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Point process bridges and weak convergence of insider trading models[*]

Umut Çetin[†]   Hao Xing[‡]

Abstract

We construct explicitly a bridge process whose distribution, in its own filtration, is the same as the difference of two independent Poisson processes with the same intensity and its time 1 value satisfies a specific constraint. This construction allows us to show the existence of Glosten-Milgrom equilibrium and its associated optimal trading strategy for the insider. In the equilibrium the insider employs a mixed strategy to randomly submit two types of orders: one type trades in the same direction as noise trades while the other cancels some of the noise trades by submitting opposite orders when noise trades arrive. The construction also allows us to prove that Glosten-Milgrom equilibria converge weakly to Kyle-Back equilibrium, without the additional assumptions imposed in *K. Back and S. Baruch, Econometrica, 72 (2004), pp. 433-465*, when the common intensity of the Poisson processes tends to infinity.

1 Introduction

In this paper we perform an explicit construction of a particular bridge process associated to a point process that arises in the solution of Glosten-Milgrom type insider trading models from Market Microstructure Theory. Our starting point is the work of Back and Baruch [4] who studies a class of equilibrium models of insider trading (of Glosten-Milgrom type) and their convergence to Kyle model.

In Glosten-Milgrom type insider trading models, there exists an insider who possesses the knowledge of the time 1 value of the asset given by the random variable $\tilde{v}$. There is also another class of traders, collectively known as noise traders, who trade without this insider knowledge. Their trades are of the same size and arrive at Poisson

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times which are assumed to be independent of $\tilde{v}$. The insider trades using her extra information in order to maximise her expected wealth at time 1 but taking into account that her trades move the prices to her disadvantage since the price is an increasing function of the total demand for the asset. Moreover, in order to hide her trades, and thus her private information, she will also submit orders that are of the same size as noise trades. The price of the asset in this market is determined by a market maker in the equilibrium whose precise definition is given in Section 2.

In the specific model that we will study (and also studied in [4]) $\tilde{v}$ takes values in $\{0, 1\}$. Since the noise buy and sell orders arrive at Poisson times and are of the same size, the net Z of cumulative buy and sell noise trades, after normalization, is given by the difference of two independent Poisson processes. Writing $Y = Z + X$ for the total demand for the asset, where X denotes the trading strategy of the insider, we will see in Theorem 3.4 that a Glosten-Milgrom equilibrium exists if

(i) Y in its own filtration has the same distribution as Z,

(ii) $[Y_1 \geq y] = [\tilde{v} = 1]$ almost surely for some y to be determined.

The second condition above implies that in the equilibrium the insider drives the process Y so that the event whether Y_1 is larger than y is predetermined at time 0 from the point of view of the insider, since the set $[\tilde{v} = 1]$ is at the disposal of the insider already at time 0. Given this characteristic of Y, it can be called (with a slight abuse of terminology) a *point process bridge*.

In Section 4, we explicitly construct a pure jump process X whose jump size is the same as that of Z and $Y = X + Z$ satisfies aforementioned conditions. From the point of view of filtering theory X can be considered as the unobserved 'drift' added to the martingale Z. The specific choice of X used in the bridge construction ensures that this drift *disappears* when we consider Y in its own filtration.

To the best of our knowledge such a bridge construction has not been studied in the literature before. On the other hand, the analogy with the enlargement of filtration theory for Brownian motion is obvious. Indeed, if Z is instead a Brownian motion and we consider the problem of finding a stochastic process X so that $Y = Z + X$ is a Brownian motion in its own filtration and $[\tilde{v} = 1] = [Y_1 \geq y]$ almost surely for some $y \in \mathbb{R}$ to be determined, the solution follows easily from the enlargement of filtration theory. The recipe is the following: Find the Doob-Meyer decomposition of Z when its natural filtration is initially enlarged with the random variable $[Z_1 \geq y]$. Then, in the finite variation part of this decomposition, replace Z with Y and $[Z_1 \geq y]$ with $[\tilde{v} = 1]$ to find X. This recipe gives

$$X = \mathbb{I}_{[\tilde{v}=1]} \int_0^{\cdot} \partial_y \log p^0(Y_s, s) \, ds + \mathbb{I}_{[\tilde{v}=0]} \int_0^{\cdot} \partial_y \log(1 - p^0(Y_s, s)) \, ds, \qquad (1.1)$$

where p^0 is the function given in (5.1). From the insider trading point of view, X defined by (1.1) is the insider's optimal trading strategy in a Kyle model, see Remark 5.2 in this respect. The counterpart of these arguments in the theory of enlargement of filtrations for jump processes also exists in the literature, see [13].

Yet the above recipe does not work when Z is the difference of independent Poisson processes. The problem is that the enlargement of filtration technique gives us the decomposition of Z as a sum of a martingale and an absolutely continuous process. This is clearly not useful for the construction that we are after, since we want to write Y as sum of Z and X which changes only by jumps. The desired jump process X is constructed explicitly in Section 4 using $[\tilde{v} = 1]$ and a sequences of iid uniformly distributed random variables independent of everything else. This amounts to say that the insider uses her private information and some additional randomness from uniformly

distributed random variables to construct her optimal strategy. Moreover, we will see in Section 5 that, after an appropriate rescaling, these jump processes converge weakly to X given by (1.1) as the intensity of the Poisson processes that constitute Z increases to infinity. Note the process X given in (1.1) does not need any extra randomness other than the set $[\tilde{v} = 1]$. This brings fore the question whether the bridge process defined in Section 4 can alternatively be constructed without the aid of the extra randomness. We believe this would be a quite interesting avenue for further research.

The construction of the point process bridge Y allows us to prove the existence of Glosten-Milgrom equilibrium (see Theorem 5.1) which was demonstrated in [4] via a numeric computation. In such an equilibrium the insider uses a mixed strategy to randomly submit two types of orders: one type trades in the same direction as noise trades while the other cancels noise trades by submitting opposite orders when noise trades arrive. Observing noise trades, the insider uses the uniformly distributed random variables to construct her strategy inductively. On the other hand, the construction of Y invites a natural application of weak convergence theory to show Glosten-Milgrom equilibria converge weakly to Kyle equilibrium when the intensity of Z increases to infinity. This convergence was first proved in [4] under strong assumption on the convergence of value functions. Utilising the theory of weak convergence, we are able to prove the result of Back and Baruch on convergence without the additional assumptions; see Theorem 5.3.

The outline of the paper is as follows. In Sections 2 and 3 we describe the Glosten-Milgrom model and characterise its equilibrium which is the motivation of this paper. Section 4 discusses the construction of the aforementioned point process bridge. In Section 5 we apply the results of Section 4 to show the existence of Glosten-Milgrom equilibria and discuss their weak convergence.

2 The model

We consider a market in continuous-time for a risky asset whose *fundamental value* is given by $\tilde{v}$. The investors in this market can also trade a riskless asset at an interest rate normalised to 0 for simplicity. Following [4] we assume that $\tilde{v}$ has two states: high and low, which correspond to two numeric representations respectively, 1 and 0. This fundamental value will be revealed to the market participants at time 1 at which point we assume the market for the risky asset will terminate[1].

The microstructure of the market, and the interaction of market participants, is modelled similarly as in [4]. There are three types of agents: noisy/liquidity traders, an informed trader (insider), and a market maker, all of whom are risk neutral. All the processes and random variables in this section are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$ satisfying the usual conditions. We assume that $\tilde{v}$ is indeed random, i.e. $\mathbb{P}(\tilde{v} = 0) \in (0, 1)$.

- *Noisy/liquidity traders* trade for liquidity reasons, and their total demand is given by the difference of two pure jump processes Z^B and Z^S, which represent their cumulative buy and sell orders, respectively. As such, the net order flow of the noise traders are given by $Z := Z^B - Z^S$. Noise traders only submit orders of fixed size δ every time they trade. As in [4], Z^B/δ and Z^S/δ are assumed to be independent Poisson processes with constant intensity β. Moreover, they are independent of $\tilde{v}$.

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[1][4] assumes that the market has a random horizon defined by an independent exponential random variable. However, one can see that this distinction is not relevant by comparing our results to those of Back and Baruch.

- *The informed trader* observes the market price process and is given the value of $\tilde{v}$ at time 0. The net order of the insider is denoted by $X := X^B - X^S$ where X^B (resp. X^S) denotes the cumulative buy (resp. sell) orders of the insider.

- A competitive *market maker* observes only the total net demand process $Y_t = X_t + Z_t$ and sets the price based solely on this information. This in particular implies that the market maker's filtration is $(\mathcal{F}_t^Y)$, the minimal filtration generated by Y satisfying the usual conditions. We assume that the market maker is risk neutral and, thus, the competitiveness means that he sets the price at $\mathbb{E}[\tilde{v}|\mathcal{F}_t^Y]$ in the equilibrium.

Although the noise traders trade for liquidity reasons exogenous to this model, the insider has the objective to maximise her expected profit out of trading. This strategic behaviour of the insider and the pricing mechanism set by the market maker as described above results in the price being determined in an equilibrium. In order to define precisely what we mean by an equilibrium between the market maker and the insider, we first need to establish the class of *admissible* actions available to both.

Definition 2.1. A function $p : \delta\mathbb{Z} \times [0,1] \to [0,1]$ is a *pricing rule* if

i) $y \mapsto p(y,t)$ is strictly increasing for each $t \in [0,1)$;
ii) $t \mapsto p(y,t)$ is continuously differentiable for each $y \in \delta\mathbb{Z}$.

This Markov assumption on the pricing functional is standard in the literature (see, e.g., [2], [6] or [8]). Given the pricing rule, the market maker sets the price to be $p(Y_t, t)$. It would be irrational for the market maker to price the asset at some value larger than 1 or less than 0 since everybody knows that the true value of the asset is 0 or 1. As we mentioned above the market maker is competitive so that in equilibrium the price equals $\mathbb{E}[\tilde{v} \mid \mathcal{F}_t^Y]$. Hence, p is typically $[0,1]$-valued. The monotonicity of $p(\cdot, t)$ implies that an increase in demand has a positive feedback on the asset price. Moreover, this leads the insider to fully observe the noise trades, Z, by simply inverting the price process and subtracting her own trades from it. Consequently, the insider's filtration, denoted with $\mathcal{F}^I$, contains the filtration generated by Z and $\tilde{v}$. We shall assume $\mathcal{F}^I$ satisfies the usual conditions. However, we refrain from setting $\mathcal{F}^I$ equal to the filtration generated by Z and initially enlarged with $\tilde{v}$ since we will only be able to show the existence of equilibrium if the insider also possess a sequence of independent random variables, which she will use in order to construct her mixed strategy. Admissible strategy of the insider is defined as follows.

Definition 2.2. The strategy $(X^B, X^S; \mathcal{F}^I)$ is *admissible*, if

i) $\mathcal{F}^I$ is a filtration satisfying the usual conditions such that $\mathcal{F}_t^I = \sigma(v, \mathcal{F}_t^Z, \mathcal{H}_t)$, where $\mathcal{H}$ is a filtration independent of v and $\mathcal{F}^Z$.
ii) X^B and X^S, with $X_0^B = X_0^S = 0$, are $\mathcal{F}^I$-adapted and integrable[2] increasing point processes with jump size δ;
iii) the $(\mathcal{F}^I, \mathbb{P})$-dual predictable projections[3] of X^B and X^S are absolutely continuous functions of time.

The first assumption on $\mathcal{F}^I$ makes the insider's filtration part of the equilibrium. This is to allow mixed strategies which will be determined in equilibrium. Note that the additional information can only come from a source that is independent of Z. This implies in particular that the insider does not have any extra information about the future

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[2]That is, $\mathbb{E}[X_1^B]$ and $\mathbb{E}[X_1^S]$ are both finite.
[3]These are simply the predictable compensators of the increasing processes X^B and X^S. See, e.g. [11] for a precise definition.

demand of the noise traders. Although we allow this additional source of information to vary in time, in the form of filtration $\mathcal{H}$, in the equilibrium that we will compute, $\mathcal{H}_t = \mathcal{H}_0$ for all $t \in [0, 1]$.

We assume that the insider can only trade δ-shares of the asset in every trade like the noise traders. This is one of the underlying assumptions of the Glosten-Milgrom model, which we keep in this paper as well. One intuitive reason for this is that a rational insider will never submit an order of a different size, since this will immediately reveal her identity and make, at least a part of, her private information public causing to lose her comparative advantage. Moreover, in order to make this argument rigorous one needs to make assumptions on the pricing rule as to how to handle the orders of sizes which are multiples of δ. One can do the pricing uniformly, i.e. every little bit of the order is priced the same, or different parts of the order is priced differently as one walks up or down in an order book (see [5] for a discussion of such issues). However, this requires different techniques for the analysis of optimal strategies given this complicated nature of pricing; thus, we leave such analysis to a future investigation.

The third assumption on the dual predictable projections implies that X^B and X^S admit $\mathcal{F}^I$-intensities θ^B and θ^S such that $X^B - \int_0^{\cdot} \theta_s^B \, ds$ and $X^S - \int_0^{\cdot} \theta_s^S \, ds$ are $\mathcal{F}^I$-martingales (see [12, Chapter 1, Theorem 3.15]). This assumption is technical and to ensure tractability.

