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Unique equilibrium in a dynamic model of crises with frictions

Working paper

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Abstract

This paper studies a dynamic model of crises with timing frictions that combines the main aspects of Morris and Shin (1998) and Frankel and Pauzner (2000). The usual arguments for existence and uniqueness of equilibrium cannot be applied. It is shown that the model has a unique equilibrium within the class of threshold equilibria.

KEYWORDS: Unique equilibrium, crises, dynamics, strategic complementarities.

JEL CLASSIFICATION: F3, D8

1 Introduction

In a currency crisis situation, the optimal strategy for an agent may depend on expectations: if everybody is expected to attack the currency, it is optimal to attack it, but if everybody is expected to refrain from doing so, that is the optimal choice. In a complete information setup, such models present multiple (Nash) equilibria. Building on Carlsson and Van Damme (1993), Morris and Shin (1998, MS hereafter) showed that a unique equilibrium arises in a model of self-fulfilling currency crises if agents observe the economic fundamentals with some private noise.

Tackling different economic problems, Frankel and Pauzner (2000, FP hereafter) and Burdzy, Frankel and Pauzner (2001) presented a dynamic model with strategic complementarities and timing frictions that has a unique equilibrium if fundamentals follow a Brownian motion. Strategic complementarities are usually thought of as the cause of multiple equilibria, but they are crucial for the uniqueness arguments in both FP and MS.

The model in this paper is essentially a dynamic version of MS with timing frictions instead of incomplete information. It can also be seen as a model of crisis using the framework developed by FP. Interestingly however, by simply combining the currency crisis model of MS and the dynamic framework developed by FP, we lose the property of strategic complementarities among agents' choices that is key for their results. Why is that? In a dynamic framework, the agents' actions influence not only the occurrence of a crisis but also the

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size of the damage. In MS, the size of a devaluation is the realization of a random variable and, thus, exogenous. Here, the stochastic process for the fundamentals is exogenous, the agents' actions depend on the level of fundamentals (as in MS) which makes the expected magnitude of a devaluation an endogenous variable (differently from MS). Thus if agents are more willing to hold the currency, fundamentals need to be worse to cause a crisis (as in MS), which reduces the probability of a crisis (as in MS) but increases the expected magnitude of the devaluation — which does not happen in the static model of MS. In some cases, the effect in the magnitude dominates the effect in the probability: it is possible to construct examples in which an agent would choose to attack the currency if all others were not doing so but would not attack if all other agents were expected to attack.

Therefore, the argument of FP cannot be employed in the context of this model. This paper provides a new argument to show the existence and uniqueness of an equilibrium in this model within the class of *threshold* equilibria.[1]

Since Krugman (1979) and Flood and Garber (1984), considerable attention has been devoted to the study of the dynamics of currency crisis. Some recent work (e.g., Broner (2001), Chamley (2003) and Dasgupta (2003)) incorporates incomplete information and learning in dynamic models of crisis. In a related work, Abreu and Brunnermeier (2003) shows how slow diffusion of information can lead to bubbles and crashes. The model in this paper does not aim at explaining what generates the frictions in the asset markets (they are there by assumption) but it provides an interesting framework for the study of problems in which coordination, dynamics and timing frictions are important. Variations of this model could be applied to several economic problems, like bank runs, currency and financial crises and riots.

A companion paper, Guimarães (2005), studies a variation of the model of Flood and Garber (1984) — with Poisson frictions — that is a particular case of the model in this paper. Guimarães (2005) aims at analyzing the impacts of asset market frictions in a standard first generation model of currency crises. It mentions the existence and uniqueness result, proved and explained here, and presents a brief discussion of the lack of strategic complementarities and the resulting technical issues covered in detail in the paper.

Section 2 sets out the model. Section 3 discusses some properties of the model and its relation with FP. Section 4 shows the existence and uniqueness of a threshold equilibrium.

2 The Model

A continuum of agents, with mass equal to 1, chooses either to go long or not in 1 unit of a currency — hereafter, we will denote agents' possible choices by *Long* and *Not*, respectively. The state of the economy is characterized by two state variables: (i) θ represents the

[1] This result is weaker than the obtained by FP and MS (they prove uniqueness of a rationalizable equilibrium) but still interesting and useful, as the assumption of Markov Perfection is very common in the applied literature.

fundamental imbalances and follows a Brownian motion ($d\theta = \mu_\theta dt + \sigma_\theta dX$); (ii) A is the fraction of agents currently long in the currency. There is a flow benefit of holding the currency, denoted by r, constant and exogenous. This benefit can be interpreted as the interest rate differential earned by choosing *Long*.[2]

Pressure for devaluation is higher when θ is higher (fundamentals are worse) and when A is lower (fewer people hold the currency). There exists an increasing function of A — $\tilde{\theta} : [0,1] \to [0,b]$, $\tilde{\theta}(0) = 0$, $\tilde{\theta}' > 0$ — such that a devaluation occurs when the curve $\tilde{\theta}$ is reached (see figure 1). A similar assumption is made by Morris and Shin (1998).[3] The function $\tilde{\theta}$ also corresponds to the zero-reserves curve of the first generation models of currency crises (e.g., Krugman (1979) and Flood and Garber (1984)) — the government is forced to abandon the pegged exchange rate regime when such bound is reached.[4] Analogously, Abreu and Brunnermeier (2003) assume that the bubble bursts when enough agents refrain from going long in the asset.

The function $\tilde{\theta}$, as well as the current values of A and θ, are common knowledge. Once a devaluation occurs, the balances of the agents long in the currency are multiplied by $D(\theta)$ and the game ends. It is assumed that $D(0) = 1$ and $D'(\theta) < 0$ (if, $\theta = 0$ there is no devaluation and worse fundamentals imply a higher devaluation).

