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The interplay of introduction and elimination rules for propositional connectives is often seen as suggesting a distinguished role for intuitionistic logic. We prove three formal results concerning intuitionistic propositional logic that bear on that perspective, and discuss their significance. First, for a range of connectives including both negation and the falsum, there are no classically or intuitionistically correct introduction rules. Second, irrespective of the choice of negation or the falsum as a primitive connective, classical and intuitionistic consequence satisfy exactly the same structural, introduction and elimination (briefly, elementary) rules. Third, for falsum as primitive only, intuitionistic consequence is the least consequence relation that satisfies all classically correct elementary rules.

1. Background and purpose

Philosophical discussions of introduction and elimination rules for the logical connectives are often conducted with a foundational agenda in mind. The plan is to build a formal justification for intuitionistic logic by (i) marking out in terms independent of any prior choice of logic a natural class of structural, introduction and elimination rules for connectives, such that (ii) the intuitionistic consequence relation may be shown to be the least consequence relation satisfying such rules, while (iii) no such construction is possible for classical consequence.

As was shown by the celebrated 'tonk' example of Prior (1960), one cannot carry out the plan simply by taking the class of all possible introduction and elimination rules; it must be a proper subclass. But the task of isolating an appropriate subclass without taking any particular logic for granted, as needed for the foundationalist project, has proven to be recalcitrant. Despite decades of work, there is no consensus of success.[1]

If, on the other hand, we abandon the foundational goal, and accept as our working system classical logic with its usual semantics, we have a quite different perspective for studying intuitionistic logic. In that context, we may deploy the notion of classical consequence with complete freedom on any level without fear of circularity.

In certain limited respects, it has been quite customary to take such a classical perspective. For example, when presenting semantics for intuitionistic logic, authors often work in a classical meta-language, perhaps noting for the record any points at which their reasoning about intuitionistic logic uses principles that are not acceptable within it. An instance of this is the standard proof of the completeness of intuitionistic logic with respect to its Kripke semantics.[2] In a certain sense, this gives us a classical account of intuitionistic logic.

Another familiar use of the classical perspective arises on the syntactic side. Proof systems for intuitionistic logic may be obtained by imposing a simple restriction on a parameter in suitable presentations of classical logic. This is already done in the work of Gentzen (1934). His system *LK* for classical logic uses generalized sequents, which allow an arbitrary finite set of formulas on the right of each sequent, where they are understood disjunctively in contrast to the conjunctive reading on the left. As Gentzen observed, we may obtain intuitionistic consequence by choosing for classical consequence a suitable rule system using generalized sequents, constraining its rules never to have more than one formula on the right of their conclusion sequents, and then selecting from the provable sequents those in which there is exactly one formula on the right.

However, this characterization of intuitionistic logic in terms of classical logic is potentially presentation-dependent. It obtains the intuitionistic consequence relation by restricting certain delicately chosen sequent postulates for its classical counterpart. Without further work, one has no guarantee that small changes in the classical postulates selected may not spoil the picture.

In contrast, the relationships that we establish in this paper are more robust. They concern the entire class of elementary rules, rather than a particular selection from among them, and focus on the consequence relations themselves, rather than specific presentations. We show that in the propositional case, with sequents understood as containing a single formula on the right, the following hold. First, there are no classically (or intuitionistically) correct introduction rules for a range of connectives, including both negation and the falsum. Second, irrespective of the choice of negation or the falsum as a primitive connective, the two consequence relations satisfy exactly the same structural, introduction and elimination (briefly, elementary) rules. Third, when the falsum rather than negation is taken as primitive, intuitionistic consequence is precisely the least consequence relation that satisfies all classically correct elementary rules. In the final section, we discuss the significance of these three results for the foundational programme.

2. Elementary rules for sequents

All definitions and results are formulated for propositional logic. The question of their extension to the first-order context is discussed briefly at the end of section 3.