Given that the insider submits orders of size δ and the assumption that the market maker observes only the net demand, we see that when the insider submits an order at the same as when an uninformed order arrives, but in the opposite direction (i.e. a trade between the informed and uninformed occurs without needing a market maker) this transaction goes unnoticed by the market maker. Thus, what we are effectively assuming is that the market maker only becomes aware of the transaction when there is a need for him. The assumption that the market maker only observes net demand is a common assumption in market microstructure literature. In particular, it is always assumed in Kyle type models (see, e.g. [3]). Henceforth, when the insider makes a trade at the same with an uninformed trader but in an opposite direction, we will say that the insider *cancels* the noise trades.

Although we allow the insider to trade at the same time with the noise traders in the same direction, we will see that in the equilibrium the insider will not carry such trades. This is intuitive. does not trade in the same direction at the same time as the uniformed trades, but she does randomly cancel part of uninformed orders. Both actions are required to hide her identity from the market maker. Indeed, when two buy orders arrive at the same time the market maker will know that one of them is an informed trade. Therefore it would be to the advantage of the insider to hide her trades by submitting randomly, but of the same size, among the uninformed trades. On the other hand, since the market maker is not aware of the transactions which consist in canceling noise trades, submitting an order at the same time with the noise traders but in the opposite direction is not necessarily suboptimal. We will in fact see that the insider does randomly cancel some trades that are placed by the noise traders in the equilibrium.

As discussed in the last paragraphs, the insider's buy orders X^B consist of three components: we denote by $X^{B,B}$ the cumulative buy orders which arrive at different time than those of Z^B, by $X^{B,T}$ the cumulative buy orders which arrive at the same time as some orders of Z^B, and by $X^{B,S}$ the cumulative buy orders which cancel some sell orders of Z^S. As such, the jump time of $X^{B,T}$ (resp. $X^{B,S}$) are contained in the set of jump times of Z^B (resp. Z^S). Sell orders $X^{S,S}, X^{S,T}$, and $X^{S,B}$ are defined analogously. Therefore $X^B = X^{B,B} + X^{B,T} + X^{B,S}$ and $X^S = X^{S,S} + X^{S,T} + X^{S,B}$.

As mentioned earlier, the insider aims to maximise her expected profit. Given an

admissible trading strategy (X^B, X^S) the associated profit at time 1 of the insider is given by

$$\int_0^1 X_{t-} \, dp(Y_t, t) + (\tilde{v} - p(Y_1, 1)) X_1.$$

The last term appears due to a potential discrepancy between the market price and the liquidation value. Since X is of finite variation, an application of integration by parts rewrites the above as

$$\int_0^1 (\tilde{v} - p(Y_t, t)) \, dX_t^B - \int_0^1 (\tilde{v} - p(Y_t, t)) \, dX_t^S$$

$$= \int_0^1 (\tilde{v} - p(Y_{t-} + \delta, t)) \, dX_t^{B,B} + \int_0^1 (\tilde{v} - p(Y_{t-} + 2\delta, t)) \, dX_t^{B,T} + \int_0^1 (\tilde{v} - p(Y_{t-}, t)) \, dX_t^{B,S}$$

$$- \int_0^1 (\tilde{v} - p(Y_{t-} - \delta, t)) \, dX_t^{S,S} - \int_0^1 (\tilde{v} - p(Y_{t-} - 2\delta, t)) \, dX_t^{S,T} - \int_0^1 (\tilde{v} - p(Y_{t-}, t)) \, dX_t^{S,B},$$

where the last line is due to the fact that Y increases by δ when $X^{B,B}$ jumps, increases by 2δ when $X^{B,T}$ jumps, and is unchanged when $X^{B,S}$ and Z^S jump at the same time but different directions. Similar situation goes for negative jumps of Y. As seen from the above formula, the profit is zero when the insider place two opposite orders as the same time, we then assume without loss of generality that insider does not do so.

Let's define

$$a(y, t) := p(y + \delta, t) \qquad \text{and} \qquad b(y, t) = p(y - \delta, t).$$

Then, the expected profit of the insider conditional on her information equals

$$\mathbb{E}_{\mathbb{P}} \left[\int_0^1 (\tilde{v} - a(Y_{t-}, t)) \, dX_t^{B,B} + \int_0^1 (\tilde{v} - a(Y_{t-} + \delta, t)) \, dX_t^{B,T} + \int_0^1 (\tilde{v} - p(Y_{t-}, t)) \, dX_t^{B,S} \right.$$

$$\left. - \int_0^1 (\tilde{v} - b(Y_{t-}, t)) \, dX_t^{S,S} - \int_0^1 (\tilde{v} - p(Y_{t-} - \delta, t)) \, dX_t^{S,T} - \int_0^1 (\tilde{v} - p(Y_{t-}, t)) \, dX^{S,B} \, \middle| \, \tilde{v} \right].$$

$$(2.1)$$

Note that the assumption $\mathbb{E}[X_1^B] < \infty$ implies $\mathbb{E}[X_1^B | \tilde{v}] < \infty$ as well since $\mathbb{E}[X_1^B] = \mathbb{E}[X_1^B | \tilde{v} = 1] \mathbb{P}[\tilde{v} = 1] + \mathbb{E}[X_1^B | \tilde{v} = 0] \mathbb{P}[\tilde{v} = 0]$, and $\mathbb{P}[\tilde{v} = 0] \in (0, 1)$. Similarly, $\mathbb{E}[X_1^S | \tilde{v}] < \infty$, too. Thus, the above expectation will be finite as soon as we assume that the pricing rule is rational in the sense that it assigns a price to the asset between 0 and 1. This will be part of the definition of equilibrium, which will be made precise below. As seen from the above formulation, when price moves, one buys (resp. sells) at a price $a(y, t)$ (resp. $b(y, t)$), where y is the cumulative order right before such trade. Thus, $a(y, t)$ (resp. $b(y, t)$) can be viewed as the ask (resp. bid) price.

Our goal is to find an equilibrium between the market maker and the insider in the following fashion:

Definition 2.3. A *Glosten-Milgrom equilibrium* is a quadruplet $(p, X^B, X^S, \mathcal{F}^I)$ such that

i) given $(X^B, X^S; \mathcal{F}^I)$, p is a *rational* pricing rule, i.e., $p(Y_t, t) = \mathbb{E}[\tilde{v} \, | \, \mathcal{F}_t^Y]$ for $t \in [0, 1]$;

ii) given p, $(X^B, X^S; \mathcal{F}^I)$ is an admissible strategy maximising (2.1).

Recall that $\tilde{v}$ takes only two values by assumption. In view of this specification we will often call the insider in the sequel of *high type* when $\tilde{v} = 1$ and *low type* when $\tilde{v} = 0$.

3 Characterisation of equilibrium

Before we give a characterisation of equilibrium, we will provide some heuristics. Due to the Markov structure of the pricing rule, we will define the informed trader's value function and derive, via a heuristic argument, the associated HJB equation. Definition 2.2 ii) implies that the $\mathcal{F}^I$-dual predictable projection of $X^{i,j}$, $i \in \{B, S\}$ and $j \in \{B, S, T\}$, is of the form $\delta \int_0^{\cdot} \theta_s^{i,j}\, ds$ so that $X^{i,j} - \delta \int_0^{\cdot} \theta_s^{i,j}\, ds$ defines an $\mathcal{F}^I$-martingale. Observe that since the set of jumps times of $X^{B,S}$ and $X^{S,T}$ (resp. $X^{S,B}$ and $X^{B,T}$) is contained in the set of jump times of Z^S (resp. Z^B), we necessarily have $\theta^{B,S} + \theta^{S,T} \leq \beta$ (resp. $\theta^{S,B} + \theta^{B,T} \leq \beta$). Moreover, Definition 2.3 i) implies that p takes values in $[0, 1]$, hence both bid and ask prices are $[0, 1]$-valued by definition. Therefore, $\int_0^{\cdot} (\tilde{v} - a(Y_{u-}, u))(dX_u^{B,B} - \delta\theta_u^{B,B}\, du)$ is an $\mathcal{F}^I$-martingale (see [7, Chapter 1, T6]). Arguing similarly with the other terms, the expected profit (2.1) can then be expressed as

$$\delta\, \mathbb{E}_{\mathbb{P}} \left[\int_0^1 (\tilde{v} - p(Y_{u-} + \delta, u))\theta_u^{B,B}\, du + \int_0^1 (\tilde{v} - p(Y_{u-} + 2\delta, u))\theta_u^{B,T}\, du + \int_0^1 (\tilde{v} - p(Y_{u-}, u))\theta_u^{B,S}\, du \right.$$
$$\left. - \int_0^1 (\tilde{v} - p(Y_{u-} - \delta, u))\theta_u^{S,S}\, du - \int_0^1 (\tilde{v} - p(Y_{u-} - 2\delta, u))\theta_u^{S,T}\, du - \int_0^1 (\tilde{v} - p(Y_{u-}, u))\theta_u^{S,B}\, du \,\middle|\, \tilde{v} \right].$$

This motivates us to define the following value function for the informed trader:

$$V(\tilde{v}, y, t) = \sup_{\theta^{i,j}\,;\, i \in \{B,S\}, j \in \{B,S,T\}}$$

$$\delta\, \mathbb{E}_{\mathbb{P}} \left[\int_t^1 (\tilde{v} - p(Y_{u-} + \delta, u))\theta_u^{B,B}\, du + \int_t^1 (\tilde{v} - p(Y_{u-} + 2\delta, u))\theta_u^{B,T} + \int_t^1 (\tilde{v} - p(Y_{u-}, u))\theta_u^{B,S}\, du \right.$$
$$\left. - \int_t^1 (\tilde{v} - p(Y_{u-} - \delta, u))\theta_u^{S,S}\, du - \int_t^1 (\tilde{v} - p(Y_{u-} - 2\delta, u))\theta_u^{S,T}\, du - \int_t^1 (\tilde{v} - p(Y_{u-}, u))\theta_u^{S,B}\, du \,\middle|\, Y_t = y, \tilde{v} \right],$$

for $\tilde{v} \in \{0, 1\}$, $t \in [0, 1)$, and $y \in \delta\mathbb{Z}$. The terminal value of V at 1 can be defined via the left limit $V(\tilde{v}, y, 1) := \lim_{t\uparrow 1} V(\tilde{v}, y, t)$. As we will see in Remark 3.3 below, $V(\tilde{v}, y, 1)$ is not always zero.

Recall that $Y = X + Z$ so that if one defines $Y^B = X^{B,B} + X^{B,T} + Z^B - X^{S,B}$ and $Y^S = X^{S,S} + X^{S,T} + Z^S - X^{B,S}$, then it is easy to see that $(Y_t^B - \delta \int_0^t (\beta - \theta_s^{B,T} - \theta_s^{S,B})\, ds - \delta \int_0^t \theta_s^{B,B}\, ds - 2\delta \int_0^t \theta_s^{B,T}\, ds)$ and $(Y_t^S - \delta \int_0^t (\beta - \theta_s^{S,T} - \theta^{B,S})\, ds - \delta \int_0^t \theta_s^{S,S}\, ds - 2\delta \int_0^t \theta_s^{S,T}\, ds)$ are $\mathcal{F}^I$-martingales. Thus, applying Ito's formula to $V(\tilde{v}, Y_t, t)$ yields the following formal HJB equation (the variable $\tilde{v}$ is omitted in V for simplicity of notation) in view of the standard dynamic programming arguments:

$$\begin{aligned}
0 = {}& V_t + (V(y + \delta, t) - 2V(y, t) + V(y - \delta, t))\, \beta \\
&+ \sup_{\theta^{B,B} \geq 0} \left[V(y + \delta, t) - V(y, t) + (\tilde{v} - p(y + \delta, t))\, \delta \right] \theta^{B,B} \\
&+ \sup_{\theta^{B,T} \geq 0} \left[V(y + 2\delta, t) - V(y + \delta, t) + \delta(\tilde{v} - p(y + 2\delta, t)) \right] \theta^{B,T} \\
&+ \sup_{\theta^{B,S} \geq 0} \left[V(y, t) - V(y - \delta, t) + (\tilde{v} - p(y, t))\delta \right] \theta^{B,S} \\
&+ \sup_{\theta^{S,S} \geq 0} \left[V(y - \delta, t) - V(y, t) - (\tilde{v} - p(y - \delta, t))\, \delta \right] \theta^{S,S} \\
&+ \sup_{\theta^{S,T} \geq 0} \left[V(y - 2\delta, t) - V(y - \delta, t) - \delta(\tilde{v} - p(y - 2\delta, t)) \right] \theta^{S,T} \\
&+ \sup_{\theta^{S,B} \geq 0} \left[V(y, t) - V(y + \delta, t) - (\tilde{v} - p(y, t))\delta \right] \theta^{S,B}, \quad (y, t) \in \delta\mathbb{Z} \times [0, 1).
\end{aligned} \tag{3.1}$$

The optimiser $(\theta^{i,j}\,; i \in \{B, S\}$ and $j \in \{B, S, T\})$ in the previous equation is expected

to be the $\mathcal{F}^I$-intensities of the insider's optimal strategy $(X^{i,j})$ when the order size is normalised to 1.

