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Competitive Economies with
Incomplete Markets**

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Abstract

A substantial literature addresses the negative effect on welfare of the release of information in a competitive market economy. We show that the value of information in this setting is typically positive if asset markets are sufficiently incomplete. More specifically, for any competitive equilibrium of a generic economy, we can find a finer information structure such that an allocation that is resource feasible and measurable with respect to this information ex-post Pareto dominates the given equilibrium allocation.

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1 Introduction

The objective of this paper is to analyze the value of information in the setup of a competitive economy under uncertainty in which agents trade in asset markets to reallocate risk. It is well known that while information always has positive value in a single-agent decision-making context, this is not necessarily the case in a market setting. Indeed, if the true state of the world is revealed before markets open, no mutually beneficial risk sharing trade is possible. Thus at a competitive equilibrium all agents are worse off, when their welfare is evaluated ex-ante (i.e. prior to the receipt of any signal). More generally, if markets are complete, information cannot have positive value in the sense that no signal, whether fully or partly informative, can lead to an improvement in ex-ante welfare.

The negative effect on welfare of an increase in publicly available information has come to be known as the *Hirshleifer effect*, after Hirshleifer (1971) who produced an early example of it. The Hirshleifer effect can be understood as follows. In the absence of a signal, agents face a single budget constraint. On the other hand, if trade occurs after the receipt of a signal, agents must satisfy a budget constraint for each realization of the signal, which restricts transfers of income across states for different values of the signal (see Gottardi and Rahi (2001)).

If markets are incomplete, a second welfare effect arises. With additional information agents can achieve a larger set of state-contingent payoffs by conditioning their portfolios on this information. We refer to this as the *Blackwell effect*, after Blackwell (1951) who compared the value of different information structures in single-agent decision problems. Roughly speaking, we can think of the value of information in a competitive market economy as having a negative component due to the Hirshleifer effect, and a positive component due to the Blackwell effect. The Hirshleifer effect is stronger the greater is the degree of market completeness. The Blackwell effect, on the other hand, is stronger the more incomplete markets are, and is absent when markets are complete.

There is an extensive literature on the value of information in a competitive pure exchange setting. A long line of papers has followed Hirshleifer's lead in comparing competitive equilibrium allocations associated with differing levels of information (see, for example, Green (1981), Hakansson et al. (1982), Milne and Shefrin (1987), and Schlee (2001)).¹ These papers either derive conditions under which better information leads to an ex-ante Pareto inferior allocation or construct special examples in which information has positive value.

In this paper we provide general conditions under which the value of information is positive. We depart from the literature cited above in that we compare agents' welfare at a competitive equilibrium allocation with their welfare at a feasible (not necessarily equilibrium) allocation that is measurable with respect to a public signal. Also in contrast to the literature, we evaluate welfare not ex-ante, but ex-post, i.e. conditionally on each realization of the signal. This captures the Hirshleifer effect in

¹Eckwert and Zilcha (2001, 2003) extend this analysis to a class of production economies.

that utility transfers across different signal realizations are entirely precluded when we consider a welfare improvement ex-post.² Such a welfare improvement can then be attributed to the Blackwell effect dominating the Hirshleifer effect.

More precisely, we consider a class of two-period exchange economies parametrized by endowments, where agents have von Neumann-Morgenstern preferences and given, symmetric information over the realization of the uncertainty. We show that, provided markets are sufficiently incomplete, for an arbitrary competitive equilibrium of a generic economy, there is a finer information structure such that a feasible allocation measurable with respect to this information structure ex-post Pareto dominates the given equilibrium allocation. In particular, we demonstrate that an ex-post Pareto improvement can generically be attained with an arbitrarily small increase in information.

Our welfare analysis is in the spirit of the literature on constrained inefficiency where welfare comparisons are made between competitive equilibrium allocations and allocations attainable subject to appropriately specified constraints (see, for example, Diamond (1967) and Greenwald and Stiglitz (1986) for some early studies in this vein). Generic inefficiency results have been obtained for incomplete market economies by Geanakoplos and Polemarchakis (1986) and Citanna et al. (1998)), among others. Our paper is the first to establish such a result with respect to changes in public information. Moreover, the proof poses some new technical difficulties. These arise primarily because first order welfare effects are zero, for reasons that we explain later. Hence we need to consider second order effects in order to show that a welfare improvement can be found.

The paper is organized as follows. Section 2 describes the economy. The welfare notion is presented in Section 3; the main inefficiency result is then stated and proved. The proofs of some instrumental Lemmas are collected in the Appendix.

2 The Economy

There are two periods, 0 and 1, and a single physical consumption good. The economy is populated by $H \geq 2$ agents, with typical agent $h \in H$ (here, and elsewhere, we use the same symbol for a set and its cardinality). No consumption takes place at date 0 and agents have no endowment in that period. Uncertainty, which is resolved at date 1, is described by S states of the world. The probability of state $s \in S$ is denoted by $\bar{\pi}_s$.

Agent $h \in H$ has an endowment at date 1 given by $\omega^h \in \mathbb{R}_{++}^S$, and preferences over date 1 consumption described by a von Neumann-Morgenstern utility function $u^h : \mathbb{R}_{++} \rightarrow \mathbb{R}$, which is assumed to satisfy the following standard conditions:

²In this regard, we should also mention Campbell (2004), who shows that with complete markets any increase of information in the sense of Blackwell (1951) has a negative effect on agents' welfare. The result is obtained by comparing welfare at allocations associated with different information structures when these allocations have to be feasible and, in addition, satisfy ex-post individual rationality constraints.

Assumption 1

- (i) u^h is \mathcal{C}^2 .
- (ii) $u^{h'} > 0$ and $u^{h''} < 0$.
- (iii) $\lim_{c \rightarrow 0} u^{h'}[c] = \infty$.

Asset markets, in which $J \geq 2$ securities are traded, open at date 0. At date 1 assets pay off, and agents consume. Asset payoffs in state s are denoted by $r_s \in \mathbb{R}^J$. Thus a portfolio $y \in \mathbb{R}^J$ yields a payoff $r_s \cdot y$ in state s . Let R be the $S \times J$ matrix whose s 'th row is r_s^\top (by default all vectors are column vectors, unless transposed). We impose the following, fairly standard, regularity conditions on the payoff matrix R :

Assumption 2

- (i) *There is an asset, say asset J , whose payoff is nonnegative in every state and positive in at least one state.*
- (ii) *R is in general position, i.e. every $J \times J$ submatrix of R is nonsingular.*

Property (i), together with the monotonicity assumption on utility functions, ensures that the equilibrium price of asset J is positive. It also guarantees that budget constraints are satisfied with equality. Property (ii) requires that asset payoffs vary sufficiently across states.

Asset prices are given by a vector $p \in \mathbb{R}^J$. By Assumption 2, we can choose asset J as the numeraire, i.e. set $p_J = 1$. Let $y^h \in \mathbb{R}^J$ denote the portfolio of agent h . Since portfolios uniquely determine consumption (the consumption of agent h in state s is given by $\omega_s^h + r_s \cdot y^h$), an allocation is completely specified by a collection of portfolios, one for each agent $h \in H$.