Consider a propositional language with sentence letters p_1, p_2, … and a collection of finitary connectives, for example, but not necessarily, &, $\vee$, $\rightarrow$, $\bot$ or &, $\vee$, $\rightarrow$, $\neg$. For this paper, a *sequent* is any expression $A \Rightarrow \beta$ where β is a formula and A is a finite and possibly empty set (rather than a multiset or sequence) of formulas of the propositional language. So defined, sequents contain a single formula on the right in contrast with generalized sequents, which may contain more than one, or none. For emphasis, we sometimes call them set/formula sequents as opposed to set/set sequents. As usual in this context, we reduce clutter by writing $\{\alpha_1, \dots, \alpha_n\}$ as $\alpha_1, \dots, \alpha_n$ and using standard conventions to minimize parentheses for grouping.

Let $n \geq 0$, and fix n+1 sequents $A_1 \Rightarrow \alpha_1, \dots, A_n \Rightarrow \alpha_n, B \Rightarrow \beta$ in that order. This determines an (n+1)-ary *rule for sequents*, consisting of all substitution instances of the (n+1)-tuple $\langle A_1 \Rightarrow \alpha_1, \dots, A_n \Rightarrow \alpha_n, B \Rightarrow \beta\rangle$. It is understood intuitively as authorizing passage from any substitution instance of the first n sequents to the corresponding substitution instance of the last one; this will be marked by writing a slash before the last sequent and calling it the *conclusion-sequent* of the rule, while the first n items are *premiss-sequents*. We omit angle brackets when no confusion is possible, and for brevity speak simply of rules, always understood as rules for sequents.

From the definition, each n-premiss rule has an underlying tuple, which clearly is not unique, but is so up to injective relettering of its sentence letters and permutation of its premiss-sequents, and we abuse terminology a little by speaking of 'the' underlying tuple.

We say that a rule is *elementary* iff its underlying tuple $A_1 \Rightarrow \alpha_1, \dots, A_n \Rightarrow \alpha_n / B \Rightarrow \beta$ satisfies the following condition: all formulas in all n+1 sequents are sentence letters except for at most one, in which case it is of the form $*(p_1, \dots, p_k)$ for some k-place connective $*$ and sentence letters $p_1, \dots, p_k$ (not necessarily distinct) and occurs exactly once, either on the left or the right of the conclusion-sequent.

When the tuple consists entirely of sentence letters, thus with no occurrence of any connective, it is called a *structural* rule. Three well-known examples of structural rules are the finite-premiss versions of reflexivity, monotonicity and cumulative transitivity (alias cut),[3] which together are customarily called the rules for *consequence relations*.[4] When the connective appears just once then, if it is on the right side of the conclusion-sequent we say we have an *introduction* rule, while if it is on the left, we have an *elimination* rule.[5]

Our requirement that there is at most one occurrence of the connective in the underlying tuple for the rule is quite standard. If the connective does occur, we require that it be in the conclusion-sequent $B \Rightarrow \beta$ of the tuple. For elimination rules, that is more restrictive than sometimes required, and we will see later (in comments on the Lemma in the next section) how the definition of an elimination rule may be relaxed by a notch in this respect without affecting the results of the paper.

A sequent $A \Rightarrow \alpha$ is understood to be *classically valid* iff α is a classical consequence of A; it is *intuitionistically acceptable* iff α is an intuitionistic consequence of A. An (n+1)-ary rule (for $n \geq 0$) is called *classically correct* iff in each of its instances the conclusion-sequent is classically valid when all of the premiss-sequents are; likewise it is *intuitionistically correct* iff in each of its instances the conclusion-sequent is intuitionistically acceptable when all of the premiss-sequents are.

Notice that that a classically or intuitionistically correct introduction or elimination rule can be *degenerate* in the sense that it is also an instance of a correspondingly correct structural rule. For example, the zero-premiss rule with underlying tuple $p, p\&q \Rightarrow p$ is simultaneously an elimination rule for &, as we have defined it, and an instance of the structural rule with underlying tuple $p, q \Rightarrow p$.

Finally, we recall a well-known fact about the relationship between intuitionistically and classically correct rules, together with its short proof.

Fact. Every intuitionistically correct rule (elementary or not) is also classically correct.