Notice that all maximisations in (3.1) are linear in θ. Therefore (3.1) reduces to the following system:

$$\begin{aligned}
V_t + \left(V(y+\delta,t) - 2V(y,t) + V(y-\delta,t)\right)\beta &= 0, \\
V(y+\delta,t) - V(y,t) + (\tilde{v} - p(y+\delta,t))\delta &\leq 0, \\
V(y-\delta,t) - V(y,t) - (\tilde{v} - p(y-\delta,t))\delta &\leq 0, \quad (y,t) \in \delta\mathbb{Z} \times [0,1).
\end{aligned} \tag{3.2}$$

Here the first inequality corresponds to the maximisation in $\theta^{B,j}$; while the second inequality corresponds to the maximisation in $\theta^{S,j}$, $j \in \{B, S, T\}$. Let's denote the optimisers in (3.1) with $(\theta^{i,j}(y,t); (y,t) \in \delta\mathbb{Z} \times [0,1))$, $i \in \{B, S\}$ and $j \in \{B, S, T\}$. Observe that the first inequality in (3.2) can be strict only if $\theta^{B,B}(y,t) = \theta^{B,S}(y+\delta,t) = \theta^{B,T}(y-\delta,t) = 0$. Similarly, the second inequality can be strict only if $\theta^{S,S}(y,t) = \theta^{S,B}(y-\delta,t) = \theta^{S,T}(y+\delta,t) = 0$. We will see later that the optimal $\theta^{B,B}$ and $\theta^{B,S}$ are never 0 for the high type insider meanwhile $\theta^{S,S}$ and $\theta^{S,B}$ are never 0 for the low type. Therefore the first inequality in (3.2) is actually an equality when $\tilde{v} = 1$ and the second inequality is an equality when $\tilde{v} = 0$. Economically speaking, these equalities imply that at every instant of time there is a non-zero probability that a high type insider will make a buy order by either contributing to uninformed buy orders or canceling uninformed sell orders, and the low type insider will do the opposite. Such actions are certainly reasonable for the insider. Indeed, a high type insider will reveal her information gradually and keep the market price strictly less than 1. Recall that p is a martingale bounded by 1, so once it hits 1, it will be stopped at that level. Therefore, since there is always a strictly positive difference between the true price, which is 1 in this case, and the market price, the insider will always want to take advantage of this discrepancy and buy with positive probability since the asset is undervalued by the market. The situation for the low type is similar.

In view of the previous discussion, let's consider the following system:

$$\begin{aligned}
V_t^H + \left(V^H(y+\delta,t) - 2V^H(y,t) + V^H(y-\delta,t)\right)\beta &= 0, \\
V^H(y+\delta,t) - V^H(y,t) + (1 - p(y+\delta,t))\delta &= 0;
\end{aligned} \tag{HJB-H}$$

$$\begin{aligned}
V_t^L + \left(V^L(y+\delta,t) - 2V^L(y,t) + V^L(y-\delta,t)\right)\beta &= 0, \\
V^L(y-\delta,t) - V^L(y,t) + p(y-\delta,t)\delta &= 0,
\end{aligned} \tag{HJB-L}$$

for $(y,t) \in \delta\mathbb{Z} \times [0,1)$. We expect that $V^H(y,t) = V(1,y,t)$ and $V^L(y,t) = V(0,y,t)$. The next lemma will construct solutions to the above system and will be useful in solving the insider's optimisation problem. However, before the statement and the proof of this lemma we need to introduce a class of functions satisfying certain boundary conditions and differential equations. We will nevertheless denote them with p since, as we shall see later, they will appear in the equilibrium as pricing rules for the market maker.

To this end, for each $z \in \delta\mathbb{Z}$, let

$$P^z(y) := \begin{cases} 0, & y < z \\ 1, & y \geq z \end{cases}, \tag{3.3}$$

and define

$$p^z(y,t) := \mathbb{E}_{\mathbb{P}}[P^z(Z_1) \,|\, Z_t = y]. \tag{3.4}$$

Observe that Z/δ is the difference of two independent Poisson processes. The Markov

property implies[4] that p^z satisfies

$$p_t^z + \left(p^z(y+\delta,t) - 2p^z(y,t) + p^z(y-\delta,t)\right)\beta = 0, \quad (y,t) \in \delta\mathbb{Z} \times [0,1),$$
$$p^z(y,1) = P^z(y), \qquad\qquad\qquad\qquad\quad y \in \delta\mathbb{Z}. \tag{3.5}$$

Lemma 3.1. *Let p^z be defined by (3.4) for some fixed $z \in \delta\mathbb{Z}$ and define*

$$H(y,1) := \delta \sum_{j=\frac{y}{\delta}}^{\frac{z-\delta}{\delta}} (1 - A(\delta j)), \quad L(y,1) := \delta \sum_{j=\frac{z}{\delta}}^{\frac{y}{\delta}} B(\delta j), \quad y \in \delta\mathbb{Z},$$

where $A(y) := P^z(y+\delta)$, $B(y) := P^z(y-\delta)$, and $\sum_{j=m}^{n} \alpha_j = -\sum_{j=n}^{m} \alpha_j$ by convention whenever $m > n$. Then, both $H(\cdot,1)$ and $L(\cdot,1)$ are nonnegative and the following equivalences hold:

$$H(y,1) = 0 \iff A(y) = 1 \iff y \geq z - \delta, \quad L(y,1) = 0 \iff B(y) = 0 \iff y < z + \delta.$$

Moreover,

$$H(y,t) := H(y,1) + \delta\beta \int_t^1 \left(p^z(y+\delta,u) - p^z(y,u)\right) du \quad \text{and} \tag{3.6}$$

$$L(y,t) := L(y,1) + \delta\beta \int_t^1 \left(p^z(y,u) - p^z(y-\delta,u)\right) du \tag{3.7}$$

solve (HJB-H) *and* (HJB-L) *respectively.*

Proof. Statements regarding $H(y,1)$ and $L(y,1)$ directly follow from the definitions. We will next show that H satsifies (HJB-H). Analogous statement for L can be proven similarly. First observe that

$$H(y+\delta,1) - H(y,1) = -\delta + \delta A(y) = -\delta + \delta P^z(y+\delta).$$

Thus,

$$H(y+\delta,t) - H(y,t) = H(y+\delta,1) - H(y,1) + \delta\beta \int_t^1 \left(p^z(y+2\delta,u) - 2p^z(y+\delta,u) + p^z(y,u)\right) du$$
$$= \delta\left(p^z(y+\delta,t) - 1\right), \tag{3.8}$$

where (3.5) is used to obtain the last line. This proves the second equation in (HJB-H).

Next, it follows from the definition of H that

$$H_t(y,t) + \delta\beta \left(p^z(y+\delta,t) - p^z(y,t)\right) = 0.$$

However, iterating (3.8) yields

$$H(y+\delta,t) + H(y-\delta,t) - 2H(y,t) = H(y+\delta,t) - H(y,t) - (H(y,t) - H(y-\delta,t))$$
$$= \delta\left(p^z(y+\delta,t) - p^z(y,t)\right),$$

and, hence, the claim. $\qquad\square$

Given a pricing rule, let us describe insider's optimal strategies.

Proposition 3.2. *Suppose that the market maker chooses p^z as a pricing rule, where z is fixed and p^z is as defined in (3.4). Then, the following holds:*

---

[4]The Markov property of Z implies that $\mathbb{P}(Z_1 = \tilde{z} \mid Z_t = y)$ satisfies $p_t + (p(y+\delta,t) - 2p(y,t) + p(y-\delta,t))\beta = 0$. Therefore summing up the previous equation for different z induces that $\sum_{\delta\mathbb{Z} \ni \tilde{z} \geq z} \partial_t \mathbb{P}(Z_1 = \tilde{z} \mid Z_t = y)$ is finite. Hence Fubini's theorem implies that the previous sum is exactly $\partial_t p^z$ and $p^{\tilde{z}}$ solves (3.5).

i) When $\tilde{v} = 1$, $(X^B, X^S; \mathcal{F}^I)$ is an optimal strategy if and only if $Y_1 \geq z - \delta$ and $X^{S,j} = 0$, $j = \{B, S, T\}$.

ii) When $\tilde{v} = 0$, $(X^B, X^S; \mathcal{F}^I)$ is an optimal strategy if and only if $Y_1 < z + \delta$ and $X^{B,j} = 0$, $j = \{B, S, T\}$.

When the previous condition holds for $\tilde{v} = 1$ (resp. $\tilde{v} = 0$), $v(1, y, t) = H(y, t)$ (resp. $v(0, y, t) = L(y, t)$) for $(y, t) \in \delta\mathbb{Z} \times [0, 1]$.

Remark 3.3. Recall that $V(\tilde{v}, y, 1) := \lim_{t\uparrow 1} V(\tilde{v}, y, t)$. Lemma 3.1 and Proposition 3.2 combined implies that $V(\tilde{v}, y, 1) \geq 0$. It is only zero when $A(y) = 1$ for the high type and $B(y) = 0$ for the low type.

Proof. The statements for $\tilde{v} = 1$ case will be proved. Similar arguments can be applied in order to prove the statement regarding $\tilde{v} = 0$. Fix $(y, t) \in \delta\mathbb{Z} \times [0, 1)$. For any admissible trading strategy $(X^{i,j}; i \in \{B, S\})$ and $j \in \{B, S, T\}$ with associated $\mathcal{F}^I$-intensities $(\delta\theta^{i,j}; i \in \{B, S\}$ and $j \in \{B, S, T\})$, applying Ito's formula to $H(Y, \cdot)$ and utilizing Lemma 3.1, we obtain

$$
\begin{aligned}
&H(Y_1, 1) \\
=\ & H(y, t) + \int_t^1 H_t(Y_{u-}, u)\, du \\
&+ \int_t^1 \left(H(Y_{u-} + \delta, u) - H(Y_{u-}, u)\right)(\beta - \theta_u^{B,T} - \theta_u^{S,B})\, du + \int_t^1 \left(H(Y_{u-} + \delta, u) - H(Y_{u-}, u)\right)\theta_u^{B,B}\, du \\
&+ \int_t^1 \left(H(Y_{u-} + 2\delta, t) - H(Y_{u-}, u)\right)\theta_u^{B,T}\, du \\
&+ \int_t^1 \left(H(Y_{u-} - \delta, u) - H(Y_{u-}, u)\right)(\beta - \theta^{S,T} - \theta_u^{B,S})\, du + \int_t^1 \left(H(Y_{u-} - \delta, u) - H(Y_{u-}, u)\right)\theta_u^{S,S}\, du \\
&+ \int_t^1 \left(H(Y_{u-} - 2\delta, u) - H(Y_{u-}, u)\right)\theta_u^{S,T}\, du + M_1 - M_t \\
=\ & H(y, t) \\
&- \int_t^1 \left(H(Y_{u-} + \delta, u) - H(Y_{u-}, u)\right)\theta_u^{S,B}\, du + \int_t^1 \left(H(Y_{u-} + \delta, u) - H(Y_{u-}, u)\right)\theta_u^{B,B}\, du \\
&+ \int_t^1 \left(H(Y_{u-} + 2\delta, t) - H(Y_{u-} + \delta, u)\right)\theta_u^{B,T}\, du \\
&- \int_t^1 \left(H(Y_{u-} - \delta, u) - H(Y_{u-}, u)\right)\theta_u^{B,S}\, du + \int_t^1 \left(H(Y_{u-} - \delta, u) - H(Y_{u-}, u)\right)\theta_u^{S,S}\, du \\
&+ \int_t^1 \left(H(Y_{u-} - 2\delta, u) - H(Y_{u-} - \delta, u)\right)\theta_u^{S,T}\, du + M_1 - M_t \\
=\ & H(y, t) + \delta \int_t^1 \left(p(Y_{u-} + \delta, u) - 1\right)\theta_u^{B,B}\, du + \delta \int_t^1 \left(1 - p(Y_{u-}, u)\right)\theta_u^{S,S}\, du \\
&- \delta \int_t^1 \left(p(Y_{u-} + \delta, u) - 1\right)\theta_u^{S,B}\, du - \delta \int_t^1 \left(1 - p(Y_{u-}, u)\right)\theta_u^{B,S}\, du \\
&- \delta \int_t^1 \left(1 - p(Y_{u-} + 2\delta, u)\right)\theta_u^{B,T}\, du + \delta \int_t^1 \left(1 - p(Y_{u-} - \delta, u)\right)\theta_u^{S,T}\, du + M_1 - M_t.
\end{aligned}
$$

Here M contains $\int_0^{\cdot} (p(Y_{u-} + \delta, u) - 1)(dX_u^{B,B} - \delta\theta_u^{B,B} du)$ and similar processes, which are all $\mathcal{F}^I$-martingales due to the bounded integrand and the martingale property of $X^{i,j} - \delta \int_0^{\cdot} \theta_u^{i,j}\, du$ for $i \in \{B, S\}$ and $j \in \{B, S, T\}$ (see [7, Chapter 1, T6]). Thus, on

