

Rohit Rahi and Jean-Pierre Zigrand Arbitrage networks

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Arbitrage Networks*

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Abstract

This paper studies the general equilibrium implications of arbitrage trades by strategic players in segmented financial markets. Arbitrageurs exploit clientèle effects and choose to specialize in one category of trades, taking into consideration all other arbitrage strategies. This results in an equilibrium network of arbitrageurs. The optimal network for arbitrageurs is of the hub-spoke kind. The equilibrium network, in contrast, is never optimal for arbitrageurs and is never hub-spoke. The reason is that equilibrium networks suffer from a Prisoner's Dilemma problem that prevents network externalities from being internalized. We show that, as the number of intermediaries grows, equilibrium allocations converge to those of the frictionless complete-markets Arrow-Debreu economy.

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1 Introduction

The Arrow-Debreu-Radner (ADR) model describes a world in which all economic actors are price-takers and in which all claims and commodities are traded on a centralized exchange, with a Walrasian auctioneer determining one market price vector clearing all markets simultaneously. The simplifications involved in this setup have allowed the model to become a useful benchmark in economic theory.

Clearly not all actual markets correspond exactly or even approximately to such an idealization, and we would like to argue in this paper that global financial markets should be modeled based upon an extension of the ADR model in at least two directions.

First, while most retail investors in financial markets can be safely considered to be price-takers, agents such as universal banks, investment banks, market makers, mutual and hedge fund managers, insurance companies and the like do exert considerable influence on markets and must be presumed to be strategic rather than price-taking.

Second, not all assets and commodities in the entire world are traded simultaneously on one single giant exchange. Assets are traded on a variety of trading posts, such as stock exchanges, options and futures exchanges, as well as over-the-counter (OTC), i.e. in direct and private arms-length transactions bypassing formal exchanges. A large fraction of trades are OTC, and in this category one can include many derivatives deals, foreign exchange dealings, upstairs trading, block trading, bank loans and deposits, private placements of securities, book building such as in primary and secondary stock issues, and so forth. We refer to such trading posts as *exchanges*. As a result, various clientèles trade on different exchanges, and very few retail clients trade on more than one exchange, let alone on all of them simultaneously. This invalidates the posit of traditional pricing theory whereby the marginal investor in every asset market is the same broadly diversified representative investor.

This market segmentation leads to asset price characteristics distinct from those that the ADR model can generate. The obvious example that comes to mind is the importance of geographic factors. Traders need to be embedded locally in order to appreciate local market conventions, local demand by clients and to gather local information in order to successfully develop a “market feel” and to respond by formulating a “market view” (see for instance the survey-based paper by Agnes (2000)). Segmentations do not exclusively arise due to geographic factors. The usefulness of a general segmentation setup has been recognized long ago, as documented for instance in the success of the market segmentation hypothesis (Culbertson (1957)) and the preferred habitat hypothesis (Modigliani and Sutch (1966)) in fixed income analytics. For example, banks and building societies concentrate a large part of their activity at the short end of the interest rate term structure, both for asset-liability and for regulatory reasons, while pension funds and insurance companies operate at the long end. More generally, the assumption of market segmentation implies that asset prices are determined locally and that as a result overall asset prices need

not be contained in the set of no-arbitrage prices. This opens up the possibility for some sophisticated players to profit from said opportunities by intermediating and facilitating trade. These large players, which we shall simply refer to as *arbitrageurs* in this paper, have a well-defined objective function even in the presence of exploitable arbitrage opportunities, since the awareness of a market impact naturally bounds their trades, for else arbitrage opportunities vanish and no profit at all can be reaped.

A series of recent papers have tried to empirically quantify the extent to which state prices differ across markets. One of the first systematic studies is the paper by Chen and Knez (1995) who consider the mean-square distance between sets of state prices on different exchanges. They find that the NYSE and NASDAQ are priced by sets of state prices which are close in that norm but do not intersect, showing that the marginal investors on the two markets are close but distinct. There is also a prolific literature on home bias, wherein a national stock market is held and priced predominantly by national investors (see for instance Lewis (1999) for a survey). More recently, there have been a number of event studies of changes in the composition of the S&P 500 index (see for instance Massa et al. (2005)). Around the time of the addition a stock to the index, mutual funds benchmarked to the index have an incentive to purchase the stock, and their marginal valuations are shown to differ from those of the market at large. Part of the resulting arbitrage is in fact performed by the managers of the company admitted to the index. Similarly, Da and Gao (2006) provide empirical evidence supporting the view that a sharp rise in a firm's default likelihood causes a change in its shareholder clientele as mutual funds decrease their holdings of the firm's shares. This liquidity shock is initially absorbed by market-makers before large traders move in to provide the liquidity. The paper by Blackburn et al. (2006) attempts to show that the marginal growth investor is distinct from the marginal value investor by measuring the different risk aversion parameters priced into the two markets. They find preliminary evidence that the marginal growth investor is indeed less risk averse than the marginal value investor. Gabaix et al. (2007), in a study of the mortgage-backed securities (MBS) market, show that collateralized mortgage obligations (CMOs) are priced not by the marginal investor of the broader market whose state prices depend on the aggregate wealth or consumption of the economy, but by investors wholly specialized in the MBS market. In particular, they find that prepayment risk is priced even though it washes out in the aggregate. The market price of prepayment risk has a systematic relation with the marginal utility proxy of the MBS specialist investor. The intermediaries who purchase the mortgages and transform them into CMOs play the role of our arbitrageurs. A further illustration of the way the stresses in the credit markets starting in May 2005, initially triggered by concerns about GM and Ford, were transmitted through the system show the complex arbitrage web in contemporary credit correlation trades. Schematically, a simplified rendition of the arbitrage network can be found in Figure 1.

The fact that markets are decentralized across various exchanges leaves open

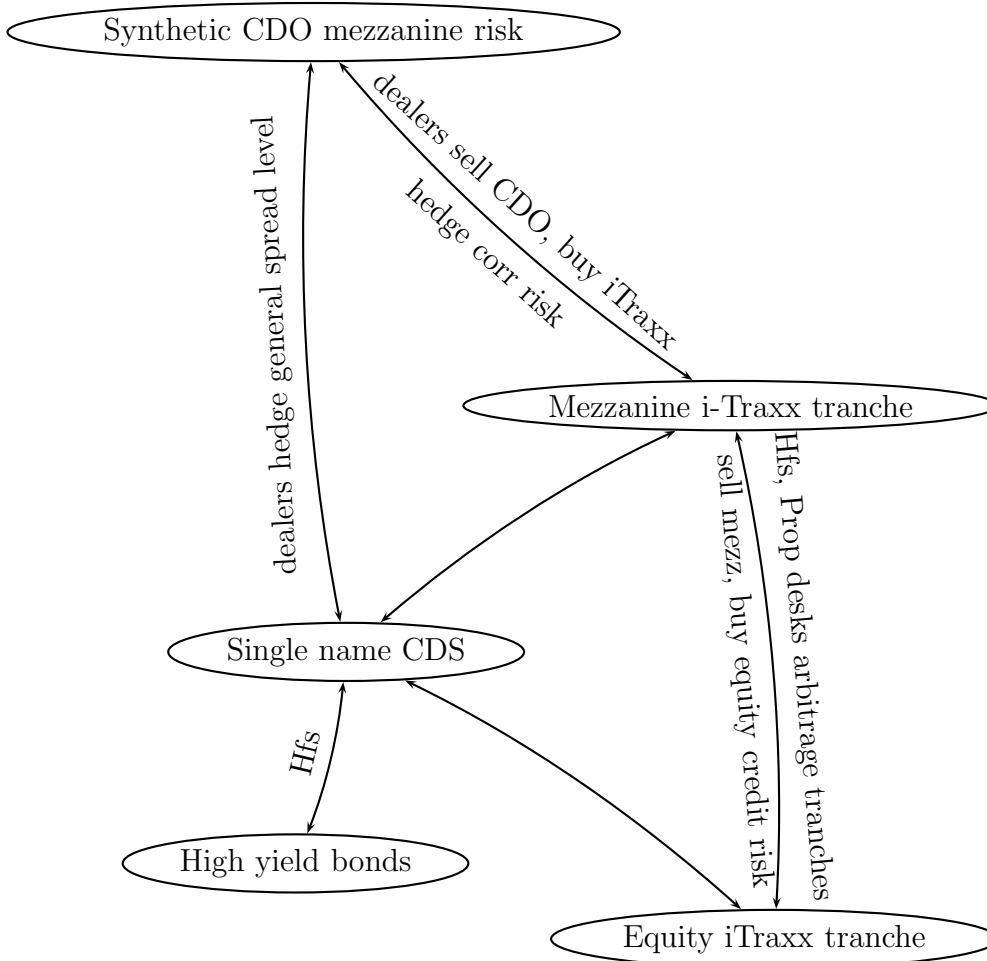


Figure 1: **Credit Correlation Network.** Continental European and Asian banks, pension funds and life insurers in their quest for higher returns buy synthetic mezzanine risk. These structured synthetic CDOs are sold to them by dealers, i.e. the dealers buy credit insurance from these clients. The dealers in turn hedge their exposures partly with single name CDS to hedge exposure to the level of spreads, and partly with mezzanine iTraxx tranches. Notice it is the dealers' arbitraging between the CDO and the iTraxx which creates a dislocation between the various iTraxx tranches, which in turn attracts hedge funds and the dealers' prop trading desks. These latter sell credit risk on mezzanine tranches and buy credit risk on equity by selling protection on the equity iTraxx tranches.

the question as to how the global arbitrageurs link exchanges and investors. In Rahi and Zigrand (2007c), all arbitrageurs are active on all exchanges. We refer to this scenario as “universal arbitrage.” In actual trading networks, however, even large traders only operate on a few exchanges at best. For instance, pairs trading is a fashionable component of equity long-short hedge funds. What is more, even if they do operate between a number of exchanges, the various desks do not seem to coordinate in general. Anecdotal evidence puts this down to informational and other frictions (for instance, Agnes (2000) cites local “market feel” as the reason for a concerted strategy among global swaps banks to decentralize non-US swaps books to their natural markets), to the fact that each desk has allocated a capital limit and operates roughly as a stand-alone profit center, as well as to the fact that compensations nearly exclusively depend on a desk’s own P&L and lead to a natural rivalry among dealers within the same global institution (refer for instance to Drobny (2000)). In the current paper, we shall from the outset allow arbitrageurs to only link two exchanges, but let them choose which ones. As a result, the active links of the network and the number of traders on each such link emerge endogenously at a Nash equilibrium of the network formation game. This is in contrast with much of the existing literature on networks in finance where interdependencies are assumed exogenously at the outset. In our framework the dependencies between intermediaries are established endogenously.

An overall equilibrium is a subgame perfect outcome of a two-stage game. The backwards order by which we solve the game is as follows. First investors solve for their portfolio demands given the asset structure and given the supplies of assets by arbitrageurs, and arbitrageur trades are determined at a Nash equilibrium of the trading game, taking as given the demand function of investors, and the network structure. In the next stage the equilibrium network (the distribution of arbitrageurs across all permissible links) is determined at a Nash equilibrium of the network formation game.

In order to focus on the network structure we assume that asset markets are complete. In actual fact, a considerable number of securities are issued by what we call arbitrageurs. In Rahi and Zigrand (2007b) we characterize equilibrium security design for an arbitrary network structure. Our characterization of equilibrium networks with complete markets in fact also holds when markets are incomplete but with asset payoffs that correspond to an equilibrium of the security design game.

The questions we would like to ask are the following. What is the equilibrium network, and how do the equilibrium asset trades depend on the network? How integrated can we expect the global economy to be? Can different exchanges be integrated to a different extent? To which equilibrium does the economy converge as the number of arbitrageurs grows without limit? When will the equilibrium in the limit be integrated and when will the global economy merely be a collection of disjoint subnetworks? How is the network related to the extent of the autarky gains from trade between trading locations and their depth (as measured by the price impact of an additional unit of trade). How is the network affected by externalities exerted

by arbitrageurs active on different links? What kind of architecture (i.e. the set of links that are permissible) aligns the interests of arbitrageurs thereby promoting efficiency? How are local shocks propagated through the entire financial system via the endogenous linkages created by intermediaries? This last point has become a focal point of financial research post LTCM, and worries about the financial stability of markets dominated by interdependencies established by derivative positions appear daily in various news forums.

We are not aware of any papers that have studied these questions in the context of asset markets. In the banking literature, the papers by Allen and Gale (2000) and Cifuentes et al. (2005) study the stability of networks formed by the borrowings of banks from each other. Both the form of the securities (debt) and the network links are assumed exogenously. There is a large literature on networks in other settings. For example, in Bala and Goyal (2000) and Goyal and Joshi (2005), agents form links with other agents in an abstract game. Incentives to form links depend solely on the number of links the player as well as the potential partner has. Here, in contrast, it crucially matters which precise links they have, as they anticipate the (subgame perfect) trades and prices of the equilibrium assets. What is more, our paper does not suffer from the indeterminacy arising in Bala and Goyal (2000) whereby the model predicts for instance that under some conditions all equilibrium networks are hub-and-spoke, but does not provide any guidance as to which of the nodes emerges as the hub.

Briefly, we derive the following results in this paper. We prove existence of equilibria. We show that network externalities give rise to networks that are suboptimal for arbitrageurs. Controlling for depth, an optimal network is a hub-spoke network. The complete network, in which all links are permissible, is always suboptimal. If the complete network is hub-and-spoke, it uses the suboptimal hub. The reason has to do with the provision of liquidity. Roughly speaking, the optimal hub is a hub whose equilibrium state-price deflator lies towards the center of all nodes so as to be used as a repository of liquidity. This allows mispricings to be exploited with as little market impact as possible, provided all arbitrageurs use the same hub. However, each arbitrageur, if given the opportunity, has an incentive to deviate and form a link across two exchanges, one on each side of the hub, since there is a larger mispricing on this link. The deviating player will therefore not only not contribute to liquidity, but will in fact use up liquidity at both ends. All other players act similarly, leading to a Prisoner's Dilemma style inefficient outcome.

As the number of arbitrageurs goes to infinity, state prices on all exchanges converge to the frictionless complete-markets Walrasian state prices of the integrated economy. In that sense, arbitrageurs connect markets and ensure securities trades in aggregate that exactly coincide with the transfers of securities that a global Walrasian auctioneer would have performed. This is true despite the inefficiencies arising from the network externalities, from market power and from the fact that each arbitrageur is only allowed to connect two exchanges. The equilibrium network may not be connected, however, even asymptotically.

A note on assumptions:

We model each trading location or exchange as a standard Arrow-Debreu economy. Arbitrageurs take the Walrasian demand function on each exchange as given and play a Cournot trading game in asset supplies. In order to characterize the Cournot-Walras equilibria of this game, we assume that the Walrasian demand functions are linear in asset supplies. More precisely, we assume that state prices on an exchange are linear in the net aggregate endowment (aggregate endowment plus asset supplies) of the exchange, i.e. the CAPM holds with respect to net aggregate endowments. Quadratic utility ensures this, so we assume quadratic utility at the outset.

2 The Setup

We consider a two-period economy with uncertainty parametrized by the state space $S := \{1, \dots, S\}$. Assets are traded in several locations or “exchanges.” They are in zero net supply. We assume that markets are complete on each exchange. Without loss of generality we can take the set of tradable securities to be the Arrow securities.

Investor $i \in I^k := \{1, \dots, I^k\}$ on exchange $k \in K := \{0, \dots, K\}$ has endowments $(\omega_0^{k,i}, \omega^{k,i}) \in \mathbb{R} \times \mathbb{R}^S$, and preferences which allow a quasilinear quadratic representation

$$U^{k,i}(x_0^{k,i}, x^{k,i}) = x_0^{k,i} + \sum_{s \in S} \pi_s \left[x_s^{k,i} - \frac{1}{2} \beta^{k,i} (x_s^{k,i})^2 \right],$$

where $x_0^{k,i} \in \mathbb{R}$ is consumption at date 0, $x^{k,i} \in \mathbb{R}^S$ is consumption at date 1, and π_s is the probability (strictly positive and common across agents) of state s . The coefficient $\beta^{k,i}$ is positive. Investors are price-taking and can trade only on their own exchange. To rule out trivial cases we assume that there are at least three exchanges, i.e. $K \geq 2$.

In addition there are N arbitrageurs who possess the trading technology which allows them to also trade across exchanges. Arbitrageurs have no endowments, so they can be interpreted as pure intermediaries. For simplicity, we assume that arbitrageurs only care about time zero consumption. They are imperfectly competitive.

Given the set of exchanges K , we specify a set \mathcal{A} of *links* (k, ℓ) , i.e. $\mathcal{A} \subset \{(k, \ell) : k, \ell \in K, k \neq \ell\}$. We will use the abbreviated notation $k\ell$ instead of (k, ℓ) . To avoid notational ambiguity, links $k\ell$ and ℓk are taken to be the same link. Each arbitrageur chooses to arbitrage one of the admissible links. Let $N^{k\ell}$ be the number of arbitrageurs on link $k\ell$. We use the same notation for the set of arbitrageurs on link $k\ell$. For notational convenience we define $N^{k\ell}$ to be zero if $k\ell \notin \mathcal{A}$. We have $\sum_{k\ell \in \mathcal{A}} N^{k\ell} = N$.

Formally, $\mathcal{G} := (K, \mathcal{A})$ is a *graph*, with *nodes* K and *links* \mathcal{A} .¹ We say that ℓ is a

¹A standard reference on graph theory is Diestel (2005). We employ the terms “node” and “link” instead of “vertex” and “edge,” reserving the latter terminology for its standard usage in the theory of polytopes, which we make extensive use of later.

neighbor of k if $k\ell \in \mathcal{A}$. The graph is *complete* if every link $k\ell$ is admissible (i.e. every node is a neighbor of every other node); if not, it is *incomplete*. We will have occasion to consider a number of incomplete graphs. If $\mathcal{A} = \mathcal{A}^{h_k} := \{k\ell : \ell \in K, \ell \neq k\}$, we say that \mathcal{G} is a *hub-spoke* graph² with node k as the hub (for brevity, we call this an h_k -graph). \mathcal{G} is *unary* if only one link is admissible (if $k\ell$ is the admissible link, we call this a $u_{k\ell}$ -graph). \mathcal{G} is a *cycle* if the $K+1$ nodes can be ordered as $\{k_1, \dots, k_{K+1}\}$ such that $\mathcal{A} = \{k_1k_2, k_2k_3, \dots, k_Kk_{K+1}, k_{K+1}k_1\}$. In a cycle, each node has precisely two neighbors. There is a *path* connecting k and ℓ if there is a sequence of distinct nodes $\{k_1, \dots, k_I\}$ in K such that $k_1 = k$, $k_I = \ell$ and $\{k_1k_2, k_2k_3, \dots, k_{I-1}k_I\} \subset \mathcal{A}$. We say that \mathcal{G} is *connected* if there is a path connecting any pair of nodes $k, \ell \in K$. $\mathcal{G}' := (K', \mathcal{A}')$ is a *subgraph* of \mathcal{G} , denoted $\mathcal{G}' \subset \mathcal{G}$, if $K' \subset K$ and $\mathcal{A}' \subset \mathcal{A}$. If, in addition, \mathcal{A}' contains all the links $k\ell \in \mathcal{A}$ for $k, \ell \in K'$, we say that \mathcal{G}' is an *induced subgraph* of \mathcal{G} , and that it is induced by K' . A maximal connected subgraph of \mathcal{G} is called a *component* of \mathcal{G} (where “maximal” is with respect to the subgraph relation).