Proof. Consider any rule with underlying tuple $\tau = A_1 \Rightarrow \alpha_1, \dots, A_n \Rightarrow \alpha_n / B \Rightarrow \beta$, and suppose that it is not classically correct, so that it has an instance $\tau' = A_1' \Rightarrow \alpha_1', \dots, A_n' \Rightarrow \alpha_n' / B' \Rightarrow \beta'$ where each premiss-sequent is classically valid and the conclusion-sequent is not. By the latter there is a valuation making all formulas in B' true but β'

false; take any one such valuation v and substitute for each letter p with $v(p) = 1$ an intuitionistically provable formula such as $p \to p$, and for each letter p with $v(p) = 0$ an intuitionistically disprovable one such as $\neg(p \to p)$ or $\bot$ according to the choice of primitive connective. This gives us another instance $\tau'' = A_1'' \Rightarrow \alpha_1'', \dots , A_n'' \Rightarrow \alpha_n'' / B'' \Rightarrow \beta''$ of the tuple, and it is easy to verify that each premiss-sequent is intuitionistically acceptable while the conclusion-sequent is not.

3. Formal results, proofs, and comments

We begin with a simple observation that must surely have been noticed somewhere in the literature, though the authors have not been able to find it mentioned. It states that there are no classically or intuitionistically correct introduction rules for a broad class of connectives that includes negation, the falsum, and other familiar connectives.

Call a k-ary truth-functional connective $*$ *contrarian* iff $*(p_1, \dots , p_k)$ receives value 0 when all of $p_1, \dots , p_k$ are assigned the value 1 (i.e. in the first row of the truth-table, when $k \geq 1$). This includes the falsum (vacuously, taking $k = 0$), negation ($k = 1$), as well as exclusive disjunction, 'neither nor' and 'not both' ($k = 2$).

Observation. There are no classically or intuitionistically correct introduction rules using set/formula sequents for any contrarian truth-functional connective.

Proof of the Observation

Given the Fact, it will suffice to establish the Observation for classical correctness. Consider any introduction rule for a contrarian k-ary connective $*$. Its underlying tuple must be of the form $A_1 \Rightarrow \alpha_1, \dots , A_n \Rightarrow \alpha_n / B \Rightarrow *(p_1, \dots , p_k)$ where all formulas in B and in the premiss-sequents are sentence letters. Substitute a tautology for all sentence letters. Then each premiss-sequent becomes classically valid while clearly the conclusion-sequent does not, so the rule is not classically correct, completing the proof.

Corollaries

(a) For every $n \geq 1$, if a connective expressing a truth-function of n arguments has a classically correct introduction rule using set/formula sequents, then the connective expressing its negation does not have one. (b) For every $n \geq 1$, there are at least as many truth-functions of n arguments whose connectives do not have a classically correct introduction rule using set/formula sequents, as there are whose connectives do.

Proof of the Corollaries

(a) If a connective for a given function has such a rule, then by the Observation it is not contrarian so the top row of its table takes 1. Hence the top row of its negation takes 0, and thus the truth-function expressed by the negation is contrarian and so by the Observation has no introduction rule. (b) The passage from a given truth function to its negation is a bijection.

Comments on the Observation

We comment on the scope, contrasts, dual form, and general significance of the Observation. Regarding scope, the restriction to set/formula sequents is essential. If empty right-hand sides are allowed in sequents then the Observation fails; witness the introduction rule for $\neg$ with underlying tuple $p, q \Rightarrow \varnothing$ / $p \Rightarrow \neg q$, and that for $\perp$ with underlying tuple $p \Rightarrow \varnothing$ / $p \Rightarrow \perp$ (noting, incidentally, that the latter introduction rule is degenerate, in the sense defined in Section 2). On the other hand, there is a respect in which the scope may be broadened: the Observation continues to hold, with the same proof, if we allow any number of non-contrarian connectives into the premiss-sequents and into the left side of the conclusion sequent of the rule.

The Observation contrasts with the well-known availability of classically correct introduction rules for the non-contrarian connectives $\&, \vee, \rightarrow$ in the context of set/formula sequents. It also contrasts with the equally familiar existence of classically correct elimination rules for the contrarian connectives $\neg, \perp$ in the same context. For example, the underlying tuple $p, \neg p \Rightarrow q$ provides a classically correct zero-premiss elimination rule for $\neg$, while $\perp \Rightarrow q$ does the same for $\perp$.