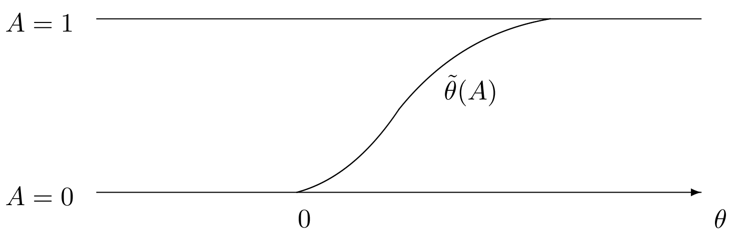

Figure 1: Threshold for occurrence of devaluation ($\tilde{\theta}(A)$)

Each agent gets the opportunity to change position according to an independent Poisson process with arrival rate δ, assumed to be greater than r. Thus: (i) an attack does not

[2]Under the assumption of risk neutrality, nothing substantial changes in the model if we allow agents to hold fractional units of the currency, as their optimal decision will always be in a corner, given the payoff structure. Also, if agents were choosing between going short or not, the analysis would be taken in the same way — A would be the fraction of agents *not short* and $\tilde{\theta}$ would also be increasing in A. Now, if the marginal utility of holding domestic deposits is decreasing, then agents' optimal decision could be much more complicated. Guimarães and Morris (2004) present a *static* model of currency crisis, similar to Morris and Shin (1998) but where agents are risk averse and have a continuous set of actions. They show that while the main results of Morris and Shin (1998) holds — there is a unique equilibrium, defined by a threshold — the equilibrium depends substantially on parameters related to risk aversion, wealth, portfolio composition and market incompleteness.

[3]In the static model of currency crises of Morris and Shin (1998), a threshold for the government behavior is obtained by assuming a fixed benefit of keeping the current regime and a cost of abandoning it increasing in θ and decreasing in A. However, in a dynamic framework, this assumption cannot be justified in the same way.

[4]See Guimarães (2005).

occurs instantaneously, it takes some (little) time until it forces the devaluation and (ii) agents are uncertain about whether they will be able to escape if there is a crisis. If, at a given moment, all agents decide to attack the currency, all are ex-ante equally likely to succeed in running before a devaluation occurs. But when $\tilde{\theta}$ is reached, some will be caught by the crisis while others will escape. As in MS, an agent's decision must take into account the risk of a devaluation, which depends on others' actions. The frictions and the assumption that the devaluation will not occur before $\tilde{\theta}$ is reached give some time for the agents to run and compete against each other.

2.1 The agent's problem

When the opportunity of changing position comes, the agent decides between *Long* or *Not*. His decision holds until the next random signal comes. Choosing *Not* yields 0 up to the next opportunity of changing portfolio.

In general, a strategy could assign a decision for every history, that is, for every path of (θ, A). However, an agent's payoff of investing depends just on the current state and on others' strategies, so past values of (θ, A) are relevant only if agents condition their actions on past events in an arbitrary way. Future work may answer if it is possible to construct equilibria in which the path of (θ, A) matters. This paper restricts its attention to strategies that depend only on the current state (Markov strategies). So, a strategy for the agent yields a decision (*Long* or *Not*) for every pair (θ, A), that is, $s : R^2 \rightarrow \{Long, Not\}$.

A process $\{x\} = \{(\theta, A)\}$ will denote a particular path of the state variables of the model, and z will denote a particular realization of the Brownian motion. Suppose that all other agents are following a strategy around a threshold θ^* as shown in figure 2. The threshold θ^* defines 2 regions: the 'L' area, at its left, where agents choose *Long* and the 'N' area, at its right, where agents choose *Not*. Let (θ_0, A_0) denote the current state of the economy. Call $\Delta t(z)$ the time it will take for the crisis and $\theta^{end}(z)$ the level of fundamental imbalances when the crisis occurs.

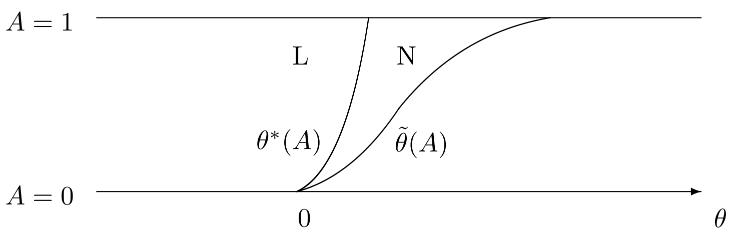

Figure 2: Investors's threshold for investing ($\theta^*(A)$)

The difference between the expected payoffs of choosing *Long* and *Not* (hereafter called 'expected payoff of choosing *Long*' to ease notation) equals the expected payoff of going long in the currency until the next opportunity of choosing — what happens after that is irrelevant because the present choice has no influence in future decisions. An agent that is long in the currency gets $\exp(r.dt)$ at every dt if a crisis has not occurred yet. But when it occurs, his balance is multiplied by $D(\theta)$. Given z and θ^*, the only source of uncertainty is the realization of the Poisson process, and the expected payoff of choosing *Long* is given by:

$$\pi(z;\theta_0,A_0,\theta^*) = \int_0^{\Delta t(z)} \delta e^{-\delta t} e^{rt} dt \ + \ e^{r\Delta t(z)}.D\left(\theta^{end}(z)\right) \int_{\Delta t(z)}^{\infty} \delta e^{-\delta t} dt \ - \ 1$$

The first term is what an agent gets if he receives a signal before time $\Delta t(z)$. The second term is the agent's return if he is caught by the crisis.

Doing the algebra, we obtain:

$$\pi(z;\theta_0,A_0,\theta^*) = \left(1 - e^{-(\delta-r)\Delta t(z)}\right)\frac{\delta}{\delta - r} + e^{-(\delta-r)\Delta t(z)}.D\left(\theta^{end}(z)\right) - 1 \tag{1}$$

Long is the optimal choice if $E\pi(\theta_0,A_0,\theta^*) \geq 0$ where:

$$E\pi(\theta_0,A_0,\theta^*) = \int_z \pi(z;\theta_0,A_0,\theta^*) f(z) dz \tag{2}$$

3 The lack of strategic complementarities

Frankel and Pauzner (2000) present a model of sectorial choice in which agents choose between 2 actions and get the opportunity to revise their decisions according to a Poisson process. The gains from using one technology depends positively on the fraction of agents that is currently employing it and on an exogenous fundamental variable, that follows a Brownian motion. It is shown that a unique strategy survives iterated deletion of strictly dominated strategies.

