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Mathematical Structures of Simple Voting Games

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Abstract

We address simple voting games (SVGs) as mathematical objects in their own right, and study structures made up of these objects, rather than focusing on SVGs primarily as co-operative games. To this end it is convenient to employ the conceptual framework and language of category theory. This enables us to uncover the underlying unity of the basic operations involving SVGs.

Mathematical Structures of Simple Voting Games[1]

1 Introduction

Simple games, the simplest kind of co-operative game studied in game theory, have been adopted by voting theory – particularly the theory of voting power, where they are referred to as 'simple voting games' (SVGs) – as the simplest kind of rule for making decisions by vote.

In this paper we address SVGs as mathematical objects in their own right, a point of view that goes back to Shapley's 1962 paper [10]; but we focus primarily on structures made up of these objects. The role of simple games in game theory proper (concerned with bargaining, coalition formation etc.) is wholly out of the picture. The use of SVGs as decision rules in voting theory is kept in the background: it is useful as a heuristic, because the truth of various propositions about SVGs is easy to see when their interpretation as decision rules is borne in mind. Also, we conform to widely accepted terminology derived from game theory (such as 'game' and 'coalition') and voting-power theory (such as 'voter' and 'assembly').

Our main aim here is not to obtain new results about SVGs, but to place the theory of SVGs in the overall architecture of mathematics at large, within the general study of abstract structures, and to reveal its interconnections with other mathematical theories. Hence our use of the conceptual framework and language of category theory. This enables us to uncover the underlying structural unity of various operations involving SVGs:

- Composition of SVGs, including the special cases of forming the meet and join of SVGs.
- Adding dummy voters to an SVG.
- Transforming an SVG by forming voter blocs, whereby coalitions of voters amalgamate to form new single voters.
- Formation of Boolean subgames, including the special cases of forming subgames and reduced games.

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- Application of an SVG as decision rule to a division of the voters into "yes" and "no" voters.

Our analysis reveals that these operations are 'natural' in the technical sense of category theory. We also show that composition is the most general of these operations, in the sense that all the others listed above can be construed in a natural way as special cases of it. This is preparatory labour, that can serve as launchpad for gaining new insights and deriving new results – which at the moment is work in progress.

For background on SVGs, the reader is referred to Felsenthal and Machover [5]. See also Taylor and Zwicker [11]. For the (rather elementary) prerequisites from category theory the reader may consult any textbook on the subject, such as the classic Mac Lane [8] or more recent texts such as McLarty [9] or Awodey [1]. We shall also use some basic results from lattice theory, for which the reader is referred to Balbes and Dwinger [2]

Since SVGs are the only games we shall consider here, we will from now on drop this acronym and refer to them simply as 'games'. Also, throughout this paper we let V be an arbitrary finite set. We kick off with our first basic definition.

1.1. Definition A *game* on V is a pair (V, G), where G is a set of subsets of V satisfying the following *closure upwards* condition:

Whenever $X \subseteq Y \subseteq V$ and $X \in \mathsf{G}$, then also $Y \in \mathsf{G}$.

In this connection, we refer to V as the *assembly* and to its members as the *voters* of the game (V, G). A set of voters (subset of V) is referred to as a *coalition*. A coalition belonging to G is said to be a *winning* coalition of (V, G); otherwise it is said to be a *losing* coalition of this game. If X is a losing coalition, its complement, $V - X$, is said to be a *blocking* coalition of the game.

For the interpretation of a game as a decision rule, see [5, Rem. 2.1.2(iii)]. The present definition agrees with that of [11] but differs materially from the conventional one used in voting-power theory – see [5, Def. 2.1.1] – in that the latter imposes two further conditions: $V \in \mathsf{G}$ and $\emptyset \notin \mathsf{G}$.

In view of the upwards closure of G, these further conditions respectively exclude the trivial cases in which G is empty or the entire power set $\wp V$ of V. Jointly, these conditions also exclude the degenerate case in which V itself is empty. These are pretty useless for practical purposes as decision rules;[2] but

[2] The conventionally excluded cases $\mathsf{G} = \emptyset$ and $\mathsf{G} = \wp V$ respectively yield a stonewalling rule under which bills are invariably rejected, and a rubber-stamp decision rule under which bills invariably pass – no matter how the voters divide on them. $V = \emptyset$ means that there are no voters.

for our purposes they play a useful role, so we do not exclude them.

1.2. Definition We denote by L_V the structure consisting of the set of all games on V, with the binary operations *join* and *meet*, denoted respectively by $\vee$ and $\wedge$, defined as

$$(1) \qquad (V, \mathsf{G}) \vee (V, \mathsf{H}) := (V, \mathsf{G} \cup \mathsf{H}), \quad (V, \mathsf{G}) \wedge (V, \mathsf{H}) := (V, \mathsf{G} \cap \mathsf{H});$$

and binary relation $\leq$ defined by

$$(2) \qquad (V, \mathsf{G}) \leq (V, \mathsf{H}) :\Leftrightarrow \mathsf{G} \subseteq \mathsf{H}.$$

We put:

$$(3) \qquad \perp_V := (V, \emptyset), \quad \top_V := (V, \wp V).$$

The following result is obvious.

1.3. Theorem L_V *is a finite (hence complete) distributive lattice, with* $\leq$ *as the associated partial ordering and with* $\perp_V$ *and* $\top_V$ *respectively as bottom (least) and top (greatest) element.* □

1.4. Remark The operations of join and meet are extended in the obvious way, so that for any set S of games[3] on V, S has a join and meet, denoted respectively by $\bigvee \mathrm{S}$ and $\bigwedge \mathrm{S}$. We shall refer to the members of S as the *terms* of the join $\bigvee \mathrm{S}$ and the *factors* of the meet $\bigwedge \mathrm{S}$. In particular, for $\mathrm{S} = \emptyset$ we have

$$(4) \qquad \bigvee \emptyset = \perp_V, \quad \bigwedge \emptyset = \top_V.$$

1.5. Definition We put

$$(V, \mathsf{G})^* := (V, \{X \subseteq V : V - X \notin \mathsf{G}\}),$$

and refer to $(V, \mathsf{G})^*$ as the *dual* of (V, G).

From now on we shall often perpetrate a slight abuse of terminology and notation: we will conflate a game with its set of winning coalitions; and omitting the first component in '(V, G)', we shall write simply 'G'.[4]

The following facts, which are easy to verify, mean that L_V, with the added operation * of duality, is a *De Morgan algebra*.

[3]It makes no difference whether S is taken to be an ordered or plain set.

[4]Note that in any case, except for the trivial $\perp_V$, the assembly of a game is uniquely determined by its set of winning coalitions as the largest such coalition.

1.6. Theorem (i) *For any game* G *on* V*,* G^* *is also a game on* V*.*
(ii) *The winning coalitions of* G^* *are just the blocking coalitions of* G*.*
(iii) *Duality is an involution:* $\mathsf{G}^{**} = \mathsf{G}$*.*
(iv) *Duality obeys the De Morgan laws: for any games* G *and* H *on* V*,*

$$(\mathsf{G} \vee \mathsf{H})^* = \mathsf{G}^* \wedge \mathsf{H}^*, \quad (\mathsf{G} \wedge \mathsf{H})^* = \mathsf{G}^* \vee \mathsf{H}^*.$$

(v) *Duality is order-reversing:* $\mathsf{G} \leq \mathsf{H} \Rightarrow \mathsf{H}^* \leq \mathsf{G}^*$*.*
(vi) *In particular,* ${\perp_V}^* = \top_V$ *and* ${\top_V}^* = \perp_V$*.* □

While we will not pursue the connection between the L_V and logic in this paper, we should point out that each L_V may be regarded as a generalized set of truth values. Indeed, $\mathrm{L}_\emptyset$ is just the set of two truth values of classical logic, with duality serving as negation. For nonempty V, duality does not work as negation – for one thing, there are self-dual games – but in the interpretation of games as decision rules duality is in some sense related to negation: see [5, Rem. 2.3.3(iii)].