The notion for connectives dual to that of being contrarian is that of a connective forming compounds with value 1 when all the components have value 0 (cf. Humberstone forthcoming 3.18, where they are discussed under the name 'compositional subcontrariety'). These connectives include negation, 'neither nor', 'not both' as well as the conditional and biconditional (but not the falsum or exclusive disjunction). We can obtain a dual form of the Observation holding for rules using only formula/set sequents, and thus in particular for those using only formula/formula ones: in the context of such sequents, there are no classically correct elimination rules for any dual-contrarian truth-functional connective. The proof is similar, substituting a contradiction instead of a tautology.

As to significance, the Observation explains why discussions of 'harmony' between introduction and elimination rules using set/formula sequents (see for example the references cited in note 1) stop short of requiring that the sets of introduction and of elimination rules for each connective must be non-empty. By the Observation, this is impossible for set/formula rules. We cannot dispense with taking some kind of contrarian connective as primitive and, whether we choose negation, the falsum, or a connective like 'neither nor', we lack a classically or intuitionistically correct introduction rule to balance the elimination rule.[6]

It also shows also that, when attention is restricted to elementary rules, there is no procedure that would, from a set of introduction rules, deliver in a one-one manner an appropriately corresponding set of elimination rules (cf. Dummett 1991 pp. 286–296 and elsewhere), since for example the set of correct introduction rules for 'neither nor' and that for 'not and' coincide (both being the empty set) while their sets of correct elimination rules do not.

We now prove a lemma that tells us that as far as elementary rules are concerned, classical and intuitionistic logic are indistinguishable.

Lemma. Irrespective of whether $\neg$ or $\bot$ is taken as primitive (along with &, $\vee$, $\rightarrow$), the intuitionistically correct elementary rules are exactly the classically correct elementary rules.

Proof of the Lemma

As already remarked in the initial Fact, every intuitionistically correct rule is classically correct so, restricting attention to elementary rules, we need only prove the converse. We first establish it for the case that $\bot$ is taken as primitive (along with &, $\vee$, $\rightarrow$) and then show how the argument may be adapted for primitive $\neg$.

Case 1: The primitives are &, $\vee$, $\rightarrow$, $\bot$. Take any elementary rule with underlying tuple $\tau = A_1 \Rightarrow \alpha_1, \ldots, A_n \Rightarrow \alpha_n \,/\, B \Rightarrow \beta$, and suppose that it is classically correct; we want to show that it is intuitionistically correct. The general strategy is to massage τ into a single classically valid sequent σ, show that σ is intuitionistically acceptable, and conclude that the rule with underlying tuple τ is intuitionistically correct.

Consider the sequent $\sigma = (\wedge A_1 \rightarrow \alpha_1), \ldots, (\wedge A_n \rightarrow \alpha_n) \Rightarrow \wedge B \rightarrow \beta$ corresponding to the tuple τ. It is constructed by transforming each sequent of the elementary rule into a conditional formula whose consequent (e.g. α_1) is the right formula of the sequent and whose antecedent (e.g. $\wedge A_1$) is the conjunction[7] of all the left formulas of the sequent, and replacing the slash sign / of the rule by a sequent separator $\Rightarrow$. We claim that this sequent is classically valid. For suppose there is a valuation v with $v(\mathit{left}(\sigma)) = 1$ and $v(\mathit{right}(\sigma)) = 0$. Take the function $'$ that substitutes a tautology for every letter that is true under v and a contradiction for every letter that is false under v (cf. the substitution in the proof of the Fact in section 2). Then each $(\wedge A_i \rightarrow \alpha_i)'$ is a tautology while $(\wedge B \rightarrow \beta)'$ is a contradiction, so that the rule with underlying tuple τ is not classically correct, contrary to hypothesis.

Since the rule is elementary, each formula α_i and all formulas in the sets A_i are sentence letters, while either (Subcase 1.1) all of the formulas in B are sentence letters but β applies a connective to sentence letters, or (Subcase 1.2) one of the formulas in B applies a connective to sentence letters while β and all other formulas in B are sentence letters.