A competitive equilibrium is defined as follows:

Definition 1 *A competitive equilibrium consists of an allocation $\{y^h\}_{h \in H}$, and a price vector p , satisfying the following two conditions:*

- (a) *Agent optimization: $\forall h \in H$, y^h solves*

$$\begin{aligned} \max_y \sum_s \bar{\pi}_s u^h[\omega_s^h + r_s \cdot y] \\ \text{subject to} \quad p \cdot y = 0. \end{aligned} \tag{1}$$

- (b) *Market clearing:*

$$\sum_h y^h = 0. \tag{2}$$

We consider a set of economies parametrized by the agents' endowments $\omega := \{\omega^h\}_{h \in H} \in \mathbb{R}_{++}^{SH}$. The space of economies is then \mathbb{R}_{++}^{SH} and by “generically” we mean “for an open, dense subset of \mathbb{R}_{++}^{SH} .”

3 Blackwell Inefficiency

At any competitive equilibrium the information of agents over the realization of the uncertainty is given by the probability distribution over S , $\{\bar{\pi}_s\}_{s \in S}$. We intend to investigate whether an increase in the information available to agents can be found such that a feasible allocation consistent with this information structure Pareto dominates a given equilibrium allocation. Taking $\{\bar{\pi}_s\}_{s \in S}$ as the prior over S , we model an increase in information about the realization of s by a public signal correlated with s . As is well-known for economies in which allocations depend on such a signal, various efficiency criteria can be defined depending on the stage at which utilities are evaluated (see Holmström and Myerson (1983)). As mentioned in the Introduction, the literature on the value of information in a market economy adopts the ex-ante efficiency criterion, where agents' welfare is evaluated unconditionally, before the receipt of any signal. In contrast, we use the notion of ex-post efficiency, where agents' welfare is evaluated conditionally on the realization of the signal.

We show that, for any competitive equilibrium of a generic economy, an appropriate public signal and a corresponding feasible portfolio allocation can be found in such a way as to make everyone better off ex-post (and hence also ex-ante). In other words, as described in the Introduction, it is generically possible to find an increase in agents' information such that the Blackwell effect dominates the Hirshleifer effect. We refer to this informational inefficiency property of competitive equilibria as Blackwell inefficiency.

We now formally describe the set of signals we consider, and the corresponding feasible allocations. Since our objective is to demonstrate the existence of a signal with an associated Pareto improvement, there is no loss of generality in imposing any restriction on the set of signals that is convenient for our analysis. Accordingly we assume that all signals have the same support Σ , with a generic element of Σ denoted by σ . Furthermore, we assume that the cardinality of Σ , also denoted by $|\Sigma|$, is at least 3, and that the marginal distribution of all signals over Σ is uniform.³ To ensure consistency with the probability structure of the underlying state space S , the marginal distribution over S must also be the same for all signals and equal to $\{\bar{\pi}_s\}_{s \in S}$. Having fixed the state spaces S and Σ , each signal is completely described by the probabilities $\pi := \{\pi_{s\sigma}\}_{s \in S, \sigma \in \Sigma} \in \mathbb{R}_{++}^{S \times \Sigma}$, where $\pi_{s\sigma}$ denotes $\text{Prob}(s, \sigma)$; let $\pi_{s|\sigma} := \text{Prob}(s|\sigma)$ and $\pi_\sigma := \text{Prob}(\sigma)$. The set of possible signals is thus given by the set:⁴

$$\Pi := \left\{ \pi \in \mathbb{R}_{++}^{S \times \Sigma} \mid \sum_{\sigma \in \Sigma} \pi_{s\sigma} = \bar{\pi}_s, \forall s \in S; \sum_{s \in S} \pi_{s\sigma} = \bar{\pi}_\sigma, \forall \sigma \in \Sigma \right\},$$

where we use the vector π to identify a signal, and $\bar{\pi}_\sigma := \frac{1}{|\Sigma|}$.⁵ The agents' infor-

³Since we are interested in evaluating agents' welfare ex-post, conditional on each $\sigma \in \Sigma$, the marginal distribution over Σ must be the same (though not necessarily uniform) for all signals in order for the welfare comparison to make sense. See also footnote 6 below.

⁴Note that the adding-up restriction $\sum_{s, \sigma} \pi_{s\sigma} = 1$ follows from the adding-up restriction on $\bar{\pi}_s$.

⁵For convenience we will continue to use the notation $\bar{\pi}_\sigma$ in the rest of the paper. The assumption

mation at any competitive equilibrium is described by the “initial” signal $\bar{\pi} \in \Pi$, which is completely uninformative about s , i.e. satisfies the independence condition $\bar{\pi}_{s\sigma} = \bar{\pi}_s \bar{\pi}_\sigma$, for all $s \in S$, $\sigma \in \Sigma$. Any other signal $\pi \in \Pi$ represents a purely informational change relative to $\bar{\pi}$ and an increase in information over $\bar{\pi}$. Notice that this construction allows us to examine smooth perturbations of the information of agents at an equilibrium, or local changes in information, by considering values of π in a neighborhood of $\bar{\pi}$.

Given any signal π , we can define an associated portfolio allocation as an allocation that is measurable with respect to the signal: $\{y_\sigma^h\}_{h \in H} \in \mathbb{R}^{JH}$, for each $\sigma \in \Sigma$. We say then that the associated portfolio allocation is feasible if it satisfies, in addition, the resource feasibility condition:

$$\sum_h y_\sigma^h = 0, \quad \forall \sigma \in \Sigma. \quad (3)$$

We are now ready to provide a formal definition of our notion of informationally inefficiency:

Definition 2 *A competitive equilibrium $\{\{y^h\}_{h \in H}, p\}$ is Blackwell inefficient if there exists a signal $\pi \in \Pi$ and a corresponding feasible allocation $\{y_\sigma^h\}_{h \in H, \sigma \in \Sigma}$ such that*

$$\sum_s \pi_{s|\sigma} u^h[\omega_s^h + r_s \cdot y_\sigma^h] \geq \sum_s \bar{\pi}_s u^h[\omega_s^h + r_s \cdot y^h], \quad \forall h \in H, \sigma \in \Sigma, \quad (4)$$

where at least one of these inequalities is strict.