We end this section with an outline of the rest of the paper. In Section 2 we shall look at lattices of the form L_V as structures, and characterize them among all bounded lattices. The material in this section is mostly borrowed from the theory of voting power and lattice theory.

In Section 3 we turn our attention to the category **G**, whose objects are the lattices L_V for all V, and whose morphisms (aka 'arrows') are bounded lattice homomorphisms. We shall see, in particular, that familiar operations on games are naturally presented as the action of such morphisms on elements of their domain objects. The underlying idea of this section is that for our purposes it is most fruitful to consider a game not as an isolated object but as an element in the lattice of all games with a given assembly. Following the example of Grassmann and Peano, who revealed the true nature of vector algebra by taking as basic objects vector spaces rather than individual vectors,[5] we take as basic objects the L_V rather than individual games. This contrasts with the viewpoint of Blass [3]: in his theory of ultrafilters – which are infinite analogues of our dictatorial games – it was sufficient to take as a basic object of his category an individual ultrafilter rather than the lattice generated by all ultrafilters on a given infinite set.

In Section 4 we consider each L_V as a category of a simple kind (order category). We then single out for each natural number n a 'canonical' $\mathrm{L}_{\widehat{n}}$ whose assembly $\widehat{n}$ has cardinality n, and describe a recursive category-theoretic construction of the $\mathrm{L}_{\widehat{n}}$.

[5]See entry 'Vector Space' in *Wikipedia*.

Finally, in Section 5, moving away from the viewpoint (inherited from game theory) that focuses on winning, and looking at games in terms of losing coalitions, we shall be able to make a connection between the present theory and a branch of combinatorics related to topology.

2 Characterization of the L_V

We start by restating the definitions of various concepts from the theory of voting power. An asterisk in the label of a clause indicates duality.

2.1. Definition Let G be a game on V.
(i) A *minimal winning coalition* (MWC) of G is a winning coalition of G that does not include any other winning coalition of G.
(i*) A *minimal blocking coalition* (MBC) of G is a blocking coalition of G that does not include any other blocking coalition of G.
(ii) If v is a voter such that the singleton $\{v\}$ is a blocking coalition of G, then v is said to be a *vetoer*[6] in G.
(ii*) If v is a voter such that the singleton $\{v\}$ is a winning coalition of G, then v is said to be a *passer* in G.
(iii) A *dictator* in G is a voter that is both a vetoer and a passer in G. A game that has a dictator is said to be a *dictatorial* game.
(iv) A *dummy* in G is a voter v such that for every losing coalition X of G, $X \cup \{v\}$ is also a losing coalition of G.

The following facts are easily established.

2.2. Proposition *Let* G *be a game on* V*.*
(i) G *is uniquely determined by its set of MWCs. Moreover, if* M *is any set of coalitions (ie, subsets of* V*) then* M *is the set of MWCs of some game iff no member of* M *is included in another member of* M*.*
(ii) *Every MWC of* G *is an MBC of* G^**, and vice versa.*
(iii) *Every vetoer in* G *is a passer in* G^**, and vice versa.*
(iv) *A dictator in* G *is also a dictator in* G^**.*
(v) *Voter* v *is a dictator in* G *iff* $\{v\}$ *is the sole MWC of* G*. Hence there can be at most one dictator in* G*.*
(vi) *Voter* v *is a dummy in* G *iff* v *does not belong to any MWC of* G*.*
(vii) *A dummy in* G *is also a dummy in* G^**.* □

We proceed to define games of a special kind that serve as building blocks for all games on V.

[6] Sometimes also called a *blocker*.

2.3. Definition (i) For any $A \subseteq V$ we denote by $(V, \lfloor A \rfloor)$ the game on V that has A as its sole MWC:

$$(V, \lfloor A \rfloor) := (V, \{X : A \subseteq X \subseteq V\}).$$

We call $(V, \lfloor A \rfloor)$ the *principal* game on V *determined by* A.
(i*) We call the game on V that has A as its sole MBC – namely, the dual $(V, \lfloor A \rfloor)^*$ of $(V, \lfloor A \rfloor)$ – the *prime* game on V *determined by* A.

As before, where there is no risk of confusion we shall abuse notation and terminology and omit reference to V. Thus we shall write '$\lfloor A \rfloor$', '$\lfloor A \rfloor^*$' and $\lfloor \{v\} \rfloor$ instead of '$(V, \lfloor A \rfloor)$', '$(V, \lfloor A \rfloor)^*$' and '$(V, \lfloor \{v\} \rfloor)$'.

As we shall see in a moment, the terms 'principal' and 'prime' are justified by the algebraic properties of these games in the lattice L_V.

2.4. Proposition (i) *In the principal game $\lfloor A \rfloor$ all members of A are vetoers, and all other voters are dummies.*
(i*) *In the prime game $\lfloor A \rfloor^*$ all members of A are passers, and all other voters are dummies.*
(ii) *For any $v \in V$, $\lfloor \{v\} \rfloor$ is self-dual, hence it is both principal and prime. Conversely, a game that is both principal and prime must be of the form $\lfloor \{v\} \rfloor$ for some $v \in V$. Moreover, in $\lfloor \{v\} \rfloor$ v is dictator and all other voters are dummies.*
(iii) *Any game G on V can be presented as a join of a set of pairwise incomparable principal games:*

$$\mathsf{G} = \bigvee_{i=1}^{k} \lfloor A_i \rfloor, \text{ where } k \geq 0 \text{ and } i \neq j \Rightarrow A_j \not\subseteq A_i. \tag{1}$$

Moreover, this presentation is unique (up to the order of the A_i).
(iv) *If $\mathsf{G} = \bigvee_{i=1}^{k} \lfloor A_i \rfloor$ and $\mathsf{H} = \bigvee_{j=1}^{m} \lfloor B_j \rfloor$ are such presentations of games G and H, then $\mathsf{G} \leq \mathsf{H}$ iff for each i $(1 \leq i \leq k)$ there is a j $(1 \leq j \leq m)$ such that $B_j \subseteq A_i$.*
(v) *A game G on V is principal iff $\mathsf{G} \neq \bot_V$ and G is not the join of strictly smaller games: $\mathsf{G} = \mathsf{H} \vee \mathsf{K} \Rightarrow \mathsf{G} = \mathsf{H}$ or $\mathsf{G} = \mathsf{K}$.*
(v*) *A game G on V is prime iff $\mathsf{G} \neq \top_V$ and G is not the meet of strictly larger games: $\mathsf{G} = \mathsf{H} \wedge \mathsf{K} \Rightarrow \mathsf{G} = \mathsf{H}$ or $\mathsf{G} = \mathsf{K}$.*

Proof (i) follows at once from Def. 2.1(ii)&(iv) and Def. 2.3(i).

(i*) follows from (i) by duality.

(ii): The self-duality of $\lfloor\{v\}\rfloor$ follows from the fact that its winning coalitions and blocking coalitions are the same, namely those subsets of V that contain v. Hence $\lfloor\{v\}\rfloor$, which is principal, equals $\lfloor\{v\}\rfloor^*$, which is prime.

Conversely, if the principal game $\lfloor A\rfloor$ is not of the form $\lfloor\{v\}\rfloor$, then either $A = \emptyset$ or A has at least two members. But $\lfloor\emptyset\rfloor = \top_V$ and is not prime because its dual, $\bot_V = \emptyset$, is not principal. On the other hand, if A has two distinct members, say v_1 and v_2, then by (i) both of them must be vetoers in $\lfloor A\rfloor$. Hence both $\{v_1\}$ and $\{v_2\}$ are MBCs of $\lfloor A\rfloor$, so it cannot be prime. That v is dictator in $\lfloor\{v\}\rfloor$ follows from Prop. 2.2(v).