In Subcase 1.1, the sequent σ is *clear* in the sense that the connective $\rightarrow$ does not occur anywhere within the scope of a disjunction or within the antecedent of a conditional. But it is known from work of Belnap, Thomason and Hazen,[8] and easily verified using the Kripke semantics for intuitionistic logic, that clear sequents (in the language with &, $\vee$, $\rightarrow$, $\bot$ as primitives) are classically valid iff they are intuitionistically acceptable. It follows that the sequent σ is intuitionistically acceptable, and thus that the rule with underlying tuple τ is also intuitionistically correct, as desired.

In Subcase 1.2, σ may not be clear, since $\rightarrow$ can occur in one of the formulas in B and thus within the antecedent of the conditional $\wedge B \rightarrow \beta$. In this situation, $\wedge B \rightarrow \beta$ may be taken to be of the form $(p_1 \& \ldots \& p_m \& (q \rightarrow r)) \rightarrow s$ where $p_1, \ldots, p_m, q, r, s$ are sentence letters. We form the sequent $\sigma*$ by moving the conjunct $q \rightarrow r$ from the right of the $\Rightarrow$ in σ

to become an element on the left. That is, we put $\sigma*$ to be the sequent $(\wedge A_1\rightarrow\alpha_1)$, … , $(\wedge A_n\rightarrow\alpha_n)$, $(q\rightarrow r) \Rightarrow (p_1\& \ldots \&p_m)\rightarrow s$, which is intuitionistically equivalent to σ. But $\sigma*$ is clear, so we may apply the same result as in Case 1 to conclude that it is intuitionistically acceptable, so again σ is intuitionistically acceptable and the rule with underlying tuple τ is also intuitionistically correct, as needed to complete the proof for the case where $\perp$ rather than $\neg$ is primitive.

Case 2: The primitives are $\&, \vee, \rightarrow, \neg$. In this case, if the definition of clarity is left unchanged then the Belnap/Thomason/Hazen equivalence fails; witness the sequent $\neg\neg p \Rightarrow p$, which has no occurrences of $\rightarrow$ and so is vacuously clear, but is not intuitionistically correct. However, one can reformulate the definition of clarity so as to recover the equivalence in that context. Take a formula α in the language with $\&, \vee, \rightarrow, \neg$ primitive to be clear iff no occurrence of $\rightarrow$ or of $\neg$ is in the scope of an occurrence of $\vee$ or of $\neg$, nor is in the left scope of an occurrence of $\rightarrow$.

Then it is not difficult to re-prove the Belnap/Thomason/Hazen equivalence in this context, either by re-running one of the available proofs, for example that in Hazen 1990 or the later one in subsection 7.14 of Humberstone forthcoming, or else via translation from the language of $\neg$ to that of $\perp$. We may then continue as in Case 1, repeating the argument for Subcase 1.2 to cover also the situation where $\neg$ occurs in one of the formulas in B. We omit the details.

Comments on the Lemma

This Lemma explains (as was remarked by a referee) why textbook natural deduction systems for classical propositional logic with set/formula inferences invariably have recourse to at least one rule (for negation or for the falsum) that is *not* elementary. For if they didn't, then by the Lemma they could not yield any more than intuitionistic logic.

Of course, recalling Gentzen's remark mentioned in section 1, if we enlarge the context from sequents as they are understood here to generalized sequents, then not all classically correct elementary sequents will be intuitionistically correct. For example, with more than one formula on the right, the classically correct sequent $\varnothing \Rightarrow \alpha, \alpha\rightarrow\beta$ counts as an introduction rule (zero-premiss, empty left hand side, multiple right hand side), although it is not intuitionistically correct.

Even within the context of set/formula sequents, one may obtain classical logic by enlarging the set of elementary rules. For example, we may do so by allowing a single connective to occur twice in an allowed position (as in the zero-premiss rule of double negation elimination, when negation is primitive). We can also do it by permitting a connective to occur once on the left of a premiss-sequent (as in the rule with underlying tuple $p,q\rightarrow r \Rightarrow s$, $p,q \Rightarrow s$ / $p \Rightarrow s$ which, when the falsum is primitive and negation defined gives us as an instance $p,\neg q \Rightarrow s$, $p,q \Rightarrow s$ / $p \Rightarrow s$).[9] And of course, one may obtain classical logic by using more than one connective in a rule.