While we have introduced the above terminology for the graph \mathcal{G} , it applies of course to any other graph that we consider in the paper (typically a subgraph of \mathcal{G}). \mathcal{G} itself will be referred to as an *architecture*, with \mathcal{A} being the set of *admissible links*. While \mathcal{G} is not necessarily complete, we assume that it is connected. This is without loss of generality as each component of \mathcal{G} can be analyzed as a separate economy.

To an graph \mathcal{G} we assign a distribution of arbitrageurs across links that are admissible in that architecture, $\{N^{k\ell}\}_{k\ell \in \mathcal{A}}$. We say that an admissible link $k\ell$ is *active* if $N^{k\ell} > 0$. Let $\mathcal{A}^* \subset \mathcal{A}$ be the set of active links. The graph $\mathcal{G}^* := (K, \mathcal{A}^*) \subset \mathcal{G}$ is called a *network*. While we have assumed that \mathcal{G} is connected, \mathcal{G}^* need not be. We denote by \mathcal{C} the set of components of \mathcal{G}^* , with typical element (C, \mathcal{A}_C^*) . Since the latter is just the subgraph of \mathcal{G}^* induced by the nodes C , we will denote the component itself by C ; no confusion should arise.

We model the strategic interaction of arbitrageurs as a two-stage extensive-form game, which we call the *network game*. We study subgame-perfect Nash equilibria of this game. It is convenient to refer to the first stage as a game in its own right, taking as given a continuation equilibrium in each subsequent subgame. In the first stage, the *network formation game*, each arbitrageur chooses a link on which to trade. The outcome is a distribution of arbitrageurs $\{N^{k\ell}\}_{k\ell \in \mathcal{A}}$. In each subgame associated with some distribution of arbitrageurs, arbitrageurs play the *trading game* in which each arbitrageur decides how much of the given assets to supply to the two exchanges on which he is active. Formally, investors are not players in this game—they simply determine the demand functions that arbitrageurs face on each exchange.

Thus, for a given architecture \mathcal{G} , the distribution of arbitrageurs $\{N^{k\ell}\}_{k\ell \in \mathcal{A}}$ is determined endogenously, as part of an equilibrium of the network game. We have defined above a network \mathcal{G}^* for an arbitrary arbitrageur distribution. If \mathcal{G}^* corresponds to an equilibrium arbitrageur distribution, it is called an *equilibrium network*.

²This is called a *star* in the graph-theoretic literature.

3 The Trading Game

We begin by studying equilibria of the trading game for a given network $\mathcal{G}^* = (K, \mathcal{A}^*)$. At this point the distribution of arbitrageurs $\{N^{k\ell}\}_{k\ell \in \mathcal{A}}$ is arbitrary. Let $y_{k\ell}^{k,n}$ be the supply of state-contingent consumption on exchange k of a typical arbitrageur n active on link $k\ell \in \mathcal{A}^*$. Let $y_{k\ell}^k := \sum_{n \in N^{k\ell}} y_{k\ell}^{k,n}$ be the aggregate supply on exchange k of all arbitrageurs active on $k\ell$, and $y^k := \sum_{k' \neq k} y_{kk'}^k$ the aggregate supply on exchange k of all arbitrageurs in the economy. Finally, let $y^{k,\setminus n}$ be the aggregate supply on exchange k of all arbitrageurs except n , i.e. $y^{k,\setminus n} = y^k - y_{k\ell}^{k,n}$.

Definition 1 Given $\{\mathcal{G}^*, \{N^{k\ell}\}\}$, a Cournot-Walras equilibrium (CWE) of the trading game is an array of asset price functions, asset demand functions, and arbitrageur supplies, $\{q^k : \mathbb{R}^S \rightarrow \mathbb{R}^S, \theta^{k,i} : \mathbb{R}^S \rightarrow \mathbb{R}^S, y_{k\ell}^{k,n} \in \mathbb{R}^S\}_{i \in I^k, n \in N^{k\ell}, k\ell \in \mathcal{A}^*}$, such that

i. *Investor optimization:* For given q^k , $\theta^{k,i}(q^k)$ solves

$$\begin{aligned} \max_{\theta^{k,i} \in \mathbb{R}^S} \quad & x_0^{k,i} + \sum_{s \in S} \pi_s \left[x_s^{k,i} - \frac{\beta^{k,i}}{2} (x_s^{k,i})^2 \right] \\ \text{s.t.} \quad & x_0^{k,i} = \omega_0^{k,i} - q^k \cdot \theta^{k,i} \\ & x^{k,i} = \omega^{k,i} + \theta^{k,i}. \end{aligned}$$

ii. *Arbitrageur optimization:* For given $q^k(\cdot), q^\ell(\cdot), y^{k,\setminus n}$, and $y^{\ell,\setminus n}$, $(y_{k\ell}^{k,n}, y_{k\ell}^{\ell,n})$ solves

$$\begin{aligned} \max_{y_{k\ell}^{k,n} \in \mathbb{R}^S, y_{k\ell}^{\ell,n} \in \mathbb{R}^S} \quad & y_{k\ell}^{k,n \top} q^k (y_{k\ell}^{k,n} + y^{k,\setminus n}) + y_{k\ell}^{\ell,n \top} q^\ell (y_{k\ell}^{\ell,n} + y^{\ell,\setminus n}) \\ \text{s.t.} \quad & y_{k\ell}^{k,n} + y_{k\ell}^{\ell,n} \leq 0. \end{aligned}$$

iii. *Market clearing:*

$$\sum_{i \in I^k} \theta^{k,i}(q^k(y^k)) = y^k, \quad \forall k \in K.$$

Note that investors take asset prices as given, while arbitrageurs compete Cournot-style. Thus a CWE is a Nash equilibrium of the trading game. This equilibrium concept is due to Gabszewicz and Vial (1972), and a review can be found in Mas-Colell (1982). Arbitrageurs maximize time zero consumption, i.e. profits from their arbitrage trades, but subject to the restriction that they are not allowed to default in any state at date 1. Equivalently, arbitrageurs need to be completely collateralized.

Let $\Pi := \text{diag}(\pi_1, \dots, \pi_S)$ and $\mathbf{1} := (1 \dots 1)^\top$. Investor (k, i) 's utility can be written as

$$U^{k,i} = \omega_0^{k,i} - q^k \cdot \theta^{k,i} + \mathbf{1}^\top \Pi (\omega^{k,i} + \theta^{k,i}) - \frac{\beta^{k,i}}{2} (\omega^{k,i} + \theta^{k,i})^\top \Pi (\omega^{k,i} + \theta^{k,i}). \quad (1)$$

The first order condition for the investor's optimization problem gives us his asset demand function:

$$\theta^{k,i}(q^k) = \frac{1}{\beta^{k,i}} [p^{k,i} - \Pi^{-1}q^k], \quad (2)$$

where $p^{k,i} := \mathbf{1} - \beta^{k,i}\omega^{k,i}$. The vector $\Pi^{-1}q^k$ is the state-price deflator (or pricing kernel) associated with the asset price vector q^k . Thus $p^{k,i}$ is agent (k, i) 's no-trade state-price deflator. We can now use the market clearing condition to deduce the inverse demand mapping, i.e. the price vector on exchange k that sets aggregate demand, $\theta^k := \sum_{i \in I^k} \theta^{k,i}$, equal to aggregate arbitrage supply y^k :

$$q^k(y^k) = \Pi[p^k - \beta^k y^k], \quad (3)$$

where $\beta^k := [\sum_i (\beta^{k,i})^{-1}]^{-1}$, $\omega^k := \sum_i \omega^{k,i}$, and $p^k := \mathbf{1} - \beta^k \omega^k$. It is convenient to think in terms of state-price deflators. Denoting the equilibrium state-price deflator on exchange k by \hat{p}^k , we can write (3) as

$$\hat{p}^k(y^k) := p^k - \beta^k y^k. \quad (4)$$

Notice that the state prices are affine in net aggregate endowments $\omega^k + y^k$, or equivalently that the CAPM relation holds with respect to net aggregate endowments. The vector p^k is exchange k 's autarky state-price deflator (autarky with respect to the rest of the world, but allowing trade within k). We assume that $p^k \geq 0$, for all $k \in K$, which says that the representative investor on each exchange is nonsatiated at the aggregate endowment point of that exchange. The parameter β^k represents the “depth” of exchange k , i.e. the price impact of a unit of arbitrageur trading.³ For instance, ceteris paribus, the market impact of a trade is smaller on exchanges with a larger population—it can be absorbed by more investors.

Our assumptions on preferences, in conjunction with the absence of nonnegativity constraints on consumption, guarantee that the equilibrium pricing function (3) on an exchange does not depend on the initial distribution of endowments, but merely on the aggregate endowment of the local investors.

We now solve the Cournot game among arbitrageurs, given the asset price function (3). It turns out that there is a unique CWE, and that this equilibrium is symmetric, i.e. supplies of all arbitrageurs on a given link $k\ell$ are the same.

Lemma 1 (Equilibrium supplies) *Given $\{\mathcal{G}^*, \{N^{k\ell}\}\}$, the equilibrium supply of arbitrageur $n \in N^{k\ell}$, $k\ell \in \mathcal{A}^*$, is given by*

$$y_{k\ell}^{k,n} = -y_{k\ell}^{\ell,n} = \frac{1}{\beta^k + \beta^\ell} \cdot (\hat{p}^k - \hat{p}^\ell). \quad (5)$$

³More precisely, the state- s value of the equilibrium state-price deflator falls by β^k for a unit increase in arbitrageur supply of s -contingent consumption.

The Cournot-Walras equilibrium is symmetric: all arbitrageurs on a given link have the same supply (as we shall see shortly, the CWE is also unique). The interpretation of (5) is straightforward. Arbitrageurs on link $k\ell$ supply consumption in state s to exchange k when the price that agents on exchange k are willing to pay for a unit of state s consumption exceeds the price at which arbitrageurs can procure that unit on exchange ℓ .

The factor of proportionality in (5) is determined by depth. The deeper the exchanges k and ℓ (i.e. the lower are β^k and β^ℓ), the more arbitrageur n trades, since he can afford to augment his supply without affecting margins as much. Notice that implicitly, as we shall see, the equilibrium mispricing $\hat{p}^k - \hat{p}^\ell$ depends on competition as well as on all other arbitrageur trades on the respective exchanges. In particular, we shall see that the supply vector is scaled to zero as competition intensifies, because at the same time the mispricing shrinks and there are more players to share the smaller pie with.

We can solve for the equilibrium state-price deflators \hat{p}^k , $k \in K$, as follows. From (5),

$$y^k = \sum_{\ell} N^{k\ell} y_{k\ell}^{k,n} = \sum_{\ell} \frac{N^{k\ell}}{\beta^k + \beta^\ell} (\hat{p}^k - \hat{p}^\ell). \quad (6)$$

Let $\alpha^{k\ell} := \frac{N^{k\ell}}{\beta^k + \beta^\ell}$, and $\alpha^k := \sum_{\ell \in K} \alpha^{k\ell}$. Using (4), $\{\hat{p}^k\}_{k \in K}$ is a solution to the following system of equations:

$$(1 + \beta^k \alpha^k) \hat{p}^k - \beta^k \sum_{\ell} \alpha^{k\ell} \hat{p}^\ell = p^k, \quad k \in K. \quad (7)$$

Lemma 2 (Equilibrium prices) *Given, $\{\mathcal{G}^*, \{N^{k\ell}\}\}$, there exists a unique profile of equilibrium state-price deflators $\{\hat{p}^k\}_{k \in K}$. For any component $C \in \mathcal{C}$, we have*

$$\hat{p}^k = p^{\eta,k} := \sum_{j \in C} \eta^{kj} p^j, \quad k \in C,$$

for some weights $\{\eta^{kj}\}_{k,j \in C}$ that depend only on $\{\beta^k\}_{k \in C}$ and $\{N^{k\ell}\}_{k\ell \in \mathcal{A}^*, k \in C}$, and satisfy (a) $\eta^{jj} > \sum_{i \neq j, i \in C} \eta^{ij}$, for all $j \in C$, and (b) $\eta^{kj} > 0$ for all $j, k \in C$ and $\sum_{j \in C} \eta^{kj} = 1$ for all $k \in C$.

The equilibrium state-price deflator on any exchange is a convex combination of the autarky state-price deflators of all exchanges to which it is linked directly or indirectly. How much p^ℓ is impounded into \hat{p}^k depends on depths as well as on $\{N^{k\ell}\}$.⁴ Intuitively, if in equilibrium state prices on k depend on preferences and endowments on other exchanges, it is due to the arbitrage trades which integrate the various exchanges. The higher a particular $N^{k\ell}$ and the lower β^k and β^ℓ , the more arbitrageurs transfer state-contingent consumption in equilibrium across k and

⁴Later we endogenize $\{N^{k\ell}\}$, so that the weights $\{\eta^{kj}\}$ become endogenous as well.

ℓ , thereby reducing the mispricing $\hat{p}^k - \hat{p}^\ell$, increasing the influence of preferences and endowments (and hence of p^ℓ) of other exchanges ℓ on the local state-price deflator \hat{p}^k . On top of that, given that yet other arbitrageurs transfer resources between $j \notin \{\ell, k\}$ and ℓ , the state prices of j also find their way into \hat{p}^k . This explains why, depending on $\{N^{k\ell}\}$, autarky state prices of all exchanges in a component are reflected in each one of them in equilibrium.

Moreover, if p_s^k , the autarky valuation of state s in exchange k , rises exogenously, then \hat{p}_s^ℓ increases on all exchanges ℓ in the component of K to which k belongs, with the largest increase occurring on k itself. In fact, the direct effect on k itself is larger than the indirect effects on all other exchanges combined. While a positive local shock affects local state prices most, it positively affects the state prices of all exchanges directly or indirectly connected to it. Much research has been focused on such contagion effects following the LTCM episode: how is a systemic shock in one part of the financial system propagated through the entire system via the endogenous cross links established by the global financial players? This question is epitomized by the saying “Why does Brazil catch a cold when Russia sneezes?”

While Lemma 2 gives us an explicit solution to (7), the solution for the general case is unwieldy and difficult to manipulate analytically. However, we can derive tractable closed-form solutions for a hub-spoke network, and also for the case of three exchanges. Consider a hub-spoke network with exchange 0 as the hub. It is straightforward to solve (7) for $\{\hat{p}^k\}_{k \in K}$:

Lemma 3 (Equilibrium prices: hub-spoke network) *Consider an h_0 -network. Then*

$$\hat{p}^0 = \sum_{k \in K} \gamma^k p^k, \quad (8)$$

where

$$\begin{aligned} \gamma^k &:= \frac{\frac{\beta^0 \alpha^{0k}}{1 + \beta^k \alpha^{0k}}}{1 + \beta^0 \sum_j \frac{\alpha^{0j}}{1 + \beta^j \alpha^{0j}}}, & k \neq 0 \\ \gamma^0 &:= \frac{1}{1 + \beta^0 \sum_j \frac{\alpha^{0j}}{1 + \beta^j \alpha^{0j}}}, \end{aligned}$$

and

$$\hat{p}^k = (1 + \beta^k \alpha^{0k})^{-1} (p^k + \beta^k \alpha^{0k} \hat{p}^0), \quad k \neq 0. \quad (9)$$

Note that $\sum_{k \in K} \gamma^k = 1$ and, for $k \neq 0$, the weight γ^k is increasing in N^{0k} , the number of arbitrageurs active on exchange k (for given $N^{0j}, j \neq k$).

Finally, we calculate equilibrium arbitrageur profits. We will be using this information in the sequel. We will need to distinguish between equilibrium arbitrageur profits for a given distribution of arbitrageurs $\{N^{k\ell}\}$, and equilibrium profits in an equilibrium network, with endogenously determined $\{N^{k\ell}\}$. Let $\varphi^{k\ell}$ be the equilibrium level of profits on link $k\ell$ for given $\{N^{k\ell}\}$. In an equilibrium network, let

Φ , Φ^{h_k} and $\Phi^{u_{k\ell}}$ denote profits associated with the complete architecture, the h_k -architecture, and the $u_{k\ell}$ -architecture, respectively. For state-contingent consumption $x \in \mathbb{R}^S$, the $L^2(\Pi)$ -norm of x is defined as follows: $\|x\|_2 := (x^\top \Pi x)^{\frac{1}{2}}$.

Lemma 4 (Equilibrium profits) *Given $\{\mathcal{G}^*, \{N^{k\ell}\}\}$, the equilibrium profit of arbitrageur $n \in N^{k\ell}$, for $k\ell \in \mathcal{A}^*$, is*

$$\varphi^{k\ell} = \frac{1}{\beta^k + \beta^\ell} \cdot \|\hat{p}^k - \hat{p}^\ell\|_2^2. \quad (10)$$

We call $\|\hat{p}^k - \hat{p}^\ell\|_2^2$ the autarky gains to trade between exchanges k and ℓ . This quantity, weighted by the depths of the two exchanges,

$$\mu_{k\ell} := \frac{1}{\beta^k + \beta^\ell} \cdot \|\hat{p}^k - \hat{p}^\ell\|_2^2,$$

is an upper bound to equilibrium profits on link $k\ell$.

4 Equilibrium Networks: Preamble

We have shown that, in each subgame corresponding to a given distribution of arbitrageurs $\{N^{k\ell}\}_{k\ell \in \mathcal{A}}$, there exists a unique equilibrium of the trading game. We are now in a position to analyze the network formation game in which $\{N^{k\ell}\}$ is determined. Let $\{N^{k\ell}(N)\}_{k\ell \in \mathcal{A}}$ be the equilibrium distribution of arbitrageurs when the number of arbitrageurs is N . All the variables introduced earlier, such as prices, profits and the equilibrium network itself, depend on $\{N^{k\ell}(N)\}$. To save on notation, we write $\hat{p}^k(N)$ instead of $\hat{p}^k(\{N^{k\ell}(N)\})$, and likewise for all other variables (notice that $\hat{p}^k(0) = p^k$). Also, we write $\Phi(\infty)$ for $\lim_{N \rightarrow \infty} \Phi(N)$, and similarly for other variables.

In an equilibrium network, the distribution of arbitrageurs is such that no arbitrageur can increase his profits by deviating to any other admissible link. Ignoring integer constraints on the number of arbitrageurs,⁵ profits must be equal on all active links, with (weakly) lower profits on all inactive admissible links. In the same spirit we define⁶ equilibrium profits $\Phi(N)$ on the entire interval $[0, \infty)$ with $\Phi(0) := \lim_{N \rightarrow 0} \Phi(N) = \frac{1}{\beta^k + \beta^\ell} \|p^k - p^\ell\|_2^2$. Similarly, $\mathcal{A}^*(0) := \lim_{N \rightarrow 0} \mathcal{A}^*(N)$.