$[\tilde{v} = 1]$

$$\delta \int_t^1 (1 - p(Y_{u-} + \delta, u)) \, \theta_u^{B,B} \, du + \delta \int_t^1 (1 - p(Y_{u-}, u)) \, \theta_u^{B,S} \, du + \delta \int_t^1 (1 - p(Y_{u-} + 2\delta, u)) \theta_u^{B,T} \, du$$

$$-\delta \int_t^1 (1 - p(Y_{u-} - \delta, u)) \, \theta_u^{S,S} \, du - \delta \int_t^1 (1 - p(Y_{u-}, u)) \, \theta_u^{S,B} \, du - \delta \int_t^1 (1 - p(Y_{u-} - 2\delta, u)) \, \theta_u^{S,T} \, du$$

$$= \quad M_1 - M_t - H(Y_1, 1) + H(y, t)$$

$$-\delta \int_t^1 (p(Y_{u-}, u) - p(Y_{u-} - \delta, u)) \, \theta_u^{S,S} \, du - \delta \int_t^1 (p(Y_{u-} + \delta, u) - p(Y_{u-}, u)) \, \theta_u^{S,B} \, du$$

$$-\delta \int_t^1 (p(Y_{u-} - \delta, u) - p(Y_{u-} - 2\delta, u)) \, \theta_u^{S,T} \, du.$$

Observe that the left side of the above equality is the wealth of the insider. Moreover, since $H \geq 0$ and p is strictly increasing in y, the expected wealth, conditioned on $\mathcal{F}_t^I$, is maximised when $H(Y_1, 1) = 0$ $\mathbb{P}$-a.s., $\theta^{S,S}$, $\theta^{S,T}$, and $\theta^{S,B}$ are identically zero. However, in view of Lemma 3.1, $H(Y_1, 1) = 0$ if and only if $Y_1 \geq z - \delta$. $\qquad \square$

We are now ready to state the conditions for equilibrium.

Theorem 3.4. $(p, X^B, X^S, \mathcal{F}^I)$ *is a Glosten-Milgrom equilibrium if there exists a* $y_\delta \in \delta \mathbb{Z}$ *such that*

  i) $[Y_1 \geq y_\delta] = [\tilde{v} = 1]$ $\mathbb{P}$*-a.s.;*
 ii) $p = p^{y_\delta}$ *which is defined by (3.4);*
iii) $(X^B, X^S; \mathcal{F}^I)$ *is an admissible strategy such that* $Y = Z + X^B - X^S = Y^B - Y^S$ *where* Y^B/δ *and* Y^S/δ *are independent,* $\mathcal{F}^Y$*-adapted Poisson processes with common intensity* β*, and* $X^S \equiv 0$ *(resp.* $X^B \equiv 0$*) on* $[\tilde{v} = 1]$ *(resp.* $[\tilde{v} = 0]$*).*

Proof. Given the pricing rule $p = p^{y_\delta}$, Proposition 3.2 implies that $(X^B, X^S; \mathcal{F}^I)$ is optimal because $[Y_1 \geq y_\delta] = [\tilde{v} = 1]$ $\mathbb{P}$-a.s. and $X^S \equiv 0$ (resp. $X^B \equiv 0$) on $[\tilde{v} = 1]$ (resp. $[\tilde{v} = 0]$). Thus it remans to show p^{y_δ} is a rational pricing rule given $(X^B, X^S; \mathcal{F}^I)$. Indeed, since Y and Z have the same distribution, it follows from (3.4) and the Markov property of Y that $\mathbb{E}_{\mathbb{P}}[\tilde{v}|\mathcal{F}_t^Y] = \mathbb{P}[Y_1 \geq y_\delta|\mathcal{F}_t^Y] = p^{y_\delta}(Y_t, t)$ for $t \in [0, 1]$. $\qquad \square$

Remark 3.5. Theorem 3.4 iii) necessarily requires that $X^{B,T} \equiv 0$ (resp. $X^{S,T} \equiv 0$) on $[\tilde{v} = 1]$ (resp. $[\tilde{v} = 0]$) since it implies that the jumps occur with magnitude δ only. Recall from the proof of Proposition 3.2 that this is not a requirement for optimality from the point of view of the insider. Rather, the insider chooses not to trade at the same time and in the same direction with the noise traders in order to make it possible that there is a rational pricing rule that the market maker can choose.

The equilibrium given in the above theorem is another manifestation of *inconspicuous trade theorem* commonly observed in the insider trading literature (see, e.g., [14], [2], [6], etc.). Indeed, when the insider is trading optimally in the above equilibrium, the distribution of the net order process is the same as that of the net orders of the noise traders, i.e. the insider is able to hide her trades among the noise trades. However, the private information is fully, albeit gradually, revealed to the public since $\tilde{v} \in \mathcal{F}_1^Y$. We will construct an admissible strategy satisfying conditions above and show the existence of Glosten-Milgrom equilibrium in the following section.

Remark 3.6. Proposition 3.2 and Theorem 3.4 indicate that 'bluffing' strategies selling for the high-type and buying for the low-type are sub-optimal. This is in contrast to the results in [4], which use numeric computations to suggest such bluffing might be optimal.

4 Construction of a point process bridge

As seen in Theorem 3.4 we are interested in the construction of a process $Y = Z + X^B - X^S$ such that, in its natural filtration, $Y = Y^B - Y^S$ such that Y^B/δ and Y^S/δ are independent Poisson processes with intensity β. To this end, we will construct explicitly a process Y on some $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$ such that

$$Y = Z^B - Z^S + X^B \, \mathbb{I}_I - X^S \, \mathbb{I}_{I^c}, \tag{4.1}$$

where $I \in \mathcal{F}_0$ with specified probability, X^B and X^S are two point processes and Z/δ is $\mathcal{F}$-adapted and is the difference of two independent Poisson processes with intensity β. In particular, I is independent of Z since Z has independent increments and $Z_0 = 0$. The set I is to be associated with the set $[\tilde{v} = 1]$. In order to comply with the conditions of the equilibrium described in the last section, we will further require $[Y_1 \geq y_\delta] = I$ $\mathbb{P}$-a.s. for a given suitable y_δ. Since Y is expected to have the same distribution as Z, the previous condition necessitates $\mathbb{P}(I) = \mathbb{P}(Z_1 \geq y_\delta)$. During the construction of the probability space and the process Y, we will take $\delta = 1$ without loss of generality since all the processes can be scaled by δ to construct the process we are after.

In order to construct such a process we first need to determine its intensity. Since Y would behave like Z in its own filtration, we can view, in the sense of equality in distributions, the decomposition in (4.1) as that of Z when its own filtration is initially enlarged with the random variable $\mathbb{I}_{[Z_1 \geq y_1]}$. Thus, the intensity of Y will be that of Z in this enlarged filtration.

Let $(\mathbb{D}([0,1], \mathbb{Z}), \mathcal{F}^1, (\mathcal{F}^1_t)_{t \in [0,1]}, \mathbb{P}^1)$ be the canonical space where $\mathbb{D}([0,1], \mathbb{Z})$ is $\mathbb{Z}$-valued càdlàg functions, $\mathbb{P}^1$ is a probability measure under which Z^B and Z^S are independent Poisson processes with intensities β, $(\mathcal{F}^1_t)_{t \in [0,1]}$ is the minimal filtration generated by Z^B and Z^S satisfying the usual conditions, and $\mathcal{F}^1 = \bigvee_{t \in [0,1]} \mathcal{F}^1_t$. Let's denote with $(\mathcal{G}^1_t)_{t \in [0,1]}$ the filtration $(\mathcal{F}^1_t)_{t \in [0,1]}$ enlarged with the random variable $\mathbb{I}_{[Z_1 \geq y_1]}$.

In order to find the $\mathcal{G}^1$-intensity of Z, we will use a standard enlargement of filtration argument which can be found, e.g., in [15]. To this end, let $h : [0,1] \times \mathbb{Z} \mapsto [0,1]$ be the function defined by

$$h(z,t) := \mathbb{P}^1[Z_1 \geq y_1 \,|\, Z_t = z]. \tag{4.2}$$

Note that h is strictly positive on $[0,1) \times \mathbb{Z}$. Moreover since $(h(Z_t, t))_{t \in [0,1]}$ is an $\mathcal{F}^1$-martingale, Ito's formula yields

$$h_t(z,t) + \beta \left(h(z+1,t) + h(z-1,t) - 2h(z,t) \right) = 0, \quad (t,z) \in [0,1) \times \mathbb{Z}. \tag{4.3}$$

Lemma 4.1. *The $\mathcal{G}^1$-intensities of Z^B and Z^S at $t \in [0,1)$ are given by*

$$\mathbb{I}_{[Z_1 \geq y_1]} \beta \frac{h(Z_{t-} + 1, t)}{h(Z_{t-}, t)} + \mathbb{I}_{[Z_1 < y_1]} \beta \frac{1 - h(Z_{t-} + 1, t)}{1 - h(Z_{t-}, t)},$$

$$\mathbb{I}_{[Z_1 \geq y_1]} \beta \frac{h(Z_{t-} - 1, t)}{h(Z_{t-}, t)} + \mathbb{I}_{[Z_1 < y_1]} \beta \frac{1 - h(Z_{t-} - 1, t)}{1 - h(Z_{t-}, t)},$$

respectively.

Proof. We will only calculate the intensity for Z^B. The intensity of Z^S can be obtained similarly. All expectations are taken under $\mathbb{P}^1$ throughout this proof. For $s \leq t < 1$, take an arbitrary $E \in \mathcal{F}^1_s$ and denote $M^B_t := Z^B_t - \beta t$. The definition of h and the

$\mathcal{F}$-martingale property of M^B imply

$$
\begin{aligned}
E\left[(M_t^B - M_s^B)\mathbb{I}_E\mathbb{I}_{[Z_1 \geq y_1]}\right] &= E\left[(M_t^B - M_s^B)\mathbb{I}_E h(Z_t, t)\right] \\
&= E\left[\mathbb{I}_E\left(\langle M^B, h(Z_., \cdot)\rangle_t - \langle M^B, h(Z_., \cdot)\rangle_s\right)\right] \\
&= E\left[\mathbb{I}_E \int_s^t (h(Z_{r-} + 1, r) - h(Z_{r-}, r))\,\beta\,dr\right] \\
&= E\left[\mathbb{I}_E \int_s^t \mathbb{I}_{[Z_1 \geq y_1]} \frac{h(Z_{r-} + 1, r) - h(Z_{r-}, r)}{h(Z_{r-}, r)}\beta\,dr\right].
\end{aligned}
$$

Since $\mathbb{P}^1(Z_1 < \delta \,|\, Z_t = z) = 1 - h(z, t)$, similar computations yield

$$
E\left[(M_t^B - M_s^B)\mathbb{I}_E\mathbb{I}_{[Z_1 < y_1]}\right] = E\left[\mathbb{I}_E \int_s^t \mathbb{I}_{[Z_1 < y_1]} \frac{h(Z_{r-}, r) - h(Z_{r-} + 1, r)}{1 - h(Z_{r-}, r)}\beta\,dr\right].
$$

These computations imply that

$$
M^B - \int_0^{\cdot} \mathbb{I}_{[Z_1 \geq y_1]} \frac{h(Z_{r-} + 1, r) - h(Z_{r-}, r)}{h(Z_{r-}, r)}\beta\,dr - \int_0^{\cdot} \mathbb{I}_{[Z_1 < y_1]} \frac{h(Z_{r-}, r) - h(Z_{r-} + 1, r)}{1 - h(Z_{r-}, r)}\beta\,dr
$$

defines a $\mathcal{G}^1$-martingale. Therefore the $\mathcal{G}^1$-intensity of Z^B follows from $Z_t^B = M_t^B + \beta t$. $\qquad\square$

In what follows, given $I \in \mathcal{F}_0$ and h as in (4.2) such that $\mathbb{P}(I) = h(0, 0)$, X^B on I and X^S on I^c will be constructed so that Y matches the intensities given in the above lemma. As a result, Proposition 4.4 ensures $I = [Y_1 \geq y_1]$ $\mathbb{P}$-a.s., which is what we are after. We will focus on the construction of X^B on I in what follows. By symmetry, X^S on I^c can be constructed by the same method but applied to $-Z$ and $-y_1$.

Recall that one of the goals of the process X^B on I is to make sure that Y_1 ends up at a value larger than or equal to y_1. In order to achieve this goal X^B will have to add some jumps in addition to the jumps coming from Z^B. However, this by itself won't be enough since Z^S will make Y jump downward. Thus, X^B will also need to cancel some of downwards jumps coming from Z^S. Of course, there are many ways in which X^B achieves this goal. However, Y is required to have the same distribution as Z. We will see in Proposition 4.4 that this distribution requirement will also be satisfied once Y has the correct intensity given by Lemma 4.1.