Nevertheless, certain enlargements may be made without destroying the Lemma. For example we may allow empty right hand sides in sequents, without affecting it, as is easily shown by replacing the empty set by the falsum and re-running the proof. Even within the context of set/formula sequents, the definition of what counts as an elementary rule may be broadened by a notch without damage to the Lemma. Specifically, we may also include all rules in which the unique occurrence of the connective is on the *right* of some premiss-sequent. Indeed, this is the most common form employed for presenting systems of natural deduction by means of sequent-to-sequent rules (as, for example, in section 4.1 of Dummett 2000). With this broadening, the unique occurrence of the connective can be *anywhere except* on the left of a premiss-sequent, and it may be checked that the above argument for the Lemma continues to goes through unchanged.

One might speculate that the Lemma can be generalized to say that exactly the same elementary rules are correct for any two intermediate consequence relations (that is, consequence relations that include intuitionistic and are included in classical consequence). However, that is not the case. Rautenberg 1986 has remarked that an intermediate consequence relation need not satisfy the intuitionistically correct elementary rule for $\rightarrow$ introduction $A, p \Rightarrow q \;/\; A \Rightarrow p \rightarrow q$. A simple illustration of this possibility is given in Humberstone 2006. Nevertheless, any elementary rule that is correct for a given intermediate consequence relation is intuitionistically correct, despite the failure of the converse. For the same argument as used in the proof of the initial Fact in section 2 shows, more generally, that any rule for sequents (whether elementary or not) that is correct for a given intermediate consequence relation is also correct for classical consequence. Then, applying the Lemma, we may conclude that every elementary rule that is satisfied by the given intermediate consequence is also intuitionistically correct.

Our third result, which focuses on the consequence relations generated by elementary rules, follows easily from the Lemma.

Theorem. For the language with $\&$, $\vee$, $\rightarrow$, $\perp$ as primitive (but not when $\neg$ is primitive): intuitionistic propositional consequence is the least consequence relation over that language that satisfies all classically correct elementary rules.

Proof of the Theorem

By the identity given in the Lemma, it suffices to show that intuitionistic propositional consequence is the least among the consequence relations over that language that satisfy all intuitionistically correct elementary rules. Evidently it is one such relation; and that it is the least (i.e. included in every such relation) follows from the well-known existence of a set/formula axiomatization of intuitionistic logic using only elementary rules with the mentioned choice of primitives. Several such presentations are available, deriving from the systems *NJ* and *LJ* of Gentzen 1934 (see for example Dummett 2000, Section 4.2, but dropping the rule of thinning on the right as well as both rules for negation, replacing the latter with $\perp \Rightarrow A$ and ignoring, for present purposes, the quantifier rules).

Comments on the Theorem

The first part of the proof of the Theorem is clearly independent of the choice of $\perp$ rather than $\neg$ as primitive. However, the second part, showing that intuitionistic consequence is included in every such relation, *does* depend on that choice.[10] Indeed, when $\neg$ is taken as primitive it fails badly. In particular, neither the intuitionistically acceptable sequent $p \Rightarrow$

$\neg\neg p$ nor even the sequent $\neg p \Rightarrow \neg(p\&p)$ can be derived by classically correct elementary rules.[11] Thus, with $\neg$ primitive instead of $\bot$ we obtain a *very* weak subrelation of intuitionistic consequence that is not even congruential with respect to negation.

Roughly speaking, the underlying reason for this sensitivity is that the definition of $\neg\alpha$ as $\alpha\to\bot$ permits the rules for $\to$ to do some of the work needed to obtain principles for $\neg$, as may be illustrated by a typical derivation of $p \Rightarrow \neg\neg p$.[12] In contrast, when $\neg$ is taken as primitive its own elementary rules do not suffice for the job. It might be interesting to study further the least consequence relation satisfying all classically correct elementary rules in the language with primitive negation. We will not do so here, as our main concern is with intuitionistic logic itself; but the relation will play a part in the philosophical discussion in the next section.