We seek to answer two sets of questions. The first relates to features of an equilibrium network, for a given architecture. Which links attract the most arbitrageur activity? What connectivity properties emerge in equilibrium? Are all admissible links active? If not, is an equilibrium network still connected?

The second set of questions pertains to the ‘‘comparative statics’’ of equilibrium networks with respect to the architecture. How are arbitrageur profits and investors’ utilities affected by the architecture? What connectivity properties do desirable architectures possess?

⁵Taking integer constraints into account makes the exposition messy without leading to any new insights.

⁶Since, strictly speaking, $\Phi(N)$ is only defined for $N \geq 1$.

5 Networks with Many Arbitrageurs

A useful benchmark in our analysis of equilibrium networks is limiting networks when the number of arbitrageurs grows without bound. We show that these networks are Walrasian: as the number of arbitrageurs goes to infinity, equilibrium state prices converge to the Walrasian equilibrium state prices of the entire integrated economy (with unrestricted participation and no arbitrageurs). This is true for an arbitrary architecture (provided it is connected, which we have assumed throughout).

For $C \in K$, let

$$p_C^\lambda := \sum_{k \in C} \lambda_C^k p^k$$

where

$$\lambda_C^k := \frac{\frac{1}{\beta^k}}{\sum_{j \in C} \frac{1}{\beta^j}}.$$

The vector p_C^λ is the average willingness to pay of all investors in C , with the willingness to pay on each exchange weighted by its relative depth. p_C^λ is the state-price deflator for the complete-markets Walrasian equilibrium when the economy is only composed of exchanges in C (see Rahi and Zigrand (2007c)). Let $p^\lambda := p_K^\lambda$ be the state-price deflator for the global complete-markets Walrasian equilibrium.

Recall that \mathcal{C} is the set of components of the equilibrium network.

Proposition 1 (Convergence) *For an equilibrium network, state prices on all exchanges converge to the complete-markets Walrasian state prices of the integrated economy, i.e. $\hat{p}^k(\infty) = p^\lambda$, for all $k \in K$. Moreover $p_C^\lambda = p^\lambda$, for all $C \in \mathcal{C}(\infty)$.*

As the number of arbitrageurs increases without bound, all mispricings across exchanges vanish. Even though no single arbitrageur ties all the markets together, fierce competition is a substitute for unrestricted access to global markets. Arbitrageurs connect markets and ensure securities trades in aggregate that exactly coincide with the transfers of state-contingent consumption across exchanges that a global Walrasian auctioneer would have performed. We shall see that convergence need not be monotone, though.

Corollary 1 *Generically in preferences or endowments, $\mathcal{C}(\infty) = K$.*

If for two arbitrary disjoint subsets K_1 and K_2 of K , we have $p_{K_1}^\lambda \neq p_{K_2}^\lambda$, then there must be a path connecting K_1 and K_2 for large enough N . So either the state-price deflator on all exchanges converges to a common state-price deflator as a result of the equilibrium interexchange of flows of funds resulting from the equilibrium $\{N^{k\ell}(N)\}$, or if in equilibrium there emerge two disjoint subnetworks (with a connection allowed by \mathcal{A} but not arising in equilibrium), then even though there are no flows between them, the equilibrium state-price deflator on each one of them must independently converge to a common state-price deflator in the limit.

Example 1 As an example, consider the following nodes forming a cross in \mathbb{R}^2 (Fig. 2): $p^0 = (a, a + 2)$, $p^1 = (a, a - 2)$, $p^2 = (a - 1, a)$ and $p^3 = (a + 1, a)$ for $a > 2$. Assume $\beta^k = \beta$, $k \in K$. For small N , all arbitrageurs are on link 01. As N grows $\hat{p}^0(N)$ and $\hat{p}^1(N)$ converge along the vertical segment linking them until $N = \bar{N}$ for which $\|\hat{p}^0(\bar{N}) - \hat{p}^1(\bar{N})\|_2^2 = \|p^2 - p^3\|_2^2$. For $N > \bar{N}$, two active links appear, 01 and 23, and the network comprises two disjoint subnetworks. As N increases without bound, all four nodes converge to $p^\lambda = (a, a)$. Even as profits on 01 and 23 converge to zero, they are higher than potential profits on any of the other links.

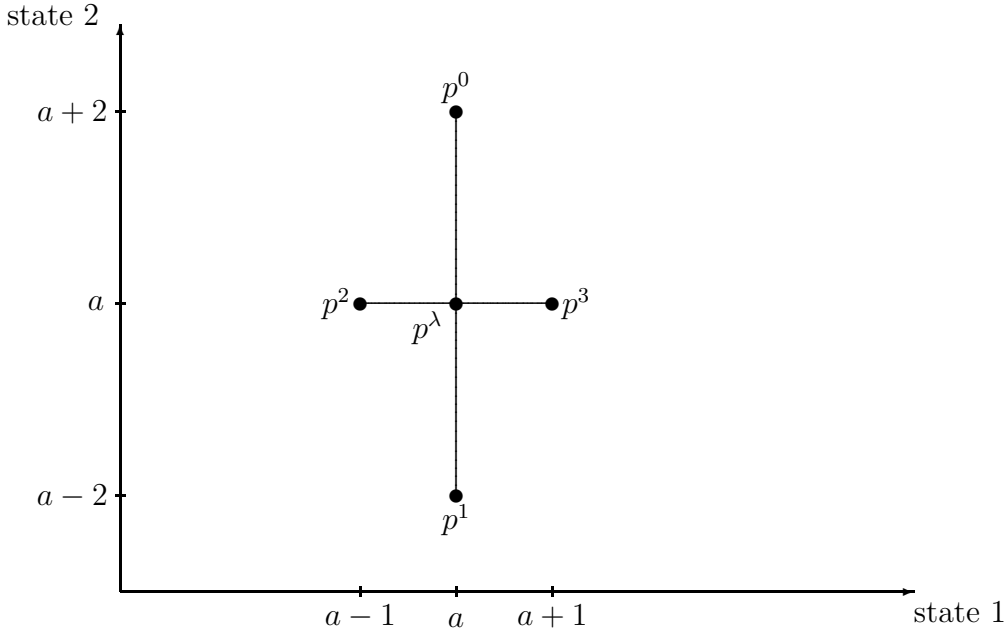


Figure 2: **The economy in Example 1**

This is an example in which the nodes $\hat{p}^k(\bar{N})$ are symmetrical with respect to p^λ . Another such case arises when the nodes are vertices of a (hyper-) cube, for which the active links correspond to the inner diagonals. We provide a precise characterization of disconnected equilibrium networks later (Proposition 3, part (ii)).

Note that if $\{p^k\}_{k \in K}$ are linearly independent, then $p_C^\lambda = p^\lambda$ only if $C = K$, i.e. the limiting graph must be connected. In this case, we also see from Lemma 2 that η^{kj} converges to λ^j , for all $k \in K$, as N goes to infinity.

There are cases where, for small N , not all admissible links are active because each arbitrageur can only arbitrage across a single link, and would therefore choose the most attractive opportunity. But as N goes to infinity, ultimately each exchange will see some arbitrage trade, as long as there is some reward to be reaped, i.e. as long as its autarky state-price deflator is not equal to p^λ . For a hub-spoke architecture, in particular, all admissible links will be active for sufficiently large N .

For the case of the h_0 -architecture, Proposition 1 tells us that the state-price

deflator on the hub \hat{p}^0 , given by (8), converges to p^λ . Indeed, it can be verified directly that γ^k converges to λ^k as N^{0k} goes to infinity for every k , $k \neq 0$.

6 The Geometry of Equilibrium Networks

Before we proceed, a few definitions: Let $A \subset \mathbb{R}^d$, and $\{x_i\}$ a finite number of points in A . An *affine combination* of $\{x_i\}$ is a linear combination $\sum \nu_i x_i$ in which the weights ν_i add up to one. The points $\{x_i\}$ are *affinely independent* if none of these points can be expressed as an affine combination of the other points. An *affine subspace* is a translate of a linear subspace, i.e. of the form $x + M$, where x is a point in \mathbb{R}^d and M is a linear subspace of \mathbb{R}^d . The *affine hull* of A , denoted $\text{aff}(A)$ is the smallest affine subspace containing A ; it is the set of all affine combinations of points in A . The *convex hull* of A , denoted $\text{conv}(A)$, is the set of all convex combinations of points in A (thus $\text{conv}(A) \subset \text{aff}(A)$). The dimension of A is the dimension of $\text{aff}(A)$, which is defined to be the dimension of the corresponding linear subspace. The convex hull of a finite number of points in Euclidean space is called a *polytope*.⁷

Geometrically we can view the nodes of a network as points in \mathbb{R}^S . The nodes are given by $\{\hat{p}^k(N)\}_{k \in K}$ (we refer to both $k \in K$ and $\hat{p}^k \in \mathbb{R}^S$ as the node corresponding to exchange k). From (10) we see that the distance between two nodes is proportional to the square-root of the equilibrium profits reaped by arbitrageurs on that link.⁸ From Lemma 2 we know that the nodes $\{\hat{p}^k(N)\}_{k \in K}$ are in $\text{conv}(\{p^k\}_{k \in K})$. Let $\hat{\mathcal{P}}(N) := \text{conv}(\{\hat{p}^k(N)\}_{k \in K})$ and $\mathcal{P} := \hat{\mathcal{P}}(0) = \text{conv}(\{p^k\}_{k \in K})$. Thus \mathcal{P} and $\hat{\mathcal{P}}(N)$ are polytopes with $\hat{\mathcal{P}}(N) \subset \mathcal{P}$ for all N , and $\hat{\mathcal{P}}(\infty) = \{p^\lambda\}$.

We now summarize some basic notation and facts about polytopes.⁹ Let P be a d -polytope (i.e. a d -dimensional polytope). A *face* of P is the intersection of P with a supporting hyperplane. Each face is itself a polytope. The 0-faces are called *vertices*, the 1-faces are called *edges*, and the $(d - 1)$ -faces are called *facets*. Thus a 1-polytope is a line segment, a 2-polytope is a polygon whose facets (which are also its edges) are segments, a 3-polytope is a three-dimensional solid, whose facets are polygons and whose edges are segments, and so forth.

A *simplex* is a polytope whose vertices are affinely independent. If the midpoints of the edges incident at a vertex v of P lie on a hyperplane, then these midpoints are the vertices of a $(d - 1)$ -polytope called the *vertex figure* of P at v . The *circumsphere* of P , if it exists, is the sphere that circumscribes P , i.e. whose surface contains all the vertices of P . If the circumsphere exists, its center is called the *center* of P ; it is

⁷Only convex polytopes are considered in this paper. The reader may consult Coxeter (1963) and Grünbaum (2003) for the background material on polytopes that we use here.

⁸When we provide Euclidean geometric intuition, we view the Hilbert space L^2 with the inner product $\langle p, p' \rangle = E[pp']$ as the Euclidean space \mathbb{R}^S with inner product $\langle x, x' \rangle = x^\top x'$ via the isomorphism $p \mapsto \Pi^{1/2} p =: x$. For notational simplicity, we will not make this transformation explicit.

⁹As mentioned in footnote 1, we employ the terms “vertex” and “edge” as is standard in the theory of polytopes. We do not use these terms in the graph-theoretic sense.

the point from which all the vertices are equidistant.

The notion of a *regular polytope* can be defined inductively as follows. A regular polygon is a polygon that is equilateral and equiangular. A regular polytope is a polytope with regular facets and vertex figures. This definition implies that the facets are in fact equal and so are the vertex figures.

We say that P is *centrally symmetric* about 0 if $P = -P$. P is centrally symmetric if it is a translate of a polytope that is centrally symmetric about 0. Centrally symmetric polytopes have an even number of vertices: each vertex is symmetric with respect to another vertex. The line segment joining such a pair of vertices is called an *axis* of P . All regular polytopes are centrally symmetric, except the simplices of dimension greater than or equal to two and the odd polygons (i.e. 2-polytopes with an odd number of vertices).

Nodes \hat{p}^k that are not vertices of $\hat{\mathcal{P}}$ are called *internal nodes* (note that the internality of a given exchange depends on N , and also that an internal node may not be in the interior of \mathcal{P}).

Most of the questions raised in this paper boil down to a combinatorial problem which, as one might expect, leads to very few clear-cut general results since so many tradeoffs must be balanced, such as the various depths, and the various initial mispricings across all active links, taking into account that the prices on each exchange depend on all flows across the network, no matter how “remote,” i.e. no matter how many links away (as long there is a path connecting them). In particular, one should not expect to obtain general results of the sort “every equilibrium network is hub-spoke,” as have been derived in Bala and Goyal (2000), for in our paper nodes are exchanges with heterogeneous intrinsic characteristics. Over and above the connectivity structure, the location of nodes (the position of the autarky state-price deflators in \mathbb{R}^S) and depths matter as well. Hence the reliance on both graph and polytope theories. Indeed, any given connectivity structure \mathcal{A}^* can be perturbed by varying the fundamental parameters of preferences and endowments, and thus scaling the depths and the autarky gains from trade. For instance, consider any equilibrium network with a particular connectivity structure. Pick a state-price deflator, say p^0 , and move it in \mathbb{R}^S space further away from the other p^k 's. At some stage the resulting equilibrium network becomes hub-spoke with 0 as the hub. On the other hand, assume a network with 0 as the hub. As we raise β^0 while holding p^0 constant (by scaling ω^0 down), at some stage trade with node 0 disappears as other links become relatively more profitable (provided some other link is admissible, and nodes other than 0 are not all identical).

This is the sense in which “scale” can always make or break any particular connectivity structure. Hence the most interesting effects are the “network effects,” which depend on the relative location of the state prices, keeping the scale fixed so as not to overpower the network effects (for example, taking the β^k 's to be equal, and/or imposing symmetry restrictions on the p^k 's such as regularity of the polytope \mathcal{P}).

Before we study network effects in more detail, the next example illustrates the

two aspects of scale.

Example 2 (Pure scale effects) *In order to focus on scale only, we study the class of unary networks, in which there is only one link, and therefore no externalities across links. This provides us with a useful benchmark. All arbitrageurs have to operate on the same link. Arbitrageur profits in the $u_{k\ell}$ -architecture are easily calculated from Lemma 4,*

$$\Phi^{u_{k\ell}} = \frac{1}{(1+N)^2(\beta^k + \beta^\ell)} \cdot \|p^k - p^\ell\|_2^2. \quad (11)$$

From this we see that the optimal unary architecture for arbitrageurs is $u_{k^\ell^*}$, where $(k^*, \ell^*) = \arg \max_{k,\ell} \frac{1}{\beta^k + \beta^\ell} \cdot \|p^k - p^\ell\|_2^2 =: \mu_{k\ell}$. In particular, if $\beta^k = \beta^\ell$ for all k, ℓ , then $(k^*, \ell^*) = \arg \max_{k,\ell} \|\omega^k - \omega^\ell\|_2^2$. And if $\omega^k = \omega^\ell$ for all k, ℓ , then $(k^*, \ell^*) = \arg \max_{k,\ell} \frac{(\beta^k - \beta^\ell)^2}{\beta^k + \beta^\ell}$. This is reminiscent of a result in Duffie and Jackson (1989) which says that the volume-maximizing futures contract maximizes the “endowment differential” of the long and short sides of the market.*

To develop some intuition consider the case of identical endowments. Then the exchanges $k \in K$ differ only with respect to their preference parameters $\{\beta^k\}$. The function $f(\beta^k, \beta^\ell) := \frac{(\beta^k - \beta^\ell)^2}{\beta^k + \beta^\ell}$ is depicted in Figure 3 for fixed β^ℓ . We see that the

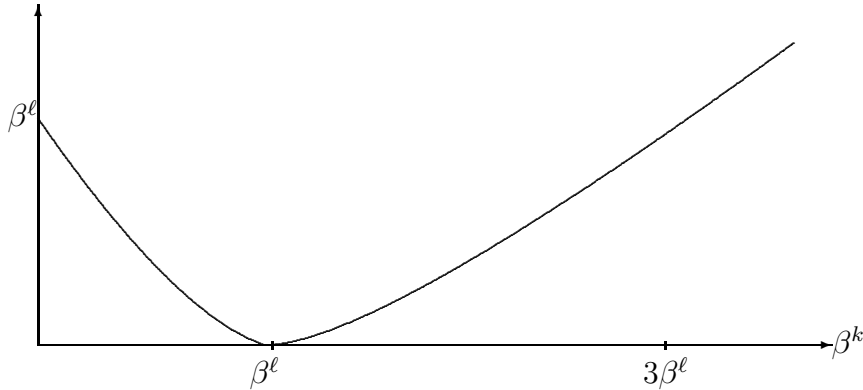


Figure 3: **The objective function of arbitrageurs in Example 2**

slope of f to the left of β^ℓ is steeper than the slope to its right. The solution is to choose the exchange with the highest β^k , provided $\beta^k > 3\beta^\ell$. If $\beta^k \leq 3\beta^\ell$ for all k , then it may be optimal to choose the exchange with the lowest β^k , provided it is close enough to zero. The reason why the optimal exchange will be either the k with the smallest or the largest β^k is as follows. The term $(\beta^k - \beta^\ell)^2$ measures the extent of the unutilized gains from trade between exchanges k and ℓ . These gains are largest the furthest from β^ℓ the new exchange is located. But β^k also determines the shallowness of the exchange. This is reflected in the denominator of the expression. The slope to the left of β^ℓ is steeper since markets with low β^k are deeper, and therefore more attractive to arbitrageurs. Thus there is a tradeoff between gains from trade and

depth. The best exchange may not be the deepest one, since the restriction $\beta^k > 0$ limits the gains from trade. The shallowest exchange may be preferred to the deepest one if the gains from trade are sufficiently large to compensate for the shallowness. We can restate these results as follows: the profit-maximizing link is the one that maximizes $\mu_{k\ell}$.

We now study network effects a bit more formally. For an arbitrary architecture, the following naive conjecture suggests itself: $N^{k\ell} > N^{k'\ell'}$ if and only if $\mu_{k\ell} > \mu_{k'\ell'}$. This is not true in general precisely because of network externalities, and one might use the negation of this statement as a definition of network effects:

Definition 2 (Strong equilibrium network effects) *We say that an equilibrium network \mathcal{G}^* exhibits strong equilibrium network effects if either of the following statements is true:*

There are $k\ell$ and $k'\ell'$ in \mathcal{A} :

$$[SENE1] \quad \mu_{k\ell} \geq \mu_{k'\ell'} \quad \text{and yet} \quad N^{k\ell} < N^{k'\ell'}, \text{ or}$$

$$[SENE2] \quad \mu_{k\ell} > \mu_{k'\ell'} \quad \text{and yet} \quad N^{k\ell} \leq N^{k'\ell'}.$$

For instance even for hub-spoke architectures, the conjecture holds only in the case of two spokes; with more than two, the relative positions of the spokes, and the resulting network effects, matter in addition to the gains from trade between a given spoke and the hub, and they can no longer be internalized. This is the object of Proposition 8. Notice that even with two spokes there is interaction of trades on exchange 0 and the profitability of 01 is affected by the extent of trade on 02 and vice versa, and $\{N^{k\ell}\}$ is affected. The definition of network effects as *equilibrium* network effects is therefore a very strong one with some externalities possibly internalized at equilibrium. The following provides an illustration of a situation where the location-induced network effects dominate scale:

Example 3 (Strong equilibrium network effects in a hub-spoke network)

Consider an architecture composed of one hub, 0, and three spokes, 1, 2 and 3. Assume $\beta^k = \beta$, all $k \in K$. Assume all autarky state-prices lie on a straight line segment with p^1 at one extremity, p^0 in the middle equidistant between the extremities, and $p^2 = p^3$ at the other extremity. We show that there are strong equilibrium network effects. The intuition is straightforward: since μ_{0k} is the same for all $k \neq 0$, the absence of strong network effects would amount to an arbitrageur distribution $N^{0k} = N/3$, $k = 1, 2, 3$. But if that was true, then there would be $2N/3$ arbitrageurs pulling \hat{p}^0 into the direction $p^2 = p^3$ and only $N/3$ pulling it into the direction of p^1 . Since only $N/3$ arbitrageurs are pulling \hat{p}^k , $k = 1, 2, 3$ towards the middle, in equilibrium the gains from trade between exchanges 0 and 1 are larger than between 0 and either 2 or 3, violating the assumption that $N^{0k} = N/3$. It is clear that the results of this example still go through even if p^0 is located closer to p^1 .

