. As in the proof of Lemma 1, we denote the inverse bid functions in the first-price auction equilibrium by $\phi_w$ and $\phi_l$. The first-price winning probabilities are then given by $Q_l^{FP}(c) = 1 - F_w(\phi_w(b_l(c)))$ and $Q_w^{FP}(c) = 1 - F_l(\phi_l(b_l(c)))$. The procurement cost difference, $D = \left[PC^e - PC^{FP}\right]/\beta$, can be written as:

$$\begin{aligned}
D &= \int_S c\left[F_w\left(\phi_w\left(b_l(c)\right)\right) - F_w(c)\right] F_l'(c)\, dc \\
&\quad + \int_S c\left[F_l\left(\phi_l\left(b_w(c)\right)\right) - F_l(c)\right] F_w'(c)\, dc \\
&\quad + 2\int_S F_l(c)\left[F_w\left(\phi_w\left(b_l(c)\right)\right) - F_w(c)\right] dc \\
&= \int_S c\left[F_w\left(\phi_w\left(b_l(c)\right)\right) - F_w(c)\right] F_l'(c)\, dc \\
&\quad + \int_S c\left[F_l\left(\phi_l\left(b_w(c)\right)\right) - F_l(c)\right] F_w'(c)\, dc \\
&\quad + \int_S F_l(c)\left[F_w\left(\phi_w\left(b_l(c)\right)\right) - F_w(c)\right] dc \\
&\quad - \int_S c\left[F_w\left(\phi_w\left(b_l(c)\right)\right) - F_w(c)\right] F_l'(c)\, dc \\
&\quad - \int_S cF_l(c)\left[F_w'\left(\phi_w\left(b_l(c)\right)\right)\phi_w'\left(b_l(c)\right) b_l'(c) - F_w'(c)\right] dc \\
&= \int_S F_l(c)\left[F_w\left(\phi_w\left(b_l(c)\right)\right) - F_w(c)\right] dc \\
&\quad + \int_S \left[c - \phi_l\left(b_w(c)\right)\right] F_l\left(\phi_l\left(b_w(c)\right)\right) F_w'(c)\, dc.
\end{aligned}$$

The second equality follows from integration by parts of the expression,

$$\begin{aligned}
\int_S F_l(c)\left[F_w\left(\phi_w\left(b_l(c)\right)\right) - F_w(c)\right] dc &= -\int_S c\left[F_w\left(\phi_w\left(b_l(c)\right)\right) - F_w(c)\right] F_l'(c)\, dc \\
&\quad - \int_S cF_l(c)\left[F_w'\left(\phi_w\left(b_l(c)\right)\right)\phi_w'\left(b_l(c)\right) b_l'(c) - F_w'(c)\right] dc.
\end{aligned}$$

The third equality cancels terms and uses the substitution $u = \phi_w(b_l(c))$ which yields:

$$\int_S cF_l(c)\, F'_w(\phi_w(b_l(c)))\, \phi'_w(b_l(c))\, b'_l(c)dc = \int_S \phi_l(b_w(c))\, F_l(\phi_l(b_w(c)))\, F'_w(c)\, dc$$

as the function $\phi_w(b_l(c))$ is from $S$ onto $S$.

By Lemma 1, condition (1) implies that $\phi_w(b_l(c)) > c$ and $c > \phi_l(b_w(c))$. Thus, $PC^e - PC^{FP} > 0$ which establishes part (i). By Lemma 1, condition (2) implies that $\phi_w(b_l(c)) < c$ and $c < \phi_l(b_w(c))$. Thus, $PC^e - PC^{FP} < 0$ which establishes part (ii).

■

**Proof of Lemma 2.** It is well known - Mas-Colell, Whinston and Greene [1995], Proposition 23.D.2 - that the allocation $(Q_i^t, T_i^t)$ is Bayesian incentive compatible if and only if, for all $i, t = 1, 2$,

(i) $Q_i^t$ is monotone decreasing, and

(ii) $\Pi_i^t(c,c) = \int_c^{\overline{C}} Q_i^t(x)\, dx + \Pi_i^t(\overline{C}, \overline{C})$ for all $c \in S$.

Notice, that integration by parts yields:

$$\int_S \Pi_i^2(x,x) F'_i(x) dx = \int_S F_i(c) Q_i^2(c)\, dc + \Pi_i^2(\overline{C}, \overline{C}) \qquad \text{for } i = l, w.$$