To end this formal section, we recall that all of our definitions and results have been formulated for propositional consequence. It is clear how to extend the definition of an elementary rule to the first-order context, and it is also clear that the Observation carries over. But it is not yet known whether the Lemma and the Theorem do so too; our conjecture is positive.

4. Philosophical discussion

What bearing do these results have on the project of formal foundations for intuitionistic logic? The answer depends in part on one's appreciation of the role of elementary rules.

On the one hand, one may take elementary rules as being of special epistemological significance, providing inferential meanings for the connectives in a manner that is philosophically preferable to truth valuations. Under this perspective, the Theorem might be said to *strengthen* the case for intuitionistic logic. With an appropriate concept of a sequent and with the falsum as a primitive connective (along with &, $\vee$, $\to$), *whether we set out from the class of all intuitionistically correct sequent rules or from the class of all classically correct ones, restriction to the elementary rules among them gives us exactly intuitionistic consequence*.

However, to keep a sense of balance, we should note that intuitionistic consequence is not unique in this respect. As we have seen in the comments on the Theorem, when negation rather than the falsum is primitive, then a much weaker consequence relation has exactly the same property. Whether we set out from the class of all sequent rules that are correct for that weak relation or from the class of all classically correct ones, restriction to the elementary rules among them gives us exactly the former. Thus, attributing a special epistemological role to elementary rules provides intuitionistic consequence with no greater claim on our allegiance than it does for the other, very weak consequence relation.

Suppose, on the other hand, that we see the distinction between elementary and non-elementary rules as merely a matter of relative simplicity: elementary sequent rules are just those dealing with the limiting case where the number of occurrences of connectives equals one and that occurrence is situated in the conclusion-sequent. From that perspective, our Theorem *explains away* an attraction of intuitionistic consequence: it is what we get when, with suitable choices of the notion of sequent and of primitive connectives, *we restrict ourselves to the simplest among the classically correct sequent rules*.

Such a restriction can have computational advantages for those working in Prolog or logic programming. Moreover, in that community a rather surprising choice of language finds favour. Rules using set/set sequents are regarded as inconvenient, since they tend to make large demands on memory. Set/formula sequents, known to logic programmers as definite clauses, are easy to work with, but they are not very expressive, particularly for formulating constraints on rules. The optimum language is seen as that of set/formula-or-empty sequents, where the right side may contain a single formula or be empty. Goldilocks style, this is more expressive than the language of set/formula sequents, yet more manageable than that of set/set sequents.

How do elementary rules behave in the set/formula-or-empty context? As we saw when commenting on the Observation, introduction rules for contrarian connectives now exist. As noted in our comments on the Lemma, the intuitionistically correct elementary rules remain just the classically correct ones. Finally, in this context our Theorem holds in an even stronger form: irrespective of the choice of falsum or negation as primitive, intuitionistic propositional consequence is the least consequence relation over that language that satisfies all classically correct elementary rules. The reason why the theorem now applies even when negation is primitive is that the empty set on the right of sequents is able to effect, in structural rules, much of the work that primitive falsum would do in its elimination rules.

To summarize the computational perspective, logic programmers find a certain practical advantage to working in a context (rules with set/formula-or-empty sequents) which, if one restricts the logical axioms to those of the simplest form (expressible as classically correct elementary rules), leads to intuitionistic rather than classical consequence irrespective of the choice of contrarian connective.

In contrast, from a purely mathematical point of view, it is generally accepted that the most elegant language to use is that of set/set sequents, permitting exploitation of the full symmetry between left and right of a sequent. In this context, as is well known, the least relation that satisfies all classically correct elementary rules is just classical consequence itself, irrespective of whether negation or the falsum is taken as primitive. Any attempt to justify intuitionistic logic in terms of elementary rules in that context thus disappears, as an artifact of a mathematically unfortunate decision over choice of formal language.