As described above X^B will consist of two components $X^{B,B}$ and $X^{B,S}$, where $X^{B,B}$ complements jumps of Z^B and $X^{B,S}$ cancels some jumps of Z^S. Let's denote by $(\tau_i)_{i \geq 1}$ the sequence of jump times for the Y process we wish to construct. These stopping times will be constructed inductively as follows. Given $\tau_{i-1} < 1$, τ_i is the minimum of the following three random times:

   i) the next jump of Z^B,

   ii) the next jump of $X^{B,B}$,

   iii) the next jump of Z^S which is not cancelled by a jump of $X^{B,S}$.

Here $X^{B,B}$ and $X^{B,S}$ are constructed so that $Y^B = Z^B + X^{B,B}$ and $Y^S = Z^S - X^{B,S}$ have the required $\mathcal{F}$-intensities on I. To achieve all these aims simultaneously, when the $(i-1)^{th}$ jump of Y happens before time 1, we will generate random variables ν_i and another sequence of Bernoulli random variables $(\xi_{j,i})_{j \geq 1}$ to determine the next jump of Y. In the context of the informed trader trying to make a decision, construction of X^B corresponds to the following pattern: place a buy order at time ν_i unless the next buy order from the uninformed trader arrives before ν_i and also buy at every sell order of the uninformed trader until $\xi_{j,i} = 1$ for the first time.

We will now make this intuitive construction rigorous. In order to perform the subsequent construction, we must assume that the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$ is large enough so that there exist $I \in \mathcal{F}_0$ with $\mathbb{P}(I) = h(0,0)$ and two independent sequences of iid $\mathcal{F}$-measurable random variables $(\eta_i)_{i \geq 1}$ and $(\zeta_i)_{i \geq 1}$ with uniform distribution on $[0,1]$, moreover $(\eta_i)_{i \geq 1}$ and $(\zeta_i)_{i \geq 1}$ are independent of both Z and I. These requirements can be easily satisfied by extending $\mathcal{F}_0$ and $\mathcal{F}$ if necessary. The sequences $(\eta_i)_{i \geq 1}$ and $(\zeta_i)_{i \geq 1}$ will be used to construct ν_i and $(\xi_{j,i})_{j \geq 1}$ in the last paragraph. As for the filtration $(\mathcal{F}_t)_{t \in [0,1]}$, we require that Z/δ, as the difference of two independent Poisson processes with intensity β, is adapted to $(\mathcal{F}_t)_{t \in [0,1]}$. Since Z has independent increments and $Z_0 = 0$, Z is independent of I. We will make one more assumption on the filtration later during the construction.

Denote by $(\sigma_i^+)_{i \geq 1}$ and $(\sigma_j^-)_{j \geq 1}$ jump times of Z^B and Z^S, respectively. We set $\sigma_i^\pm = \infty$ when $\sigma_i^\pm > 1$, since we are only interested in processes before time 1. In what follows, we will inductively define two sequences of $[0,1] \cup \{\infty\}$-valued random variables $(\tau_i^+)_{i \geq 1}$ and $(\tau_i^-)_{i \geq 1}$ on I. τ_{i+1}^+ (resp. τ_{i+1}^-) will denote the first potential upward (resp. downward) jump of Y after time τ_i starting with $\tau_0 = 0$. The process Y on I thus jumps at each $\tau_i := \tau_i^+ \wedge \tau_i^-$. In particular, when $\tau_i^+ < \tau_i^-$, $\Delta Y_{\tau_i} = \Delta Y_{\tau_i}^B = 1$; when $\tau_i^- < \tau_i^+$, $\Delta Y_{\tau_i} = -\Delta Y_{\tau_i}^S = -1$.

Let's start with the construction until the first jump of Y. Recall that, in view of Lemma 4.1, we want to construct Y^B (resp. Y^S) so that its intensity until its first jump is given by

$$\beta \frac{h(1,t)}{h(0,t)} \quad \left(\text{resp. } \beta \frac{h(-1,t)}{h(0,t)} \right).$$

Hence τ_1 is constructed to match this intensity.

To define τ_1^+, set

$$f_1(t) := 1 - \exp\left(\beta \int_0^t \frac{h(0,u) - h(1,u)}{h(0,u)} du \right), \quad t \in [0,1).$$

Since $z \mapsto h(z,t)$ is strictly increasing, f_1 is strictly increasing. We consider the inverse function $f_1^{-1}(y) := \inf\{t \in [0,1) : f(t) > y\}$, where the value is ∞ if the indicated set is empty. Now define

$$\nu_1 := f_1^{-1}(\eta_1) \quad \text{and} \quad \tau_1^+ := \nu_1 \wedge \sigma_1^+ \quad \text{on } I.$$

Then τ_1^+ is potentially the first jump time of Y^B. It follows from the definition of ν_1 that $\mathbb{P}(\tau_1^+ < 1) > 0$. Such τ_1^+ is constructed to match the intensity of Y^B before the first jump of Y. On the other hand, in order to define τ_1^-, consider

$$\xi_{j,1} := \mathbb{I}_{\left[\zeta_j \leq \frac{h(-1,\sigma_j^-)}{h(0,\sigma_j^-)}, \sigma_j^- < 1 \right]} + \mathbb{I}_{[\sigma_j^- \geq 1]} \quad \text{for } j \geq 1. \tag{4.4}$$

This indicator random variable determines whether the jth jump of Z^S will be cancelled by an opposite jump of $X^{B,S}$. When $\xi_{j,1} = 0$, which only happens when the jth jump of Z^S happens before 1, this jump of Z^S will be cancelled by a jump of $X^{B,S}$. Such cancelation is performed at a rate $h(-1,\sigma_j^-)/h(0,\sigma_j^-)$ so as to match the intensity of Y^S before the first jump of Y. Therefore, τ_1^-, which is potentially the first negative jump, is the first jump time σ_j^- of Z^S which is *not* cancelled. That is,

$$\tau_1^- := \min\{\sigma_j^- : \xi_{j,1} = 1\}.$$

Consequently, we define the first jump time of Y on I as

$$\tau_1 := \tau_1^+ \wedge \tau_1^-.$$

This construction yields $\mathbb{P}(\tau_1 < 1) > 0$. On $[t \leq \tau_1, I]$ with $t \leq 1$, we define $X^{B,B}$ and $X^{B,S}$ as

$$X_t^{B,B} := \mathbb{I}_{[\nu_1 < \sigma_1^+]} \mathbb{I}_{[\tau_1^+ \leq t]} \quad \text{and} \quad X_t^{B,S} := \sum_{j=1}^{\infty} (1 - \xi_{j,1}) \mathbb{I}_{[\sigma_j^- \leq t]}.$$

Now suppose that τ_{i-1} with $\mathbb{P}(\tau_{i-1} < 1) > 0$ and Y_t for $t \leq \tau_{i-1} \wedge 1$ have been defined. We will define in this paragraph τ_i and Y_t for $t \in (\tau_{i-1} \wedge 1, \tau_i \wedge 1]$. To this end, when $\tau_{i-1} < 1$, consider the random function

$$f_i(t) := 1 - \exp\left(\lambda \int_{\tau_{i-1}}^t \frac{h(Y_{u-}, u) - h(Y_{u-} + 1, u)}{h(Y_{u-}, u)} du\right), \quad t \in [\tau_{i-1}, 1).$$

Since f_i is strictly increasing, the inverse function $f_i^{-1}(y) := \inf\{t \in [\tau_{i-1}, 1) : f(t) > y\}$ is well-defined. When $\tau_{i-1} \geq 1$, set $f_i^{-1}(y) = \infty$. Now define

$$\nu_i := f_i^{-1}(\eta_i) \quad \text{and} \quad \tau_i^+ := \nu_i \wedge \sigma_{Z_{\tau_{i-1}}^B + 1}^+ \quad \text{on } I.$$

To ease notation, we denote $\widetilde{\sigma}_i^+ := \sigma_{Z_{\tau_{i-1}}^B + 1}^+$, where $Z_{\tau_{i-1}}^B$ counts the number of Z^B jumps until τ_{i-1}. Hence $\widetilde{\sigma}_i^+$ indicates which jumps of Z^B could be the next jump of Y^B after τ_{i-1}. Similarly, define

$$\xi_{j,i} := \mathbb{I}_{\left[\zeta_j \leq \frac{h(Y_{\tau_{i-1}} - 1, \sigma_j^-)}{h(Y_{\tau_{i-1}}, \sigma_j^-)}, \tau_{i-1} \leq \sigma_j^- < 1\right]} + \mathbb{I}_{[\sigma_j^- \geq 1]},$$

and set

$$\tau_i^- := \min\{\sigma_j^- > \tau_{i-1} : \xi_{j,i} = 1\}.$$

The i-th jump of Y on I is then defined as

$$\tau_i := \tau_i^+ \wedge \tau_i^-.$$

Since $\mathbb{P}(\tau_{i-1} < 1) > 0$, the above construction yields $\mathbb{P}(\tau_i < 1) > 0$. The increment of $X^{B,B}$ and $X^{B,S}$ on $(\tau_{i-1} \wedge 1, \tau_i \wedge 1]$ are defined as

$$X_t^{B,B} - X_{\tau_{i-1}}^{B,B} = \mathbb{I}_{[\widetilde{\sigma}_i^+ > \nu_i]} \mathbb{I}_{[\tau_i^+ \leq t]} \quad \text{and} \quad X_t^{B,S} - X_{\tau_{i-1}}^{B,S} = \sum_{j=1}^{\infty} (1 - \xi_{j,i}) \mathbb{I}_{[\tau_{i-1} \leq \sigma_j^- \leq t]},$$

for $t \in (\tau_{i-1} \wedge 1, \tau_i \wedge 1]$.

This completes the construction of X^B since $X^B = X^{B,B} + X^{B,S}$ and we thus obtain the decomposition (4.1) on I for $t \in [0, 1 \wedge \lim_{i \to \infty} \tau_i]$. As mentioned earlier, the construction on I^c can be performed analogously.

Remark 4.2. A natural question on whether $\tau := \lim_{i \to \infty} \tau_i \geq 1$ or not arises at this point. Observe that since Z^B and Z^S are finite processes, $\mathbb{P}(\tau < 1, I) > 0$ implies that there are infinitely many jumps in $X^{B,B}$ so that $\lim_{i \to \infty} Y_{\tau_i}^B = \infty$, in which case we define $Y^B = \infty$ after τ. A similar explosion on I^c will result in Y becoming $-\infty$. However, we will see in Proposition 4.4 that Y is $\mathbb{P}$-a.s. a finite process and, thus, $\tau \geq 1$, $\mathbb{P}$-a.s..

In order to be able to perform the construction above on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$, in addition to the assumptions already imposed on the filtration, we add one more assumption that $(\mathcal{F}_t)_{t \in [0,1]}$ is right continuous and complete such that X^B and X^S are $\mathcal{F}$-adapted and $(\tau_i^+)_{i \geq 1}$, $(\tau_i^-)_{i \geq 1}$ and $(\nu_i)_{i \geq 1}$ are $\mathcal{F}$-stopping times. This completes our assumptions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$. We now return to verify that the process Y just constructed satisfies

i) $[Y_1 \geq y_1] = I$, $\mathbb{P}$-a.s., and

ii) In its own filtration $Y = Y^B - Y^S$ where Y^B and Y^S are $\mathcal{F}^Y$-adapted independent Poisson processes with intensity β.

We first establish that the $\mathcal{F}$-intensity of Y is of the same form as the $\mathcal{G}^1$-intensity of Z computed in Lemma 4.1.

Lemma 4.3. *The $\mathcal{F}$-intensities of Y^B and Y^S at $t \in [0,1)$ are given by*

$$\mathbb{I}_I \beta \frac{h(Y_{t-}+1,t)}{h(Y_{t-},t)} + \mathbb{I}_{I^c}\beta \frac{1-h(Y_{t-}+1,t)}{1-h(Y_{t-},t)} \quad and \quad \mathbb{I}_I\beta\frac{h(Y_{t-}-1,t)}{h(Y_{t-},t)} + \mathbb{I}_{I^c}\beta\frac{1-h(Y_{t-}-1,t)}{1-h(Y_{t-},t)}.$$

Proof. We will calculate the $\mathcal{F}$-intensities of Y^B and Y^S on I. Their intensities on I^c can be similarly verified. First, observe that the construction of ν_i implies that on I and $[\tau_{i-1} < 1]$

$$\begin{aligned}
\mathbb{P}\left(\nu_i > t \vee \tau_{i-1} \,|\, \mathcal{F}_{\tau_{i-1}}\right) \\
= \mathbb{P}\left(\eta_i > f_i(t \vee \tau_{i-1}) \,|\, \mathcal{F}_{\tau_{i-1}}\right) \\
= \exp\left(\beta \int_{\tau_{i-1}}^{t \vee \tau_{i-1}} \frac{h(Y_{u-},u) - h(Y_{u-}+1,u)}{h(Y_{u-},u)} du\right), \quad \text{for } t \in [0,1).
\end{aligned} \tag{4.5}$$

We will make repeated use of (4.5) in order to obtain the $\mathcal{F}$-intensity of Y^B on I. To this end, note that $[\tau_i^+ > t \geq \tau_{i-1}, I] = [\widetilde{\sigma}_i^+ > t, \nu_i > t, t \geq \tau_{i-1}, I]$. Therefore we have on $[t \geq \tau_{i-1}, I]$ that

$$\begin{aligned}
\mathbb{P}(\tau_i^+ > t \,|\, \mathcal{F}_{\tau_{i-1}}) &= \mathbb{P}(\widetilde{\sigma}_i^+ > t \,|\, \mathcal{F}_{\tau_{i-1}})\mathbb{P}(\nu_i > t \,|\, \mathcal{F}_{\tau_{i-1}}) \\
&= \mathbb{P}(Z_t = Z_{\tau_{i-1}} \,|\, \mathcal{F}_{\tau_{i-1}}) \exp\left(\beta \int_{\tau_{i-1}}^t \frac{h(Y_{u-},u) - h(Y_{u-}+1,u)}{h(Y_{u-},u)} du\right) \\
&= \exp(-\beta(t-\tau_{i-1})) \exp\left(\beta \int_{\tau_{i-1}}^t \frac{h(Y_{u-},u) - h(Y_{u-}+1,u)}{h(Y_{u-},u)} du\right),
\end{aligned} \tag{4.6}$$

where the first line is due to the independence of Z^B and ν_i and the last line follows from the strong Markov property of Z^B and the fact that τ_{i-1} is an $\mathcal{F}$-stopping time.