We will prove that, generically, competitive equilibria are Blackwell inefficient. But first we wish to show that the welfare improvement allowed by the increase in the agents’ information is indeed the consequence of the larger hedging possibilities available to agents. The next Lemma establishes that when asset markets are complete, so that agents’ hedging possibilities cannot be expanded by an increase in information, competitive equilibria are always Blackwell efficient. In fact, an improvement cannot be found even according to the weaker, ex-ante welfare criterion. To this end we introduce the following variant of Definition 2:

Definition 3 *A competitive equilibrium $\{\{y^h\}_{h \in H}, p\}$ is ex-ante Blackwell inefficient if there exists a signal $\pi \in \Pi$ and a corresponding feasible allocation $\{y_\sigma^h\}_{h \in H, \sigma \in \Sigma}$ such that*

$$\sum_{s,\sigma} \pi_{s\sigma} u^h[\omega_s^h + r_s \cdot y_\sigma^h] \geq \sum_{s,\sigma} \bar{\pi}_{s\sigma} u^h[\omega_s^h + r_s \cdot y^h], \quad \forall h \in H, \quad (5)$$

where at least one of these inequalities is strict.

that the marginal distribution of the signals over Σ is uniform, i.e. $\bar{\pi}_\sigma := \frac{1}{\Sigma}$, $\sigma \in \Sigma$, is used only once, in the proof of Lemma A.3 in the Appendix.

Given the restriction, implied by $\pi \in \Pi$, that $\pi_\sigma = \bar{\pi}_\sigma$, for all $\sigma \in \Sigma$, (4) can be written as

$$\sum_s \pi_{s\sigma} u^h[\omega_s^h + r_s \cdot y_\sigma^h] \geq \sum_s \bar{\pi}_{s\sigma} u^h[\omega^h + r_s \cdot y^h], \quad \forall h \in H, \sigma \in \Sigma, \quad (6)$$

which clearly implies (5). Hence ex-ante Blackwell efficiency implies Blackwell efficiency.⁶

Lemma 1 *Suppose markets are complete. Then a competitive equilibrium is ex-ante Blackwell efficient.*

Note that, since probabilities vary in the welfare comparison considered in the Blackwell efficiency notion, the result is not an immediate consequence of the first welfare theorem, and requires an additional argument.⁷

Proof of Lemma 1:

Consider a competitive equilibrium $\{\{y^h\}_{h \in H}, p\}$, and suppose it is ex-ante Blackwell inefficient. Then there is a $\hat{\pi} \in \Pi$ and a feasible allocation $\{y_\sigma^h\}_{h \in H, \sigma \in \Sigma}$ such that condition (5) holds at $\pi = \hat{\pi}$, i.e.

$$\sum_{s,\sigma} \hat{\pi}_{s\sigma} u^h[\omega_s^h + r_s \cdot y_\sigma^h] \geq \sum_{s,\sigma} \bar{\pi}_{s\sigma} u^h[\omega^h + r_s \cdot y^h], \quad \forall h \in H, \quad (7)$$

with at least one strict inequality. Since

$$\begin{aligned} \sum_{s,\sigma} \bar{\pi}_{s\sigma} u^h[\omega^h + r_s \cdot y^h] &= \sum_s \bar{\pi}_s u^h[\omega^h + r_s \cdot y^h] \\ &= \sum_s \hat{\pi}_s u^h[\omega^h + r_s \cdot y^h] \sum_\sigma \hat{\pi}_{\sigma|s} \\ &= \sum_{s,\sigma} \hat{\pi}_{s\sigma} u^h[\omega^h + r_s \cdot y^h], \end{aligned}$$

condition (7) can be equivalently written as

$$\sum_{s,\sigma} \hat{\pi}_{s\sigma} u^h[\omega_s^h + r_s \cdot y_\sigma^h] \geq \sum_{s,\sigma} \hat{\pi}_{s\sigma} u^h[\omega^h + r_s \cdot y^h], \quad \forall h \in H.$$

This means that, keeping π fixed at $\hat{\pi}$, the (random) allocation $\{y_\sigma^h\}_{h \in H, \sigma \in \Sigma}$ ex-ante Pareto dominates $\{y^h\}_{h \in H}$. However, since by assumption markets are complete, the competitive equilibrium allocation $\{y^h\}_{h \in H}$ is ex-ante Pareto efficient and this

⁶This property does not hold in the absence of the invariance property of the marginal distribution of π over Σ , which we have employed in order to get (6).

⁷This result also differs from sunspot ineffectivity results (e.g. Cass and Shell (1983)), since the public signal σ is typically correlated with the uncertainty over fundamentals described by s and hence does not constitute sunspot uncertainty.

property, characterized by the equality of the agents' marginal rates of substitution across states,

$$\frac{u^h'[\omega_s^h + r_s \cdot y^h]}{u^h'[\omega_{\hat{s}}^h + r_{\hat{s}} \cdot y^h]} = \frac{u^{\hat{h}}'[\omega_s^{\hat{h}} + r_s \cdot y^{\hat{h}}]}{u^{\hat{h}}'[\omega_{\hat{s}}^{\hat{h}} + r_{\hat{s}} \cdot y^{\hat{h}}]}, \quad \forall h, \hat{h} \in H; s, \hat{s} \in S,$$

is independent of the value of π . Hence, there cannot be an allocation which ex-ante Pareto dominates $\{y^h\}_{h \in H}$, for any π . This is a contradiction. \square

We can now state and prove our main result.

Theorem *Suppose $S > J + H$ and $J > H + 1$. Then, for a generic subset of economies, every competitive equilibrium is Blackwell inefficient.*

The Theorem states that for any equilibrium allocation of a generic economy, there exists a signal such that a feasible allocation measurable with respect to that signal is ex-post Pareto improving. The argument in the proof shows that an ex-post Pareto improvement can in fact be attained, for a generic economy, with an infinitesimal increase in information. The result requires markets to be sufficiently incomplete.

The proof is based on the following idea. We identify the conditions an allocation must satisfy to be locally Blackwell efficient. Then we evaluate these conditions at a competitive equilibrium allocation and show that, generically, they cannot hold. From this it follows that it is generically possible to achieve a Pareto improvement by considering a local perturbation of the signal structure away from a completely uninformative one.

The main difficulty we face in the proof lies in the fact that the first order necessary conditions (FONCs) for Blackwell efficiency are always satisfied at an equilibrium allocation (for reasons explained in Step 3 of the proof, first order changes in information and in agents' portfolios cannot lead to a welfare improvement). We have then to turn our attention to the second order necessary conditions (SONCs) and show that they, generically, cannot hold. While FONCs can be employed without having to establish constraint qualification (as in Citanna et al. (1998), for example), this is not the case for SONCs. Indeed, for SONCs, there are no usable constraint qualification conditions other than nondegeneracy of the constraint set, i.e. full rank of the Jacobian of the constraints (which we establish in Lemma A.3 in the Appendix).⁸

Proof of Theorem:

The proof requires that two full rank properties, involving agents' utilities and marginal utilities, be satisfied at a competitive equilibrium. These properties are established in Lemmas A.1 and A.2 in the Appendix for a generic subset of economies.

⁸In particular, the Karush-Kuhn-Tucker conditions do not suffice. For details see Simon (1986) or, for a full treatment, Hestenes (1975) or Berkovitz (2002).

For our analysis below we consider an economy in the intersection of these two generic subsets, which we denote by $\hat{\Omega}$. Clearly, $\hat{\Omega}$ is also a generic subset.