Formally, assume to the contrary that $N^{0k} = N/3$. Then from Lemmas 3 and 4 we find that equilibrium profits are $\varphi^{0k} = \frac{2}{2+N/3} \|p^k - \hat{p}^0\|_2^2$ (also see (15) below). Since $\alpha^{0k} = \frac{N}{6\beta}$, we find that $\gamma^0 = \frac{6+N}{6+N(K+1)}$ and $\gamma^k = \gamma^0 \frac{N}{6+N}$, $k \neq 0$. It follows that $\hat{p}^0 := \sum_{k \in K} \gamma^k p^k = \frac{6}{6+N(K+1)} p^0 + \frac{N(K+1)}{6+N(K+1)} p^\lambda$. Using the fact that all exchanges lie on a straight line with p^0 in the middle, we see that $p^2 = 2p^0 - p^1$. We now use these expressions to verify under which conditions $\|p^1 - \hat{p}^0\|_2^2 = \|p^2 - \hat{p}^0\|_2^2$. After a series of manipulations, replacing p^λ by $\frac{1}{4}(5p^0 - p^1)$, this equality holds iff $2N(K+1) = 0$, i.e. iff $N = 0$.

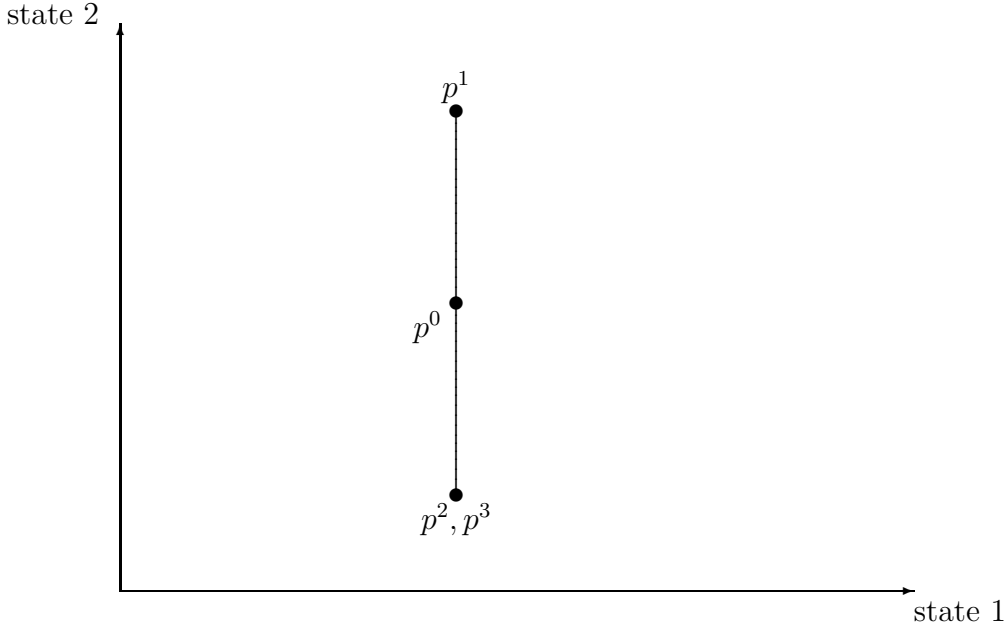


Figure 4: **The economy in Example 3**

We now derive a few general results about equilibrium networks. The main geometric intuition can be summarized as follows: rather than study a network for a given N' , it turns out to be more fruitful to first consider the case where $N = 0$ and then to “trace out” the evolution of the network by increasing N to the desired N' . Consider the polytope $\hat{\mathcal{P}}(N)$ and assume $\beta^k = \beta$, all $k \in K$. As N is increased from zero, arbitrageurs locate on the graph. Arbitrageurs never trade with internal nodes, as will be shown in Proposition 2 below, for better profit opportunities exist with vertices. The diameter of the polytope, $\text{diam}(\hat{\mathcal{P}}(N)) := \max_{x, y \in \hat{\mathcal{P}}(N)} \|x - y\|_2$, is equal to (2β) times the square-root of the equilibrium profit of an arbitrageur. If $N^{k\ell} > 0$ then $\|\hat{p}^k - \hat{p}^\ell\|_2 = \text{diam}(\hat{\mathcal{P}}(N))$. As we have noted before, $\hat{\mathcal{P}}(N)$ is a subset of \mathcal{P} . Among the nodes K , only those pairs of nodes that are furthest apart in \mathbb{R}^S are directly connected by an active link. All such links generate the same profits in equilibrium, i.e. the linked nodes are equally far apart. There can be vertices that are not connected to any other nodes. But as N increases, the equilibrium polytope

contracts along those edges that are active, while the polytope remains centered in the sense that p^λ is always in the interior of $\hat{\mathcal{P}}(N)$, $p^\lambda = \sum_{k \in K} \lambda^k \hat{p}^k(N)$, all $N \geq 0$ (see (28)). For N large enough this implies that hitherto inactive vertices become active as the length of the active edges contracts to the length of the largest edge emanating from the hitherto inactive vertex. At the same time, as the polytope contracts, an internal node becomes itself a vertex for some N large enough, and at some yet higher N becomes an active node. This procedure is repeated until, as seen in Proposition 1, the polytope converges to the singleton point $\{p^\lambda\}$. We provide some general results that provide the basis of our intuition, with finer points illustrated in particular examples later on.

Proposition 2 (Internal nodes) *Suppose that \mathcal{A} is either complete or hub-spoke and that $\beta^k = \beta$, all $k \in K$. Then the equilibrium network (K, \mathcal{A}^*) never involves trade with an internal node unless \mathcal{A} is hub-spoke with the internal node being the hub.*

In particular, if the architecture is complete, the equilibrium network can never be hub-spoke with an internal hub. It can, however, be hub-spoke with a vertex hub. For instance, assume that $p^k = p^1$, $k = 1, \dots, K+1$ and $p^0 \neq p^1$. Then no arbitrageur chooses to arbitrage $k\ell$ if $k \neq 0$ and $\ell \neq 0$, and in equilibrium $N^{0k} = N/K$, $k \neq 0$.

We call a polytope *strongly symmetric* if it is either a regular polytope, or a centrally symmetric polytope with equal axes. Of course, it can be both, e.g. the cube. Examples of non-regular polytopes that are centrally symmetric with equal axes are the rectangle, the bipyramid (with equal axes) whose basis is a regular even polygon with six or more vertices,¹⁰ or any prism based upon an even regular polygon. A strongly symmetric polytope can be circumscribed by a sphere, so its center, as defined above, always exists. In case the polytope is centrally symmetric, the center coincides with its center of symmetry. We denote the family of strongly symmetric polytopes by \mathbb{P}^{ss} . Of all the polytopes, this family provides us with a clear intuition as well as tractable closed-form solutions due to the symmetry between vertices. Intuitively it amounts to assuming that state prices are evenly distributed in state space.

Proposition 3 (Strongly symmetric complete networks) *Suppose that \mathcal{A} is complete and that $\beta^k = \beta$, all $k \in K$. Suppose further that there is an $\bar{N} \geq 0$, with equilibrium arbitrageur distribution $\{\bar{N}^{k\ell}\}$, such that $\hat{\mathcal{P}}(\bar{N}) \in \mathbb{P}^{ss}$ with no internal nodes. Then the center of $\hat{\mathcal{P}}(\bar{N})$ is p^λ , and the equilibrium network can be characterized as follows, for all $N \geq \bar{N}$:*

¹⁰A bipyramid is a pyramid pasted on both sides of the basis. Indeed we can take such a 3-bipyramid itself as the basis and construct a d -bipyramid, by repeatedly pasting s -dimensional pyramids on both “sides” of the $(s-1)$ -dimensional pyramid. Provided we keep the axes equal each time we increase the dimension, the result is a centrally symmetric polytope with equal axes. See Grünbaum (2003) for details.

- i. If $\hat{\mathcal{P}}(\bar{N})$ is a regular simplex, the equilibrium network is complete, with $|\mathcal{A}^*| = K(K+1)/2$.
- ii. If $\hat{\mathcal{P}}(\bar{N})$ is centrally symmetric with equal axes, the equilibrium network is not connected for $K > 1$. It has $\frac{K+1}{2}$ components, each consisting of two nodes which are symmetric with respect to p^λ .
- iii. If $\hat{\mathcal{P}}(\bar{N})$ is a regular odd polygon, the equilibrium network is a cycle: the neighbors of node $k, k \in K$, are the two vertices of the segment opposite to k . Also, $|\mathcal{A}^*| = K + 1$.

In each case, $\hat{\mathcal{P}}(N)$ belongs to the same family of polytopes as $\hat{\mathcal{P}}(\bar{N})$, i.e. (i), (ii) or (iii), with the same center p^λ , but with a smaller circumsphere. For $N \geq \bar{N}$, $\mathcal{A}^*(N) = \mathcal{A}^*(\bar{N})$, and

$$\hat{p}^k(N) = \nu(N)\hat{p}^k(\bar{N}) + (1 - \nu(N))p^\lambda, \quad (12)$$

$$N^{k\ell}(N) = [\nu(N)]^{-1}\bar{N}^{k\ell} + b(N), \quad k\ell \in \mathcal{A}^* \quad (13)$$

$$\Phi(N) = [\nu(N)]^2\Phi(\bar{N}), \quad (14)$$

where $\nu(N)$ is strictly decreasing in N , with $\nu(\bar{N}) = 1$, $\nu(\infty) = 0$, and $b(N)$ is strictly increasing in N , with $b(\bar{N}) = 0$, $b(\infty) = \infty$. Moreover, $N^{k\ell}(N)$ is an affine function of N .

Note that the cases (i)–(iii) in the proposition cover all possible regular polytopes. In addition, (ii) covers some non-regular polytopes as well. For the case in which $\hat{\mathcal{P}}(\bar{N})$ is a polygon with r vertices, with r odd and $r \geq 5$, the equilibrium network is connected but not complete (for $r = 3$ the polygon is in fact a simplex). The cycle should obviously not be visualized as the polygon itself, since the neighbors of k are not the adjacent nodes in the polytope but the ones that are maximally distant from k .

We can specialize Proposition 3 to the case where $\bar{N} = 0$, so that \mathcal{P} is strongly symmetric with vertex set $\{p^k\}_{k \in K}$. Then $\hat{\mathcal{P}}(N)$ is a smaller strongly symmetric polytope within the autarky polytope \mathcal{P} , and $\hat{\mathcal{P}}(N)$ contracts evenly to the singleton $\{p^\lambda\}$ as $N \rightarrow \infty$, with each state-price deflator \hat{p}^k converging on a straight line segment towards p^λ , the center of the polytope. Equilibrium profits converge monotonically to zero. Each active link attracts the same number of arbitrageurs.

For arbitrary polytopes convergence need not be along a linear trajectory but can be along a nonlinear curve, either globally or piecewise. Examples will be given in the sequel. However, even if \mathcal{P} is not strongly symmetric, $\hat{\mathcal{P}}(\bar{N})$ may be for some \bar{N} . Interestingly, convergence is linear from that \bar{N} onwards, with equilibrium numbers of arbitrageurs spread out according to the rules laid out in the proposition. For instance, suppose there are three exchanges and \mathcal{P} is an isosceles triangle with the two sides that are equal making an angle greater than 60° . Then the third side is longer and initially all arbitrageurs will concentrate on that link. As N increases, this side contracts, until the triangle becomes equilateral, i.e. a regular simplex. As

another example, reconsider Example 1. Here the autarky polytope is a rhombus which converges to a square. If we modify Example 1, so that the two lines in Figure 2 do not cross at a right angle, then the autarky polytope is a parallelogram which converges to a rectangle. In both cases, there is an \bar{N} for which the polytope $\hat{\mathcal{P}}(\bar{N})$ is centrally symmetric with equal axes, so the proposition applies (in particular, part (ii)). In fact, this observation generalizes to all centrally symmetric polytopes:

Lemma 5 *Suppose that \mathcal{A} is complete and that $\beta^k = \beta$, all $k \in K$. If \mathcal{P} is centrally symmetric, then there is an \bar{N} such that $\hat{\mathcal{P}}(\bar{N})$ is centrally symmetric with equal axes.*

Arbitrageurs gravitate to the links that correspond to the longest axes of the polytope. This causes these axes to shorten until there is activity on all the axes. Then they must be equal.

If $\hat{\mathcal{P}}(\bar{N}) \in \mathbb{P}^{ss}$ with some internal nodes, the proposition still applies for $N > \bar{N}$ as long as the nodes that are internal for $\hat{\mathcal{P}}(\bar{N})$ are also internal for $\hat{\mathcal{P}}(N)$. Let N_{\max} be the maximum N for which this is the case and let C be the vertex set of $\hat{\mathcal{P}}(\bar{N})$. Then we have linear convergence of \hat{p}^k to p_C^λ , for $N \in [\bar{N}, N_{\max}]$, for all $k \in C$.

In view of the importance of hub-spoke networks in delivering higher payoffs, it is worth emphasizing the following corollary of Proposition 3:

Corollary 2 *Suppose that \mathcal{A} is complete and that $\beta^k = \beta$, all $k \in K$. If $\hat{\mathcal{P}}(\bar{N}) \in \mathbb{P}^{ss}$, then the equilibrium network is not hub-spoke for any $N \geq \bar{N}$.*

In particular if \mathcal{P} is strongly symmetric with no internal nodes, then the equilibrium network is not hub-spoke for any N . The intuition here is that if the architecture is complete and if equilibrium network is hub-spoke, then the autarky state-prices must be distributed “irregularly” or asymmetrically in \mathbb{R}^S space, with some nodes further out (but with the mirror-nodes not also further out in a symmetrical fashion) or some other nodes in clusters.

The strong symmetry imposed in the previous proposition also implies that there cannot be strong network effects:

Corollary 3 (No strong network effects in \mathbb{P}^{ss}) *Under the assumptions of Proposition 3, there are no strong network effects.*

Intuitively, strong network externalities imposed upon one node cannot emerge in equilibrium because by symmetry those same externalities are also imposed on the other relevant nodes. This causes them to mutually cancel each other out in equilibrium, and this is precisely what contributes to the relatively succinct closed form expressions. This does not say that there are no network externalities, of course. For instance, in cases (i) and (iii) the state-prices and the state-contingent consumptions on exchange k are affected by arbitrageur actions (and ultimately by preferences and endowments of investors located) on any other exchange ℓ , no matter how remote and indirectly connected.

7 Hub-Spoke Geometry

We now analyze the geometry of hub-spoke networks in a bit more detail. Arbitrageur profits in an h_0 -architecture are easily calculated, using (9):

$$\varphi^{0k} = \frac{\beta^k + \beta^0}{[(1 + N^{0k})\beta^k + \beta^0]^2} \cdot \|p^k - \hat{p}^0\|_2^2, \quad (15)$$

with \hat{p}^0 given by (8). In equilibrium (with endogenous $\{N^{0k}\}$) we must have $\varphi^{0k} = \Phi^{h_0}$, for all k on which there is some arbitrage activity. The profit φ^{0k} is a product of two terms. The first term is decreasing in β^k , and therefore increasing in the depth of exchange k , while the second term captures the net gains from trade on exchange k , taking into account the fact that all other arbitrageurs, including those linking 0 to $j \neq k$, also trade on 0 (hence \hat{p}^0 appears in lieu of p^0). Consequently, there is a tradeoff between depth and equilibrium gains from trade. The following result is immediate from an inspection of (15):

Proposition 4 (Equalizing differences: hub-spoke architecture) *Suppose $\mathcal{A} = \mathcal{A}^{h_0}$. Then:*

- i. Suppose $N^{0k} = N^{0\ell} > 0$. $\|p^k - \hat{p}^0\|_2^2 > \|p^\ell - \hat{p}^0\|_2^2$ iff $\beta^k > \beta^\ell$.*
- ii. Suppose $\|p^k - \hat{p}^0\|_2^2 = \|p^\ell - \hat{p}^0\|_2^2$. Then $N^{0k} > N^{0\ell}$ iff $\beta^k < \beta^\ell$.*
- iii. Suppose $\beta^k = \beta^\ell$. Then $N^{0k} > N^{0\ell}$ iff $\|p^k - \hat{p}^0\|_2^2 > \|p^\ell - \hat{p}^0\|_2^2$.*

Proposition 4 does not give us an explicit characterization of $\{N^{0k}\}$ since \hat{p}^0 itself depends on $\{N^{0k}\}$ in a complex manner. The symmetric case, however, is amenable to further analysis. We say that the nodes $\{p^k\}_{k \in K}$ are symmetric with respect to p^0 if $\|p^k - p^0\|_2$ does not depend on k , for $k \neq 0$. Let $L := \text{aff}(\{p^0, p^\lambda\})$. If $p^0 \neq p^\lambda$, then L is the line passing through p^0 and p^λ ; if $p^0 = p^\lambda$, then L is just the point p^λ . We say that the nodes $\{p^k\}_{k \in K}$ are symmetric with respect to L if for every k , there is a $p^{\ell k}$, such that $p^{\ell k}$ is the reflection of p^k through L .

Proposition 5 (Symmetric hub-spoke networks) *Suppose $\mathcal{A} = \mathcal{A}^{h_0}$, and $\beta^k = \beta$, all $k \in K$. Suppose further that the nodes $\{p^k\}$ are symmetric with respect to p^0 and L . Then, $\hat{p}^0(N) = \nu(N)p^0 + (1 - \nu(N))p^\lambda$, where $\nu(N)$ is strictly decreasing in N , with $\nu(0) = 1$ and $\nu(\infty) = 0$.*

Suppose the above hypotheses hold except that the nodes $\{p^k\}$ are not necessarily symmetric with respect to p^0 . Then, $\hat{p}^0(N)$ is in $\text{aff}(\{p^0, p^\lambda\})$ and converges to p^λ . There is an array of exchanges $\{K_i\}_{i=1}^n$ and arbitrageurs $\{N_i\}_{i=1}^n$, with $N_i > N_{i-1}$, $p^0 \in K_1$, $K_{i-1} \subset K_i$, $K_n = K$, such that for $N \in [N_{i-1}, N_i)$, $\hat{p}^0(N) \in \nu(N_{i-1})\hat{p}^0(N_{i-1}) + (1 - \nu(N_{i-1}))\hat{p}_{K_i}^\lambda$. We have $\nu(N_{i-1}) \in [0, 1]$, and for $N' > N \geq N_{i-1}$, $\nu(N') < \nu(N)$.