(iii) follows at once from Def. 2.1(i), Prop. 2.2(i) and Def. 2.3(i). The A_i in (1) are just the MWCs of G.

Note, in particular, that in the presentation (1) if $\mathsf{G} = \bot_V$, then $k = 0$; and if $\mathsf{G} = \top_V$ then $k = 1$ and $A_1 = \emptyset$.

(iv): Observe that by Def. 1.2(2), $\mathsf{G} \leq \mathsf{H}$ means that every winning coalition of G is also a winning coalition of H. Clearly, this holds iff every MWC of G includes an MWC of H.

(v) Suppose G is a principal game on V. Thus $\mathsf{G} = \lfloor A\rfloor$ for some $A \subseteq V$. Clearly, $\lfloor A\rfloor \neq \bot_V$. If $\lfloor A\rfloor = \mathsf{H} \vee \mathsf{K}$, then $\lfloor A\rfloor \geq \mathsf{H}$ and $\lfloor A\rfloor \geq \mathsf{K}$. We must show that the sharp inequalities $\lfloor A\rfloor > \mathsf{H}$ and $\lfloor A\rfloor > \mathsf{K}$ cannot both hold.

If $\lfloor A\rfloor > \mathsf{H}$, then there must exist some X such that $A \subseteq X \subseteq V$ but X is a losing coalition of H. Similarly, if $\lfloor A\rfloor > \mathsf{K}$, then there must exist some Y such that $A \subseteq Y \subseteq V$ but Y is a losing coalition of K. Hence $A \subseteq X \cap Y$ but $X \cap Y$ is a losing coalition of $\mathsf{H} \vee \mathsf{K}$ – contrary to our assumption that $\lfloor A\rfloor = \mathsf{H} \vee \mathsf{K}$.

Conversely, suppose that the game G is not principal. If $\mathsf{G} = \bot_V$, we have nothing to prove. If $\mathsf{G} \neq \bot_V$, then G must have at least two MWCs. Thus, in the presentation (1) $k \geq 2$. Hence $\mathsf{G} = \lfloor A_1\rfloor \vee \bigvee_{i=2}^{k} \lfloor A_i\rfloor$, which shows that G is the join of two strictly smaller games.

(v*) follows from (v) by duality. □

While Prop. 2.4(iii) shows that the principal games can serve as building blocks for all games on V, we shall now show that the dictatorial games are the *ultimate* components of all these games.

First observe that any principal game $\lfloor A\rfloor$ can be presented as a meet of dictatorial games:

$$\lfloor A\rfloor = \bigwedge_{x\in A} \lfloor\{x\}\rfloor.$$

Moreover, this presentation is unique (up to the order of the dictatorial games). And if $\lfloor A \rfloor = \bigwedge \mathrm{R}$ and $\lfloor B \rfloor = \bigwedge \mathrm{S}$, where R and S are sets of dictatorial games, then

$$B \subseteq A \Leftrightarrow \lfloor A \rfloor \leq \lfloor B \rfloor \Leftrightarrow \mathrm{S} \subseteq \mathrm{R}.$$

In view of this observation, Prop. 2.4(iii)&(iv) yields the following theorem.

2.5. Theorem (i) *Any game* G *on* V *can be presented as*

$$\mathsf{G} = \bigvee_{i=1}^{k} \bigwedge \mathrm{R}_i, \text{ where } k \geq 0 \text{ and each } \mathrm{R}_i \text{ is a set of dictatorial games such that } i \neq j \Rightarrow \mathrm{R}_j \not\subseteq \mathrm{R}_i. \tag{2}$$

Moreover, this presentation is unique (up to the order of the R_i *and the order of the dictatorial games in each* R_i*).*
(ii) *if* $\mathsf{G} = \bigvee_{i=1}^{k} \bigwedge \mathrm{R}_i$ *and* $\mathsf{H} = \bigvee_{j=1}^{m} \bigwedge \mathrm{S}_j$ *are such presentations of games* G *and* H*, then* $\mathsf{G} \leq \mathsf{H}$ *iff for each* i $(1 \leq i \leq k)$ *there is a* j $(1 \leq j \leq m)$ *such that* $\mathrm{S}_j \subseteq \mathrm{R}_i$*.* □

We shall refer to the right-hand side of (2) as the *join normal form* (JNF) of G, in analogy with the disjunctive normal form of propositional logic.

Obviously, the dual of Thm. 2.5 yields a *meet normal form* (MNF) for each game on V.

2.6. Remark In view of Proposition 2.2(vi), voter v is a dummy in G iff $\lfloor \{v\} \rfloor$ does not occur in the JNF of G.

Thm. 2.5 means that the dictatorial games in L_V constitute a set of *independent generators*, or a *basis*, of L_V.

The following corollary of Thm. 2.5 uses the existence (but not the uniqueness) of a JNF.

2.7. Corollary (Proof by structural induction) *To prove that all games on* V *possess a property* $\mathfrak{P}$*, it is sufficient to show that* $\bot_V$*,* $\top_V$ *and all dictatorial games on* V *possess* $\mathfrak{P}$*; and that whenever games* G *and* H *on* V *both possess* $\mathfrak{P}$*, then so do* $\mathsf{G} \vee \mathsf{H}$ *and* $\mathsf{G} \wedge \mathsf{H}$*.* □

Thm. 2.5 provides the following structural characterization of the L_V among all bounded lattices.

2.8. Theorem *Let* L *be a bounded lattice. Suppose there are* n *elements in* L *– call them 'atoms' – such that any element* g *of* L *has a unique JNF* *presentation in terms of the atoms; i.e., it can be presented in the form*

$$g = \bigvee_{i=1}^{k} \bigwedge \mathrm{R}_i, \ \textit{where } k \geq 0 \textit{ and each } \mathrm{R}_i \textit{ is a set of atoms} \tag{3}$$
$$\textit{such that } i \neq j \Rightarrow \mathrm{R}_j \not\subseteq \mathrm{R}_i;$$

and this presentation is unique (up to the order of the R_i *and the order of the atoms in each* R_i*).*

Then L *is isomorphic (in the category of all bounded lattices) to* L_V *with* $|V| = n$.

Proof First, let $g = \bigvee_{i=1}^{k} \bigwedge \mathrm{R}_i$ and $h = \bigvee_{j=1}^{m} \bigwedge \mathrm{S}_j$ be JNFs of elements g and h of L.

We claim: $g \leq h$ *iff for each* i $(1 \leq i \leq k)$ *there is a* j $(1 \leq j \leq m)$ *such that* $\mathrm{S}_j \subseteq \mathrm{R}_i$.

The sufficiency of the condition is obvious. To prove its necessity, suppose $g \leq h$. Then

$$h = g \vee h = \bigvee_{i=1}^{k} \bigwedge \mathrm{R}_i \vee \bigvee_{j=1}^{m} \bigwedge \mathrm{S}_j.$$

We can re-write this as

$$h = g \vee h = \bigvee_{i=1}^{k+m} \bigwedge \mathrm{T}_i, \tag{4}$$

where $\mathrm{T}_i = \mathrm{R}_i$ for $1 \leq i \leq k$ and $\mathrm{T}_j = \mathrm{S}_j$ for $k+1 \leq j \leq k+m$.

We can reduce (4) back to JNF as follows. If for some $i \neq j$ we have $\mathrm{T}_j \subseteq \mathrm{T}_i$, then the ith term in (4) is redundant and we delete it. Proceeding in this way, we end up with what must be the JNF $h = \bigvee_{j=1}^{m} \bigwedge \mathrm{S}_j$.