The proof is organized in five steps. In Step 1 we write down the system of equations that a competitive equilibrium must satisfy. In Step 2 we derive the first and second order necessary conditions for an equilibrium to be Blackwell efficient. Then, in Step 3, we show that the FONCs are always satisfied at an equilibrium. Step 4 is devoted to a detailed study of the SONCs, which are shown to imply a key condition, labeled condition **(C)**. Finally, in Step 5, we show that condition **(C)** is never satisfied at an equilibrium of an economy in $\hat{\Omega}$. Therefore, the SONCs do not hold, so an equilibrium cannot be Blackwell efficient.

We first introduce some shorthand notation for matrices which will be used in the proof. Given an index set \mathcal{I} with typical element i , and a collection $\{z_i\}_{i \in \mathcal{I}}$ of vectors or matrices, we denote by $\text{diag}_{i \in \mathcal{I}}[z_i]$ the (block) diagonal matrix with typical entry z_i , where i varies across all elements of \mathcal{I} . For a given vector or matrix z , $\text{diag}_{i \in \mathcal{I}}[z]$ is then the diagonal matrix with the term z repeated $\#\mathcal{I}$ times. In a similar fashion, we write $[\dots z_i \dots]_{i \in \mathcal{I}}$ to denote the row block with typical element z_i , and analogously for column blocks.⁹ We denote by I_K the $K \times K$ identity matrix, and let $\mathbf{1}_K^\top := (1 \ 1 \dots 1)_{1 \times K}$. We use the same symbol 0 for the zero scalar and the zero matrix; in the latter case we occasionally indicate the dimension in order to clarify the argument. A “*” stands for any term whose value is immaterial to the analysis.

We will sometimes need to consider states or agents in a particular sequence. For this purpose we order the set S (and similarly the sets Σ and H) as $\{s_1, s_2, \dots\}$, s_1 being the first state and so on.

Step 1: Characterization of equilibria

The first order conditions of the utility-maximization program (1) are:¹⁰

$$\sum_s \bar{\pi}_s u^{h'}[\omega_s^h + r_s \cdot y^h] r_s - \lambda^h p = 0, \quad \forall h \in H \quad (8)$$

$$p \cdot y^h = 0, \quad \forall h \in H, \quad (9)$$

where λ^h is the Lagrange multiplier associated with the budget constraint. By Walras' law, the market-clearing equation for one asset is redundant. Hence, the market-clearing condition (2) reduces to

$$\sum_h \hat{y}^h = 0, \quad (10)$$

where \hat{y}^h is the vector obtained from y^h by deleting the last element.

⁹We drop reference to the index set if it is obvious from the context: for example $\text{diag}_{h \in H}$ is shortened to diag_h , and $[\dots z_s \dots]_{s \neq s_1}$ is shorthand for $[\dots z_s \dots]_{s \in S, s \neq s_1}$.

¹⁰Due to Assumption 1, equilibrium consumption is strictly positive in every state.

Any competitive equilibrium $\{\{y^h\}_{h \in H}, p\}$, together with the associated Lagrange multipliers $\{\lambda^h\}_{h \in H}$, must satisfy the equation system (8)–(10).

Step 2: Characterization of Blackwell efficiency

Take an arbitrary equilibrium of an economy in the generic set $\hat{\Omega}$ and denote it by $\{\{\bar{y}^h\}_{h \in H}, \bar{p}\}$, with corresponding Lagrange multipliers $\{\bar{\lambda}^h\}_{h \in H}$. Denote the utility levels of agent h at the equilibrium allocation $\{\bar{y}^h\}_{h \in H}$ by $\bar{u}_s^h := u^h[\omega_s^h + r_s \cdot \bar{y}^h]$ in state s , with expected utility $\bar{u}^h := \sum_s \bar{\pi}_s u^h[\omega_s^h + r_s \cdot \bar{y}^h]$. Analogously, let $\bar{u}_s^{h'} := u^{h'}[\omega_s^h + r_s \cdot \bar{y}^h]$, and $\bar{u}_s^{h''} := u^{h''}[\omega_s^h + r_s \cdot \bar{y}^h]$.

Let $y := \{y_\sigma^h\}_{h \in H, \sigma \in \Sigma}$, with $y = \bar{y}$ meaning that $y_\sigma^h = \bar{y}^h$, for all h, σ , and $\xi := (\pi, y) \in \Pi \times \mathbb{R}^{JH\Sigma}$. If the equilibrium $\{\{\bar{y}^h\}_{h \in H}, \bar{p}\}$ is Blackwell efficient, then $\bar{\xi} := (\bar{\pi}, \bar{y})$ is a solution to the following program:

$$\max_{\xi} \sum_s \pi_{s\sigma_1} \left(u^{h_1}[\omega_s^{h_1} + r_s \cdot y_{\sigma_1}^{h_1}] - \bar{u}^{h_1} \right) \quad (\mathbf{P})$$

subject to

$$\begin{aligned} \Phi_1 &:= \sum_s \pi_{s\sigma} \left(u^h[\omega_s^h + r_s \cdot y_\sigma^h] - \bar{u}^h \right) \geq 0, & \forall h \in H, \sigma \in \Sigma, (h, \sigma) \neq (h_1, \sigma_1) \\ \Phi_2 &:= \sum_h y_\sigma^h = 0, & \forall \sigma \in \Sigma \\ \Phi_3 &:= \sum_\sigma \pi_{s\sigma} - \bar{\pi}_s = 0, & \forall s \in S \\ \Phi_4 &:= \sum_s \pi_{s\sigma} - \bar{\pi}_\sigma = 0, & \forall \sigma \in \Sigma, \sigma \neq \sigma_1 \end{aligned}$$

where, for notational convenience, we have multiplied each agent's ex-post expected utility conditional on σ by the constant $\pi_\sigma = \bar{\pi}_\sigma$, yielding the expression $\sum_s \pi_{s\sigma} u^h[\omega_s^h + r_s \cdot y_\sigma^h]$ for the “agent-type” (h, σ) . In program (\mathbf{P}) , both the informativeness of the signal, as described by π , and the allocation of assets, are chosen to maximize the ex-post expected utility of agent-type (h_1, σ_1) ,¹¹ subject to the constraint that the ex-post utility levels of all other agent-types are not lower than at the competitive equilibrium ($\Phi_1 \geq 0$), the resource feasibility constraints ($\Phi_2 = 0$), and the admissibility constraints on probabilities ($\Phi_3 = 0$ and $\Phi_4 = 0$; note that these constraints imply that $\sum_s \pi_{s\sigma_1} = \bar{\pi}_{\sigma_1}$). Let $\Phi := (\Phi_1, \dots, \Phi_4)$.