However, their results cannot be applied here because, in their framework, player's actions are *strategic complements*: the incentives to choose an action depend positively on the fraction of agents that are expected to choose it in the future. It could seem that such property would hold in this model. It is true that the likelihood of a crisis depends negatively on the fraction of agents that will choose *Long*. However, the magnitude of the devaluation depends *positively* on the fraction of agents expected to go long in the currency in the future. The choice of *Not* works as a 'discipline device', by preventing fundamental imbalances to get too high.

Figure 3-a shows an example to illustrate why the property of strategic complementarities does not hold in this model. All examples in this section assume $D(\theta) = \exp(-\theta)$.[5] Suppose that all other agents will choose *Not*. Then, an agent choosing *Long* at (θ =

[5] The other parameters used in this example are: $\delta = 1, r = 0.01, \mu_\theta = 0.1, \sigma_\theta = 0.1$ and $\tilde{\theta} = 0.25A^{\frac{1}{3}}$.

$-0.05, A = 0)$ has a positive expected payoff of around 0.0025, because the crisis will occur when θ hits 0, so there is no risk of losing money. On the other hand, if all other agents are choosing *Long*, her expected payoff drops to around -0.0200. The graph at figure 3-a shows an example of the path of θ when everybody picks *Long*, starting from the point marked with $*$: the possibility of getting a high devaluation makes going long in the currency a risky business.

The existence of *strategic complementarities* is key to the results of Frankel and Pauzner (2000) and also to the global games literature (see, for example, Morris and Shin, 2003).[6] If that property holds, the fact that *Long* is the optimal choice in a given state if everybody else is choosing *Not* implies that *Long* is a dominant strategy. Here that is not true. That is why the proof by Frankel and Pauzner (2000) based on iterative deletion of strictly dominated strategies cannot be applied here.

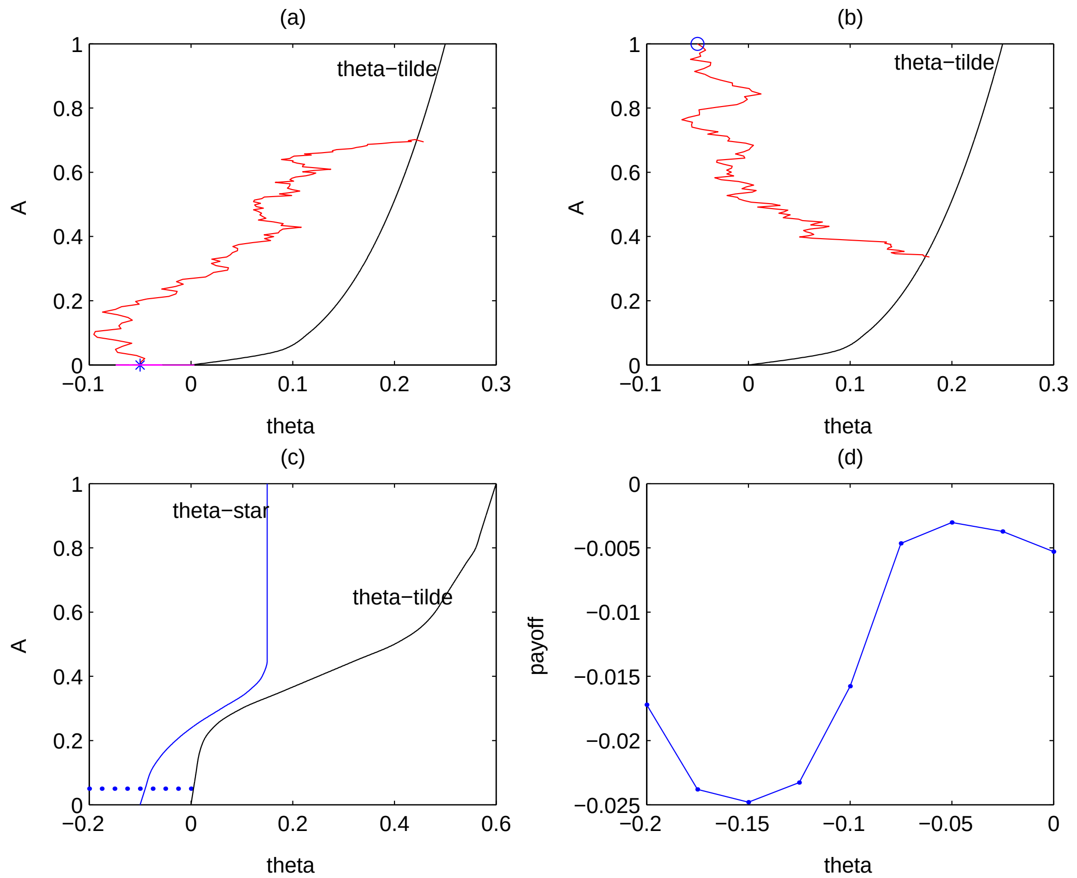

Figure 3: Counter examples for some usual assumptions

[6]Indeed, the argument for uniqueness in the global games model of Frankel, Morris and Pauzner (2003) and in the dynamic model of Frankel and Pauzner (2000) are similar.

It is worth mentioning some other complicating features of this model. First, $E\pi$ is not always increasing in A. Suppose the same parameter values used in figure 3-a. Assume that everybody else will choose *Not* and $\theta = -0.05$. If $A = 0$, choosing *Long* yields a positive payoff, but if $A = 1$, the expected payoff of *Long* is around -0.0200. The graph at 3-b shows an example of the path of θ when everybody picks *Not*, starting from the point marked with a circle. The large devaluation would not happen if nobody was long in the currency at the beginning.