Thus it is precisely the first k terms in (4) that have been deleted. But this means that for each i $(1 \leq i \leq k)$ there was a j $(1 \leq j \leq m)$ such that $\mathrm{S}_j \subseteq \mathrm{R}_i$ – as claimed.

Now let $|V| = n$, and map the set of n atoms of L bijectively onto the set of dictatorial games on V. Extend the map to the whole of L in the obvious way via the presentations (3) and (2). This extended map is clearly a bijection of L onto L_V; and it respects the bottom and top elements of these lattices. Also, by the claim we have just proved and Thm. 2.5(ii), it is an order isomorphism, and hence also a lattice isomorphism. □

3 The category G

In this section we shall study the category **G** whose objects are the lattices L_V for all finite sets V, and whose morphisms, or arrows, are bounded lattice homomorphisms: maps between these objects that respect bottom and top elements and the operations of join and meet. More formally:

3.1. Definition A *morphism*, or *arrow*, of the category **G** is a map

$$f : \mathrm{L}_V \to \mathrm{L}_W,$$

where V and W are any finite sets, such that

$$f : \bot_V \mapsto \bot_W, \quad f : \top_V \mapsto \top_W,$$

and for all G and H in L_V

$$f(\mathsf{G} \vee \mathsf{H}) = f\mathsf{G} \vee f\mathsf{H}, \quad f(\mathsf{G} \wedge \mathsf{H}) = f\mathsf{G} \wedge f\mathsf{H}.$$

In this connection we say that L_V and L_W are respectively the *domain* of f, briefly $\operatorname{dom} f$, and its *codomain*, briefly $\operatorname{cod} f$.

These maps necessarily respect the ordering (which is uniquely determined by the lattice operations); in other words, they are monotone. On the other hand, we do not require them to respect duality; but of course those that do are of special interest.

3.2. Definition Let $f : \mathrm{L}_V \to \mathrm{L}_W$ be a morphism of **G**. The *dual* of f is the mapping f^* from L_V to L_W defined by

$$f^*\mathsf{G} := (f(\mathsf{G}^*))^*,$$

for all $\mathsf{G} \in \mathrm{L}_V$.

It is easy to verify that f^* is in fact a morphism of **G**. Clearly, the definition of f^* is equivalent to the identity

$$f^*\mathsf{G}^* = (f\mathsf{G})^*,$$

for all $\mathsf{G} \in \mathrm{L}_V$. Thus f respects duality iff it is self-dual. Moreover, we now have:

3.3. Theorem *A morphism $f : \mathrm{L}_V \to \mathrm{L}_W$ of* **G** *is self-dual (i.e., respects duality) iff for every $v \in V$, $f\lfloor\{v\}\rfloor$ is self-dual in L_W.*

Proof The condition is clearly necessary, because the $\lfloor\{v\}\rfloor$ are self-dual (Prop. 2.4(ii)).

For the converse, suppose all the $f\lfloor\{v\}\rfloor$ are self-dual. It follows easily by structural induction (Cor. 2.7) that $f(\mathsf{G}^*) = (f\mathsf{G})^*$ for all $\mathsf{G} \in \mathrm{L}_V$. $\square$

Several results obtained in the present section have the following form: an operation that is commonly applied to games in voting theory (where the games are used as decision rules) can be conceptualized as the 'natural' action of some morphism f of $\mathbf{G}$ on elements of dom f. This is an important insight: operations that voting theory applies to individual games are, so to speak, the uniform retail effects of a wholesale morphism, a template, acting on an L_V.

The following result is of crucial importance for the rest of this section. For one thing, it provides a useful starting point for constructing morphisms in $\mathbf{G}$. It also helps to reveal the fundamental nature of composition as an operation on games. Finally, readers with more than superficial knowledge of category theory will no doubt recognize it as encapsulating an adjunction relation, which will indeed be made explicit below in Thm. 3.12.

3.4. Main Lemma *Let φ be an arbitrary map from V into L_W. Then there exists a unique morphism $f : \mathrm{L}_V \to \mathrm{L}_W$ of $\mathbf{G}$ such that $f\lfloor\{v\}\rfloor = \varphi v$ for all $v \in V$. This f is given, for every game G on V, by*

$$f\mathsf{G} := \{Y \subseteq W : \{x \in V : Y \in \varphi x\} \in \mathsf{G}\}\,. \tag{1}$$

Proof We can re-write (1) as:

$$\forall Y \subseteq W : Y \in f\mathsf{G} \Leftrightarrow \{x \in V : Y \in \varphi x\} \in \mathsf{G}. \tag{2}$$

Using (1) or (2) it is easy to verify that $f\mathsf{G}$ satisfies the upward closure condition of Def. 1.1 and is therefore a game on W. And using (2) it is also easy to check that f respects the bottom and top elements and the operations of meet and join. Thus it is a morphism of $\mathbf{G}$.

Now let $v \in V$. Putting $\lfloor\{v\}\rfloor$ for G in (2) we have, for all $Y \subseteq W$,

$$\begin{aligned} Y \in f\lfloor\{v\}\rfloor &\Leftrightarrow \{x \in V : Y \in \varphi x\} \in \lfloor\{v\}\rfloor \\ &\Leftrightarrow v \in \{x \in V : Y \in \varphi x\} \\ &\Leftrightarrow Y \in \varphi v. \end{aligned}$$

Thus $f\lfloor\{v\}\rfloor = \varphi v$.

To prove the uniqueness of f, suppose that $f' : \mathrm{L}_V \to \mathrm{L}_W$ and that f' agrees with f on $\{\lfloor\{v\}\rfloor : v \in V\}$. Then by routine structural induction on G (Cor. 2.7) it follows that f' coincides with f on the whole of L_V. $\square$

3.5. Remark The statement of our Main Lemma – except for the explicit form of f given in equation (1) – remains true if L_W is replaced by *any* bounded distributive lattice, and f is required to be a homomorphism of such lattices. The proof, which is quite easy if somewhat messy, is left to the interested reader. This result means that the L_V are precisely (up to isomorphism) *the* free finitely generated bounded distributive lattices, with the dictatorial games of L_V as free generators.

Let us now go back to Def. 3.1 and relate it to an operation on games used in the theory of voting power.

Observe that by Lemma 3.4 the values $f\lfloor\{v\}\rfloor$ can be chosen arbitrarily as any games on W, and the morphism f uniquely determined by this choice then satisfies

$$f\mathsf{G} := \{Y \subseteq W : \{x \in V : Y \in f\lfloor\{x\}\rfloor\} \in \mathsf{G}\}\,, \tag{3}$$

for any game G on V. Thus a morphism $f : \mathrm{L}_V \to \mathrm{L}_W$ yields some kind of operation involving an arbitrary game G on V and n games $f\lfloor\{v\}\rfloor$ on W, where $n = |V|$.

To work out what operation this is, it will be convenient to introduce the following definition (which will also be much needed in Section 4).

3.6. Definition For any natural number n, we put $\widehat{n} := \{1, 2, \ldots, n\}$, and refer to $\widehat{n}$ as the *canonical assembly of size* n. In particular, $\widehat{0} = \emptyset$.