The Lagrangian of the program (\mathbf{P}) is

$$\begin{aligned} \mathcal{L}(\xi; \theta) &= \sum_{h, \sigma} \mu_\sigma^h \sum_s \pi_{s\sigma} \left(u^h[\omega_s^h + r_s \cdot y_\sigma^h] - \bar{u}^h \right) \\ &\quad - \sum_\sigma \gamma_\sigma^\top \sum_h y_\sigma^h - \sum_s \eta_s \left[\sum_\sigma \pi_{s\sigma} - \bar{\pi}_s \right] - \sum_\sigma \nu_\sigma \left[\sum_s \pi_{s\sigma} - \bar{\pi}_\sigma \right], \end{aligned}$$

¹¹For notational convenience, we have subtracted the constant $\bar{\pi}_{\sigma_1} \bar{u}^{h_1}$ from the objective function of the program (\mathbf{P}) .

where

$$\theta := \{\mu_\sigma^h, \gamma_\sigma, \eta_s, \nu_\sigma\}_{h \in H, s \in S, \sigma \in \Sigma} \in \mathbb{R}^{H\Sigma} \times \mathbb{R}^{J\Sigma} \times \mathbb{R}^S \times \mathbb{R}^\Sigma$$

is the vector of Lagrange multipliers, except for the elements $\mu_{\sigma_1}^{h_1}$ and ν_{σ_1} , which are set equal to 1 and 0 respectively.

We now apply Theorem 3.3 in Simon (1986) to the program (\mathbf{P}) . In particular, constraint qualification holds at $\bar{\xi}$: the Jacobian of the constraints, $D_\xi \Phi(\bar{\xi})$, has full row rank by Lemma A.3. Lemma A.3 also establishes that $D_\xi \Phi(\bar{\xi})$ is row-equivalent to the matrix M defined as

$$\left(\begin{array}{c|c|c} 0 & 0 & \left(\frac{\dots \{0_{(H-1) \times J} \text{diag}_{h \neq h_1} [\bar{p}^\top]\} \dots \sigma}{\text{diag}_\sigma \{\dots I_J \dots h\}} \right) \\ \hline 0 & \text{diag}_{\sigma \neq \sigma_1} \left(\begin{array}{c} \vdots \\ \dots (\bar{u}_s^h - \bar{u}^h) \dots_s \\ \vdots \\ \mathbf{1}_S^\top \end{array} \right) & * \\ \hline I_S & \dots I_S \dots_{\sigma \neq \sigma_1} & 0 \end{array} \right) \quad (11)$$

A vector $d\xi = (d\pi, dy) \in \mathbb{R}^{S\Sigma} \times \mathbb{R}^{JH\Sigma}$ satisfying $Md\xi = 0$ is called a *second order test vector*¹² for the program (\mathbf{P}) . Let $\mathcal{M} := \{d\xi : Md\xi = 0\}$ be the subspace of second order test vectors. Then, if $\bar{\xi}$ is a solution of (\mathbf{P}) , there exists a unique θ such that

$$\begin{aligned} D_\xi \mathcal{L}(\bar{\xi}; \theta) &= 0 && (\mathbf{P}^{\text{foc}}) \\ (d\xi)^\top [D_{\xi\xi}^2 \mathcal{L}(\bar{\xi}; \theta)] d\xi &\leq 0, \quad \forall d\xi \in \mathcal{M} && (\mathbf{P}^{\text{soc}}) \end{aligned}$$

These are respectively the first and second order necessary conditions for Blackwell efficiency of the equilibrium $\{\{\bar{y}^h\}_{h \in H}, \bar{p}\}$. We analyze these in turn.

¹²This is standard terminology in the optimization literature. See, for example, Berkovitz (2002).

Step 3: First order necessary conditions for Blackwell efficiency

Writing $(\mathbf{P}^{\text{foc}})$ more explicitly, we have:

$$\frac{\partial \mathcal{L}}{\partial \pi_{s\sigma}} = \sum_h \mu_\sigma^h \left(u^h[\omega_s^h + r_s \cdot y_\sigma^h] - \bar{u}^h \right) - \eta_s - \nu_\sigma = 0 \quad (12)$$

$$\frac{\partial \mathcal{L}}{\partial y_\sigma^h} = \mu_\sigma^h \sum_s \pi_{s\sigma} u^{h'}[\omega_s^h + r_s \cdot y_\sigma^h] r_s - \gamma_\sigma = 0 \quad (13)$$

It is straightforward to check that these equations are satisfied at $\theta = \bar{\theta}$, where $\bar{\theta}$ is given by:

$$\mu_\sigma^h = \frac{\bar{\lambda}^{h_1}}{\bar{\lambda}^h}, \quad \gamma_\sigma = \bar{\pi}_\sigma \bar{\lambda}^{h_1} \bar{p}, \quad \eta_s = \sum_h \frac{\bar{\lambda}^{h_1}}{\bar{\lambda}^h} \left(\bar{u}_s^h - \bar{u}^h \right), \quad \nu_s = 0.$$

Since the value of the Lagrange multipliers θ is unique, we can use $\theta = \bar{\theta}$ in the remainder of our analysis.

The reason why the FONCs are satisfied at $\bar{\xi}$ is as follows. The agents' ex-ante utility levels, evaluated at $y = \bar{y}$, depend on π only via the marginal distribution over S , $\{\pi_s\}_{s \in S}$, which does not change as we vary π in Π . Hence the first order effect of a change in information on agents' ex-ante utilities must be zero. A feasible change in the portfolio allocation, on the other hand, may affect individual utilities but cannot produce a Pareto improvement since, with $\pi = \bar{\pi}$, the allocation \bar{y} is constrained efficient. This rules out the possibility of an ex-ante, and hence also ex-post, Pareto improvement with respect to first order changes in π and y .

Step 4: Second order necessary conditions for Blackwell efficiency

We now consider the second order necessary conditions $(\mathbf{P}^{\text{soc}})$. We evaluate all second derivatives of \mathcal{L} at $(\bar{\xi}, \bar{\theta})$, dropping any explicit reference to $(\bar{\xi}, \bar{\theta})$ for notational ease. From (12) we see that $D_{\pi\pi}^2 \mathcal{L} = 0$. Therefore,

$$(d\xi)^\top (D_{\xi\xi}^2 \mathcal{L})(d\xi) = (dy)^\top (D_{yy}^2 \mathcal{L})dy + 2(d\pi)^\top (D_{\pi y}^2 \mathcal{L})dy. \quad (14)$$

Consider a second order test vector $\delta\xi = (\delta\pi, \delta y)$ with the following properties:

$$\delta y = 0, \quad (15)$$

$$\delta\pi_{s\sigma} = 0, \quad \forall s \geq J+1, \forall \sigma \neq \sigma_1. \quad (16)$$

It is apparent from an inspection of the equation system $M\delta\xi = 0$, together with

(15) and (16), that $\delta\pi$ solves the equation system $L\delta\pi = 0$ where

$$L := \left(\begin{array}{c|c} 0 & \text{diag}_{\mathcal{S}\sigma \neq \sigma_1} \left(\begin{array}{c} \vdots \\ \dots (\bar{u}_s^h - \bar{u}^h) \dots_s \\ \vdots_h \\ \mathbf{1}_S^\top \end{array} \right) \\ \hline I_S & \dots I_S \dots_{\sigma \neq \sigma_1} \\ \hline 0 & \text{diag}_{\mathcal{S}\sigma \neq \sigma_1} \{0 \quad I_{S-J}\} \end{array} \right)$$

which is row-equivalent to

$$\left(\begin{array}{c|c} 0 & \text{diag}_{\mathcal{S}\sigma \neq \sigma_1} \left(\begin{array}{c|c} \vdots & \\ \dots (\bar{u}_s^h - \bar{u}^h) \dots_{s \leq J} & * \\ \vdots_h & \\ \hline \mathbf{1}_J^\top & * \\ \hline 0 & I_{S-J} \end{array} \right) \\ \hline I_S & \dots I_S \dots_{\sigma \neq \sigma_1} \end{array} \right) \quad (17)$$

Given the dimensionality condition $J > H + 1$ imposed in the statement of the Theorem, the upper right block of (17) has full row rank by Lemma A.1. Hence, the set of second order test vectors $\delta\xi$ that satisfy (15)–(16) is a subspace of $\mathcal{M}' \subset \mathcal{M}$ of dimension $[J - (H + 1)](\Sigma - 1)$.