Second, it is also possible to build examples in which $E\pi$ is not monotonically decreasing in θ. This assumption sounds very natural: the higher is θ (more fundamental imbalances), the less incentives an agent has to choose *Long*. That indeed holds *in equilibrium*, but it does not work for any threshold in any situation.

Graph 3-c shows an example of a situation in which payoff is sometimes *increasing* with respect to θ.[7] Graph 3-d shows the payoffs at points marked with a dot in graph 3-c. The key for this example are the high values of μ_θ and σ_θ. When the economy is close enough to $\tilde{\theta}$, in the 'L' area, the payoff is negative but small in absolute value because a small devaluation is very likely to come quickly. However, at those points further from 0, A and θ are likely to increase at the beginning and the devaluation is likely to be much greater and take not much time. So, choosing *Long* is worse in this case.

The last example may sound a bit contrived. Indeed, in equilibrium, payoffs are always decreasing in θ as we would expect. So, the failure of the such assumption may not have so interesting economic impacts. However, it presents theoretical difficulties to prove the existence and uniqueness of equilibrium.

Last, if the size of the devaluation is a known constant ($\forall\theta, D(\theta) = c < 1$) and θ influences only the occurrence of a crisis, there are strategic complementarities in the model and we can apply Frankel-Pauzner tools (the other 2 assumptions mentioned above also hold).

4 Equilibrium

In a threshold equilibrium, choosing according to the threshold is the optimal decision for an agent given that all others are doing so.

Definition 1 *A Threshold equilibrium is characterized by a continuous function $\theta^* : [0,1] \to (-\infty, \tilde{\theta}(1)]$. An agent deciding at time t (optimally) chooses* Long *if $\theta_t < \theta^*(A_t)$ and* Not *if $\theta_t > \theta^*(A_t)$.*

Theorem 1 *If $\sigma_\theta \neq 0$, there exists a unique threshold equilibrium.*

The proof of the theorem is fully presented in the appendix. It is divided in 3 propositions, that can be summarized as following:

[7] The parameters are: $\delta = 1, r = .01, \mu_\theta = 0.2, \sigma_\theta = 0.1$, and θ^* and $\tilde{\theta}$ are shown in the figure.

1. There exists a function θ^* such that $\forall a \in [0,1], E\pi(\theta^*(a), a; \theta^*) = 0$.
2. If lemma 1 holds, *Long* is the optimal strategy at the 'L' area and *Not* is the optimal strategy at the 'N' area. So, a threshold equilibrium exists.
3. There can't be more than one threshold equilibrium.

The proof of proposition 1 frames the problem in a way that allows it to apply the Schauder's fixed point theorem.[8] The existence of dominant regions and continuity of $E\pi(\theta^*(A), A; \theta^*)$ on θ^* are the 2 key factors of the proof.

The proof of proposition 2 starts by assuming that all agents are playing according to a threshold equilibrium around θ^*, such that $\forall a, E\pi(\theta^*(a), a; \theta^*) = 0$. At the 'L' area, players must have positive payoffs because when the economy gets to θ^*, they will be indifferent and, up to that moment, they will be profiting from the positive interest rate r. A key assumption for the argument is that the opportunity of changing position is a Poisson event.

At the 'N' area, the proof is trickier. It compares two processes: x', starting at $x'_0 = (\theta'_0, A)$, and x, starting at $x_0 = (\theta_0, A)$, such that $\theta_0 > \theta'_0 \geq \theta^*(A)$, as indicated at figure 4. The problem is that x' can get a lower payoff than x by achieving the 'L' area, going up in A, returning to the 'N'-area and getting a higher devaluation. It is shown that although that can indeed happen for some realizations of the Brownian motion z, the *expected* payoff of x' is always higher than the *expected* payoff of x. In order to get a higher devaluation, x' needs to go to the 'L' area and, then, cross again the threshold θ^* to enter the 'N' area. The key to the argument is that when x' crosses the threshold, its expected payoff is zero, *as if it was crossing* θ^* *at any other point.*

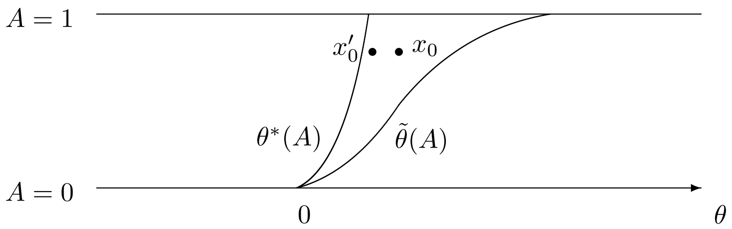

Figure 4: Points x_0 and x'_0

As the argument above illustrates, the proof of proposition 2 relies on the expected payoff at any point of θ^* being equal to zero. That does not hold in the examples shown at figure 3-c-d.

The proof of proposition 3 supposes the existence of 2 threshold equilibria, characterized by $\underline{\theta}$ and $\bar{\theta}$, respectively, and finds a contradiction. Denote by a a point that maximizes

[8] Schauder's fixed point theorem extends Brouwer's theorem to more generic metric spaces.

the horizontal distance between the two curves — indicated at figure 5 — and consider a process x' starting at $x'_0 = (a, \underline{\theta}(a))$ when all agents follow a switching strategy around $\underline{\theta}(A)$ and a process x starting at $x_0 = (a, \bar{\theta}(a))$ when all agents follow a switching strategy around $\bar{\theta}(A)$. The proof shows that the expected payoff of x is lower than the expected payoff of x', which contradicts the assumption that the expected payoff is zero at the equilibrium threshold.