There is no real loss of generality if we let $\mathrm{L}_{\widehat{n}}$ here stand in for any L_V with $|V| = n$, which is of course isomorphic to $\mathrm{L}_{\widehat{n}}$. But one advantage of using $\mathrm{L}_{\widehat{n}}$ is that its canonical assembly comes with a ready-made ordering. So let us rewrite (3) for the case where $V = \widehat{n}$ and $f : \mathrm{L}_{\widehat{n}} \to \mathrm{L}_W$, and let us put $\mathsf{H}_i := f\lfloor\{i\}\rfloor$ for all $i \in \widehat{n}$. We obtain

$$\forall Y \subseteq W : Y \in f\mathsf{G} \Leftrightarrow \{i \in \widehat{n} : Y \in \mathsf{H}_i\} \in \mathsf{G}. \tag{4}$$

Referring to Felsenthal and Machover [5, Def. 2.3.12], we see that (4) means that $f\mathsf{G}$ is the *composite of* $\mathsf{H}_1, \mathsf{H}_2, \ldots, \mathsf{H}_n$ (in this order) *under* G; or, using the notation of [5, Def. 2.3.12]:

$$f\mathsf{G} = \mathsf{G}[\mathsf{H}_1, \mathsf{H}_2, \ldots, \mathsf{H}_n]. \tag{5}$$

For the use of $\mathsf{G}[\mathsf{H}_1, \mathsf{H}_2, \ldots, \mathsf{H}_n]$ to model a 'federal' or two-tier voting system, see [5, Rem. 2.3.13(ii)]. (The definition of composition and its use in modelling voting systems go back to Shapley [10], but he makes the unnecessarily restrictive assumption that the assemblies of H_i are pairwise disjoint.)

The definition of $\mathsf{G}[\mathsf{H}_1, \mathsf{H}_2, \ldots, \mathsf{H}_n]$ in [5] is apparently more general, in that it allows the H_i to have different assemblies, and the assembly of the composite $\mathsf{G}[\mathsf{H}_1, \mathsf{H}_2, \ldots, \mathsf{H}_n]$ is then the union of these n assemblies; whereas here all the H_i are games on the same assembly, W. But that apparent greater generality is not essential: as we shall see later (Rem. 3.11 below), any L_W can be embedded (in a sense to be made precise) in $\mathrm{L}_{W'}$, where W' is any finite superset of W. Hence L_{W_i}, where $i \in \widehat{n}$, can all be embedded in L_W, where $W = \bigcup_{i=1}^n W_i$.

So, ignoring the minor and superficial differences between (2) and the definition of game composition in [5] – namely, that the V in (2) is not necessarily canonical; and that the definition in [5] allows the H_i to have different assemblies – it transpires that any morphism $f : \mathrm{L}_V \to \mathrm{L}_W$ acts on any game G on V by forming the composite of the arbitrarily chosen games $\{f\lfloor\{x\}\rfloor : x \in V\}$ under G.

Thus composition, far from being a rather specialized operation on games, is a very general one. Indeed, we shall see that particular kinds of morphisms of $\mathbf{G}$ yield various operations on games that are familiar from the theory of voting power. Since composition is the general form of an operation on games yielded by a morphism of $\mathbf{G}$, these operations turn out to be special cases of composition. This applies not only to fairly obvious operations such as taking the meet or join of several games – which are noted as special cases of composition in [5, Def. 2.3.12] – but also to rather less obvious cases.

An important class of morphisms of $\mathbf{G}$, which we now proceed to define, are induced in a natural way by mappings between assemblies.

3.7. Definition Let $\varphi : V \to W$ be an arbitrary map from V to the finite set W. The morphism $\mathrm{L}\varphi : \mathrm{L}_V \to \mathrm{L}_W$ *induced* by φ is defined by putting, for all $v \in V$,

$$(\mathrm{L}\varphi)\lfloor\{v\}\rfloor := \lfloor\{\varphi v\}\rfloor. \tag{6}$$

Note that by Lemma 3.4 this defines $\mathrm{L}\varphi$ uniquely.

3.8. Theorem *Let φ and $\mathrm{L}\varphi$ be as in Def. 3.7. Then*
(i) *For all $\mathsf{G} \in \mathrm{L}_V$, $\mathrm{L}\varphi\mathsf{G} = \{Y \subseteq W : \varphi^{-1}[Y] \in \mathsf{G}\}$.*
(ii) *$\mathrm{L}\varphi$ respects duality.*
(iii) *If $w \in W - \varphi[V]$ (ie, $w \in W$ but not in the range of φ), then for any $\mathsf{G} \in \mathrm{L}_V$ w is a dummy in $\mathrm{L}\varphi\mathsf{G}$.*

Proof (i) Substituting $\mathrm{L}\varphi$ for f in (1) and using (6) we have

$$\begin{aligned}\mathrm{L}\varphi\mathsf{G} &= \{Y \subseteq W : \{v \in V : Y \in \lfloor\{\varphi v\}\rfloor\} \in \mathsf{G}\} \\ &= \{Y \subseteq W : \{v \in V : \varphi v \in Y\} \in \mathsf{G}\} \\ &= \{Y \subseteq W : \varphi^{-1}[Y] \in \mathsf{G}\}.\end{aligned}$$

(ii) follows from Thm. 3.3 and Prop. 2.4(ii).

To prove (iii), write G in JNF:

$$\mathsf{G} = \bigvee_{i=1}^{k} \bigwedge_{x \in A_i} \lfloor\{x\}\rfloor, \text{ where } i \neq j \Rightarrow A_j \nsubseteq A_i.$$

Hence by (6)

$$\mathrm{L}\varphi\mathsf{G} = \bigvee_{i=1}^{k} \bigwedge_{x \in A_i} \lfloor\{\varphi x\}\rfloor.$$

This may not be the JNF of $\mathrm{L}\varphi\mathsf{G}$, because φ need not be injective. However, the JNF of $\mathrm{L}\varphi\mathsf{G}$ can be obtained from it by eliminating duplication and redundancy. It follows that the JNF of $\mathrm{L}\varphi\mathsf{G}$ contains only dictatorial games of the form $\lfloor\{\varphi x\}\rfloor$. Therefore by Rem. 2.6, if w is not in the range of φ it must be a dummy in $\mathrm{L}\varphi\mathsf{G}$. □

It is easy to see that if

$$U \xrightarrow{\psi} V \xrightarrow{\varphi} W$$

then $\mathrm{L}(\varphi\psi) = \mathrm{L}\varphi\mathrm{L}\psi$. Thus Def. 3.7 yields a functor L from the category **FinSet** of finite sets to **G**. This is stated more precisely and fully in the following theorem.

3.9. Theorem *For any V, let $\mathrm{L}V := \mathrm{L}_V$ and for any mapping φ between finite sets let $\mathrm{L}\varphi$ be as defined in Def. 3.7. Then $\mathrm{L} :$ **FinSet** $\to$ **G** is a faithful functor from **FinSet** to **G**.* □

The case in which $\varphi : V \to W$ is injective is of special importance. The following facts are easily established.

3.10. Theorem *Let W be a finite set, and let $\varphi : V \to W$ be an injective map. Then $\mathrm{L}\varphi : \mathrm{L}_V \to \mathrm{L}_W$ is a subobject of L_W in* **G**.

Also, the image $\mathrm{L}\varphi[\mathrm{L}_V]$ of L_V is a sublattice of L_W, isomorphic to L_V. □

3.11. Remark If $V \subseteq W$ and $\varphi : V \hookrightarrow W$ is the insertion map, then $\mathrm{L}\varphi[\mathrm{L}_V]$ is essentially a replica of L_V. If G is any game on V, then G and its image $\mathrm{L}\varphi\mathsf{G}$ have *formally* exactly the same JNF. Of course, this expression and the factors occurring in it do not denote exactly the same games on W as they do on V: if $v \in V$, the members (ie winning coalitions) of $\lfloor\{v\}\rfloor$ as a game on W may include extra elements, belonging to $W - V$. But by Thm. 3.8(iii) all these extra elements are dummies in $\mathrm{L}\varphi\mathsf{G}$; and for all practical purposes –

as well as in applications in voting-power theory – addition of dummies does not essentially alter a game, since they play no active role in it and are, as it were, mere spectators. So we may regard $\mathrm{L}\varphi\mathsf{G}$ as essentially a replica of G.