Now, if $d\xi \in \mathcal{M}$ and $\delta\xi \in \mathcal{M}'$, then $(d\xi + x\delta\xi) \in \mathcal{M}$, for all $x \in \mathbb{R}$. For second order test vectors of this form, $(\mathbf{P}^{\text{soc}})$ reduces to (using (14)):

$$(dy)^\top (D_{yy}^2 \mathcal{L}) dy + 2(d\pi + x\delta\pi)^\top (D_{\pi y}^2 \mathcal{L}) dy \leq 0, \quad \forall d\xi \in \mathcal{M}, \delta\xi \in \mathcal{M}', x \in \mathbb{R}. \quad (18)$$

This condition is satisfied only if

$$v(d\xi, \delta\xi) := (\delta\pi)^\top (D_{\pi y}^2 \mathcal{L}) dy = 0, \quad \forall d\xi \in \mathcal{M}, \delta\xi \in \mathcal{M}'.$$

If not, x can be chosen so that the inequality in (18) is violated. The matrix $D_{\pi y}^2 \mathcal{L}$ is easily calculated from (13), giving us the following condition:

$$v(d\xi, \delta\xi) = \bar{\lambda}^{h_1} \cdot \sum_{h,s,\sigma} \frac{1}{\bar{\lambda}^h} \delta\pi_{s\sigma} \bar{u}_s^{h'} r_s \cdot dy_\sigma^h = 0, \quad \forall d\xi \in \mathcal{M}, \delta\xi \in \mathcal{M}'. \quad (\mathbf{C})$$

We have shown that $(\mathbf{P}^{\text{soc}})$ requires that condition (C) holds. Therefore (C) is a necessary condition for Blackwell efficiency of the equilibrium $\{\{\bar{y}^h\}_{h \in H}, \bar{p}\}$.

Step 5: Analysis of the necessary condition (C)

Condition (C) implies that $D_{d\xi}v$ lies in the row span of M . From Lemma A.3, all the diagonal blocks of M have full row rank. Since $D_{d\pi}v = 0$, $D_{dy}v$ must then be in the row span of the top right block of M , i.e. there must be scalars $a^h \in \mathbb{R}$, with $a^{h_1} = 0$, and vectors $b_\sigma \in \mathbb{R}^J$ such that

$$\frac{1}{\bar{\lambda}^h} \sum_s \delta\pi_{s\sigma} \bar{u}_s^{h'} r_s = b_\sigma + a^h \bar{p}, \quad \forall h \in H, \sigma \in \Sigma$$

so that

$$\sum_s \delta\pi_{s\sigma} \left[\frac{\bar{u}_s^{h'}}{\bar{\lambda}^h} - \frac{\bar{u}_s^{h_1'}}{\bar{\lambda}^{h_1}} \right] r_s = a^h \bar{p}, \quad \forall h \in H, \sigma \in \Sigma.$$

Since the term on the right hand side does not depend on σ we have

$$\sum_{s \leq J} (\delta\pi_{s\sigma_2} - \delta\pi_{s\sigma_3}) \left[\frac{\bar{u}_s^{h'}}{\bar{\lambda}^h} - \frac{\bar{u}_s^{h_1'}}{\bar{\lambda}^{h_1}} \right] r_s = 0, \quad \forall h \in H, \quad (19)$$

where we have used (16) to truncate the summation beyond the first J terms.

The vectors $\{r_s\}_{s \leq J}$ are linearly independent due to the general position of R (Assumption 2). Furthermore, the term in square brackets is always nonzero, by Lemma A.2. Therefore, $\delta\pi_{s\sigma_2} = \delta\pi_{s\sigma_3}$, for the first J states. It suffices to consider the single restriction that applies for the J 'th state, namely

$$\delta\pi_{s_J\sigma_2} = \delta\pi_{s_J\sigma_3}. \quad (20)$$

By the above argument, condition (C) implies that (20) is satisfied for every second order test vector $\delta\pi \in \mathcal{M}'$. Adding this restriction to the equation system $L\delta\pi = 0$ (see (17)), we get the augmented system $\hat{L}\delta\pi = 0$, where

$$\hat{L} := \left(\begin{array}{c|c} 0 & \hat{W} \\ \hline I_S & \dots I_S \dots_{\sigma \neq \sigma_1} \end{array} \right)$$

and \hat{W} is given by

$$\left(\begin{array}{c|c|c} \begin{array}{c} \vdots \\ \dots (\bar{u}_s^h - \bar{u}^h) \dots_{s \leq J-1} \\ \vdots_h \\ \mathbf{1}_{J-1}^\top \\ 0 \\ 0 \end{array} & \begin{array}{c} * \\ * \\ 0 \\ 1 \end{array} & \begin{array}{c} * \\ * \\ I_{S-J} \\ 0 \end{array} \\ \hline 0 & \text{diag}_{\sigma \notin \{\sigma_1, \sigma_2\}} & \begin{array}{c|c} \begin{array}{c} \vdots \\ \dots (\bar{u}_s^h - \bar{u}^h) \dots_{s \leq J} \\ \vdots_h \\ \mathbf{1}_J^\top \\ 0 \end{array} & \begin{array}{c} * \\ * \\ I_{S-J} \end{array} \end{array} \end{array} \right)$$

From Lemma A.1, and the dimensionality restriction $J-1 \geq H+1$, it follows that \hat{W} has full row rank, and therefore so does \hat{L} . Thus (20) is an independent restriction on $\delta\pi$ in addition to the restriction $L\delta\pi = 0$, implying that there exists a vector $\delta\xi^*$ in \mathcal{M}' which does not satisfy (20). But then $\delta\xi^*$ does not satisfy condition (C) either. Therefore, the equilibrium $\{\{\bar{y}^h\}_{h \in H}, \bar{p}\}$ cannot be Blackwell efficient. \square

A Appendix

Consider the system of equations (8)–(10) which any competitive equilibrium must satisfy. We denote the endogenous variables of this equation system by

$$\zeta := [\{y^h, \lambda^h\}_{h \in H}, \hat{p}] \in \mathbb{R}^{JH} \times \mathbb{R}^H \times \mathbb{R}^{J-1}$$

where \hat{p} is the vector obtained from p by deleting the last element. It is convenient to write the first set of these equations, given by (8), as $f(\zeta; \omega) = 0$, and the remaining equations, given by (9)–(10), as $g(\zeta; \omega) = 0$. Under Assumptions 1 and 2, ζ is an equilibrium if and only if

$$F(\zeta; \omega) := \begin{pmatrix} f(\zeta; \omega) \\ g(\zeta; \omega) \end{pmatrix} = 0.$$

This system has $(J+1)H + (J-1)$ equations, equal to the number of unknowns $\#\zeta$.