Under the symmetry assumptions of the proposition, the equilibrium state-price deflator on the hub is pulled evenly “from both sides” and follows a linear trajectory towards p^λ . With symmetry with respect to both p^0 and L , $\hat{p}^0(N)$ converges to p^λ monotonically along the line segment joining p^0 and p^λ . If $p^0 = p^\lambda$, then in fact $\hat{p}^0(N) = p^\lambda$ for all N . If symmetry fails and more nodes are on one side than on another one (but with similar scale properties), then the trajectory is an arc bent towards the more plentiful, attracting, side.

Once $\hat{p}^0(N)$ is determined, $\hat{p}^k(N)$ is a convex combination of $\hat{p}^0(N)$ and p^k , $\hat{p}^k(N) = \hat{p}^0(N) + \frac{2}{N+2K} \frac{\sum_\ell \|p^\ell - \hat{p}^0(N)\|_2}{\|p^k - \hat{p}^0(N)\|_2} (p^k - \hat{p}^0(N))$. When replacing $\hat{p}^0(N)$ by $\nu(N)p^0 + (1 - \nu(N))p^\lambda$ for instance we see that $\hat{p}^k(N)$ converges to p^λ on an arc turning its concavity towards p^0 .

Intuitively, assume N is initially zero. As N becomes positive, only the exchanges furthest from p^0 get connected. Those exchanges form K_1 . As N further increases, $\hat{p}^0(N)$ monotonically tends towards $p_{K_1}^\lambda := \sum_{k \in K_1} \frac{1}{K_1} p^k$ until N equals N_1 . N_1 is the critical number for which the new equilibrium gains from trade between 0 and any of the exchanges in K_1 equal those between 0 and one or more exchanges not in K_1 . Those exchanges when added to the ones in K_1 form K_2 . As N further increases, $\hat{p}^0(N)$ converges towards $p_{K_2}^\lambda$, and the argument can be repeated until $K_i = K_n = K$. Whereas convergence on each segment $[N_i, N_{i-1}]$ is monotonic, overall convergence of \hat{p}^0 from p^0 to p^λ need not be monotonic because $p_{K_i}^\lambda$ is in the convex hull of p^0 and $p_{K_{i-1}}^\lambda$. This can be illustrated in a simple example:

Example 4 (Non-monotonicity in a hub-spoke network) *Consider the h_0 - architecture with four exchanges whose autarky state-prices lie on a straight line in \mathbb{R}^S . By construction this architecture is symmetric with respect to L . p^0 and p^1 form the extremes of the polytope, with $p^2 = p^3$ assumed to be located in $\text{conv}(\{p^0, p_{K_1}^\lambda\})$, $K_1 = \{0, 1\}$. Since the gains from trade between p^0 and p^1 are largest, for small enough N we have $N^{01} = N$. This will be true until N^* which is so that $\hat{p}^0(N^*) - p^2 = \hat{p}^1(N^*) - \hat{p}^0(N^*) = 2[p_{K_1}^\lambda - \hat{p}^0(N^*)]$. In other words, at N^* all links will become active, and for $N > N^*$, $\hat{p}^0(N)$ converges to p_K^λ . Solving for N^* we get $\hat{p}^0(N^*) = \frac{2}{3}p_{K_1}^\lambda + \frac{1}{3}p^2$, and it can be verified that $p_K^\lambda = \frac{1}{2}p_{K_1}^\lambda + \frac{1}{2}p^2$. But p_K^λ lies between $\hat{p}^0(N^*)$ and p^0 , so for $N > N^*$ $\hat{p}^0(N)$ reverts back in the direction of p^0 towards its limit p_K^λ .*

One can get further results by requiring yet more symmetry and assume that autarky state prices form a polytope with the hub at the center:

Proposition 6 (Strongly symmetric networks with central hub) *Suppose $\mathcal{A} = \mathcal{A}^{h_0}$, β^0 arbitrary, $\beta^k = \beta$ for all $k = 1, \dots, K$, and $p^0 = p^\lambda$. Suppose further that $\mathcal{P} \in \mathbb{P}^{\text{sss}}$ with no internal nodes other than node 0. Then the center of \mathcal{P} is p^λ . $\hat{\mathcal{P}}(N)$ belongs to the same family of polytopes as \mathcal{P} , with the same center p^λ , but with a smaller circumsphere. We have*

$$\hat{p}^0(N) = p^0 = p^\lambda \tag{16}$$

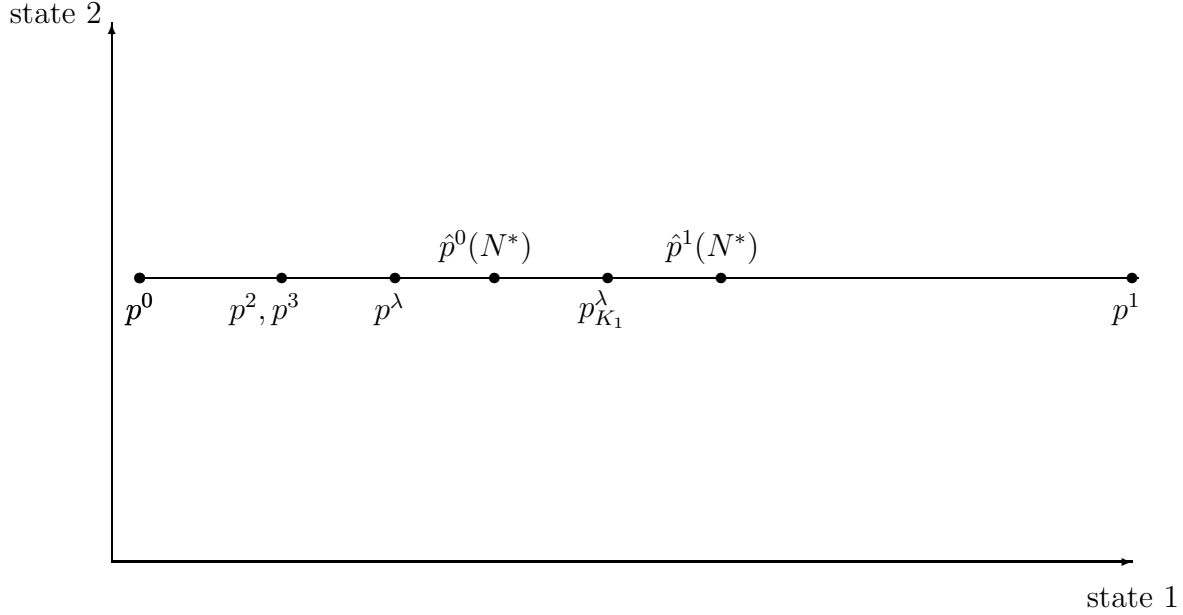


Figure 5: **The economy in Example 4**

and, for $k \neq 0$,

$$\hat{p}^k(N) = \frac{(\beta + \beta^0)K}{(\beta + \beta^0)K + \beta N} p^k + \frac{\beta N}{(\beta + \beta^0)K + \beta N} p^\lambda, \quad (17)$$

$$N^{0k}(N) = \frac{N}{K}, \quad (18)$$

$$\Phi^{h_0}(N) = \frac{\beta + \beta^0}{[(1 + N/K)\beta + \beta^0]^2} \|p^k - p^0\|_2^2. \quad (19)$$

Equilibrium profits given by (19) do not depend on k because $p^0 = p^\lambda$ is the center of \mathcal{P} .¹¹

When we consider vertex hubs we will adopt the convention of choosing exchange 1 as the hub. This will be particularly useful when we compare central and vertex hubs later.

Proposition 7 (Simplex networks with vertex hub) *Suppose $\mathcal{A} = \mathcal{A}^{h_1}$, and $\beta^k = \beta$ for all $k \in K$. Suppose further that \mathcal{P} is a regular simplex with no internal nodes. Then:*

$$\hat{p}^1(N) = \frac{2K}{N(K+1) + 2K} p^1 + \frac{N(K+1)}{N(K+1) + 2K} p^\lambda \quad (20)$$

¹¹The only property of strongly symmetric polytopes that we use in the proof is that the circum-center (which is the center as we have defined it) and the centroid (which is the equally weighted convex combination of the vertices) coincide. Hence Proposition 6 holds in fact for the larger family of polytopes that have this property.

and, for $k \neq 1$,

$$\hat{p}^k(N) = \frac{2K}{N+2K}p^k + \frac{N}{N+2K}\hat{p}^1(N), \quad (21)$$

$$N^{1k}(N) = \frac{N}{K}, \quad (22)$$

$$\Phi^{h_1}(N) = \frac{K^3[N(N+4)(K+1) + 8K]}{\beta(N+2K)^2[N(K+1) + 2K]^2} \|p^k - p^1\|_2^2. \quad (23)$$

$\Phi^{h_1}(N)$ is strictly decreasing in N . Furthermore,

$$\|\hat{p}^k(N) - \hat{p}^1(N)\|_2 < \|\hat{p}^k(N) - \hat{p}^\ell(N)\|_2, \quad \text{for } \ell \neq 1, N > 0.$$

Equilibrium profits given by (23) do not depend on k because all the edges of a regular simplex are equal. Note that $\hat{\mathcal{P}}(N)$ is not a regular simplex for $N > 0$. Nodes other than the hub are pulled closer to the hub than to other nodes.

The following proposition says roughly that for hub-spoke networks that are small enough, network effects are absent. ‘‘Small’’ may be either in terms of N small, or K small, or in the sense that $(\lambda^0)^{-1}$ is small. The latter condition would apply for instance if all investors have identical risk parameters $\beta^{k,i}$, $k \in K$, and if the number of investors on 0, I^0 , is large compared to the aggregate population. In that sense the spokes are small compared to the hub.

Proposition 8 (No strong network effects in small hub-spoke networks)

Suppose $\mathcal{A} = \mathcal{A}^{h_0}$.

Assume that (i) $\beta^k = \beta$ for all $k \in K$ and that N is small. Then there are no strong equilibrium networks effects of type 2.

Alternatively, assume either that (ii) $\beta^k = \beta$ for all $k = 1, \dots, K$ and λ^0 is large, or alternatively that (iii) $K = \{0, 1, 2\}$ with no further restrictions. Then there are no strong equilibrium network effects.

For hub-spoke networks with either large N or large K , examples can easily be constructed where network effects matter. The foregoing discussions taken together fully characterize the case of three exchanges.

Example 5 (Three exchanges) Propositions 3 and 5 allow us to completely characterize the case $K = \{0, 1, 2\}$. This example also serves as an illustration for the principle that in order to understand a network for some given N' , proposition 3 suggests that one fruitful alternative is to start N at zero and then raise N to N' . That the case with $K = \{0, 1, 2\}$ is fully characterized follows from the observation that one must be in either of the following three situations. Either only one link is active, say 01, in which case we know that equilibrium state prices lie on the segment between the two autarky state-prices and they move on that segment towards the middle point p^λ . Unless $p^2 = p^\lambda$, at some point for larger N there will be trade with exchange 2. Assume without loss of generality that the active links are 01 and 02. Then we are

in a necessarily symmetric hub-spoke network. Then \hat{p}^0 converges linearly to p^λ as N increases (and therefore also N^{01} and N^{02}) while \hat{p}^1 and \hat{p}^2 converge towards p^λ along a parabola. Either N^{12} always stays at zero, or there might be a point where $\|\hat{p}^0 - \hat{p}^1\|_2 = \|\hat{p}^0 - \hat{p}^2\|_2 = \|\hat{p}^1 - \hat{p}^2\|_2$, in which case Proposition 3 implies that all three links become active and equilibrium state prices all converge linearly towards p^λ .

8 Equilibrium vs Profit Maximizing Networks

In this section we study equilibrium networks further and contrast them to profit maximizing networks.

Proposition 9 (Profits: central hub vs complete architecture) *Suppose $\beta^k = \beta$ for all $k \neq 0$, and $p^0 = p^\lambda$. Suppose further that $\mathcal{P} \in \mathbb{P}^{ss}$ with no internal nodes except for node 0. Then, provided $\beta^0 \geq \beta$, there is no activity on node 0 in the complete architecture for any N . If $\beta^0 = \beta$, then $\Phi^{h_0}(N) > \Phi(N)$, provided $N > 2|\mathcal{A}^*|$, where \mathcal{A}^* is the set of active links for the complete architecture. For arbitrary $\beta^0 > \beta$, we have $\Phi^{h_0}(N) > \Phi(N)$, provided N is sufficiently large.*

Thus, if all exchanges have equal depth, the complete architecture is dominated as long as there are at least two arbitrageurs on each active link. Moreover, the h_0 -architecture leads to higher payoffs for arbitrageurs no matter how illiquid it is, as measured by β^0 , provided there are sufficiently many arbitrageurs.

For the sake of intuition let us first focus on the case where \mathcal{P} is centrally symmetric. Then the vertices come in pairs (k, ℓ_k) that are symmetrical with respect to p^λ . Equilibrium profits are higher with a central hub. This is so even though the autarky gains from trade are higher on link $k\ell_k$ than on the spokes of the h_0 -architecture (i.e. $\Phi(0) > \Phi^{h_0}(0)$). The reason is the positive externality that arbitrageurs on spoke $0k$ exert on arbitrageurs on the symmetrical spoke $0\ell_k$. Arbitrageurs on $0k$ pull \hat{p}^0 towards p^k , thereby increasing $\|\hat{p}^0 - \hat{p}^{\ell_k}\|_2$. Arbitrageurs on $0\ell_k$ pull \hat{p}^0 in the opposite direction, towards p^{ℓ_k} . Due to symmetry, the net impact on state prices on the hub is in fact zero, i.e. $\hat{p}^0 = p^0$. The aggregate supply of both families of arbitrageurs on the hub is also zero—any state-contingent consumption that is supplied to the hub by arbitrageurs on one spoke is absorbed by arbitrageurs on the other spoke. Thus the hub acts as a liquidity repository,¹² channeling trades in such a manner that the two groups of arbitrageurs complement each other. The network-induced complementarity is sufficient to compensate for the fact that the autarky gains from

¹²We do not define liquidity rigorously in this paper, which is left to a companion paper (Rahi and Zigrand (2007d)). Do notice however that the common measure of liquidity as depth ($1/\beta^0$) does not do justice to exchange 0. If exchange 0 is the hub, it will attract a lot of trade with zero equilibrium price impact, irrespective of β^0 . On the other hand, for any other architecture, there is no trade with exchange 0, even if $1/\beta^k$ approaches infinity. It is purely the position in the network which determines liquidity, rather than any of the standard metrics of liquidity in isolation.

trade between the center and any one of the extremes is considerably less than the gains from trade between the two extremes, ignoring the central exchange altogether.

The h_0 -architecture exhibits strategic complements. The more arbitrageurs on $0k$ trade, the more profitable each trade becomes for the arbitrageurs on $0\ell_k$ because on the one hand mispricings increase and on the other hand the arbitrageurs on link $0k$ provide liquidity to the ones on $0\ell_k$ and vice versa.

When the architecture is complete the market mechanism fails to achieve this outcome due to a Prisoner's Dilemma. If arbitrageurs could agree to not trade on $k\ell_k$, they would be able to minimize price impact and increase profits. But, given the opportunity, each arbitrageur would rather arbitrage $k\ell_k$ due to the much greater gains from trade. The result is a suboptimal arrangement with all arbitrageurs on $k\ell_k$.¹³

For the case where \mathcal{P} is not centrally symmetric, for example a simplex, essentially the same intuition applies. For each node k there is a facet that is opposite to it. The vertices of this facet pull \hat{p}^0 away from p^k . Due to symmetry, the net impact on \hat{p}^0 is zero.

In fact the complete architecture is suboptimal for arbitrageurs even if there is no central exchange. The following proposition shows that a vertex hub delivers higher profits as well. While a vertex of \mathcal{P} is not central relative to the other vertices of \mathcal{P} , it gets pulled towards the center after some level of arbitrageur activity on the spokes.

Proposition 10 (Profits: central/vertex hub) *Suppose $K \geq 3$, $\beta^k = \beta$ for all $k \in K$, and $p^0 = p^\lambda$. Suppose further that \mathcal{P} is a regular simplex with no internal nodes except for node 0. Then, for the complete and h_1 -architectures, there is no activity on node 0 for any N . We have $\Phi^{h_0}(N) > \Phi^{h_1}(N) > \Phi(N)$, provided $N \geq 2K - 1$.*

Since node 0 is always isolated in the complete and h_1 -architectures, the ranking of profits in these architectures also holds if \mathcal{P} has no internal nodes (replacing K by $K + 1$ in the proposition).

The reason why a vertex hub delivers higher profits is similar to the one given for a central hub. If $k\ell$ is left unarbitraged, there is less pressure for the prices on k and ℓ to converge, leaving the arbitrages on $1k$ and 1ℓ more differentiated, and therefore more profitable. As N increases, equilibrium state-price deflators on k and ℓ converge to p^λ along an arc while \hat{p}^1 moves more quickly into the middle between \hat{p}^k and \hat{p}^ℓ . The effect of this is again to induce strategic complements in that arbitrageurs provide liquidity to each other to some extent (but less than in the central hub case). It is therefore not hard to see that the distance between equilibrium state prices on k and ℓ is high, so that if trade on $k\ell$ was now allowed, some arbitrageurs would deviate to that link. Again a Prisoner's Dilemma result obtains.

¹³Interestingly, even if arbitrageurs were allowed to arbitrage any number of links, they would only arbitrage $k\ell_k$. Arbitrageurs facing the twin restrictions of not being able to trade on $k\ell_k$ and of trading on only one link, are better off than completely unrestricted arbitrageurs.

Thus, for both the central and vertex hub-spoke architectures, the restrictions implicit in the architecture coordinate arbitrageur actions by pooling liquidity and by preventing Prisoner's Dilemma type deviations. The state prices of the hub are in equilibrium (not necessarily in autarky) to some extent "in-between" the other state prices, therefore acting as a liquidity pool.

One way to interpret Propositions 9 and 10 is that profits are lower in the complete architecture because convergence to the Walrasian state-price deflator p^λ is faster. Let $d^k(N)$ and $d^{k,h_\ell}(N)$ denote $\|\hat{p}^k(N) - p^\lambda\|_2^2$ for the complete and h_ℓ -architectures respectively.

Proposition 11 (Speed of convergence: central hub vs complete architecture) *Suppose $\beta^k = \beta$ for all $k \in K$, and $p^0 = p^\lambda$. Suppose further that $\mathcal{P} \in \mathbb{P}^{ss}$ with no internal nodes except for node 0. Then, for all $N > 0$,*

$$\begin{aligned} d^0(N) &= d^{0,h_0}(N) = 0, \\ d^k(N) &< d^{k,h_0}(N), \quad k \neq 0. \end{aligned}$$

Also, $d^k(N)$ and $d^{k,h_0}(N)$ are strictly decreasing in N for $k \neq 0$.