Thus if we wish to operate on games H_i, where $i \in \widehat{n}$, with different respective assemblies W_i, we can always embed these assemblies in their union $W = \bigcup_{i=1}^{n} W_i$, and operate on replicas of the H_i, which have W as their common assembly.

The following theorem assumes familiarity with the notion of adjunction, which plays an important role in category theory. Readers with no particular interest in this topic may skip this theorem and its corollary.

3.12. Theorem L *has a right adjoint, namely the forgetful functor*

$$F : \mathbf{G} \to \mathbf{FinSet}.$$

Proof Let V and L be variables ranging over the objects of **FinSet** and **G** respectively. It is enough to exhibit a family of bijections

$$\Phi_{V,L} : \hom_{\mathbf{G}}(\mathrm{L}_V, L) \cong \hom_{\mathbf{FinSet}}(V, FL)$$

that is natural in both V and L.

Lemma 3.4 provides us with such a family of bijections: it tells us that a unique $f \in \hom_{\mathbf{G}}(\mathrm{L}_V, L)$ is determined by an arbitrary choice of $f\lfloor\{v\}\rfloor \in FL$ for all $v \in V$. We define $\Phi_{V,L}f \in \hom_{\mathbf{FinSet}}(V, FL)$ by putting

$$(\Phi_{V,L}f)v := f\lfloor\{v\}\rfloor \text{ for all } v \in V.$$

It is easy to see that $\Phi_{V,L}$ is the required family of bijections. $\square$

3.13. Corollary G *has all finite colimits. Thus, it has an initial object; any two objects have a coproduct; any two parallel arrows have a coequalizer and any corner of arrows has a pushout.*

Proof We know (see, for example, McLarty [9, Ex. 10.11]) that a left-adjoint functor respects all small colimits. But L is the left adjoint of the forgetful functor F, and is bijective on objects. Since **FinSet** has all finite colimits, **G** also has them. The remaining parts of our corollary follow as they refer to special cases of colimits.

Thus, for example, as $\emptyset$ is the initial object of **FinSet**, $\mathrm{L}_\emptyset$ is the initial object of **G**. As another example, let V and W be disjoint finite sets, and let $\varphi : V \to V \cup W$ and $\psi : W \to V \cup W$ be the respective insertions of V and W into $V \cup W$. Then

$$V \xrightarrow{\ \varphi\ } V \cup W \xleftarrow{\ \psi\ } W$$

is a coproduct diagram in **FinSet**; and

$$\mathrm{L}_V \xrightarrow{\mathrm{L}\varphi} \mathrm{L}_{V\cup W} \xleftarrow{\mathrm{L}\psi} \mathrm{L}_W$$

is a the corresponding coproduct diagram in **G**. □

3.14. Remark From Lemma 3.4 it follows that **G** has no terminal object, because if $V \neq \emptyset$ and W is an arbitrary finite set, there is more than one way of choosing the $f\lfloor\{v\}\rfloor$ in L_W. However, **G** has a subcategory **GD**, whose objects are the same as those of **G** and whose morphisms are the duality-respecting morphisms of **G**. $\mathrm{L}_\emptyset$ is clearly also an initial object of **GD**. And from Thm. 3.3 it follows that if V is a singleton then L_V is a terminal object of **GD**, because in this case L_V has just one self-dual game (which is the sole non-trivial game on V).

Returning to Def. 3.7, we will now show that if $\varphi : V \to W$ is an arbitrary map from V to the finite set W, then the application of the induced morphism $\mathrm{L}\varphi$ to any game G in L_V yields a game in L_W that arises from G by bloc formation.

The operation of bloc formation is defined formally by Felsenthal and Machover [5, Def. 2.3.23] for the special case where just one bloc is formed: members of a coalition of voters $S \subseteq V$ amalgamate and henceforth vote as one. This creates a new single bloc voter, which is denoted by $\&_S$ in [5], that replaces all the members of S in V, thus resulting in a new assembly, $(V - S) \cup \{\&_S\}$, and a new game on this assembly.

Taylor and Zwicker [11, p. 24] define a natural generalization, whereby several blocs are formed simultaneously or successively by mutually disjoint coalitions. In fact, taking any finite W as an index set, we can consider an arbitrary partition $\{S_w : w \in W\}$ of V into disjoint sets S_w, such that $V = \bigcup_{w\in W} S_w$, and replace all the voters in each S_w by a new single bloc voter, whom we take to be w. (Thus we are using w as proxy for $\&_{S_w}$.)[7]

Note that some of the S_w may be singletons; so our general bloc formation embraces also the case where some voters in V do not amalgamate with other voters but remain single. For even greater generality, we allow some of the S_w to be empty,[8] so that the corresponding blocs are degenerate.

[7]This operation is the exact finite analogue of the the *Rudin–Keisler projection* used to define the *Rudin–Keisler ordering* on ultrafilters. For this observation and references to the relevant literature see Taylor and Zwicker [11, p. 25].

[8]This is not normally allowed by the usual definition of the term *partition*, but here it is convenient to relax this restriction; among other things, it allows us to apply the operation to the degenerate empty assembly.

Let G be any game on V and let G' be the game on W resulting from G by this simultaneous bloc formation. What are the members (ie, winning coalitions) of G'? A straightforward extension of the definition in [5, Def. 2.3.23] to the present setting leads to the following definition:

$$\mathsf{G}' := \{Y \subseteq W : \bigcup_{w \in Y} S_w \in \mathsf{G}\}. \tag{7}$$

Now, specifying an arbitrary partition $\{S_w : w \in W\}$ of V (in the present permissive sense) is equivalent to specifying an arbitrary map $\varphi : V \to W$, where $S_w = \varphi^{-1}[\{w\}]$ for each $w \in W$. Rewriting (7) in terms of φ, we have

$$\mathsf{G}' := \{Y \subseteq W : \varphi^{-1}[Y] \in \mathsf{G}\}.$$

Therefore by Thm. 3.8(i), $\mathsf{G}' = \mathrm{L}\varphi\mathsf{G}$. Note also that S_w is empty iff w is not in the range of φ – in which case it follows from Thm. 3.8(iii) that w is a dummy in $\mathrm{L}\varphi\mathsf{G}$. Thus we have the following result.

3.15. Theorem *Let $\varphi : V \to W$ be an arbitrary map from V to the finite set W and let G be any game on V. Then $\mathrm{L}\varphi\mathsf{G}$ is the game on W resulting from G by formation of the blocs $\{\varphi^{-1}[\{w\}] : w \in W\}$. Moreover, if $w \in W - \varphi[V]$ (ie, the bloc $\varphi^{-1}[\{w\}]$ is degenerate) then w is a dummy in $\mathrm{L}\varphi\mathsf{G}$.* □

In the notation of equation (5), and assuming (without loss of generality) that $V = \widehat{n}$, we can write $\mathrm{L}\varphi\mathsf{G}$ as obtained by composition:

$$\mathrm{L}\varphi\mathsf{G} = \mathsf{G}[\lfloor\{\varphi 1\}\rfloor, \lfloor\{\varphi 2\}\rfloor, \ldots, \lfloor\{\varphi n\}\rfloor].$$

We now turn to another class of morphisms of $\mathbf{G}$, which – except for degenerate cases – do not respect duality, and therefore cannot be induced by morphisms of **FinSet**.