The Jacobian of the equilibrium system can be written as follows:

$$D_{\zeta, \omega} F = \begin{pmatrix} D_{\zeta} f & D_{\omega} f \\ D_{\zeta} g & 0 \end{pmatrix},$$

with

$$D_{\omega} f = \text{diag}_h \{ \dots \bar{\pi}_s u^{h''} [\omega_s^h + r_s \cdot y^h] r_s \dots \}$$

and

$$D_{\zeta} g = \begin{pmatrix} \text{diag}_h [p^\top] & | & 0 & | & [\dots \hat{y}^h \dots h]^\top \\ \dots \hat{I}^\top \dots h & | & 0 & | & 0 \end{pmatrix},$$

where \hat{I} is the $(J \times (J-1))$ matrix defined by

$$\hat{I} := \begin{pmatrix} I_{J-1} \\ 0 \end{pmatrix}.$$

By a standard argument, at any zero of F , both $D_{\omega} f$ and $D_{\zeta} g$ have full row rank, and hence so does $D_{\zeta; \omega} F$.

Lemma A.1 *Suppose $S > J + H$. Let*

$$W := \begin{pmatrix} \vdots \\ \dots \left(u^h [\omega_s^h + r_s \cdot y^h] - \sum_{s \in S} \bar{\pi}_s u^h [\omega_s^h + r_s \cdot y^h] \right) \dots_{s \leq H+1} \\ \vdots_h \end{pmatrix}.$$

Then, for a generic subset of economies, at an equilibrium allocation $\{y^h\}_{h \in H}$, the (square) matrix

$$\begin{pmatrix} W \\ \mathbf{1}_{H+1}^\top \end{pmatrix}$$

has full rank.

Proof :

For $\psi \in \mathbb{R}^{H+1}$ we will show that, generically, there is no solution to

$$\Psi(\zeta, \psi; \omega) := \begin{pmatrix} F(\zeta; \omega) \\ W(\zeta; \omega)\psi \\ \psi \cdot \mathbf{1}_{H+1} \\ \psi \cdot \psi - 1 \end{pmatrix} = 0.$$

The Jacobian, $D_{\zeta, \psi, \omega} \Psi$, is row-equivalent to

$$\left(\begin{array}{c|c|c} * & * & D_{\omega} \begin{pmatrix} f \\ W\psi \end{pmatrix} \\ \hline 0 & \begin{matrix} \mathbf{1}_{H+1}^{\top} \\ 2\psi^{\top} \end{matrix} & 0 \\ \hline D_{\zeta} g & 0 & 0 \end{array} \right). \quad (21)$$

We wish to show that this matrix has full row rank at any zero of Ψ . As we have seen already, $D_{\zeta} g$ has full row rank. Also, ψ is orthogonal to $\mathbf{1}_{H+1}$ and nonzero (since $\psi \cdot \psi = 1$). Hence, due to the triangular structure of (21), it suffices to show that the upper right block, given by

$$D_{\omega} \begin{pmatrix} f \\ W\psi \end{pmatrix} = \left(\frac{\text{diag}_h \{ \dots \bar{\pi}_s u^{h''} [\omega_s^h + r_s \cdot y^h] r_s \dots_s \}}{\text{diag}_h \left\{ \left(\dots \psi_s (1 - \bar{\pi}_s) u^{h'} [\omega_s^h + r_s \cdot y^h] \dots_{s \leq H+1} \right) (0 \dots 0) \right\}} \right),$$

has full row rank. The above matrix is row-equivalent to a block-diagonal matrix, with blocks indexed by h . The h 'th block is:

$$\left(\begin{array}{c|c} \dots \bar{\pi}_s u^{h''} [\omega_s^h + r_s \cdot y^h] r_s \dots_{s \leq H+1} & \dots \bar{\pi}_s u^{h''} [\omega_s^h + r_s \cdot y^h] r_s \dots_{s > H+1} \\ \hline \dots \psi_s (1 - \bar{\pi}_s) u^{h'} [\omega_s^h + r_s \cdot y^h] \dots_{s \leq H+1} & 0 \end{array} \right) \quad (22)$$

This matrix is triangular as well. Its upper right block has full row rank since it has at least J columns and, by Assumption 2, R is in general position. The lower left block is a single row which is nonzero.

We have shown that the Jacobian $D_{\zeta, \psi, \omega} \Psi$ has full row rank. Thus $\Psi(\zeta, \psi; \omega)$ is transverse to 0. By the transversality theorem, there is an open, dense subset of endowments such that, for each ω in this set, $\Psi_{\omega}(\zeta, \psi)$ is transverse to zero.¹³ But this is an overdetermined system of equations (with one extra equation relative to the number of unknowns). Hence, $\Psi_{\omega}^{-1}(0) = \emptyset$ and this establishes the result. \square

¹³Openness follows from a standard argument; see, for example, Citanna et al. (1998).

Lemma A.2 *Suppose $S > J$. Then, for a generic subset of economies, at an equilibrium $\{\{y^h, \lambda^h\}_{h \in H}, p\}$, we have*

$$\frac{u^{\hat{h}'}[\omega_{s_1}^{\hat{h}} + r_{s_1} \cdot y^{\hat{h}}]}{\lambda^{\hat{h}}} \neq \frac{u^{\check{h}'}[\omega_{s_1}^{\check{h}} + r_{s_1} \cdot y^{\check{h}}]}{\lambda^{\check{h}}},$$

for any pair of agents \hat{h} and \check{h} , for all $s \in S$.