Proposition 12 (Speed of convergence: central/vertex hub) *Suppose $K \geq 3$, $\beta^k = \beta$ for all $k \in K$, and $p^0 = p^\lambda$. Suppose further that \mathcal{P} is a regular simplex with no internal nodes except for node 0. Then, for all $N > 0$,*

$$\begin{aligned} d^0(N) &= d^{0,h_1}(N) = d^{0,h_0}(N) = 0, \\ d^{1,h_1}(N) &< d^1(N) < d^{1,h_0}(N), \\ d^k(N) &< d^{k,h_1}(N) < d^{k,h_0}(N), \quad k \notin \{0, 1\}. \end{aligned}$$

Also, the distance of each node k , $k \neq 0$, from p^λ is strictly decreasing in N for all three architectures.

Like Proposition 9, this result holds for the complete and h_1 -architectures even if \mathcal{P} has no internal nodes.

9 Networks and Social Welfare

In this section we analyze how the equilibrium utilities of investors depend on the architecture. Using investor (k, i) 's first order condition, we can write his utility (1) as:

$$U^{k,i} = \omega_0^{k,i} + \mathbf{1}^\top \Pi \omega^{k,i} - \frac{\beta^{k,i}}{2} \omega^{k,i \top} \Pi \omega^{k,i} + \frac{\beta^{k,i}}{2} \|\theta^{k,i}\|_2^2.$$

Note that $U^{k,i}$ depends on the asset structure only through the term $W^{k,i} := \beta^{k,i} \|\theta^{k,i}\|_2^2$. We will find it convenient to refer to $W^{k,i}$ as the equilibrium utility of agent (k, i) . From (2), we see that

$$\theta^{k,i} = \frac{1}{\beta^{k,i}} (p^{k,i} - \hat{p}^k), \quad k \in K.$$

Hence, we have

Lemma 6 (Equilibrium utilities) *Given a network \mathcal{G}^* , the equilibrium utility of investor (k, i) , $k \in K$, is*

$$W^{k,i} = \frac{1}{\beta^{k,i}} \|p^{k,i} - \hat{p}^k\|_2^2.$$

An architecture is optimal for an investor if it results in the highest equilibrium utility for the investor among all possible architectures. An architecture is Pareto optimal for a group of agents if there is no alternative architecture that Pareto dominates it in equilibrium for this group. An architecture is socially optimal if it is Pareto optimal for the set of all agents, arbitrageurs and investors.

We say that investors on exchange k are homogeneous if they have the same no-trade valuations, i.e. $p^{k,i} = p^k$, for all $i \in I^k$. We refer to an economy in which investors are homogeneous within each exchange as a *clientèle economy*. From the point of view of arbitrageurs, each clientèle $k \in K$ consists of agents with identical characteristics.

We will focus now on a clientèle economy. Lemma 6 gives us the following welfare index for clientèle $k \in K$:

$$W^k := \sum_{i \in I^k} W^{k,i} = \frac{1}{\beta^k} \|p^k - \hat{p}^k\|_2^2. \quad (24)$$

We can think of W^k as the inter-exchange gains from trade reaped by exchange k in moving from autarky to the arbitrated equilibrium. Note that $W^k = \beta^k \|y^k\|_2^2$, so that the gains from trade are proportional to the magnitude of state-contingent consumption trading volume. In particular this implies that if there are only two exchanges, the outcome of the game is Pareto optimal in view of the fact that $y^k = -y^\ell$. All the subtleties arise from network externalities across links.

In order to distinguish welfare across networks, we will reserve the notation W^k for welfare on exchange k in the complete architecture, and denote welfare on exchange k in the h_ℓ -architecture by W^{k,h_ℓ} .

Proposition 13 (Welfare: central hub vs complete architecture) *Suppose $\beta^k = \beta$ for all $k \in K$, and $p^0 = p^\lambda$. Suppose further that $\mathcal{P} \in \mathbb{P}^{ss}$ with no internal nodes except for node 0. Then, for all $N > 0$,*

$$\begin{aligned} W^0(N) &= W^{0,h_0}(N) = 0, \\ W^k(N) &> W^{k,h_0}(N), \quad k \neq 0. \end{aligned}$$

Also, $W^k(N)$ and $W^{k,h_0}(N)$ are strictly increasing in N for $k \neq 0$.

Thus the complete architecture Pareto dominates the h_0 -architecture for investors. This is intuitive given our earlier result that, in the complete architecture, the state-price deflator on any exchange converges faster to the Walrasian state-price deflator (Proposition 11).

Proposition 14 (Welfare: central/vertex hub) *Suppose $K \geq 3$, $\beta^k = \beta$ for all $k \in K$, and $p^0 = p^\lambda$. Suppose further that \mathcal{P} is a regular simplex with no internal nodes except for node 0. Then, for all $N > 0$,*

$$\begin{aligned} W^0(N) &= W^{0,h_1}(N) = W^{0,h_0}(N) = 0, \\ W^{1,h_1}(N) &> W^1(N) > W^{1,h_0}(N), \\ W^k(N) &> W^{k,h_1}(N) > W^{k,h_0}(N), \quad k \notin \{0, 1\}. \end{aligned}$$

Also, for $k \neq 0$, welfare is strictly increasing in N for all three architectures.

Like Propositions 9 and 12, this result holds for the complete and h_1 -architectures even if \mathcal{P} has no internal nodes.

When node 0 is the center of \mathcal{P} , the interests of arbitrageurs and investors are directly opposed. Arbitrageurs benefit when the trades between exchanges k and ℓ are “split” by having to pass through exchange 0, while agents on exchanges k and ℓ appropriate these gains themselves when they are linked directly.

Investors on node 1 (a vertex of \mathcal{P}) are better off if their exchange is the hub compared to the complete architecture. The reason is that, if exchange 1 is the hub, the other exchanges cannot trade with each other directly. All trading must be routed through exchange 1. However, if the other nodes are arranged symmetrically around the hub, as in the central hub case, the hub becomes just a conduit for liquidity, with arbitrageur trades on the hub being exactly offsetting. More generally, investors on the hub are better off the more asymmetric the location of the spokes (see Rahi and Zigrand (2007a) where we provide a complete analysis of the three-exchange case).

Such results cannot be obtained in the case where agents within an exchange are heterogeneous. This is because, for a given p^k , the deflator $p^{k,i}$ is essentially unrestricted, so any result that holds for the representative investor on exchange k can be reversed for an individual agent on k .

10 Conclusion

*** To be written ***

A Appendix: Proofs

Proof of Lemma 1 Using (3), we can write the Lagrangian for arbitrageur $n \in N^{k\ell}$ as follows:

$$\mathcal{L} = \sum_{m \in \{k, \ell\}} [p^m - \beta^m y_{k\ell}^{m,n} - \beta^m y_{k\ell}^{m, \setminus n}]^\top \Pi y_{k\ell}^{m,n} - \psi^\top \Pi \sum_{m \in \{k, \ell\}} y_{k\ell}^{m,n},$$

where ψ is the Lagrange multiplier vector attached to the no-default constraints, and can be interpreted as a (shadow) state-price deflator of the arbitrageur. The first order conditions are:

$$p^m - \beta^m y_{k\ell}^{m, \setminus n} - 2\beta^m y_{k\ell}^{m,n} - \psi = 0, \quad m \in \{k, \ell\}$$

together with complementary slackness:

$$\psi \geq 0, \quad \sum_{m \in \{k, \ell\}} y_{k\ell}^{m,n} \leq 0, \quad \text{and} \quad \psi^\top \left[\sum_{m \in \{k, \ell\}} y_{k\ell}^{m,n} \right] = 0. \quad (25)$$

We can rewrite the first order conditions as follows:

$$\hat{p}^m - \beta^m y_{k\ell}^{m,n} - \psi = 0, \quad m \in \{k, \ell\}. \quad (26)$$

It is easy to check that a solution to (25) and (26) is given by (5), with

$$\psi = \left(\frac{1}{\beta^k} + \frac{1}{\beta^\ell} \right)^{-1} \left(\frac{1}{\beta^k} \hat{p}^k + \frac{1}{\beta^\ell} \hat{p}^\ell \right). \quad (27)$$

The Lagrange multiplier vector ψ is nonnegative if \hat{p}^k and \hat{p}^ℓ are both nonnegative. This is indeed the case, as we will verify later (Lemma 2). The no-default constraints hold with equality. This argmax is in fact unique since the program is globally concave. Thus the CWE is symmetric, i.e. $y_{k\ell}^{m,n}$ does not depend on n . ■

Proof of Lemma 2 Consider a component $C \in \mathcal{C}$. Then $\{\hat{p}^k\}_{k \in C}$ solves the system (7) restricted to C . Define the matrix $\underline{m} = [\underline{m}_{k\ell}]_{k, \ell \in C}$ by $\underline{m}_{kk} := 1 + \beta^k \alpha^k$ and $\underline{m}_{k\ell} := -\beta^k \alpha^{k\ell}$ for $k \neq \ell$. Let $M := \underline{m} \otimes I_{S \times S}$. Then, letting $\hat{p} := \{\hat{p}^k\}_{k \in C}$ and $p := \{p^k\}_{k \in C}$, (7) can be written as $M\hat{p} = p$. Noting the diagonal structure of each block of M , we have $|M_{ii}| = 1 + \beta^k \alpha^k$, for some k , and $\sum_{j \neq i} |M_{ij}| = \sum_{\ell \neq k} |-\beta^k \alpha^{k\ell}| = \beta^k \alpha^k$. Therefore, $|M_{ii}| > \sum_{j \neq i} |M_{ij}|$, i.e. M is strictly (row)diagonally dominant (and so is \underline{m}).

We will appeal to the theory of M-matrices; see Berman and Plemmons (1979), henceforth BP. An M-matrix is a square matrix of the form $sI - B$, where $B \geq 0$, and $s \geq \text{rad}(B)$, the spectral radius of B . These matrices also have the property that the diagonal elements are positive and the off-diagonal elements are nonpositive. By Theorem 6.2.3 in BP, both M and \underline{m} are nonsingular M-matrices. Hence, there

exists a unique \hat{p} solving $M\hat{p} = p$, namely $\hat{p} = M^{-1}p$. Since $M^{-1} = (\underline{m}^{-1} \otimes I_{S \times S})$, we can write $\hat{p}^k = p^{n,k} := \sum_{j \in C} \eta^{kj} p^j$, where η^{kj} is element (k, j) of \underline{m}^{-1} . The matrix \underline{m} is irreducible, also called indecomposable (indeed it is irreducible if and only if C is connected; see Theorem 2.2.7 in BP). Hence $\underline{m}^{-1} \gg 0$ by Theorem 6.2.7 in BP, i.e. $\eta^{kj} > 0$, all $k, j \in C$. Since $\underline{m}\mathbf{1} = \mathbf{1}$ we also have $\underline{m}^{-1}\mathbf{1} = \mathbf{1}$, i.e. $\sum_{j \in C} \eta^{kj} = 1$, all $k \in C$. In other words, \underline{m}^{-1} is a stochastic matrix. Finally, from Theorem 2.5.12 in Horn and Johnson (1985), \underline{m}^{-1} is strictly column diagonally dominant: $\eta^{jj} > \sum_{i \neq j} \eta^{ij}$. \blacksquare

Proof of Lemma 4 Using Lemma 1, the equilibrium profit of arbitrageur $n \in N^{k\ell}$ is

$$\begin{aligned} \varphi^{k\ell} &= (q^k - q^\ell)^\top y_{k\ell}^{k,n} \\ &= (\hat{p}^k - \hat{p}^\ell)^\top \Pi y_{k\ell}^{k,n} \\ &= \frac{1}{\beta^k + \beta^\ell} \cdot \|\hat{p}^k - \hat{p}^\ell\|_2^2. \end{aligned}$$

\blacksquare

Proof of Proposition 1 Let $\Phi(N)$ denote the equilibrium profit of an arbitrageur when the number of arbitrageurs is N . Then $\Phi(N) = \frac{1}{\beta^k + \beta^\ell} \|\hat{p}^k(N) - \hat{p}^\ell(N)\|_2^2$, for all $k\ell \in \mathcal{A}^*$. We claim that $\Phi(\infty) = 0$. If not, there exists a constant $\bar{c} > 0$ such that $\Phi(N) > \bar{c}$ for arbitrarily large N . Now $\lim_{N \rightarrow \infty} N^{k\ell} = \infty$, for some $k\ell \in \mathcal{A}$. Consider such a link $k\ell$. We have $\|\hat{p}^k(N) - \hat{p}^\ell(N)\|_2^2 > (\beta^k + \beta^\ell)\bar{c}$, for arbitrarily large $N, N^{k\ell}$. Therefore, total arbitrageur supply on k by arbitrageurs on $k\ell$, given by $N^{k\ell} y_{k\ell}^{k,n} = \frac{N^{k\ell}}{\beta^k + \beta^\ell} [\hat{p}^k(N) - \hat{p}^\ell(N)]$, is unbounded: for any constant \bar{C} , howsoever large, there is an N and a state s for which this supply is greater than \bar{C} in absolute value. Suppose that the supply is in fact positive (if it is negative, then we can consider instead the state- s supply by arbitrageurs on $k\ell$ to ℓ). Since $\hat{p}^k \geq 0$, (4) implies that y^k is bounded above. Due to (6), $N^{kk_1} y_{kk_1}^{k,n}$ must be unboundedly negative in state s , for some k_1 . But then the state- s supply on exchange k_1 by arbitrageurs active on kk_1 is unboundedly positive, which places k_1 in the same situation as k was. Eventually the unboundedly large supply in state s must end up on some exchange k_m . If $k_m \neq k$, the condition that $\hat{p}^{k_m} \geq 0$ will be violated due to (4). If $k_m = k$, consider the following sequence of inequalities that must hold: $\hat{p}_s^k < \hat{p}_s^{k_1} < \hat{p}_s^{k_2} < \dots < \hat{p}_s^{k_m} = \hat{p}_s^k$, a contradiction. Basically, arbitrageurs cannot be trading unboundedly large amounts without running afoul of the fact that at equilibrium agents are nonsatiated on every exchange.

Since $\Phi(N) \rightarrow 0$, and $\Phi(N) \geq \frac{1}{\beta^k + \beta^\ell} \|\hat{p}^k(N) - \hat{p}^\ell(N)\|_2^2$, for all $k\ell \in \mathcal{A}$, we must have $\hat{p}^k(N) - \hat{p}^\ell(N)$ converging to zero, for all $k\ell \in \mathcal{A}$. We claim that this is in fact true for all $k, \ell \in K$. For arbitrary k and ℓ , there is a path connecting them, since \mathcal{G} is connected, i.e. there is a sequence of distinct vertices $\{k_1, \dots, k_I\}$ in K such that $k_1 = k$, $k_I = \ell$ and $(k_i, k_{i+1}) \in \mathcal{A}$ for all $i = 1, \dots, I - 1$. Now

$\|\hat{p}^k(N) - \hat{p}^\ell(N)\|_2^2 \leq \sum_{i=1}^{I-1} \|\hat{p}^i(N) - \hat{p}^{i+1}(N)\|_2^2$. Since each of the terms in the sum converges to zero, $\hat{p}^k(N) - \hat{p}^\ell(N)$ converges to zero as well.

Multiplying equation (7) by λ^k and summing over $k \in K$ we get

$$\sum_{k \in K} \lambda^k \hat{p}^k(N) = p^\lambda. \quad (28)$$

Hence, for all $k \in K$,

$$\begin{aligned} \|\hat{p}^k(N) - p^\lambda\|_2^2 &= \left\| \sum_{j \in K} \lambda^j [\hat{p}^k(N) - \hat{p}^j(N)] \right\|_2^2 \\ &\leq \sum_{j \in K} \lambda^j \|\hat{p}^k(N) - \hat{p}^j(N)\|_2^2 \end{aligned}$$

where the inequality follows again from the triangle inequality followed by Jensen's inequality. Since each term in the last sum converges to zero, $\hat{p}^k(N)$ converges to p^λ , for all $k \in K$.

Now consider $C \in \mathcal{C}(\infty)$. Define \bar{N} large enough so that for all $N > \bar{N}$, $\mathcal{C}(N) = \mathcal{C}(\infty)$. Multiplying equation (7) by λ_C^k and summing over $k \in C$ we get, for all $N > \bar{N}$,

$$\sum_{k \in C} \lambda_C^k \hat{p}^k(N) = p_C^\lambda.$$

Taking limits as N goes to infinity, we get $p_C^\lambda = p^\lambda$. ■

Proof of Proposition 2 Consider first the complete architecture and suppose \hat{p}^j is internal. Then $\hat{p}^j = \sum_{k \in K^*} \nu^k \hat{p}^k$, where K^* is a subset of the vertex set of $\hat{\mathcal{P}}$, $|K^*| \geq 2$, $\nu^k > 0$ for all k , and $\sum_{k \in K^*} \nu^k = 1$. For any $\ell \in K$, we have

$$\begin{aligned} \|\hat{p}^j - \hat{p}^\ell\|_2^2 &= \left\| \sum_{k \in K^*} \nu^k (\hat{p}^k - \hat{p}^\ell) \right\|_2^2 \\ &\leq \left[\sum_{k \in K^*} \nu^k \|\hat{p}^k - \hat{p}^\ell\|_2 \right]^2 \end{aligned} \quad (29)$$

$$\begin{aligned} &< \sum_{k \in K^*} \nu^k \|\hat{p}^k - \hat{p}^\ell\|_2^2 \\ &\leq \max_{k \in K^*} \|\hat{p}^k - \hat{p}^\ell\|_2^2 \end{aligned} \quad (30)$$

where (29) follows from the triangle inequality and (30) from Jensen's inequality.

For a hub-spoke architecture the same argument goes through taking ℓ to be the hub. ■

Proof of Proposition 3 Since $\beta^k = \beta$, all k , (28) implies that

$$p^\lambda = \frac{1}{K+1} \sum_{k \in K} \hat{p}^k(N), \quad (31)$$

for all N . Moreover, since $\hat{\mathcal{P}}(\bar{N}) \in \mathbb{P}^{ss}$ with vertex set $\{\hat{p}^k(\bar{N})\}_{k \in K}$, p^λ is the center of $\hat{\mathcal{P}}(\bar{N})$.