It is well-known (see, e.g. Proposition 3.1 in [11]) that the $\mathcal{F}$-intensity of Y^B on I is given by

$$\frac{\mathbb{P}(\tau_i^+ \in dt \,|\, \mathcal{F}_{\tau_{i-1}})}{\mathbb{P}(\tau_i^+ > t \,|\, \mathcal{F}_{\tau_{i-1}})dt}, \quad t \in (\tau_{i-1}, \tau_i].$$

Utilising (4.3) and (4.6), it follows from direct calculations and the observation that Y_{t-} is constant in $(\tau_{i-1}, \tau_i]$ that the above intensity is indeed

$$\beta\frac{h(Y_{t-}+1,t)}{h(Y_{t-},t)} \quad \text{on } I \text{ for } t \in (\tau_{i-1}, \tau_i].$$

To calculate the intensity of Y^S on I and in the time interval $(\tau_{i-1}, \tau_i]$, we will treat the evolution of Y^S as that of a marked point process with points $(\sigma_j^- \vee \tau_{i-1})_{j \geq 1}$ and marks $(\xi_{j,i})_{j \geq 1}$. Let $\widetilde{\sigma}_i^- := \sigma_{Z^S_{\tau_{i-1}}+1}^-$ and $\widetilde{\xi}_i$ be the associated mark. Define $G_i(dt, 1) = \mathbb{P}\left(\widetilde{\sigma}_i^- \in dt, \widetilde{\xi}_i = 1 \,|\, \mathcal{F}_{\tau_{i-1}}\right)$ and $H_i(dt) = \mathbb{P}(\widetilde{\sigma}_i^- \in dt \,|\, \mathcal{F}_{\tau_{i-1}})$. It then follows from Proposi-

tion 3.1 in [11] that the intensity of Y^S at $t \in (\tau_{i-1}, \tau_i]$ is

$$
\begin{aligned}
\frac{G_i(dt, 1)}{H_i([t, \infty])dt} &= \frac{E_{\mathbb{P}}\left[\mathbb{I}_{[\widetilde{\sigma}_i^- \in dt]}\mathbb{P}(\widetilde{\xi}_i = 1 \mid \mathcal{F}_{\widetilde{\sigma}_i^-}) \mid \mathcal{F}_{\tau_{i-1}}\right]}{\mathbb{P}(\widetilde{\sigma}_i^- > t \mid \mathcal{F}_{\tau_{i-1}})dt} \\
&= \frac{h(Y_{t-} - 1, t)}{h(Y_{t-}, t)} \frac{\mathbb{P}(\widetilde{\sigma}_i^- \in dt \mid \mathcal{F}_{\tau_{i-1}})}{\mathbb{P}(\widetilde{\sigma}_i^- > t \mid \mathcal{F}_{\tau_{i-1}})dt} \\
&= \frac{h(Y_{t-} - 1, t)}{h(Y_{t-}, t)} \beta,
\end{aligned}
$$

due to the strong Markov property of Z^S. This verifies the intensity of Y^S on I. $\qquad \square$

We are now ready to prove that our construction as desired.

Proposition 4.4. *The process $(Y_t; t \in [0, 1])$ as constructed above satisfies the following properties:*

i) $[Y_1 \geq y_1] = I$, $\mathbb{P}$-*a.s.;*

ii) Y^B *and* Y^S *are independent Poisson processes with intensity β with respect to the natural filtration $(\mathcal{F}_t^Y)_{t \in [0,1]}$ of Y. In particular, Y is finite $\mathbb{P}$-a.s. over $[0, 1]$.*

iii) $\mathbb{E}[X_1^B]$ *and* $\mathbb{E}[X_1^S]$ *are finite. Hence the constructed strategy $(X^B, X^S; \mathcal{F}^I)$ is admissible.*

Proof. To verify that Y satisfies the desired properties, let us introduce an auxiliary process $(\ell_t)_{t \in [0,1)}$ via

$$
\ell_t := \mathbb{I}_I \frac{h(0, 0)}{h(Y_t, t)} + \mathbb{I}_{I^c} \frac{1 - h(0, 0)}{1 - h(Y_t, t)}, \quad t \in [0, 1).
$$

The construction of Y^S on I (resp. Y^B on I^c) implies that there are only a finite number of jumps before a fixed time $t < 1$. Therefore $Y_t > -\infty$ on I (resp. $Y_t < \infty$ on I^c) for $t \in [0, 1)$, which implies $h(Y_t, t) > 0$ on I (resp. $h(Y_t, t) < 1$ on I^c) for $t \in [0, 1)$. As a result, $(\ell_t)_{t \in [0,1)}$ is a well-defined positive process with $\ell_0 = 1$. To prove the first statement, we first show that ℓ is a positive $\mathcal{F}$-local martingale on $[0, 1)$. To this end, Ito's formula yields that

$$
\begin{aligned}
d\ell_t = \mathbb{I}_I \ell_{t-} &\left[\frac{h(Y_{t-}, t) - h(Y_{t-} + 1, t)}{h(Y_{t-} + 1, t)}dM_t^B + \frac{h(Y_{t-}, t) - h(Y_{t-} - 1, t)}{h(Y_{t-} - 1, t)}dM_t^S\right] \\
+ \mathbb{I}_{I^c} \ell_{t-} &\left[\frac{h(Y_{t-} + 1, t) - h(Y_{t-}, t)}{1 - h(Y_{t-} + 1, t)}dM_t^{B,c} + \frac{h(Y_{t-} - 1, t) - h(Y_{t-}, t)}{1 - h(Y_{t-} - 1, t)}dM_t^{S,c}\right], \quad t \in [0, 1).
\end{aligned}
$$

Here

$$
M^B = \mathbb{I}_I Y^B - \mathbb{I}_I \beta \int_0^{\cdot} \frac{h(Y_{u-} + 1, u)}{h(Y_{u-}, u)}du, \qquad M^S = \mathbb{I}_I Y^S - \mathbb{I}_I \beta \int_0^{\cdot} \frac{h(Y_{u-} - 1, u)}{h(Y_{u-}, u)}du,
$$

$$
M^{B,c} = \mathbb{I}_{I^c} Y^B - \mathbb{I}_{I^c} \beta \int_0^{\cdot} \frac{1 - h(Y_{u-} + 1, u)}{1 - h(Y_{u-}, u)}du, \quad M^{S,c} = \mathbb{I}_{I^c} Y^S - \mathbb{I}_{I^c} \beta \int_0^{\cdot} \frac{1 - h(Y_{u-} - 1, u)}{1 - h(Y_{u-}, u)}du
$$

are all $\mathcal{F}$-local martingales. Define $\zeta_n^+ = \inf\{t \in [0, 1] : Y_t = n\}$ and $\zeta_n^- = \inf\{t \in [0, 1] : Y_t = -n\}$. Consider the sequence of stopping times $(\eta_n)_{n \geq 1}$, where

$$
\eta_n := \left(\mathbb{I}_I \zeta_n^- + \mathbb{I}_{I^c} \zeta_n^+\right) \wedge (1 - 1/n).
$$

It follows from the definition of h that $h(Y_t, t)$ on I (resp. $1 - h(Y_t, t)$ on I^c) is bounded away from zero uniformly in $t \in [0, \eta_n]$. This implies that ℓ^{η_n} is bounded, hence ℓ^{η_n} is

a $\mathcal{F}$-martingale. The construction of Y^S on I (resp. Y^B on I^c) yields $\lim_{n\to\infty}\eta_n = 1$. Therefore, ℓ is a positive $\mathcal{F}$-local martingale, hence also a supermartingale, on $[0,1)$.

Define $\ell_1 := \lim_{t\to1}\ell_t$, which exists and is finite due to Doob's supermartingale convergence theorem. This implies $h(Y_{1-},1) > 0$ on I (resp. $1 - h(Y_{1-},1) > 0$ on I^c). Recall that Y^S on I (resp. Y^B on I^c) does not jump at time 1 almost surely. Therefore $h(Y_1,1) > 0$ on I (resp. $1 - h(Y_1,1) > 0$ on I^c), which yields $Y_1 \geq y_1$ on I (resp. $Y_1 < y_1$ on I^c).

Let us now prove the second statement for Y^B. The statement for Y^S can be shown similarly. In view of the $\mathcal{F}$-intensity of Y^B calculated in Lemma 4.3, one has that, for each $i \geq 1$

$$
Y^B_{\cdot\wedge\tau_i\wedge1} - \beta\left(\mathbb{I}_I\int_0^{\cdot\wedge\tau_i\wedge1}\frac{h(Y_{u-}+1,u)}{h(Y_{u-},u)}du + \mathbb{I}_{I^c}\int_0^{\cdot\wedge\tau_i\wedge1}\frac{1-h(Y_{u-}+1,u)}{1-h(Y_{u-},u)}du\right)
$$

is an $\mathcal{F}$-martingale. We will show in the next paragraph that, when stopped at $\tau_i\wedge1$, Y^B is a Poisson process in $\mathcal{F}^Y$ by showing that $(Y^B_{t\wedge\tau_i} - \beta(t\wedge\tau_i))_{t\in[0,1]}$ is a $\mathcal{F}^Y$-martingale– recall that τ_i is an $\mathcal{F}^Y$-stopping time. This in turn will imply that Y^B is a Poisson process with intensity β on $[0,\tau\wedge1)$ where $\tau = \lim_{i\to\infty}\tau_i$ is the explosion time. Since Poisson process does not explode, this will further imply $Y^B_{\tau\wedge1} < \infty$ and, therefore, $\tau \geq 1$, $\mathbb{P}$-a.s. in view of Remark 4.2.

We proceed by projecting the above martingale into $\mathcal{F}^Y$ to see that

$$
Y^B - \beta\int_0^{\cdot}\left[\mathbb{P}(I\,|\,\mathcal{F}^Y_u)\frac{h(Y_{u-}+1,u)}{h(Y_{u-},u)} + \mathbb{P}(I^c\,|\,\mathcal{F}^Y_u)\frac{1-h(Y_{u-}+1,u)}{1-h(Y_{u-},u)}\right]du
$$

is a $\mathcal{F}^Y$-martingale when stopped at $\tau_i\wedge1$. Therefore, it remains to show that, for almost all $t\in[0,1)$, on $[t \leq \tau_i]$

$$
\mathbb{P}(I\,|\,\mathcal{F}^Y_t)\frac{h(Y_{t-}+1,t)}{h(Y_{t-},t)} + \mathbb{P}(I^c\,|\,\mathcal{F}^Y_t)\frac{1-h(Y_{t-}+1,t)}{1-h(Y_{t-},t)} = 1.
$$

In the remaining of the proof, we will show that on $[t \leq \tau_i]$

$$
\mathbb{P}(I\,|\,\mathcal{F}^Y_t) = h(Y_t,t) \quad\text{and}\quad \mathbb{P}(I^c\,|\,\mathcal{F}^Y_t) = 1 - h(Y_t,t), \quad\text{for } t\in[0,1). \tag{4.7}
$$

The statement then follows since $Y_t \neq Y_{t-}$ only for countably many times.