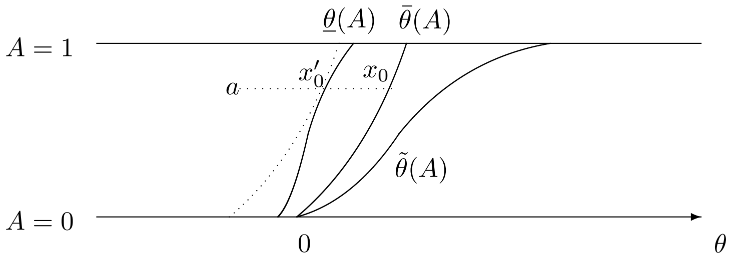

Figure 5: Functions $\bar{\theta}(A)$ and $\underline{\theta}(A)$ and point a

For any realization of the Brownian motion, at the first time x and x' are not at the same side of their own thresholds, the expected payoff of x is lower: either x' is over $\underline{\theta}$ (expected payoff is 0) and x is at its 'N' area (expected payoff is negative) or x' is at its 'L' area (expected payoff is positive) and x is over $\bar{\theta}$ (expected payoff is 0). If x and x' are always at the same side of the threshold until x gets to $\tilde{\theta}$, then, at that moment, x' will be at its 'N' area and will either get a smaller devaluation or hit $\underline{\theta}$ (which is the same as getting devaluation equal to 0).

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A Proof of Theorem 1

The proof starts with a few auxiliary lemmas.

Lemma 1 *Let x be a process starting at $x_0 = \{\bar{\theta}, a\}$ and let $\hat{\theta} : [0,1] \to (-\infty, \tilde{\theta}(1)]$ be a continuous function. Define t as the time that x hits the curve $\hat{\theta}$ for the first time. Let $\bar{\theta} \to \hat{\theta}(a)$. Then, $\forall\ \epsilon > 0$:*

$$\lim_{\bar{\theta} \to \hat{\theta}(a)} \Pr(t \leq \epsilon) \to 1 \tag{3}$$

Proof: this follows automatically from the fact that a standard Brownian motion changes sign infinitely many times in any time interval $[0, \epsilon]$, $\epsilon > 0$ (see, e.g., Karatzas and Shreve, 2000).

Lemma 2 *Consider a process x starting at time 0. Suppose that at time $\bar{t}$ the economy is at (θ, A). Then the (conditional) expected payoff of x is:*

$$E\pi(x;\theta^*) = \left(1 - e^{-(\delta-r)\bar{t}}\right)\frac{\delta}{\delta - r} + e^{-(\delta-r)\bar{t}}\left[E\pi(\theta, A;\theta^*) + 1\right] - 1 \tag{4}$$

Proof:

If the agent hasn't got the Poisson signal until $\bar{t}$, his expected payoff at $\bar{t}$ is $[E\pi(\theta, A;\theta^*)]$. But up to $\bar{t}$, the agent earns $e^{r\bar{t}}$. So, conditional on getting the Poisson signal after $\bar{t}$, the agent's expected payoff is:

$$e^{r\bar{t}}\left[E\pi(\theta, A;\theta^*) + 1\right] - 1$$

If the agent gets the Poisson signal at some $t < \bar{t}$, his payoff is $e^{rt} - 1$. So, conditional on getting the Poisson signal before $\bar{t}$, the agent's expected payoff is:

$$\frac{1}{1 - e^{-\delta\bar{t}}}\int_0^{\bar{t}} \delta e^{-\delta t}(e^{rt} - 1) = \frac{1}{1 - e^{-\delta\bar{t}}}\frac{\delta}{\delta - r}\left(1 - e^{-(\delta-r)\bar{t}}\right) - 1$$

The agent gets the signal before $\bar{t}$ with probability $1 - e^{-\delta\bar{t}}$ (and after $\bar{t}$ with probability $e^{-\delta\bar{t}}$). Summing the products of the conditional payoffs and their respective probabilities, we get the claim.$\square$

Equation 4 is similar to equation 1. The difference is that instead of having the value obtained at a terminal point, we have the expected payoff at a given state. This friendly formulation relies on the assumption of a Poisson process for δ.

Lemma 3 shows that we can work with a modified form of equation 2:

Lemma 3 *The expected payoff function can be written as:*

$$E\pi(\theta_0, A_0, \theta^*) = \int_z \pi^{mod}(z;\theta_0, A_0, \theta^*)f(z)dz \tag{5}$$

where $\pi^{mod}(z;\theta_0, A_0, \theta^*)$ is given by:

$$\pi^{mod}(z;\theta_0, A_0, \theta^*) = \left(1 - e^{-(\delta-r)\Delta t^{mod}(z)}\right)\frac{\delta}{\delta - r} + e^{-(\delta-r)\Delta t^{mod}(z)}D\left(\theta^{mod}(z)\right) - 1 \tag{6}$$

where $\Delta t^{mod}(z)$ is the time it takes for reaching either $\tilde{\theta}$ or *θ^* for the first time, and θ^{mod} is defined below:*

$$\theta^{mod} = \begin{cases} \theta^{end} & \textit{if } \tilde{\theta} \textit{ is reached before } \theta^* \\ 0 & \textit{if } \theta^* \textit{ is reached before } \tilde{\theta} \end{cases}$$

Proof:

Consider a process x starting at (θ, A) and a particular realization z of the Brownian motion. Clearly, if θ^* is not reached until the process ends at $\tilde{\theta}$, equation 6 is equal to equation 1. Now, suppose θ^* is reached before $\tilde{\theta}$. Call $\Delta t^{mod}(z)$ the time it takes for the economy to get to θ^* and a_z the value of A when θ^* is reached. Using equation 4, the expected payoff of x conditional on hitting θ^* at $(\theta^*(a_z), a_z)$ is:

$$\begin{aligned} E\pi(x;\theta^*) &= \left(1 - e^{-(\delta-r)\Delta t^{mod}(z)}\right)\frac{\delta}{\delta - r} + e^{-(\delta-r)\Delta t^{mod}(z)}\left[E\pi(\theta^*(a_z), a_z;\theta^*) + 1\right] - 1 \\ &= \left(1 - e^{-(\delta-r)\Delta t^{mod}(z)}\right)\frac{\delta}{\delta - r} + e^{-(\delta-r)\Delta t^{mod}(z)} - 1 \\ &= \left(1 - e^{-(\delta-r)\Delta t^{mod}(z)}\right)\frac{\delta}{\delta - r} + e^{-(\delta-r)\Delta t^{mod}(z)} D\left(\theta^{mod}(z)\right) - 1 \end{aligned}$$

if we define $\theta^{mod}(z) = 0$. The second equality comes from the expected payoff being equal to 0 at any point of θ^*.$\square$

Corollary 1 $\forall\ \theta, A, \theta^*,\ E\pi(\theta, A; \theta^*) > D\left(\tilde{\theta}(A)\right) - 1$

Corollary 1 comes from applying equation 6, noting that $\theta^{mod} \leq \tilde{\theta}(A)$.