In turn this implies that the expected transfer payment of a bidder with cost $c$ equals:

$$\begin{aligned}
T_i^2(c) &= cQ_i^2(c) + \int_c^{\overline{C}} Q_i^2(x)\, dx + \Pi_i^2(\overline{C}, \overline{C}); \\
T_i^1(c) &= cQ_i^1(c) + \int_c^{\overline{C}} Q_i^1(x)\, dx - \beta \left[ \int F_l(x) Q_l^2(x)\, dx + \Pi_l^2(\overline{C}, \overline{C}) \right] \\
&\quad + \beta Q_i^1(c) \left[ \int F_l(x) Q_l^2(x)\, dx + \Pi_l^2(\overline{C}, \overline{C}) - \int F_w(x) Q_w^2(x)\, dx - \Pi_w^2(\overline{C}, \overline{C}) \right] \\
&\quad + \Pi_i^1(\overline{C}, \overline{C});
\end{aligned}$$

and then, the expected sum of discounted transfer payments equals,

$$\sum_{i=1,2} \int_S \left[ \left( c + \frac{F(c)}{F'(c)} \right) Q_i^1(c) \right] F'(c)\, dc + \sum_{i=1,2} \Pi_i^1(\overline{C}, \overline{C})$$
$$+\beta \int_S c Q_l^2(c)\, F_l'(c)\, dc + \beta \int_S c Q_w^2(c)\, F_w'(c)\, dc.$$

To obtain the final expression stated in the Lemma, we add and subtract the term $2\beta \int F_l(x) Q_l^2(x)\, dx$. Observe also, that the voluntary participation constraint in period one requires that $\Pi_i^1(\overline{C}, \overline{C}) \geq \beta \int F_l(x) Q_l^2(x)\, dx$. ■

**Proof of Proposition 3.** Pointwise maximization of the procurement cost expression implies the stated allocation rule. The cost minimizing first period expected continuation payoff equals $\Pi_i^1(\overline{C}, \overline{C}) = \beta \int F_l(x) Q_l^2(x)\, dx$. ■

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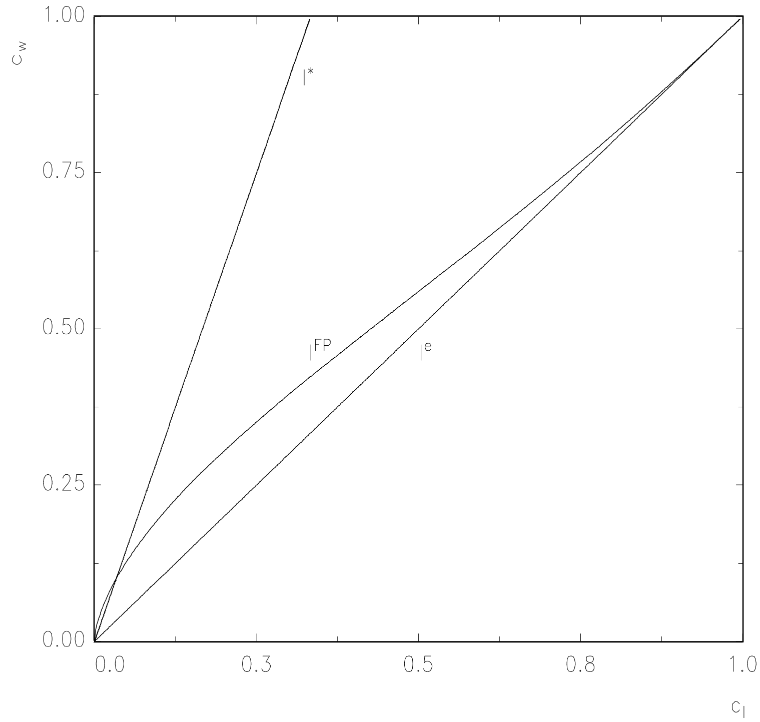


Figure 1: Substitutes

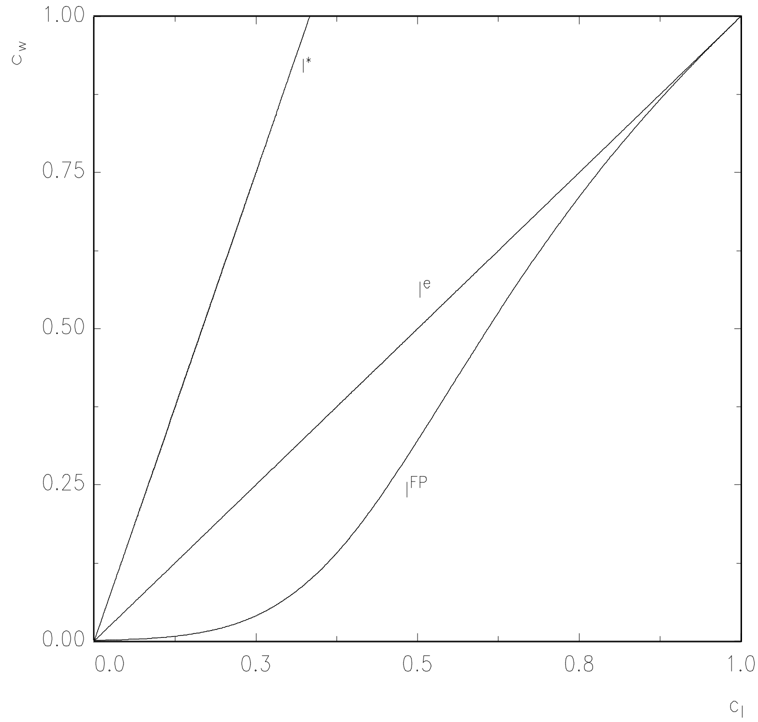


Figure 2: Complements