Proof :

We will prove the result for the first two agents, h_1 and h_2 , and the first state s_1 . The same argument applies to any other pair of agents and any other state. Let

$$q(\zeta; \omega) := \frac{u^{h_1'}[\omega_{s_1}^{h_1} + r_{s_1} \cdot y^{h_1}]}{\lambda^{h_1}} - \frac{u^{h_2'}[\omega_{s_1}^{h_2} + r_{s_1} \cdot y^{h_2}]}{\lambda^{h_2}}.$$

We will show that, generically, there is no solution to

$$\mathcal{Q}(\zeta; \omega) := \begin{pmatrix} q(\zeta; \omega) \\ F(\zeta; \omega) \end{pmatrix} = 0.$$

The Jacobian of \mathcal{Q} is

$$D_{\zeta, \omega} \mathcal{Q} = \begin{pmatrix} D_{\zeta} q & | & D_{\omega} q \\ D_{\zeta} f & | & D_{\omega} f \\ D_{\zeta} g & | & 0 \end{pmatrix},$$

where the top right block is given by

$$\left(\begin{array}{c|c} \frac{u^{h_1''}[\omega_{s_1}^{h_1} + r_{s_1} \cdot y^{h_1}]}{\lambda^{h_1}} & 0 \\ \hline \bar{\pi}_{s_1} u^{h_1''}[\omega_{s_1}^{h_1} + r_{s_1} \cdot y^{h_1}] r_{s_1} & \dots \bar{\pi}_s u^{h_1''}[\omega_s^{h_1} + r_s \cdot y^{h_1}] r_s \dots_{s \neq s_1} \end{array} \right) \begin{array}{c} * \\ \hline \text{diag}_{h \neq h_1} \{ \dots \bar{\pi}_s u^{h''}[\omega_s^h + r_s \cdot y^h] r_s \dots_s \} \end{array} \quad (23)$$

By Assumption 2, and the dimensionality condition $S > J$, it follows that the matrix $[\dots r_s \dots_{s \neq s_1}]$ has rank J . Hence the top left block of (23) has full row rank. The bottom right block of (23) clearly has full row rank. This establishes full row rank of the whole matrix (23). Furthermore, as we noted earlier, $D_{\zeta} g$ always has full row rank, so the Jacobian $D_{\zeta, \omega} \mathcal{Q}$ has full row rank as well. Thus $\mathcal{Q}(\zeta; \omega)$ is transverse to 0. By the transversality theorem, there is an open, dense subset of endowments such that, for each ω in this set, $\mathcal{Q}_{\omega}(\zeta)$ is transverse to zero. Hence, $\mathcal{Q}_{\omega}^{-1}(0) = \emptyset$ and this establishes the result. \square

For the next lemma, we consider an economy in the generic set $\hat{\Omega}$, and focus on an equilibrium of this economy, $\{\{\bar{y}^h, \bar{\lambda}^h\}_{h \in H}, \bar{p}\}$. Recall that $D_{\xi} \Phi(\bar{\xi})$ is the Jacobian of the constraints of program (P) evaluated at $\bar{\xi} = (\bar{\pi}, \bar{y})$.

Lemma A.3 *Suppose $S > J + H$. Then $D_\xi \Phi(\bar{\xi})$ has full row rank. Moreover, $D_\xi \Phi(\bar{\xi})$ is row-equivalent to M (where M is defined by (11)) and all the diagonal blocks of M have full row rank.*

Proof :

The Jacobian $D_\xi \Phi(\bar{\xi})$ is given by

$$\left(\begin{array}{c|c|c|c} \begin{array}{c} \vdots \\ \dots (\bar{u}_s^h - \bar{u}^h) \dots_s \\ \vdots_{h \neq h_1} \end{array} & 0 & 0 \text{ diag}_{h \neq h_1} [\bar{\pi}_{\sigma_1} \bar{\lambda}^h \bar{p}^\top] & 0 \\ \hline 0 & \text{diag}_{\sigma \neq \sigma_1} \left(\begin{array}{c} \vdots \\ \dots (\bar{u}_s^h - \bar{u}^h) \dots_s \\ \vdots_h \end{array} \right) & 0 & \text{diag}_{\sigma \neq \sigma_1, h} [\bar{\pi}_\sigma \bar{\lambda}^h \bar{p}^\top] \\ \hline 0 & 0 & \{\dots I_J \dots_h\} & 0 \\ \hline 0 & 0 & 0 & \text{diag}_{\sigma \neq \sigma_1} \{\dots I_J \dots_h\} \\ \hline I_S & \dots I_S \dots_{\sigma \neq \sigma_1} & 0 & 0 \\ \hline 0 & \text{diag}_{\sigma \neq \sigma_1} (\mathbf{1}_S^\top) & 0 & 0 \end{array} \right)$$

where we have used (8) to evaluate the expressions in the (1,3) and (2,4) blocks.

This matrix is row equivalent to

$$\left(\begin{array}{c|c|c|c} \begin{array}{c} \vdots \\ \dots (\bar{u}_s^h - \bar{u}^h) \dots_s \\ \vdots_{h \neq h_1} \end{array} & \begin{array}{c} \vdots \\ \dots (\bar{u}_s^h - \bar{u}^h) \dots_{s, \sigma \neq \sigma_1} \\ \vdots_{h \neq h_1} \end{array} & 0 \text{ diag}_{h \neq h_1} [\bar{\pi}_{\sigma_1} \bar{\lambda}^h \bar{p}^\top] & \dots \left\{ 0 \text{ diag}_{h \neq h_1} [\bar{\pi}_\sigma \bar{\lambda}^h \bar{p}^\top] \right\} \dots_{\sigma \neq \sigma_1} \\ \hline 0 & \text{diag}_{\sigma \neq \sigma_1} \left(\begin{array}{c} \vdots \\ \dots (\bar{u}_s^h - \bar{u}^h) \dots_s \\ \vdots_h \end{array} \right) & 0 & \text{diag}_{\sigma \neq \sigma_1, h} [\bar{\pi}_\sigma \bar{\lambda}^h \bar{p}^\top] \\ \hline 0 & 0 & \{\dots I_J \dots_h\} & 0 \\ \hline 0 & 0 & 0 & \text{diag}_{\sigma \neq \sigma_1} \{\dots I_J \dots_h\} \\ \hline I_S & \dots I_S \dots_{\sigma \neq \sigma_1} & 0 & 0 \\ \hline 0 & \text{diag}_{\sigma \neq \sigma_1} (\mathbf{1}_S^\top) & 0 & 0 \end{array} \right)$$

We perform further row operations on the above matrix. Using the second last row block, we can set the first two elements of the top row block equal to zero. Then, since $\bar{\pi}_\sigma$ is invariant with respect to σ , we can eliminate from the top row block the

terms $\bar{\pi}_\sigma \bar{\lambda}^h$, for all h and σ . It follows that $D_\xi \Phi(\bar{\xi})$ is row-equivalent to the matrix M defined by (11).

The matrix M is lower triangular. The bottom left block of M clearly has full row rank. The middle block has full row rank by Lemma A.1.¹⁴ The top right block can be written as

$$\left(\begin{array}{c|c} \left(\begin{array}{c|c} 0 & \text{diag}_{h \neq h_1}[\bar{p}^\top] \\ \hline I_J & \dots I_J \dots_{h \neq h_1} \end{array} \right) & * \\ \hline 0 & \text{diag}_{\sigma \neq \sigma_1} \{ \dots I_J \dots_h \} \end{array} \right)$$

which also has full row rank. Hence M has full row rank, and consequently so does $D_\xi \Phi(\bar{\xi})$. \square

¹⁴The dimensionality condition $S > J + H$ is needed in order to invoke Lemma A.1. We do not use Lemma A.2 for this result.

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