Now, we will prove Proposition 1.

Proposition 1 *There exists a continuous function* $\theta^* : [0,1] \to (-\infty, \tilde{\theta}(1)]$*, such that* $E\pi(\theta^*(A), A; \theta^*) = 0$ *for any* $A \in [0,1]$.

The expected payoff function ($E\pi$) is defined only at the left of $\tilde{\theta}$. For analytical convenience, we will define a functional q that coincides with the expected payoff of investing at each point over the threshold $\hat{\theta}$ when all other agents are following a strategy according to $\hat{\theta}$ if the payoff is defined at that point:

$$q(\hat{\theta}, A) = \begin{cases} E\pi(\hat{\theta}(A), A; \hat{\theta}) & \text{if } \hat{\theta}(A) < \tilde{\theta}(A) \\ D\left(\tilde{\theta}(A)\right) - 1 - \left(\hat{\theta}(A) - \tilde{\theta}(A)\right) & \text{if } \hat{\theta}(A) \geq \tilde{\theta}(A) \end{cases}$$

First, it will be shown the existence of a function $\theta^* : [0,1] \to [\underline{\theta}, \bar{\theta}]$ such that $q(\theta^*, A) = 0$ for all values of $A \in [0,1]$.

The proof starts by arguing that q is a continuous mapping:

Lemma 4 *The mapping q is continuous in θ. That is:*

$$\lim_{\theta' \to \hat{\theta}} q(\theta', A) = q(\hat{\theta}, A)$$

Clearly, for a given A, $q(., A)$ is continuous whenever $\hat{\theta}(A) > \tilde{\theta}(A)$. Now, note that $q(., A)$ is continuous at $\tilde{\theta}(A)$. Due to lemma 1, as $\theta(A) \to \tilde{\theta}(A)$, the probability of an 'instantaneous' devaluation approaches 1. That is, when $\theta(A) \to \tilde{\theta}(A)$, for almost all realizations of z, $\Delta t(z) \to 0$ and $\theta^{end}(z) \to \tilde{\theta}(A)$. Using equations 1 and 2, we get that:

$$lim_{\theta(A)\to\tilde{\theta}(A)} E\pi\left(\theta(A), A; \hat{\theta}\right) = D\left(\tilde{\theta}(A)\right) - 1 \text{ for any } \hat{\theta}$$

Now, it needs to be shown that:

$$lim_{\hat{\theta}'\to\hat{\theta}} E\pi\left(\hat{\theta}'(A), A; \hat{\theta}'\right) = E\pi\left(\hat{\theta}(A), A; \hat{\theta}\right)$$

First, note that for any $\epsilon_1 > 0$, there exists some $\bar{\Delta}t$ such that $\forall\ \Delta t(z) > \bar{\Delta}t$, $\pi(z, \theta_0, A_0; \hat{\theta}) \in \left(\frac{\delta}{\delta - r} - \epsilon_1, \frac{\delta}{\delta - r}\right)$ (see equation 1).

Now, consider two processes, $x = (\theta, A)$ starting at $x_0 = \left(\hat{\theta}(A_0), A_0\right)$ and moving according to the threshold $\hat{\theta}$, $x' = (\theta', A')$ starting at $x_0' = \left(\hat{\theta}'(A_0), A_0\right)$ and moving according to the threshold $\hat{\theta}'$, both following a given realization of the Brownian motion, z. Let $\hat{\theta}' \to \hat{\theta}$. Then, for almost all z, and for processes lasting less than $\bar{\Delta}t$, the amount of time x and x' will be at different sides of their own thresholds is arbitrarily small because, due lemma 1, whenever one of the processes crosses its threshold, the other will do the same in arbitrarily small time with probability one. Thus, $\forall\ \epsilon_2 > 0$, after some time $\Delta t < \bar{\Delta}t$, $\Pr\left(|A - A'| \le \epsilon_2\right) \to 1$. Clearly, $\theta - \theta' = \hat{\theta}(A_0) - \hat{\theta}'(A_0)$, which approaches zero by assumption.

Now, suppose WLOG that x reaches the threshold $\tilde{\theta}$ before x'. Suppose $\Delta t(z) < \bar{\Delta}t$. Then, given that $(\theta', A') \to (\theta, A)$ and lemma 1, $\Delta t'(z) \to \Delta t(z)$ and $\theta'^{end} \to \theta^{end}$ for almost all z, which implies, for almost all z, that $\pi(z, \hat{\theta}'(A_0), A_0; \hat{\theta}) \to \pi(z, \hat{\theta}(A_0), A_0; \hat{\theta})$ (see equation 1). Using equation 2, $E\pi(\hat{\theta}'(A_0), A_0; \hat{\theta}) \to E\pi(\hat{\theta}(A_0), A_0; \hat{\theta})$. $\Box$

A similar argument can show that for a given continuous function $\hat{\theta}$, q is continuous in A.

Now, note that:

1. There exists a number θ^+ such that $\forall\ A$ and $\hat{\theta}$, if $\hat{\theta}(A) < \theta^+$, $q(\hat{\theta}, A) > 0$.
2. There exists a number θ^- such that $\forall\ A$ and $\hat{\theta}$, if $\hat{\theta}(A) > \theta^-$, $q(\hat{\theta}, A) < 0$.