We have seen that $(\ell_{u\wedge\tau_i})_{u\in[0,t]}$ is a strictly positive $\mathcal{F}$-martingale for each i. Define a probability measure $\mathbb{Q}^i \sim \mathbb{P}$ on $\mathcal{F}_t$ via $d\mathbb{Q}^i/d\mathbb{P}|_{\mathcal{F}_t} = \ell_{\tau_i\wedge t}$. It follows from a simple application of Girsanov's theorem that $(Y^B_{\cdot})$ and $(Y^S_{\cdot})$ are Poisson processes when stopped at $\tau_i\wedge t$ and with intensity β under $\mathbb{Q}^i$. Therefore, they are independent from I under $\mathbb{Q}^i$. Then, for $t < 1$ we obtain from Bayes' formula that

$$
\begin{aligned}
\mathbb{I}_{[u\leq\tau_i\wedge t]}\mathbb{P}(I\,|\,\mathcal{F}^Y_u) &= \mathbb{I}_{\{u\leq\tau_i\wedge t\}}\frac{E_{\mathbb{Q}^i}\left[\mathbb{I}_I\ell_u^{-1}\,|\,\mathcal{F}^Y_u\right]}{E_{\mathbb{Q}^i}\left[\ell_u^{-1}\,|\,\mathcal{F}^Y_u\right]}\\
&= \mathbb{I}_{[u\leq\tau_i\wedge t]}\frac{E_{\mathbb{Q}^i}\left[\mathbb{I}_I\frac{h(Y_u,u)}{h(0,0)}\,|\,\mathcal{F}^Y_u\right]}{E_{\mathbb{Q}^i}\left[\mathbb{I}_I\frac{h(Y_u,u)}{h(0,0)} + \mathbb{I}_{I^c}\frac{1-h(Y_u,u)}{1-h(0,0)}\,|\,\mathcal{F}^Y_u\right]}\\
&= \mathbb{I}_{[u\leq\tau_i\wedge t]}h(Y_u,u),
\end{aligned}\tag{4.8}
$$

where the third identity follows from the aforementioned independence of Y and I under $\mathbb{Q}^i$ along with the fact that $\mathbb{Q}^i$ does not change the probability of $\mathcal{F}_0$ measurable events, so that $\mathbb{Q}^i(I) = \mathbb{P}(I) = h(0,0)$. As a result, (4.7) follows from (4.8) after sending $i \to \infty$.

Finally, since Y^B and Y^S are Poisson processes in $\mathcal{F}^Y$ and they do not jump simultaneously by their construction, they are independent (see [9, Proposition 5.3]). Since Y^B and Y^S are independent Poisson processes, it also follows $\mathbb{E}[X_1^B] < \infty$. Indeed, since $X_1^B \mathbb{I}_{I^c} = 0$, we have $\mathbb{E}[X_1^B] = \mathbb{E}[X_1^B \mathbb{I}_I] = \mathbb{E}[(Y_1^B - Z_1^B + X^{B,S}) \mathbb{I}_I] \leq \mathbb{E}[Y_1^B] + \mathbb{E}[Z_1^B] + \mathbb{E}[Z_1^S] < \infty$. Similar arguments also show that $\mathbb{E}[X_1^S] < \infty$. Hence, the constructed strategy $(X^B, X^S; \mathcal{F}^I)$ is admissible. $\qquad\square$

5 Existence and convergence of Glosten-Milgrom equilibria

In view of the results of Section 4, we can now show that a Glosten-Milgrom equilibrium exists for the market model under consideration.

Theorem 5.1. *Suppose that $(\eta_i)_{i \geq 1}$ and $(\zeta_i)_{i \geq 1}$ are two sequences of independent $\mathcal{F}$-measurable random variables uniformly distributed over $[0,1]$ that are independent from each other, Z and $\tilde{v}$. If there exists a y_δ such that*

$$\mathbb{P}(Z_1 \geq y_\delta) = \mathbb{P}(\tilde{v} = 1),$$

and $\mathcal{F}^I$ is the right continuous augmentation of $(\sigma(\tilde{v}, Z_s, \eta_i, \zeta_i; s \leq t, i \geq 1))_{t \in [0,1]}$ with the $\mathbb{P}$-null sets, then there exists a Glosten-Milgrom equilibrium.

Proof. In view of Theorem 3.4, an equilibrium exists if Y satisfies the conditions stated in Theorem 3.4 and the high type (resp. low type) insider never sells (resp. buys). However, the insider can use the uniform random variables available in her filtration to perform the construction described in Section 4 so that Y satisfies the desired properties, due to Proposition 4.4, without having to sell (resp. buy) when high type (resp. low type). $\qquad\square$

In the remainder of this section we will analyse what happens when the trade size becomes small ($\delta \to 0$) and the noise trades arrive more frequently ($\beta \to \infty$). A similar convergence has also been studied by Back and Baruch in [4] who have established that the limiting economy can be described by a Kyle-Back equilibrium. We would like to mention at this point that Back and Baruch have proved their convergence results under some extra hypotheses on the convergence of value functions which may be hard to verify. As we shall see below, we will verify the convergence via a weak convergence approach and we do not need any extra assumptions in addition to the ones which have already been assumed. Before performing a weak convergence analysis of Glosten-Milgrom equilibria, whose existence is justified by Theorem 5.1, let's first briefly describe what we mean by a Kyle-Back equilibrium.

The continuous-time model of Kyle [14], which was later extended by Back [2], studies the equilibrium pricing of a risky asset whose liquidation value at time 1 is given by $\tilde{v}$. In this model, the cumulative noise trades is modelled by a Brownian motion, denoted with W, independent of $\tilde{v}$. The risk neutral insider knows the true liquidation value from the beginning and competition among the risk neutral market makers forces them to quote prices as conditional expectations of $\tilde{v}$ based on their information. The price is again set in a Markovian manner, i.e. there exists a function $p^0 : \mathbb{R} \mapsto [0,1]$ so that the market price is given by $p^0(Y_t, t)$ at time t where Y is, as before, the cumulative demand at time t.

Let $\Omega^0 = \mathbb{D}([0,1], \mathbb{R})$ be the space of $\mathbb{R}$-valued càdlàg functions on $[0,1]$ with the coordinate process Y^0 and $\mathbb{P}^0$ be the Wiener measure. In view of the results of [2] and [8], the equilibrium price of the risky asset in this economy is given by

$$p^0(y, t) := \mathbb{P}_y^0 \left[Y_{1-t}^0 \geq y_0 \right], \tag{5.1}$$

where

$$y_0 := \Phi^{-1}(1 - \mathbb{P}(\tilde{v} = 1)),$$

and $\Phi(\cdot) = \int_{-\infty}^{\cdot} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$. The equilibrium demand satisfies the SDE

$$Y = W + \mathbb{I}_{[\tilde{v}=1]} \int_0^{\cdot} \partial_y \log p^0(Y_s, s)\, ds + \mathbb{I}_{[\tilde{v}=0]} \int_0^{\cdot} \partial_y \log(1 - p^0(Y_s, s))\, ds. \qquad (5.2)$$

Remark 5.2. Strictly speaking, the equilibrium price (5.1) and demand (5.2) in this economy do not follow directly from the results of [2] and [8] since in their framework $\tilde{v}$ has a continuous distribution. However, if one follows the arguments for the description of equilibrium given in [8], it follows that in equilibrium the insider trades so that $\tilde{v} = p^0(Y_1, 1)$ and Y is a Brownian motion in its own filtration. This immediately gives (5.1) as the equilibrium price, since the price follows a martingale with respect to the filtration of the market maker, which is the same as the filtration generated by Y. Moreover, the same characterisation gives that the SDE satisfied by Y with respect to the filtration of the insider is the same as the SDE satisfied by a standard Brownian motion when its natural filtration is initially enlarged with the random variable corresponding to its time 1 value being larger than y_0. The standard arguments contained in, e.g. Section 1.3 of [15], gives (5.2).

In view of the well-known results on the weak convergence of a sequence of difference of Poisson processes to Brownian motion (see, e.g., Theorem 5.4 in Chapter 6 of [10]), it is easy to see that the cumulative demand of noise traders in a Kyle-Back model can be considered as the weak limit of noise demands in a sequence of Glosten-Milgrom models. Based on this observation it is natural to ask whether the Kyle-Back equilibrium is the weak limit of Glosten-Milgrom equilibria.

We now return to give an affirmative answer to this question. More precisely, we consider the convergence of Glosten-Milgrom equilibria to the Kyle-Back equilibrium described by (5.1) and (5.2). In what follows, the superscript $\delta \geq 0$ indicates the trade size associated to different processes, probabilities, random variables, and functions.

Let $(\Omega^\delta, \mathcal{F}^\delta, (\mathcal{F}_t^\delta)_{t\in[0,1]}, \mathbb{P}^\delta)_{\delta \geq 0}$ be a sequence of probability spaces on which the Glosten-Milgrom models of different order sizes are defined. When $\delta > 0$, $\Omega^\delta = \mathbb{D}([0,1], \delta\mathbb{Z})$ is the space of $\delta\mathbb{Z}$-valued càdlàg functions on $[0,1]$ with the coordinate process Y^δ, $(\mathcal{F}_t^\delta)_{t\in[0,1]}$ is the minimal right continuous and complete filtration generated by Y^δ, and $\mathbb{P}^\delta$ for $\delta > 0$ is the probability measure under which Y^δ is the difference of two independent Poisson processes with the same intensity β^δ. $\mathbb{P}^0$ is the Wiener measure as mentioned in the earlier paragraphs.

To construct a sequence of pricing rules in Glosten-Milgrom equilibria which converges to the Kyle-Back equilibrium, set

$$y_\delta := \inf\{y \in \delta\mathbb{Z}, \mathbb{P}^\delta(Y_1^\delta \leq y) \geq 1 - \mathbb{P}(\tilde{v} = 1)\}, \quad \text{for } \delta > 0,$$

and denote p^{y_δ}, defined in (3.4), by p^δ for simplicity. To ensure the existence of Glosten-Milgrom equilibria with pricing rules $(p^\delta)_{\delta>0}$, we introduce a sequence of Bernoulli random variables $(\tilde{v}^\delta)_{\delta>0}$ whose distribution is

$$\mathbb{P}(\tilde{v}^\delta = 1) = \mathbb{P}^\delta(Y_1^\delta \geq y_\delta). \qquad (5.3)$$

These $(\tilde{v}^\delta)_{\delta>0}$ will be the liquidation values of the risky asset in the sequence of Glosten-Milgrom models which converges to the Kyle-Back model.

Theorem 5.3. *For any $\tilde{v}$ satisfying $\mathbb{P}(\tilde{v} = 1) \in (0, 1)$, there exists a sequence of admissible strategies $(X^{B,\delta}, X^{S,\delta})_{\delta>0}$ such that, for each $\delta > 0$, $(p^\delta, X^{B,\delta}, X^{S,\delta})$ is a Glosten-Milgrom equilibrium whose fundamental value of the risky asset is $\tilde{v}^\delta$.*

When the intensity of Poisson process is given by $\beta^\delta = (2\delta^2)^{-1}$ in the Glosten-Milgrom model, as $\delta \to 0$, the sequence of Glosten-Milgrom equilibria converge to the Kyle-Back equilibrium in the following sense:

i) The bid and ask prices in these Glosten-Milgrom equilibria converge to the price in the Kyle-Back equilibrium. That is, $\lim_{\delta\downarrow0} a^\delta(y,t) = \lim_{\delta\downarrow0} b^\delta(y,t) = \lim_{\delta\downarrow0} p^\delta(y,t) = p^0(y,t)$ for $(y,t) \in \mathbb{R} \times [0,1)$. Moreover, the corresponding market depths in the Glosten-Milgrom equilibria converges to the market depth in the Kyle-Back equilibrium:

$$\lim_{\delta\downarrow0} \frac{1}{\delta}\left(a^\delta(y,t) - p^\delta(y,t)\right) = \lim_{\delta\downarrow0} \frac{1}{\delta}\left(p^\delta(y,t) - b^\delta(y,t)\right) = \partial_y p^0(y,t), \quad for (y,t) \in \mathbb{R}\times[0,1).$$

ii) Let $Y^{0,H}$ and $Y^{0,L}$ be the solutions to the following two SDEs, respectively,

$$dY_t = \frac{\partial_y p^0(Y_t,t)}{p^0(Y_t,t)}\, dt + dW_t \quad and \quad dY_t = -\frac{\partial_y p^0(Y_t,t)}{1 - p^0(Y_t,t)}\, dt + dW_t, \qquad t \in [0,1),$$

where W is a Brownian motion under $(\Omega, \mathcal{F}^0, (\mathcal{F}^0_t)_{t\in[0,1]}, \mathbb{P}^0)$. Define

$$B^0_\cdot = \int_0^\cdot \frac{\partial_y p^0(Y^{0,H}_t,t)}{p^0(Y^{0,H}_t,t)}\, dt \quad and \quad S^0_\cdot = \int_0^\cdot \frac{\partial_y p^0(Y^{0,L}_t,t)}{1 - p^0(Y^{0,L}_t,t)}\, dt.$$

Then,

- When $\tilde{v} = 1$, $X^{B,\delta} \overset{\mathscr{L}}{\to} B^0$;
- When $\tilde{v} = 0$, $X^{S,\delta} \overset{\mathscr{L}}{\to} S^0$,

where $\overset{\mathscr{L}}{\to}$ represents the convergence in law.

iii) (p^0, Y^0) satisfies (5.1) and (5.2) where

$$Y^0 = \mathbb{I}_{[v=1]}Y^{0,H} + \mathbb{I}_{[v=0]}Y^{0,L}.$$

As such, p^0 and Y^0 are the equilibrium price and demand in the Kyle-Back equilibrium, respectively.