3.16. Definition Let T and B be disjoint subsets of V and let $W = V - (T \cup B)$. We define a morphism $\sqsubset_B^T : \mathrm{L}_V \to \mathrm{L}_W$ by putting

$$\sqsubset_B^T \lfloor\{v\}\rfloor := \begin{cases} \top_W & \text{if } v \in T, \\ \bot_W & \text{if } v \in B, \\ \lfloor\{v\}\rfloor \text{ on } W & \text{if } v \in W. \end{cases}$$

Note that here $\lfloor\{v\}\rfloor$ on the left-hand side is a game on V, which in non-sloppy notation should be written as $(V, \lfloor\{v\}\rfloor)$; whereas the $\lfloor\{v\}\rfloor$ on the right-hand side (third case) is a game on W, written more properly as $(W, \lfloor\{v\}\rfloor)$.

3.17. Theorem *For all* $\mathsf{G} \in \mathrm{L}_V$,

$$\sqsubset_B^T \mathsf{G} = \{Y \subseteq W : Y \cup T \in \mathsf{G}\}; \tag{8}$$

$$\sqsubset_B^T (\mathsf{G}^*) = (\sqsubset_T^B \mathsf{G})^*. \tag{9}$$

Proof Routine, by structural induction on G (Cor. 2.7).[9] □

3.18. Remark Equation (9) means that $(\sqsubset_B^T)^* = \sqsubset_T^B$. Equation (8) means that $\sqsubset_B^T \mathsf{G}$ is what Taylor and Zwicker [11, Def. 1.4.4] call *the Boolean subgame of* G *determined by* B *and* T.

The special case $\sqsubset_B^\emptyset \mathsf{G}$ is *the subgame of* G *determined by* $V - B$, which they denote by G_B.

The special case $\sqsubset_\emptyset^T \mathsf{G}$ is *the reduced game of* G *determined by* $V - T$, which they denote by G^T.

Heuristically, it is helpful to keep in mind the meaning of $\sqsubset_B^T \mathsf{G}$ as a decision rule. Suppose G is a decision rule with V as its set of voters. Suppose also that voters belonging to subsets T and B of V are committed in advance to voting "yes" and "no" respectively, come what may. When a bill is put to the vote, the outcome will then depend only on the votes of the remaining voters, members of $W = V - (T \cup B)$. We are left with a decision rule with W as the *de facto* set of voters. This rule is precisely $\sqsubset_B^T \mathsf{G}$. For further details see [11, pp. 21–23].

In the notation of equation (5), assuming (without loss of generality) that $V = \widehat{n}$, $W = \widehat{m}$, $T = \{m+1, \ldots, m+t\}$ and $B = \{m+t+1, \ldots, m+t+b\}$, we can write $\sqsubset_B^T \mathsf{G}$ as obtained by composition:

$$\sqsubset_B^T \mathsf{G} = \mathsf{G}[\lfloor\{1\}\rfloor, \ldots, \lfloor\{m\}\rfloor, \underbrace{\top_W, \ldots, \top_W}_{t}, \underbrace{\bot_W, \ldots, \bot_W}_{b}].$$

3.19. Remark A special case of Def. 3.16 is that in which W is empty. In this case we have a morphism $f : \mathrm{L}_V \to \mathrm{L}_\emptyset$. This morphism corresponds to a partition of V into two sets, T and B, such that $f\lfloor\{v\}\rfloor = \top_\emptyset$ or $\bot_\emptyset$ according as $v \in T$ or $v \in B$.

Where games are interpreted as decision rules, this partition is essentially what Felsenthal and Machover [5, Def. 2.1.5] call a *bipartition*, a division of the assembly V into "yes" and "no" voters – members of T and B respectively. It is then easy to show by structural induction that, for any game G on V, $f\mathsf{G}$ represents the outcome of the bipartition corresponding to f under the decision rule G; thus $f\mathsf{G} = \top_\emptyset$ or $\bot_\emptyset$ according as the proposed bill is passed

[9] Equation (8) can also be deduced directly from (1).

or blocked. This special case is of particular importance because applying a decision rule to a division of the assembly is the *raison d'être* of games in voting theory.

4 The categories L_V and a skeleton of $\mathbf{G}$

If $|V| = |W| = n$, then the objects L_V and L_W are isomorphic in the category $\mathbf{G}$. In fact, it follows from Lemma 3.4 that there are exactly $n!$ isomorphisms between them, because the dictatorial (ie, principal and prime) games on V must map bijectively to the dictatorial games on W.

Choosing $\mathrm{L}_{\widehat{n}}$ (see Def. 3.6) as the canonical representative of this isomorphism type, we obtain a *skeleton* of $\mathbf{G}$: a full subcategory of $\mathbf{G}$, such that each object of $\mathbf{G}$ is isomorphic in $\mathbf{G}$ to a unique object of the skeleton.

In this section we shall present a recursive category-theoretic construction of the $\mathrm{L}_{\widehat{n}}$; but in preparation for this we first need to observe that each L_V – and in particular each $\mathrm{L}_{\widehat{n}}$ – being a partially ordered set, can be regarded as a category of a very simple kind: for any games G and H on V, an arrow $\mathsf{G} \to \mathsf{H}$ exists iff $\mathsf{G} \leq \mathsf{H}$, in which case this arrow is unique. In this context we write $\mathsf{G} \to \mathsf{H}$ not only to denote this arrow but also to state that it exists, in other words that $\mathsf{G} \leq \mathsf{H}$. This ambiguity should not cause any confusion. Here is a brief résumé of the basic facts about the category L_V.

- All diagrams in L_V commute.
- $\bot_V$ and $\top_V$ are respectively the initial and terminal objects of L_V.
- Products and coproducts exist, with
$$\mathsf{G} \leftarrow \mathsf{G} \wedge \mathsf{H} \rightarrow \mathsf{H} \text{ and } \mathsf{G} \rightarrow \mathsf{G} \vee \mathsf{H} \leftarrow \mathsf{H}$$
respectively as product and coproduct diagrams.
- As L_V is a De Morgan algebra, the duality operation * is a contravariant functor mapping L_V to its opposite, ${\mathrm{L}_V}^{\mathrm{op}}$.

Now let us return to the $\mathrm{L}_{\widehat{n}}$. Consider two special cases of Thm. 3.17, in both of which $V = \widehat{n+1}$ and $W = \widehat{n}$. In the first case we put $B = \{n+1\}$ and $T = \emptyset$; and in the second case we put $T = \{n+1\}$ and $B = \emptyset$. We obtain for any game G in $\mathrm{L}_{\widehat{n+1}}$ the following two games in $\mathrm{L}_{\widehat{n}}$:

$$\sqsubset^{\emptyset}_{\{n+1\}} \mathsf{G} = \{Y \subseteq \widehat{n} : Y \in \mathsf{G}\},$$

$$\sqsubset^{\{n+1\}}_{\emptyset} \mathsf{G} = \{Y \subseteq \widehat{n} : Y \cup \{n+1\} \in \mathsf{G}\}.$$

The upwards closure of G implies that in the category $\mathrm{L}_{\widehat{n}}$

$$\sqsubset^{\emptyset}_{\{n+1\}} \mathsf{G} \longrightarrow \sqsubset^{\{n+1\}}_{\emptyset} \mathsf{G}.$$

Thus we have a map from objects in $\mathrm{L}_{\widehat{n+1}}$ to arrows in $\mathrm{L}_{\widehat{n}}$. This map is bijective. Indeed, given any arrow $\underline{\mathsf{G}} \to \overline{\mathsf{G}}$ in $\mathrm{L}_{\widehat{n}}$, we put

$$\mathsf{G} := \underline{\mathsf{G}} \cup \{Y \cup \{n+1\} : Y \in \overline{\mathsf{G}}\}.$$

Then it is straightforward to verify that

$$\underline{\mathsf{G}} = \sqsubset^{\emptyset}_{\{n+1\}} \mathsf{G} \text{ and } \overline{\mathsf{G}} = \sqsubset^{\{n+1\}}_{\emptyset} \mathsf{G}.$$