Define:

$$\mathcal{F} = \left\{\hat{\theta} \mid \hat{\theta} \text{ is continuous and } \hat{\theta} : [0,1] \to [\underline{\theta}, \bar{\theta}]\right\} \text{ and}$$

$$y : \mathcal{F} \to \mathcal{F}\ ,\ y(\hat{\theta}) = \hat{\theta} + \alpha q(\hat{\theta}) \text{ for some small } \alpha$$

where $\underline{\theta} < \theta^+$ and $\bar{\theta} > \theta^-$ as indicated in figure 6. For small enough α, the image of the function $y(\hat{\theta})$ is always inside $[\underline{\theta}, \bar{\theta}]$ given the behavior of function q.

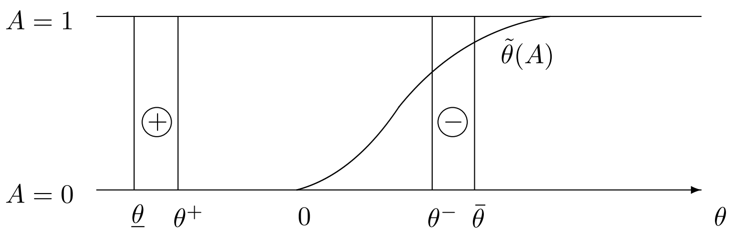

Figure 6: θ^+, θ^-, $\underline{\theta}$ and $\bar{\theta}$

By lemma 4, y is a continuous mapping of $\mathcal{F}$. The set $\mathcal{F}$ is convex and compact. Thus, we can apply Schauder's fixed point Theorem.[9] So there exists a continuous θ^* such that:

$$y(\theta^*) = \theta^* + \alpha q(\theta^*) = \theta^* \ \Rightarrow \ q(\theta^*) = 0$$

Now, $\forall\, A > 0$, if $\hat{\theta}(A) > \tilde{\theta}(A)$, $q(\hat{\theta}, A) < 0$. Thus, there must exist a continuous function $\theta^* : [0,1] \rightarrow (-\infty, \tilde{\theta}(1)]$, such that $E\pi(\theta^*(A), A; \theta^*) = 0$ for any $A \in [0,1]$, which is the claim. $\square$

Proposition 2 *If Proposition 1 holds,* Long *is the optimal choice at the 'L' area and* Not *is the optimal choice at the 'N' area. At θ^*, the agent is indifferent.*

Proposition 2 is a consequence of 2 lemmas. Lemma 5 shows that the payoff at the 'L' area is always positive and decreasing in θ. Lemma 6 shows that the payoff at the 'N' area is always negative and decreasing in θ.

Lemma 5 *For all (θ, A) in the 'L' area:*

1. *$E\pi(\theta, A, \theta^*) > 0$,*
2. *$E\pi(\theta, A, \theta^*)$ is decreasing in θ.*

Proof:

Any process starting in the 'L' area will reach θ^* before reaching $\tilde{\theta}$. For any process z, call $t_1(z)$ the time it takes for reaching θ^*. Equation 6 simplifies to:

$$\pi(z; \theta_0, A_0, \theta^*) = \left(1 - e^{-(\delta - r)t_1(z)}\right) \frac{\delta}{\delta - r} + e^{-(\delta - r)t_1(z)} - 1$$

Using equation 5, we get:

$$E\pi(\theta, A; \theta^*) = \int_z \frac{r}{\delta - r} \left(1 - e^{-(\delta - r)t_1(z)}\right) f(z) dz \ > \ 0 \tag{7}$$

[9] see, e.g., Smart (1974).

which proves the first statement.

Now, consider two processes: x, starting at (θ, A) and x', starting at (θ', A), such that $\theta' > \theta > \theta^*(A)$. Call $t_1(z)$ and $t_1'(z)$, the time it takes for x and x' to reach the threshold θ^*, respectively. It is easy to see that for any z, $t_1(z) < t_1'(z)$. The difference of expected payoffs is given by:

$$\begin{aligned}
\Delta\pi &= E\pi(\theta', A; \theta^*) - E\pi(\theta, A; \theta^*) \\
&= \int_z \frac{r}{\delta - r} \left(e^{-(\delta-r)t_1(z)} - e^{-(\delta-r)t_1'(z)} \right) f(z) dz \\
&> 0
\end{aligned}$$

which completes the proof. □

Lemma 6 *For all (θ, A) in the 'N' area:*

1. $E\pi(\theta, A, \theta^*) < 0$,
2. $E\pi(\theta, A, \theta^*)$ *is decreasing in θ.*

Proof:

Consider two processes: x, starting at (θ, A) and x', starting at (θ', A), such that $\theta^*(A) \geq \theta' > \theta$.

Consider also a process x'' that starts at starting at (θ', A) (with x') but moves in a different way, following the *x"-rules*: until x reaches either θ^* or $\tilde{\theta}$ for the first time, A'' always decreases as if the economy was at the 'N' area (*x"-rule with respect to A*); and θ follows the Brownian motion z except that it never crosses the threshold θ^* to get to the 'L' area, as shown at figure 7. Whenever the Brownian motion would push θ to the 'L' area, θ stays over θ^* (*x"-rule with respect to θ*). After x hits θ^* or $\tilde{\theta}$, x'' behaves in the regular way (as x and x').

The proof of proposition 6 comes in 2 parts:

1. $E\pi(x'; \theta^*) \ > \ E\pi(x'', \theta^*)$
2. $E\pi(x''; \theta^*) \ > \ E\pi(x, \theta^*)$

First part: $E\pi(x'; \theta^*) \ > \ E\pi(x'', \theta^*)$

By lemma 5, the expected payoff at any point in the 'L'-area is positive. By lemma 1, the expected payoff at any point over θ^* is zero. Thus, the *x"-rule with respect to θ* is automatically substituting states that yield a positive payoff by states that yield zero payoff. The x'' *rule with respect to A* has no effect on payoffs because the payoff at any point over θ^* is zero.