We see from (7) that equilibrium prices solve the following system of equations:

$$\Lambda_k := \hat{p}^k(N) - p^k + \frac{1}{2} \sum_{\ell \in K} N^{k\ell} [\hat{p}^k(N) - \hat{p}^\ell(N)] = 0, \quad k \in K. \quad (32)$$

Using (12) and (13):

$$\begin{aligned} \Lambda_k &= \nu(N) \hat{p}^k(\bar{N}) + (1 - \nu(N)) p^\lambda - p^k + \frac{1}{2} \sum_{\ell \in K} \bar{N}^{k\ell} [\hat{p}^k(\bar{N}) - \hat{p}^\ell(\bar{N})] \\ &\quad + \frac{b(N)\nu(N)}{2} \sum_{\ell \in C_k} [\hat{p}^k(\bar{N}) - \hat{p}^\ell(\bar{N})], \end{aligned} \quad (33)$$

where $C_k := \{k\} \cup \{\ell \mid k\ell \in \mathcal{A}^*\}$ (note that the inclusion of k in C_k is just a matter of convenience, as the corresponding term of the sum is zero). Using (32) evaluated at $N = \bar{N}$, (33) simplifies to:

$$\Lambda_k = (1 - \nu(N)) [p^\lambda - \hat{p}^k(\bar{N})] + \frac{b(N)\nu(N)}{2} \sum_{\ell \in C_k} [\hat{p}^k(\bar{N}) - \hat{p}^\ell(\bar{N})] \quad (34)$$

If $\hat{\mathcal{P}}(\bar{N})$ is a regular simplex, we claim that all links are active, so that $|C_k| = K + 1$. Using (31), $\sum_{\ell \in C_k} [\hat{p}^k(\bar{N}) - \hat{p}^\ell(\bar{N})] = (K + 1) [\hat{p}^k(\bar{N}) - p^\lambda]$. Now it is easy to check that $\Lambda_k = 0$, for all k , for the following choice of $\nu(N)$ and $b(N)$ (by direct substitution into (34)):

$$\nu(N) = \frac{K + \bar{N}}{N + K}, \quad b(N) = \frac{2(N - \bar{N})}{(K + 1)(K + \bar{N})}. \quad (35)$$

It follows from (12) that $\|\hat{p}^k(N) - \hat{p}^\ell(N)\|_2^2 = [\nu(N)]^2 \|\hat{p}^k(\bar{N}) - \hat{p}^\ell(\bar{N})\|_2^2$, which is independent of $k\ell$ due to the regularity of the simplex $\hat{\mathcal{P}}(\bar{N})$. Therefore, profits are equalized across all links, with $\Phi(N) = [\nu(N)]^2 \Phi(\bar{N})$, and the network is complete, for all $N \geq \bar{N}$. Indeed, as N increases, all the edges of $\hat{\mathcal{P}}(N)$ contract uniformly, so that $\hat{\mathcal{P}}(N)$ is a smaller K -simplex within $\hat{\mathcal{P}}(\bar{N})$, with the same center p^λ .

Now suppose $\hat{\mathcal{P}}(\bar{N})$ is centrally symmetric with equal axes. For $N = \bar{N}$, let ℓ_k be the exchange whose equilibrium state-price deflator is centrally symmetric to that of k , i.e. $\hat{p}^{\ell_k}(\bar{N}) = -\hat{p}^k(\bar{N}) + 2p^\lambda$. We claim that, for $\ell \neq \ell_k$, $N^{k\ell} = 0$, for all $N \geq \bar{N}$. This implies that $\sum_{\ell \in C_k} [\hat{p}^k(\bar{N}) - \hat{p}^\ell(\bar{N})] = \hat{p}^k(\bar{N}) - \hat{p}^{\ell_k}(\bar{N}) = 2(\hat{p}^k(\bar{N}) - p^\lambda)$. We can verify that $\Lambda_k = 0$ for

$$\nu(N) = \frac{K + 1 + 2\bar{N}}{K + 1 + 2N}, \quad b(N) = \frac{2(N - \bar{N})}{K + 1 + 2\bar{N}}.$$

Also, from (12), $\|\hat{p}^k(N) - \hat{p}^{\ell_k}(N)\|_2^2 = [\nu(N)]^2 \|\hat{p}^k(\bar{N}) - \hat{p}^{\ell_k}(\bar{N})\|_2^2$, which is independent of k since the axes of $\hat{\mathcal{P}}(\bar{N})$ are equal. Therefore, profits are equalized

across all such links, which are $\frac{K+1}{2}$ in number. Clearly, given (12) and (13), $\hat{\mathcal{P}}(N)$ will contract uniformly as N increases beyond \bar{N} . So the same pairs of nodes will remain symmetric. It remains to show that links between non-symmetric nodes will be inactive. If $\ell \neq \ell_k$,

$$\|\hat{p}^k(N) - \hat{p}^{\ell_k}(N)\|_2 = \|\hat{p}^k(N) - p^\lambda\|_2 + \|\hat{p}^{\ell_k}(N) - p^\lambda\|_2 \quad (36)$$

$$= \|\hat{p}^k(N) - p^\lambda\|_2 + \|\hat{p}^\ell(N) - p^\lambda\|_2 \quad (37)$$

$$> \|(\hat{p}^k(N) - p^\lambda) + (p^\lambda - \hat{p}^\ell(N))\|_2 \quad (38)$$

$$= \|\hat{p}^k(N) - \hat{p}^\ell(N)\|_2,$$

where (36) follows from the symmetry of k and ℓ_k with respect to the center p^λ , (37) from the equality of the axes, and (38) from the triangle inequality, which is a strict inequality because $\hat{p}^k(N) - p^\lambda$ is not proportional to $\hat{p}^\ell(N) - p^\lambda$. Hence $N^{k\ell} = 0$ unless $\ell = \ell_k$.

Finally, suppose $\hat{\mathcal{P}}(\bar{N})$ is a polygon with r vertices, r odd. Clearly, the nodes that have maximal distance from k are the vertices of the segment that is opposite to k . Let us denote these by ℓ_k and m_k . Then we have

$$\begin{aligned} \sum_{\ell \in C_k} [\hat{p}^k(\bar{N}) - \hat{p}^\ell(\bar{N})] &= 2 \left[\hat{p}^k(\bar{N}) - \frac{1}{2} [\hat{p}^{\ell_k}(\bar{N}) + \hat{p}^{m_k}(\bar{N})] \right] \\ &= 2[\hat{p}^k(\bar{N}) - p^\lambda] \left[1 + \cos\left(\frac{\pi}{r}\right) \right] \end{aligned}$$

where the last equality follows from a simple trigonometric calculation (see Coxeter (1963), Fig. 1.1A). The values of $\nu(N)$ and $b(N)$ for which $\Lambda_k = 0$ are:

$$\nu(N) = \frac{K + 1 + \bar{N} \left[1 + \cos\left(\frac{\pi}{r}\right) \right]}{K + 1 + N \left[1 + \cos\left(\frac{\pi}{r}\right) \right]}, \quad b(N) = \frac{N - \bar{N}}{K + 1 + \bar{N} \left[1 + \cos\left(\frac{\pi}{r}\right) \right]}.$$

That the active links are as postulated is easily seen by the same argument as for the previous two cases.

In all the three cases, it can be verified that $\sum_{k\ell \in \mathcal{A}^*} N^{k\ell} = [\nu(N)]^{-1} \bar{N} + |\mathcal{A}^*| b(N) = N$. $|\mathcal{A}^*|$ is $\frac{K(K+1)}{2}$ for the simplex, $\frac{K+1}{2}$ for the centrally symmetric case, and $K + 1$ for the odd polygon. \blacksquare

Proof of Lemma 5 Consider a link $k\ell_k$ corresponding to an axis that is maximal in length (there can be many such axes). Clearly there is no activity on any of the shorter axes. We need to show that there is also no activity on any link $m\ell$ that does not correspond to an axis. A simple extension of the argument in the proof of Proposition 3 (equations (36)–(38)) gives us:

$$\begin{aligned} \|p^k - p^{\ell_k}\|_2 &= \|p^k - p^\lambda\|_2 + \|p^{\ell_k} - p^\lambda\|_2 \\ &\geq \|p^m - p^\lambda\|_2 + \|p^\ell - p^\lambda\|_2 \end{aligned} \quad (39)$$

$$> \|(p^m - p^\lambda) + (p^\lambda - p^\ell)\|_2 \quad (40)$$

$$= \|p^m - p^\ell\|_2.$$

(39) holds as an equality only if both m and ℓ correspond to one end of an axis of maximal length. The inequality (40) is strict because m and ℓ are not the end points of an axis. \blacksquare

Proof of Proposition 5 Suppose the nodes $\{p^k\}$ are symmetric with respect to L . The reflection of p^k through L is given by

$$p^{\ell_k} = -p^k + 2p^\lambda + 2 \left\langle p^k - p^\lambda, \frac{p^0 - p^\lambda}{\|p^0 - p^\lambda\|_2^2} \right\rangle (p^0 - p^\lambda), \quad k \in K. \quad (41)$$

Note that this formula holds for all $k \in K$ (e.g. $\ell_0 = 0$), and also for the case where $p^0 = p^\lambda$. For the moment we assume that $\gamma^k(N) = \gamma^{\ell_k}(N)$. We will show later that this is implied by profit equalization across links. Multiplying both sides of (41) by γ^k and summing over $k \in K$, and using (8), we see that $\hat{p}^0(N) - p^\lambda = \nu(N)(p^0 - p^\lambda)$, where

$$\nu(N) = \left\langle \hat{p}^0(N) - p^\lambda, \frac{p^0 - p^\lambda}{\|p^0 - p^\lambda\|_2^2} \right\rangle.$$

Hence $\hat{p}^0(N) = \nu(N)p^0 + (1 - \nu(N))p^\lambda$. Since $\hat{p}^0(N)$ is equal to p^0 at $N = 0$ and converges to p^λ as N goes to infinity, we have $\nu(0) = 1$ and $\nu(\infty) = 0$. From (9),

$$\hat{p}^k(N) - \hat{p}^0(N) = \frac{2}{2 + N^{0k}}(p^k - \hat{p}^0(N)), \quad k \neq 0. \quad (42)$$

By profit equalization, $\frac{2}{2 + N^{0k}}\|p^k - \hat{p}^0(N)\|_2$ does not depend on k , for $k \neq 0$, from which we can deduce that

$$2 + N^{0k} = (N + 2K) \frac{\|p^k - \hat{p}^0(N)\|_2}{\sum_{\ell \geq 1} \|p^\ell - \hat{p}^0(N)\|_2}, \quad k \neq 0. \quad (43)$$

In particular $N^{0k} = N^{0\ell_k}$, thus verifying that $\gamma^k = \gamma^{\ell_k}$.

If we provisionally assume that $\nu(N)$ is continuous in N , then so is $\hat{p}^0(N)$. This in turn implies that $\{N^{0k}\}$ is continuous in N , and hence so is $\{\gamma^k\}$. Therefore $\nu(N)$ is continuous in N , verifying our provisional assumption.

We now establish monotonicity of ν . We argue by contradiction. Suppose ν is not strictly monotone. Then as N increases, $\hat{p}^0(N)$ crosses the same point \bar{p} on the segment $\text{conv}(\{p^0, p^\lambda\})$ more than once, i.e. there is an N' and N'' , with $0 \leq N' < N'' < \infty$ such that $\hat{p}^0(N') = \hat{p}^0(N'') = \bar{p}$. From (42) and (43),

$$\hat{p}^k(N) - \bar{p} = \frac{2}{N + 2K} \frac{\sum_{\ell \geq 1} \|p^\ell - \bar{p}\|_2}{\|p^k - \bar{p}\|_2} (p^k - \bar{p}); \quad k \neq 1, N = N', N''.$$

Summing both sides over $k \geq 1$ we get (using (31)):

$$(K + 1)p^\lambda - \bar{p} - K\bar{p} = \frac{2}{N + 2K} \left(\sum_{\ell \geq 1} \|p^\ell - \bar{p}\|_2 \right) \sum_{k \geq 1} \frac{p^k - \bar{p}}{\|p^k - \bar{p}\|_2}, \quad N = N', N''. \quad (44)$$

Notice that the LHS does not depend on N . Therefore we have

$$\frac{1}{N' + 2K} \sum_{k \geq 1} \frac{p^k - \bar{p}}{\|p^k - \bar{p}\|_2} = \frac{1}{N'' + 2K} \sum_{k \geq 1} \frac{p^k - \bar{p}}{\|p^k - \bar{p}\|_2},$$

implying that $\sum_{k \geq 1} \frac{p^k - \bar{p}}{\|p^k - \bar{p}\|_2} = 0$. But then $\bar{p} = p^\lambda$ from (44). So if $\bar{p} \neq p^\lambda$, then $N = N'$, a contradiction. This establishes monotonicity.

Now we remove the assumption of symmetry with respect to p^0 . Define $K_1 := \{k \in K : \|p^0 - p^k\|_2 \geq \max_{\ell \in K} \|p^0 - p^\ell\|_2\} \cup \{0\}$ and $p_{K_1}^\lambda := \frac{1}{|K_1|} \sum_{k \in K_1} p^k$. For the set of nodes K_1 , ignoring all other nodes, the previous proof applies with the obvious changes. In particular, symmetry with respect to L implies symmetry with respect to $L_1 := \text{aff}(\{p^0, p_{K_1}^\lambda\})$ (indeed, $L_1 = L$). Therefore $\hat{p}^0(N)$ converges monotonically to $p_{K_1}^\lambda$ as $N \rightarrow \infty$. Now there will be a smallest $N_1 > 0$ such that $\|\hat{p}^k(N_1) - \hat{p}^0(N_1)\|_2 = \|p^\ell - \hat{p}^0(N_1)\|_2$ for $k \in K_1$ and $\ell \notin K_1$. Define K_2 recursively as the union of K_1 with all nodes $\ell \notin K_1$ for which the previous condition applies, and define $p_{K_2}^\lambda := \frac{1}{|K_2|} \sum_{k \in K_2} p^k$. A modification of the proof above again applies to the world with nodes in K_2 only. It can easily be seen that $p_{K_i}^\lambda$ is in the convex hull of p^0 and $p_{K_{i-1}}^\lambda$. This argument is repeated until the finite N_n for which $K_n = K$. ■

Proof of Proposition 6 We have $\lambda^0 = \frac{1}{1+K\beta^0/\beta}$ and $\lambda^k = \frac{1}{K+\beta/\beta^0}$. Using the fact that $p^0 = p^\lambda$, this implies that $p^\lambda = \frac{1}{K} \sum_{k \geq 1} p^k$. Therefore p^λ is the center of \mathcal{P} . Plugging $N^{0k} = \frac{N}{K}$ into the γ^j weights in Lemma 3 we get $\gamma^0 = \frac{K\beta^0 + \beta(K+N)}{K\beta^0(1+N) + \beta(K+N)}$ and $\gamma^k = \frac{\beta^0 N}{K\beta^0(1+N) + \beta(K+N)}$, $k \neq 0$, so that

$$\hat{p}^0(N) = \gamma^0 p^0 + \gamma^1 K p^\lambda \quad (45)$$

which is equal to p^λ since $p^0 = p^\lambda$. Using (15),

$$\Phi^{h_0}(N) = \frac{\beta + \beta^0}{[(1 + N/K)\beta + \beta^0]^2} \|p^k - \hat{p}^0(N)\|_2^2, \quad (46)$$

from which (19) follows since $\hat{p}^0(N) = p^0$. This expression does not depend on k because the p^k 's are equidistant from the center $p^0 = p^\lambda$. The formula for $\hat{p}^k(N)$ is easy to check from (9). ■

Proof of Proposition 7 When $N^{1k} = \frac{N}{K}$, (45) and (46) hold, with node 0 replaced by node 1. Hence (20) is verified, and

$$\Phi^{h_1}(N) = \frac{2}{\beta} \left[\frac{K}{N + 2K} \right]^2 \|p^k - \hat{p}^1(N)\|_2^2, \quad (47)$$

Denoting the coefficient of p^1 in (20) by $\nu(N)$, we have (suppressing the dependence of \hat{p}^1 and ν on N):

$$\begin{aligned} \|p^k - \hat{p}^1\|_2^2 &= \|p^k - p^1 + (1 - \nu)(p^1 - p^\lambda)\|_2^2 \\ &= \|p^k - p^1\|_2^2 + (1 - \nu)^2 \|p^1 - p^\lambda\|_2^2 + 2(1 - \nu) \langle p^k - p^1, p^1 - p^\lambda \rangle. \end{aligned} \quad (48)$$

For vectors x, y and z in \mathbb{R}^S , we have

$$\begin{aligned}\|x - z\|_2^2 &= \|x - y + y - z\|_2^2 \\ &= \|x - y\|_2^2 + \|y - z\|_2^2 + 2\langle x - y, y - z \rangle,\end{aligned}$$

so that

$$2\langle x - y, y - z \rangle = \|x - z\|_2^2 - \|x - y\|_2^2 - \|y - z\|_2^2. \quad (49)$$

Using this to evaluate the inner product in (48), we get

$$\|p^k - \hat{p}^1\|_2^2 = (1 - \nu)^2 \|p^1 - p^\lambda\|_2^2 + \nu \|p^k - p^1\|_2^2 \quad (50)$$

The (squared) ratio of the circumradius and the edge length of the simplex \mathcal{P} is given by a standard formula (see Coxeter (1963), p. 292–295):

$$\frac{\|p^k - p^\lambda\|_2^2}{\|p^k - p^\ell\|_2^2} = \frac{K}{2(K+1)}, \quad k, \ell \in K. \quad (51)$$

In particular, $\|p^1 - p^\lambda\|_2^2 = \frac{K}{2(K+1)} \|p^k - p^1\|_2^2$. Substituting this, as well as the value of ν , into (50), we get

$$\|p^k - \hat{p}^1(N)\|_2^2 = \frac{K[N(N+4)(K+1) + 8K]}{2[N(K+1) + 2K]^2} \|p^k - p^1\|_2^2 \quad (52)$$

$$< \|p^k - p^1\|_2^2, \quad \forall N > 0, \quad (53)$$

where the inequality (53) holds because $\|p^k - \hat{p}^1(N)\|_2^2$ is strictly decreasing in N for all $N \geq 0$ (this is easily verified from (52)). This also implies, from (47), that $\Phi^{h_1}(N)$ is strictly decreasing in N . The expression for profits (23) can be obtained by substituting (52) into (46). The formula for $\hat{p}^k(N)$, (21), follows from (9). Using (21) and (53), we have for $N > 0$ and $\ell \neq 1$:

$$\begin{aligned}\|\hat{p}^k(N) - \hat{p}^1(N)\|_2 &= \frac{2K}{N + 2K} \|p^k - \hat{p}^1(N)\|_2 \\ &< \frac{2K}{N + 2K} \|p^k - p^1\|_2 \\ &= \frac{2K}{N + 2K} \|p^k - p^\ell\|_2 \\ &= \|\hat{p}^k(N) - \hat{p}^\ell(N)\|_2.\end{aligned}$$

■

Proof of Proposition 8 Statement (i) can be seen as follows. Assume $\|p^k - p^0\|_2 > \|p^\ell - p^0\|_2$. From Proposition 4 we know that $N^{0k} > N^{0\ell}$ iff $\|p^k - \hat{p}^0(N)\|_2 > \|p^\ell - \hat{p}^0(N)\|_2$. Lemma 3 has shown that $\hat{p}^0(N)$ can be made arbitrarily close to p^0 by choosing N small enough, from which we can deduce that $\|p^k - p^0\|_2 > \|p^\ell - p^0\|_2$ implies for small N that $\|p^k - \hat{p}^0(N)\|_2 > \|p^\ell - \hat{p}^0(N)\|_2$ as well.

As to (ii), λ^0 large is equivalent to β^0 small relative to all other β^k , $k \neq 0$. From Lemma 3 it follows again that $\hat{p}^0(N)$ can be made arbitrarily close to p^0 by choosing β^0 small enough, and Proposition 4 (iii) establishes the result.