The above theorem tells us that Kyle-Back model with Bernoulli distributed $\tilde{v}$ can be approximated by a sequence of Glosten-Milgrom models whose risky asset fundamental price converges to $\tilde{v}$ in distribution. Since there is no bid-ask spread in the Kyle-Back equilibrium, the above convergence results in particular tell us that the bid-ask spread gets smaller and vanish in the limit as the frequency of noise trades increase. Moreover the rate the convergence is $O(\delta)$.

To show the desired convergence results contained in the theorem above, let us first prove the convergence in law of the cumulative order processes as seen in the filtration, say $\overline{\mathcal{F}}^\delta$, with respect to which X^δ and Z^δ are adapted and $\tilde{v} \in \overline{\mathcal{F}}^\delta_0$. Note that this filtration is smaller than the filtration that is assumed to be contained in the insider's filtration in Theorem 5.1, however, it contains all the relevant processes and random variables describing insider's strategy and the informational advantage. Moreover, it will be enough to limit ourselves to these filtrations in order to prove Theorem 5.3. Recall from Section 4 that, for each $\delta > 0$, the distribution of the cumulative order process in $\overline{\mathcal{F}}^\delta$ is the same as the distribution of Y^δ conditioned on $Y^\delta_1 \geq y_\delta$ or $Y^\delta_1 < y_\delta$. Here Y^δ/δ is the difference of two Poisson processes in its own filtration.

Lemma 5.4. Let $\beta^\delta = (2\delta^2)^{-1}$. We have

$$Law(Y^\delta\,|\,Y^\delta_1 \geq y_\delta) \Rightarrow Law(Y^{0,H}) \quad and \quad Law(Y^\delta\,|\,Y^\delta_1 < y_\delta) \Rightarrow Law(Y^{0,L}), \quad as\ \delta \to 0,$$

where $Y^{0,H}$ and $Y^{0,L}$ are defined in Theorem 5.3 ii) and $\Rightarrow$ represents the weak convergence of probability measures.

Proof. The first convergence will be proved. The second convergence can be shown similarly. Since $\beta^\delta = (2\delta^2)^{-1}$, it follows from [10, Theorem 5.4 in Chapter 6] that $\mathbb{P}^\delta \Rightarrow \mathbb{P}^0$ and, in particular, $Law(Y_1^\delta) \Rightarrow Law(Y_1^0)$. Observe that y_δ is the $(1 - \mathbb{P}(\tilde{v} = 1))^{th}$ quantile of the distribution for Y_1^δ and the distribution of Y_1^0 is continuous. It then follows

$$\lim_{\delta \downarrow 0} y_\delta = y_0. \tag{5.4}$$

Meanwhile the conditional distribution $Law(Y^\delta \mid Y_1^\delta \geq y_\delta)$ is defined via

$$\mathbb{P}^{\delta,H}(A) := \frac{\mathbb{P}^\delta(A, Y_1^\delta \geq y_\delta)}{\mathbb{P}^\delta(Y_1^\delta \geq y_\delta)}, \quad \text{for } A \in \mathcal{F}^\delta. \tag{5.5}$$

We will show $\mathbb{P}^{\delta,H} \Rightarrow \mathbb{P}^{0,H}$ as $\delta \downarrow 0$. This statement will follow once we show the finite dimensional distributions of Y^δ converge weakly to the finite dimensional distributions of Y^0, and $(\mathbb{P}^{\delta,H})_{\delta > 0}$ is tight (see e.g. [12, VI.3.20]). We will prove both of these conditions using the already observed convergence of $\mathbb{P}^\delta$ to $\mathbb{P}^0$.

To this end, we will first establish the the convergence of $\mathbb{P}^\delta(Y_1^\delta \geq y_\delta)$ to $\mathbb{P}^0(Y_1^0 \geq y_0)$. Indeed, due to (5.4), there exists a sufficiently small δ_ϵ such that $y_\delta \geq y_0 - \epsilon$ for $\delta \leq \delta_\epsilon$. Thus,

$$\mathbb{P}^\delta(Y_1^\delta \geq y_\delta) \leq \mathbb{P}^\delta(Y_1^\delta \geq y_0 - \epsilon) \to \mathbb{P}^0(Y_1^0 \geq y_0 - \epsilon), \quad \text{as } \delta \downarrow 0,$$

where the convergence follows from $Law(Y_1^\delta) \Rightarrow Law(Y_1^0)$ and the fact that the distribution of Y_1^0 is continuous at $y_0 - \epsilon$. Then the previous inequality yields $\limsup_{\delta \downarrow 0} \mathbb{P}^\delta(Y_1^\delta \geq y_\delta) \leq \mathbb{P}^0(Y_1^0 \geq y_0)$ since the choice of ϵ is arbitrary. Combining the previous inequality with $\liminf_{\delta \downarrow 0} \mathbb{P}^\delta(Y_1^\delta \geq y_\delta) \geq \mathbb{P}^0(Y_1^0 \geq y_0)$, which can be similarly proved, we obtain

$$\lim_{\delta \downarrow 0} \mathbb{P}^\delta(Y_1^\delta \geq y_\delta) = \mathbb{P}^0(Y_1^0 \geq y_0) = \mathbb{P}(\tilde{v} = 1) > 0. \tag{5.6}$$

In order to prove the convergence of the finite dimensional distributions of Y^δ, we are first going to show

$$\lim_{\delta \downarrow 0} \mathbb{E}^{\mathbb{P}^{\delta,H}} \left[f(Y_{t_1}^\delta, \cdots, Y_{t_n}^\delta) \right] = \mathbb{E}^{\mathbb{P}^{0,H}} \left[f(Y_{t_1}^0, \cdots, Y_{t_n}^0) \right],$$

for arbitrary bounded continuous function $f : \mathbb{R}^n \to \mathbb{R}$ and $0 \leq t_1 \leq \cdots < t_n \leq 1$. However, similar arguments as those employed in the last paragraph yield

$$\lim_{\delta \downarrow 0} \mathbb{E}^{\mathbb{P}^\delta} \left[f(Y_{t_1}^\delta, \cdots, Y_{t_n}^\delta) \mathbb{I}_{[Y_1^\delta \geq y_\delta]} \right] = \mathbb{E}^{\mathbb{P}^0} \left[f(Y_{t_1}^0, \cdots, Y_{t_n}^0) \mathbb{I}_{[Y_1^0 \geq \delta_0]} \right].$$

The claim then follows from combining the previous convergence with (5.5) and (5.6).

To verify the tightness of $(\mathbb{P}^{\delta,H})_{\delta > 0}$, it is equivalent to prove the following two conditions (see [12, Theorem VI.3.21]):

1. for any $\epsilon > 0$, there exist δ_ϵ and $K \in \mathbb{R}$ with

$$\mathbb{P}^{\delta,H} \left(\sup_{0 \leq t \leq 1} |Y_t^\delta| > K \right) \leq \epsilon, \quad \text{for all } \delta \leq \delta_\epsilon;$$

2. for any $\epsilon > 0$ and $\eta > 0$, there exists $\delta_{\epsilon,\eta}$ and $\theta_{\epsilon,\eta}$ such that

$$\mathbb{P}^{\delta,H} \left(w_1'(Y^\delta, \theta_{\epsilon,\eta}) \geq \eta \right) \leq \epsilon, \quad \text{for all } \delta \leq \delta_{\epsilon,\eta}.$$

We refer reader to [12, Chapter VI, Section 1a] for the definition of w_1'.

Observe that, since $\mathbb{P}^\delta \Rightarrow \mathbb{P}^0$, $(\mathbb{P}^\delta)_{\delta>0}$ is tight which implies that the two conditions above hold when $\mathbb{P}^{\delta,H}$ is replaced by $\mathbb{P}^\delta$. Moreover, if A stands for the event $[\sup_{0\le t\le 1}|Y_t^\delta| > K]$ or $[w_1'(Y^\delta, \theta_{\epsilon,\eta}) \ge \eta]$, (5.5) along with (5.6) yields

$$\mathbb{P}^{\delta,H}(A) = \frac{\mathbb{P}^\delta(A, Y_1^\delta \ge y_\delta)}{\mathbb{P}^\delta(Y_1^\delta \ge y_\delta)} \le \frac{\mathbb{P}^\delta(A)}{\mathbb{P}^\delta(Y_1^\delta \ge y_\delta)} \le \frac{\epsilon}{\mathbb{P}^\delta(Y_1^\delta \ge y_\delta)} \le \frac{2\epsilon}{\mathbb{P}^0(Y_1^0 \ge y_0)} \quad \text{for sufficiently small } \delta,$$

which confirms the aforementioned conditions for $\mathbb{P}^{\delta,H}$.

Finally, it remains to verify that $\mathbb{P}^{0,H}$ is the law of $Y^{0,H}$. To this end, note that $\mathbb{P}^{0,H}$ is the law of a Brownian motion conditioned on its time 1 value being larger than y_0. A standard calculation using the well-known h-transform technique gives the following semimartingale decomposition of Y^0 under $\mathbb{P}^{0,H}$:

$$Y_t^0 = \int_0^t \frac{\partial_y p^0(Y_u^0, u)}{p^0(Y_u^0, u)} \, du + \widetilde{W}_t, \quad t \in [0,1),$$

where $\widetilde{W}$ is a $\mathbb{P}^{0,H}$-Brownian motion. Since $\partial_y p^0 / p^0$ is locally Lipschitz, the previous SDE has a unique solution in law, therefore $\mathbb{P}^{0,H}$ must be the law of $Y^{0,H}$. □

We are now ready to prove the convergence results.

Proof of Theorem 5.3. The existence of Glosten-Milgrom equilibria follows from (5.3) and Theorem 5.1 directly. We will prove the statements on convergence in what follows.

i) First note that $\lim_{\delta\downarrow 0} p^\delta(y,t) = p^0(y,t)$ follows from the argument which leads to (5.6). Moreover, this immediately implies the convergence of bid and ask prices as given in i) since $a^\delta(y,t) = p^\delta(y+\delta,t)$ and $b^\delta(y,t) = p^\delta(y-\delta,t)$. To verify the convergence of the market depth, observe that

$$a^\delta(y,t) - p^\delta(y,t) = \mathbb{P}_{y+\delta}^\delta[Y_{1-t}^\delta \ge y_\delta] - \mathbb{P}_y^\delta[Y_{1-t}^\delta \ge y_\delta] = \mathbb{P}_0^\delta[Y_{1-t}^\delta = y_\delta - y - \delta]$$
$$= \overline{\mathbb{P}}\left[\overline{Y}_{1-t} = \frac{y_\delta - y - \delta}{\delta}\right],$$

where $\overline{Y}_{1-t}$ is the difference of two independent Poisson random variables with the common parameter $(1-t)\beta = (1-t)(2\delta^2)^{-1}$ under $\overline{\mathbb{P}}$. Recall that the difference of two independent Poissons has the so-called *Skellam* distribution (see [16]). Thus, $\overline{\mathbb{P}}(\overline{Y}_{1-t} = k) = e^{-2\mu} I_{|k|}(2\mu)$, where $I_{|k|}(\cdot)$ is the *modified Bessel function of the second kind* and $\mu = (1-t)(2\delta^2)^{-1}$. As a result

$$\frac{1}{\delta}(a^\delta(y,t) - p^\delta(y,t)) = \frac{1}{\delta}\overline{\mathbb{P}}\left[\overline{Y}_{1-t} = \frac{y_\delta - y - \delta}{\delta}\right]$$
$$= \frac{1}{\delta} \exp\left(-\frac{1-t}{\delta^2}\right) I_{\left|\frac{y_\delta - y - \delta}{\delta}\right|}\left(\frac{1-t}{\delta^2}\right)$$
$$\to \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(y_0-y)^2}{2(1-t)}\right), \quad \text{as } \delta \downarrow 0.$$

Here the convergence follows from (5.4) and [1, Theorem 2], which states that the density of the Skellam distribution converges to the density of the normal after appropriate rescaling. Similar argument shows that $(p^\delta(y,t) - b^\delta(y,t))/\delta$ converges to the same function. This establishes the convergence of market depths given in i) since $\partial_y p^0(y,t)$ is exactly the normal density above.

ii) Recall from Section 4 and the discussion preceding Lemma 5.4 that, for each $\delta > 0$, the distribution of the cumulative order process in $\overline{\mathcal{F}}^\delta$ on the set $[\tilde{v} = 1]$ (resp. $[\tilde{v} =$

0]) is the same as the distribution of Y^δ conditioned on $Y_1^\delta \geq y_\delta$ (resp. $Y_1^\delta < y_\delta$). However, Lemma 5.4 has already shown that $Law(Y^\delta \,|\, Y_1^\delta \geq y_\delta) \Rightarrow Law(Y^{0,H})$, where $Y^{0,H} = B^0 + W$. Since $Law(Z^\delta) \Rightarrow Law(W)$ as $\delta \downarrow 0$, it follows from [12, Proposition VI.1.23] that $Law(X^{B,\delta}) \Rightarrow Law(B^0)$ as $\delta \downarrow 0$. The convergence of $Law(X^{S,\delta})$ can be similarly proved.

iii) This now follows from Remark 5.2.

$\square$

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