Moreover, this map is functorial. Indeed, if $\mathsf{G} \to \mathsf{H}$ is any arrow in $\mathrm{L}_{\widehat{n+1}}$, then in $\mathrm{L}_{\widehat{n}}$ we have the commutative diagram:

$$\begin{array}{ccc}
\sqsubset^{\emptyset}_{\{n+1\}} \mathsf{G} & \longrightarrow & \sqsubset^{\{n+1\}}_{\emptyset} \mathsf{G} \\
\Big\downarrow & & \Big\downarrow \\
\sqsubset^{\emptyset}_{\{n+1\}} \mathsf{H} & \longrightarrow & \sqsubset^{\{n+1\}}_{\emptyset} \mathsf{H}
\end{array}$$

Thus $\mathrm{L}_{\widehat{n+1}}$ is essentially (ie, canonically isomorphic to) the category of arrows of $\mathrm{L}_{\widehat{n}}$. Another way of putting it is that $\mathrm{L}_{\widehat{n+1}}$ is the functor category $\mathrm{Funct}(\mathrm{L}_{\widehat{0}}, \mathrm{L}_{\widehat{n}})$. (Recall that $\mathrm{L}_{\widehat{0}}$ is our old $\mathrm{L}_{\emptyset}$, now seen as the category with a single non-identity arrow, which is commonly denoted by $\mathbf{2}$.)

The canonical isomorphism (in the category of bounded lattices) from the category of arrows of $\mathrm{L}_{\widehat{n}}$ to $\mathrm{L}_{\widehat{n+1}}$ is easy to figure out. According to Thm. 2.8 we need only specify the arrows of $\mathrm{L}_{\widehat{n}}$ that correspond to the dictatorial games on $\widehat{n+1}$. And it is easy to see that for $i = 1, 2, \ldots, n$ the identity arrow $\lfloor\{i\}\rfloor \to \lfloor\{i\}\rfloor$ in $\mathrm{L}_{\widehat{n}}$ corresponds to the dictatorial game $\lfloor\{i\}\rfloor$ on $\widehat{n+1}$, whereas $\perp_{\widehat{n}} \to \top_{\widehat{n}}$ corresponds to the dictatorial game $\lfloor\{n+1\}\rfloor$.

To sum up: starting with the category $\mathrm{L}_{\widehat{0}}$ (commonly known as $\mathbf{2}$), we obtain all the $\mathrm{L}_{\widehat{n}}$ recursively in the sense that $\mathrm{L}_{\widehat{n+1}}$ is canonically isomorphic to the functor category $\mathrm{Funct}(\mathrm{L}_{\widehat{0}}, \mathrm{L}_{\widehat{n}})$.

5 Losing a game and gaining insight

Games are played to win, and game theory accordingly focuses on winning, which it privileges over losing. Voting-power theory inherited this bias, and we have also followed this convention so far, referring to a game (V, G) in terms of its assembly V and set G of *winning* coalitions (Def. 1.1).[10] However, it would be equally possible to refer to a game in terms of its assembly and set of *losing* coalitions.

Note that if (in the notation of Def. 1.1) (V, G) is a game, then its set $\wp V - \mathsf{G}$ of losing coalitions is closed *downwards*:

$$Y \subseteq X \in \wp V - \mathsf{G} \Rightarrow Y \in \wp V - \mathsf{G}.$$

Conversely, if C is a downwards closed set of subsets of V, then it is the set of losing coalitions of a unique game on V, namely $(V, \wp V - \mathsf{C})$. This legitimizes the following definition.

5.1. Definition For any downwards closed set C of subsets of V, we put

$$\langle V, \mathsf{C} \rangle := (V, \wp V - \mathsf{C}).$$

In other words, we introduce '$\langle V, \mathsf{C} \rangle$' as a synonym for '$(V, \wp V - \mathsf{C})$'. We use angled brackets to distinguish the new notation from the old one.

Note that we cannot use a sloppy version of the new notation: whereas – except for the trivial case of $\bot_V$ – a game's set of winning coalitions uniquely determines its assembly (as its biggest member), no such thing holds for the set of losing coalitions.

Note also that if we wish to stick to the operations and ordering of Def. 1.1, then

$$\langle V, \mathsf{C} \rangle \vee \langle V, \mathsf{D} \rangle = \langle V, \mathsf{C} \cap \mathsf{D} \rangle, \quad \langle V, \mathsf{C} \rangle \wedge \langle V, \mathsf{D} \rangle = \langle V, \mathsf{C} \cup \mathsf{D} \rangle;$$

and

$$\langle V, \mathsf{C} \rangle \leq \langle V, \mathsf{D} \rangle \Leftrightarrow \mathsf{D} \subseteq \mathsf{C}.$$

An advantage of this representation of games is that it reveals a communicating door between the theory of games as decision rules and a branch of combinatorics concerned with abstract simplicial complexes.

In the latter theory – which is closely related to combinatorial topology[11] – an *abstract simplicial complex* (briefly: *complex*) is a set C of sets that is closed downwards. Here we shall admit $\emptyset$ as a trivial complex. The union set $\bigcup \mathsf{C}$ is the set of *vertices* of C. Thus a vertex of C is any member of a member of C.

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[10] For reasons that will soon become clear, we avoid in this section the sloppy practice of conflating a game with its set of winning coalitions, and insist on the strict notation of Def. 1.1.

[11] See, for example, Lee [6].

A special case is the power set $\wp V$ of the finite set V. This complex is an *abstract simplex* (briefly: *simplex*) with V as its set of vertices. Its *dimension* is defined as $\dim \wp V := |V|-1$, because the usual geometric realization of $\wp V$ is as a $(|V|-1)$-dimensional Euclidean simplex. For example, if $|V|=4$, $\wp V$ is realized as a tetrahedron. This has the somewhat awkward consequence that in the case $V=\emptyset$, which we shall admit, $\dim \wp V = -1$; but we can live with this.

Thus the lattice L_V is identical with the lattice of all sub-complexes of the simplex $\wp V$ (with their order of inclusion reversed). The voters are the vertices of that simplex, and the coalitions are its faces. The duality * is the well-known *Alexander duality.*

Some of the concepts defined in Section 2 assume a particularly simple and easily visualizable form in this guise; for example:

- A passer in a game $\langle V, \mathsf{C}\rangle$ is a member of V (ie vertex of $\wp V$) that is not a vertex of C.

- The prime game on V determined by the coalition/face A is

$$(V, \lfloor A \rfloor)^* = \langle V, \wp(V - A)\rangle.$$

  This is the sub-simplex $\wp(V-A)$ of the simplex $\wp V$. Its dimension is $|V-A|-1$.

- In particular, the dictatorial game $(V, \lfloor \{v\} \rfloor) = \langle V, \wp(V - \{v\})\rangle$ is the sub-simplex whose set of vertices is $V-\{v\}$. Its dimension is $|V|-2$; in other words, it is a maximal proper sub-simplex of $\wp V$.

- The principal game on V determined by the coalition/face A is

$$(V, \lfloor A \rfloor) = \langle V, \bigcup_{v\in A} \wp(V - \{v\})\rangle.$$

  It is a union of maximal proper sub-simplexes of $\wp V$.

The connection between games and abstract simplicial complexes – they are in fact the same, albeit differently represented – has been noted by several authors, for example by Klivans [7] and Edelman et al. [4]. This connection between the theory of games as decision rules and the theory of finite abstract simplicial complexes allows a transfer of ideas and results from one to the other. We believe that to derive full benefit from this connection it is best to consider as basic objects not individual simplicial complexes but lattices, each of which has as elements all abstract simplicial complexes of a given abstract simplex. This is a jumping off point for further research, and is work in progress.

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