Now to (iii), so suppose $K = \{0, 1, 2\}$. Let $x = N^{01}$ and $y = N^{02}$, and define $b_k := \frac{\beta^0}{\beta^0 + \beta^k}$, for $k = 1, 2$. Using (15), we find that

$$\begin{aligned}\varphi^{01}(x, y) &= \frac{[1 + (1 - b_2)y][(1 + y)\mu_{01} - b_1y\mu_{02}] + y(1 + y)(b_1 + b_2 - 2b_1b_2)\mu_{12}}{[1 + N + xy(1 - b_1b_2)]^2}, \\ \varphi^{02}(x, y) &= \frac{[1 + (1 - b_1)x][(1 + x)\mu_{02} - b_2x\mu_{01}] + x(1 + x)(b_1 + b_2 - 2b_1b_2)\mu_{12}}{[1 + N + xy(1 - b_1b_2)]^2}.\end{aligned}$$

The equilibrium distribution of arbitrageurs (x, y) is given (ignoring integer constraints) by the solution to $\varphi^{01}(x, y) = \varphi^{02}(x, y)$, with $x + y = N$, a quadratic equation, provided a real solution exists and is an element of $[0, N]$:

$$x^2(1 - b_1b_2)(\mu_{02} - \mu_{01}) + (2x - N)H + (N + 1)(\mu_{02} - \mu_{01}) = 0, \quad (54)$$

where

$$H := (N + 1)(1 - b_2)\mu_{01} + [1 - b_1 - Nb_1(1 - b_2)]\mu_{02} + (N + 1)(b_1 + b_2 - 2b_1b_2)\mu_{12}.$$

It can be verified that $H > 0$ unless $\mu_{01} = \mu_{02} = \mu_{12} = 0$, in which case $H = 0$. The result follows: assume wlg that $\mu_{02} > \mu_{01}$, then (54) implies that $2x - N < 0$, i.e. $N^{01} < N/2 < N^{02}$. \blacksquare

Proof of Proposition 9 For the complete architecture, $p^0 = p^\lambda$ is the center of $\hat{\mathcal{P}}(N)$ for all N . In particular, it is internal for all N . Moreover $\beta^0 \geq \beta$. By Proposition 2, no trade occurs with exchange 0 and we can simply ignore it. From Proposition 3, equilibrium profits for the complete architecture are

$$\begin{aligned}\Phi(N) &= [\nu(N)]^2 \Phi(0) \\ &= \frac{1}{2\beta} [\nu(N)]^2 \|p^k - p^\ell\|_2^2, \quad k\ell \in \mathcal{A}^* \\ &\leq \frac{1}{2\beta} [\nu(N)]^2 [\|p^k - p^0\|_2 + \|p^\ell - p^0\|_2]^2, \quad k\ell \in \mathcal{A}^* \quad (55)\end{aligned}$$

$$= \frac{2}{\beta} [\nu(N)]^2 \|p^k - p^0\|_2^2 \quad (56)$$

where (55) follows from the triangle inequality ((55) holds as an equality if and only if the polytope \mathcal{P} is centrally symmetric), and (56) from the centrality of p^0 .

For the h_0 -architecture, profits do depend on β^0 . Consider first the case of $\beta^0 = \beta$. Comparing (56) with (19), and using the appropriate expression for $\nu(N)$ from the proof of Proposition 3 (recall that we are applying this proposition for K , not $K + 1$, nodes), we see that $\Phi^0(N) > \Phi(N)$ if and only if $N \geq K(K - 1) = 2|\mathcal{A}^*|$ for the case

of the simplex, $N > K = 2|\mathcal{A}^*|$ for the centrally symmetric case, and $N \geq \frac{K}{\cos(\frac{\pi}{r})}$ for the case of the odd polygon with r vertices. In the last case, the condition is most stringent for $r = 3$, i.e. $N \geq 2K = 2|\mathcal{A}^*|$.

Now suppose $\beta^0 > \beta$. It is immediate from (15) that $N^{0k} = \frac{N}{K}$, so that

$$\begin{aligned}\Phi^{h_0}(N) &= \frac{\beta + \beta^0}{\left[\left(1 + \frac{N}{K}\right)\beta + \beta^0\right]^2} \|p^k - p^0\|_2^2 \\ &= \frac{K^2(\beta + \beta^0)}{[K(\beta + \beta^0) + N\beta]^2} \|p^k - p^0\|_2^2.\end{aligned}\tag{57}$$

Comparing (56) and (57) for the case of the simplex, it suffices to show that, for sufficiently large N ,

$$\frac{K^2(\beta + \beta^0)}{[K(\beta + \beta^0) + N\beta]^2} > \frac{2}{\beta} \left[\frac{K-1}{N+K-1} \right]^2$$

or

$$\left[\frac{N+K-1}{K(\beta + \beta^0) + N\beta} \right]^2 > \frac{2}{\beta(\beta + \beta^0)} \left[\frac{K-1}{K} \right]^2.$$

This condition is satisfied for large N , since the limit of the LHS as N goes to infinity is $\frac{1}{\beta^2}$. Cases (ii) and (iii) of Proposition 3 can be dealt with in analogous fashion, and we leave the details to the reader. \blacksquare

Proof of Proposition 10 For the complete architecture, there is no trade on node 0 by Proposition 9. We claim the same is true for the h_1 -architecture. From (20),

$$\hat{p}^1(N) - p^\lambda = \frac{2(K-1)}{NK + 2(K-1)} (p^1 - p^\lambda).\tag{58}$$

Since $p^0 = p^\lambda$,

$$\|\hat{p}^1(N) - p^0\|_2^2 = \left[\frac{2(K-1)}{NK + 2(K-1)} \right]^2 \|p^1 - p^0\|_2^2.\tag{59}$$

Using (21), (52) and (51), in that order, we have for $k \notin \{0, 1\}$:

$$\begin{aligned}\|\hat{p}^1(N) - \hat{p}^k(N)\|_2^2 &= \left[\frac{2(K-1)}{N+2(K-1)} \right]^2 \|\hat{p}^1(N) - p^k\|_2^2 \\ &= \frac{2(K-1)^3 [NK(N+4) + 8(K-1)]}{[N+2(K-1)]^2 [NK+2(K-1)]^2} \|p^1 - p^k\|_2^2 \\ &= \frac{4K(K-1)^2 [NK(N+4) + 8(K-1)]}{[N+2(K-1)]^2 [NK+2(K-1)]^2} \|p^1 - p^0\|_2^2.\end{aligned}\tag{60}$$

It is easy to verify that (59) is strictly less than (60), i.e. node 0 will not see any trade in the h_1 -architecture. Therefore, for all computations concerning the complete and

h_1 -architectures, we can ignore node 0 and apply our results for these architectures replacing K by $K - 1$.

From (23) and (51), and the fact that $p^0 = p^\lambda$,

$$\Phi^{h_1}(N) = \frac{(K-1)^3[NK(N+4) + 8(K-1)]}{\beta[N+2(K-1)]^2[NK+2(K-1)]^2} \|p^k - p^1\|_2^2 \quad (61)$$

$$= \frac{2K(K-1)^2[NK(N+4) + 8(K-1)]}{\beta[N+2(K-1)]^2[NK+2(K-1)]^2} \|p^k - p^0\|_2^2. \quad (62)$$

Comparing (62) with (19), $\Phi^{h_0}(N) > \Phi^{h_1}(N)$ if and only if

$$K[N+2(K-1)]^2[NK+2(K-1)]^2 > (K-1)^2(N+2K)^2[NK(N+4) + 8(K-1)].$$

After some laborious (but straightforward) algebraic manipulations, this condition can be rewritten as:

$$\begin{aligned} & \frac{1}{9} \left[4K(K^2 - 9)(N^3 + 8K) + 3NK(K-3)(N^3 + 20N + 64K) + 144K(N+1) \right. \\ & \quad + 12N(7NK^2 + 6N + 44K^3) + K^2(7N^4 - 144K^3 + 256K^2) \\ & \quad \left. + N^2K^2(5N^2 - 36K) + 32NK^3(N^2 - 8K) + NK^2(3N^3 - 32K^2) \right] > 0, \end{aligned}$$

which is satisfied for $K \geq 3$ and $N \geq 2K - 1$ (all the bracketed terms are nonnegative).

For the complete architecture, using (14) and (35), with $K - 1$ instead of K :

$$\Phi(N) = \frac{1}{2\beta} \left[\frac{K-1}{N+K-1} \right]^2 \|p^k - p^1\|_2^2.$$

Comparing with (61), $\Phi^{h_1}(N) > \Phi(N)$ if and only if

$$\frac{2(K-1)[NK(N+4) + 8(K-1)]}{[N+2(K-1)]^2[NK+2(K-1)]^2} > \frac{1}{(N+K-1)^2}.$$

After some tedious algebra it can be shown that this condition is equivalent to (since $K \geq 3$)

$$N^3K - 2N(K-1)^2(K+3) - 8(K-1)^3 > 0.$$

It is easy to check that this condition is satisfied if $N \geq 2K - 1$. In fact, the weaker condition $N \geq 2(K-1)$ suffices for $K \geq 5$. \blacksquare

Proof of Proposition 11 Using (12) and (17):

$$d^k(N) = [\nu(N)]^2 \|p^k - p^\lambda\|_2^2, \quad k \in K, \quad (63)$$

$$d^{k,h_0}(N) = \left[\frac{2K}{N+2K} \right]^2 \|p^k - p^\lambda\|_2^2, \quad k \in K. \quad (64)$$

These equations hold trivially for $k = 0$ since $\hat{p}^0 = p^0 = p^\lambda$. In the complete architecture there is no activity on node 0 (Proposition 10), while in the h_0 -architecture the equilibrium price is equal to p^0 for all N due to offsetting trades with the other nodes (Proposition 6). It is easy to check that

$$\nu(N) < \frac{2K}{N + 2K} \quad (65)$$

in all the three cases studied in Proposition 3 (as in the proof of Proposition 9, we apply Proposition 3 for K nodes, not $K + 1$). The result follows. \blacksquare

Proof of Proposition 12 For the complete and h_1 -architectures, there is no activity on node 0 (Proposition 10). So we apply our results for these architectures for $K - 1$ instead of K . The result for exchange 0 follows as in Proposition 11.

Consider the h_1 -architecture. From (58),

$$d^{1,h_1}(N) = \left[\frac{2(K-1)}{NK + 2(K-1)} \right]^2 \|p^1 - p^\lambda\|_2^2. \quad (66)$$

For $k \notin \{0, 1\}$,

$$\begin{aligned} \hat{p}^k(N) - p^\lambda &= \frac{2(K-1)}{N + 2(K-1)}(p^k - p^\lambda) + \frac{N}{N + 2(K-1)}[\hat{p}^1(N) - p^\lambda] \\ &= \frac{2(K-1)}{N + 2(K-1)} \left[(p^k - p^\lambda) + \frac{N}{NK + 2(K-1)}(p^1 - p^\lambda) \right] \end{aligned}$$

from (21) and (58). Therefore,

$$\begin{aligned} d^{k,h_1}(N) &= \left[\frac{2(K-1)}{N + 2(K-1)} \right]^2 \left[\|p^k - p^\lambda\|_2^2 + \left(\frac{N}{NK + 2(K-1)} \right)^2 \|p^1 - p^\lambda\|_2^2 \right. \\ &\quad \left. + \left(\frac{2N}{NK + 2(K-1)} \right) \langle p^k - p^\lambda, p^1 - p^\lambda \rangle \right], \quad k \notin \{0, 1\}. \quad (67) \end{aligned}$$

Using (49) and (51), and the fact that all vertices of \mathcal{P} are equidistant from p^λ , we see that $\langle p^k - p^\lambda, p^\lambda - p^1 \rangle = \frac{1}{K-1} \|p^k - p^\lambda\|_2^2$. Substituting into (67), some algebraic manipulations give us:

$$d^{k,h_1}(N) = \frac{4(K-1)^2 \left([NK + 2(K-1)]^2 - N(N+4) \right)}{[N + 2(K-1)]^2 [NK + 2(K-1)]^2} \|p^k - p^\lambda\|_2^2, \quad k \notin \{0, 1\}. \quad (68)$$

From (63) and (35),

$$d^k(N) = \left[\frac{K-1}{N + K-1} \right]^2 \|p^k - p^\lambda\|_2^2, \quad k \in K. \quad (69)$$

The ranking for exchange 1 follows from (64), (66) and (69). Comparing (68) with (69), we see after some tedious algebra that $d^k(N) < d^{k,h_1}(N)$ for all $N > 0$, since $K \geq 3$. Finally, $d^{k,h_1}(N) < d^{k,h_0}(N)$, for $N > 0$, follows from (64) and (68).

Monotonicity with respect to N can be deduced from the relevant expressions (64), (66), (68) and (69). \blacksquare

Proof of Proposition 13 The welfare index of clientéle $k \in K$ is given by (24). Using (12) and (17):

$$W^k(N) = \frac{1}{\beta} [1 - \nu(N)]^2 \|p^k - p^\lambda\|_2^2, \quad k \in K, \quad (70)$$

$$W^{k,h_0}(N) = \frac{1}{\beta} \left[\frac{N}{N + 2K} \right]^2 \|p^k - p^\lambda\|_2^2, \quad k \in K. \quad (71)$$

Now the result follows from (65). \blacksquare

Proof of Proposition 14 The result for exchange 0 follows from the corresponding result in Proposition 12. From (20),

$$W^{1,h_1}(N) = \frac{1}{\beta} \left[\frac{NK}{NK + 2(K-1)} \right]^2 \|p^1 - p^\lambda\|_2^2.$$

For $k \notin \{0, 1\}$, we use (21), (52) and (51), in that order, to get

$$\begin{aligned} W^{k,h_1}(N) &= \frac{1}{\beta} \left[\frac{N}{N + 2(K-1)} \right]^2 \|p^k - \hat{p}^1(N)\|_2^2 \\ &= \frac{1}{\beta} \cdot \frac{N^2(K-1)[NK(N+4) + 8(K-1)]}{2[N + 2(K-1)]^2[NK + 2(K-1)]^2} \|p^k - p^1\|_2^2 \\ &= \frac{1}{\beta} \cdot \frac{N^2K[NK(N+4) + 8(K-1)]}{[N + 2(K-1)]^2[NK + 2(K-1)]^2} \|p^k - p^\lambda\|_2^2, \quad k \notin \{0, 1\}. \end{aligned} \quad (72)$$

From (70) and (35),

$$W^k(N) = \frac{1}{\beta} \left[\frac{N}{N + K - 1} \right]^2 \|p^k - p^\lambda\|_2^2, \quad k \in K. \quad (74)$$

We now compare (71), (73) and (74). It is easy to check $W^{1,h_1}(N) > W^1(N) > W^{1,h_0}(N)$. A tedious computation also verifies that $W^k(N) > W^{k,h_1}(N)$ for $k \notin \{0, 1\}$. Also for $k \notin \{0, 1\}$, $W^{k,h_1}(N) > W^{k,h_0}(N)$ provided

$$K[NK(N+4) + 8(K-1)] \geq [NK + 2(K-1)]^2,$$

a condition that is easily verified.

Monotonicity of welfare with respect to N is straightforward to check (for the case of $W^{k,h_1}(N)$ it follows from (72) and the observation made in the proof of Proposition 7 that $\|p^k - \hat{p}^1(N)\|_2^2$ is decreasing in N). \blacksquare

References

- Agnes, P. (2000). The “End of geography” in financial services? Local embeddedness and territorialization in the interest rate swaps industry. *Economic Geography*, pages 347–366.
- Allen, F. and Gale, D. (2000). Financial contagion. *Journal of Political Economy*, 108(1):1–33.
- Bala, V. and Goyal, S. (2000). A noncooperative model of network formation. *Econometrica*, 68(5):1181–1229.
- Berman, A. and Plemmons, R. (1979). *Nonnegative Matrices in the Mathematical Sciences*. Academic Press, New York.
- Blackburn, D., Goetzmann, W., and Ukhov, A. (2006). Risk aversion and clientèle effects. Mimeo.
- Chen, Z. and Knez, P. J. (1995). Measurement of market integration and arbitrage. *Review of Financial Studies*, 8(2):287–325.
- Cifuentes, R., Ferrucci, G., and Shin, H. S. (2005). Liquidity risk and contagion. *Journal of the European Economic Association*, 3(2-3):556–566.
- Coxeter, H. (1963). *Regular Polytopes*. Macmillan, 2nd edition.
- Culbertson, J. (1957). The term-structure of interest rates. *Quarterly Journal of Economics*, 71:485–517.
- Da, Z. and Gao, P. (2006). Clientele change, liquidity shock, and the return on financially distressed stocks. Mimeo, Kellogg School of Management, Northwestern University.
- Diestel, R. (2005). *Graph Theory, 3rd edition*. Springer.
- Drobny, S. (2000). *Inside the House of Money: Top Hedge Fund Traders on Profiting in the Global Markets*. Wiley.
- Duffie, D. and Jackson, M. (1989). Optimal innovation of futures contracts. *Review of Financial Studies*, 2(3):275–296.
- Gabaix, X., Krishnamurthy, A., and Vigneron, O. (2007). Limits of arbitrage: Theory and evidence from the mortgage-backed securities market. *Journal of Finance*, 62(2):557–595.
- Gabszewicz, J. J. and Vial, J.-P. (1972). Oligopoly ‘a la Cournot’ in a general equilibrium analysis. *Journal of Economic Theory*, 4(3):381–400.

- Galeotti, A., Goyal, S., Jackson, M. O., Vega-Redondo, F., and Yariv, L. (2006). Network games. Mimeo, University of Essex and Caltech.
- Goyal, S. and Joshi, S. (2005). Unequal connections. Mimeo, University of Essex.
- Grünbaum, B. (2003). *Convex Polytopes*. Springer-Verlag, 2nd edition.
- Horn, R. and Johnson, C. (1985). *Matrix Analysis*. Cambridge University Press.
- Lewis, K. (1999). Trying to explain home bias in equities and consumption. *Journal of Economic Literature*, 37(2):571–608.
- Mas-Colell, A. (1982). The Cournotian foundations of Walrasian equilibrium theory: an exposition of recent theory. In Hildenbrand, W., editor, *Advances in Economic Theory*. Cambridge University Press.
- Massa, M., Peyer, U., and Tong, Z. (2005). Limits of arbitrage and corporate financial policy. CEPR Discussion Paper, No. 4829.
- Modigliani, F. and Sutch, R. (1966). Innovations in interest rate policy. *American Economic Review*, 56(2):178–197.
- Rahi, R. and Zigrand, J.-P. (2007a). Arbitrage networks: The three-exchange case. Mimeo, London School of Economics.
- Rahi, R. and Zigrand, J.-P. (2007b). Financial innovation in arbitrage networks. Mimeo, London School of Economics.
- Rahi, R. and Zigrand, J.-P. (2007c). Strategic financial innovation in segmented markets. *Review of Financial Studies*. Forthcoming.
- Rahi, R. and Zigrand, J.-P. (2007d). A theory of strategic intermediation and endogenous liquidity. Mimeo, London School of Economics.