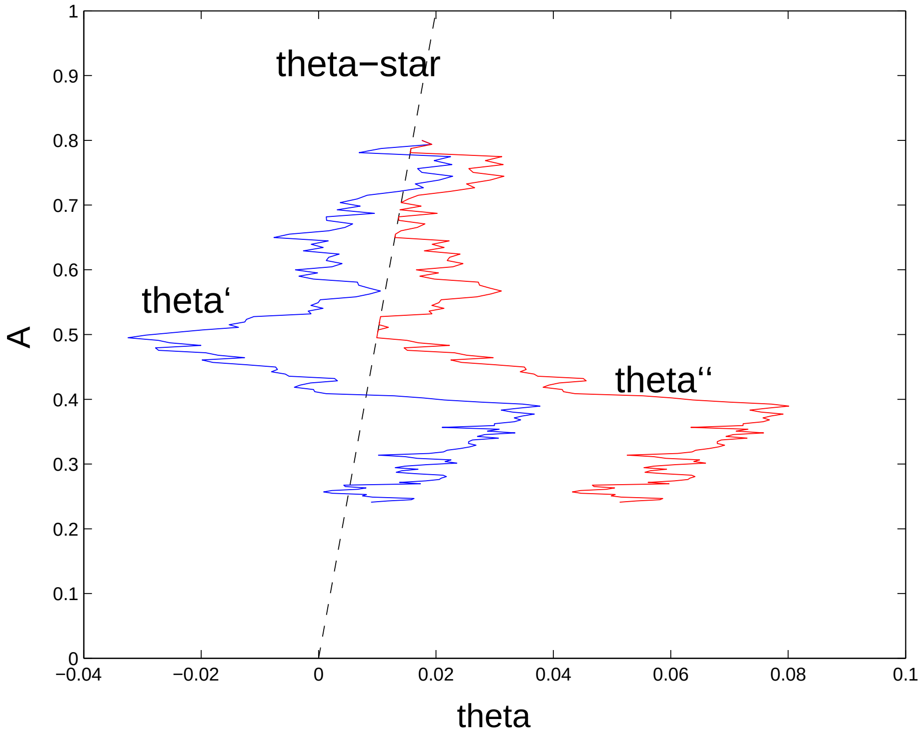

Figure 7: Path of θ' and θ''

Due to the assumption of a Poisson process, the expected payoff at any point can be written as a function of the expected payoff of its reachable states (weighted by the probability of reaching each state). Thus, the expected payoff of x' must be greater than the expected payoff of x''. The strict inequality depends on the fact that x' will reach θ^* with positive probability.

Second part: $E\pi(x''; \theta^*) \ > \ E\pi(x, \theta^*)$

Suppose that x reaches θ^*. Then, from that point on, paths of x and x'' coincide.

Suppose that x gets to $\tilde{\theta}$ (without never hitting θ^*) at time t. From that point on, x'' follows the regular laws of motion of the game. From equation 4 and corollary 1, expected payoff of x'' in this case is greater than expected payoff of x.

The strict inequality depends on the fact that x' will reach θ^* with probability smaller than one.

Combining both parts, we get the claim. $\square$

Proposition 3 *There is not more than one threshold equilibrium.*

Proof: Suppose 2 equilibria, one with threshold $\underline{\theta}(A)$ and other with threshold $\bar{\theta}(A)$. Define the point a such that $a = \text{argmax}\{|\bar{\theta}(A) - \underline{\theta}(A)|\}$ as shown in figure 5.

Suppose a process x' starting at $x'_0 = (a, \underline{\theta}(a))$ when all agents follows a switching strategy around $\underline{\theta}(A)$ and a process x starting at $x_0 = (a, \bar{\theta}(a))$ when all agents follow a switching strategy around $\bar{\theta}(A)$. If both curves do not coincide, the expected payoff of x' is strictly greater that the expected payoff of x, which contradicts the fact that investors are indifferent at all points of those thresholds.

Let Γ be the set of realizations of the Brownian motion (z) such that x and x' are never at different sides of their own thresholds and Γ' its complement.

For any $z \in \Gamma$, we know that x will get to $\tilde{\theta}$ before x' and the value of A will be the same for both processes (as x and x' were always at the same side of their own thresholds). By corollary 1 and equation 4, we get:

$$\int_{z\in\Gamma} \left[\pi(z; x'_0, \underline{\theta}) - \pi(z; x_0, \bar{\theta})\right] \; f(z)dz \; > \; 0 \tag{8}$$

For all $z \in \Gamma'$, call $t^1(z)$ the first moment in which x and x' are not at the same side of their own thresholds. It will happen when one process (x or x') hits its own threshold while the other has not reached it. As a maximizes the distance between curves, it must be true that $x'_{z,t^1(z)}$ will be at the 'L' area or over $\underline{\theta}$, which implies:

$$\int_{y\in\Omega} \pi(y; x'_{z,t^1(z)}, \underline{\theta}) f(y)dy > 0$$

while $x_{z,t^1(z)}$ will be at the 'N' area or over $\bar{\theta}(A)$ and so:

$$\int_{y\in\Omega} \pi(y; x_{z,t^1(z)}, \bar{\theta}) f(y)dy \; < 0$$

Combining both inequalities and considering there is no difference in payoffs for agents that got a signal before $t^1(z)$, we get:

$$\int_{z\in\Gamma'} \left[\pi(z; x'_0, \underline{\theta}) - \pi(z; x_0, \bar{\theta})\right] \; f(z)dz = \tag{9}$$
$$\int_{z\in\Gamma'} e^{-(\delta-r)t^1(z)} \left(\int_{y\in\Omega} \left[\pi(y; x_{z,t^1(z)}, \underline{\theta}) - \pi(y; x_{z,t^1(z)}, \bar{\theta})\right] f(y)dy \right) f(z)dz \; > \; 0$$

Summing (8) and (9), we get a contradiction. $\square$