But it is also well known that the human mind has some difficulty manipulating set/set inferences with a disjunctive reading of the conclusion, perhaps because the disjunctive conceptualization requires some kind of parallel processing. This empirical fact is reflected in the practice of almost all introductory textbooks, whether written for students of philosophy, computer science or mathematics, of avoiding the set/set (or even set/formula-or-empty) mode of presentation in favour of the set/formula one.[13]

Given this situation, it is of interest to obtain a clear picture of the formal behaviour of elementary rules in the traditional set/formula context and in particular their connections with classical and intuitionistic consequence, as we have sought to do here.[14]

Notes

1. See, for example, Sundholm 1983, Dummett 1991, Tennant 1997 and forthcoming, Read 2010, Milne 2002, Prawitz 2006, each with references to earlier work by the same and other authors.

2. See for example van Dalen 1997 Sects 5.3–5.4, especially the completeness theorem 5.3.10 and his remark on it in the last paragraph of Sect. 5.4.

3. Strictly speaking, under our definitions, monotonicity is not a single rule but a countable *collection* of such rules with underlying tuples $p_1, \dots, p_m \Rightarrow q$ / $p_1, \dots, p_m, p \Rightarrow q$, one for each choice of $m \geq 0$. Similarly for all other rules using arbitrary finite sets of formulas. This multiplicity might be seen as rather awkward. It disappears if one takes a sequent-rule in a more customary manner as a syntactic item from the metalanguage, often known as a 'rule scheme', rather than as a set of items from the level of the object-language (specifically, as the set of all substitution instances of a tuple of sequents). However, working with 'rule schemes' as metalanguage items brings its own inconvenience. As soon as one begins talking about rule schemes in general, one is forced to choose between pedantry and laxity, according to whether or not one systematically introduces *meta-metalinguistic* variables to range over those schemes and their various ingredients. The present formulation of the notion of a sequent rule permits a clean and rigorous proof of our desired theorem with minimal notational fuss. It should be emphasized, however, that the results of this paper continue to hold under the more customary 'rule scheme' formulation.

4. In the context of logics where the rule of monotonicity fails, the term 'consequence relation' is often used in a broader sense, to cover any relation for which at least reflexivity holds. Those satisfying all three conditions of reflexivity, monotonicity and cumulative transitivity are then termed 'closure relations'. However, in the present discussion, where we compare only classical and intuitionistic logic, we retain the more traditional terminology.

5. Curiously, Prawitz 1965, Sect. II.1 describes the intuitionistically correct rule $\bot \Rightarrow \alpha$ as falling 'outside the division into I- and E-rules'. Elementary rules as defined here are both 'pure' and 'simple' in the sense of Dummett 1991, and correspond, in the context of sequent-to-sequent rules, to the 'immediate' rules in Tennant 1997, p. 316.

6. Care needs to be taken in interpreting statements in this area. For example, Tennant has declared that '*Every* logical operator enjoys an Introduction rule and a corresponding Elimination rule' (Tennant forthcoming, Sect. 6.2, with italics in the original). At first sight, this appears to be falsified by the absence of any classically correct introduction rule for contrarian connectives. But reading the claim in the light of Tennant 1997 Sect. 10.3 suggests that it is intended in the context of set/formula-or-empty sequents, with $\bot$ used on the right of a sequent as a non-standard notation for the empty set. In that context our Observation does *not* hold, as mentioned in the text. Nevertheless, even there Tennant's claim loses much of its philosophical bite when we consider the cases of $\bot$ and $\top$. It is easy to show that for set/formula-or-empty sequents, any classically correct introduction rule for $\bot$ (or elimination rule for $\top$) is degenerate in the sense defined in section 2: it is already an instance of the classically correct *structural* rule obtained by

taking the underlying tuple for the given rule and replacing $\perp$ (or $\top$) by a sentence letter not occurring elsewhere in the tuple.

7. In the limiting case that A is empty, $\wedge A \rightarrow \alpha$ is understood to be the formula α. For the conjunction of individual formulas α, β, the standard notation is of course $\alpha \wedge \beta$, but *Mind* house rules have required the authors to use an ampersand.

8. This result is established for rules governing and & and $\vee$ in Belnap and Thomason 1963 using proof-theoretic arguments, and is extended in Hazen 1990 to cover the case of implications between $\{\&, \vee\}$-formulas. A simple proof using Kripke models for the $\{\&, \vee, \rightarrow\}$-fragment, which extends unproblematically to the addition of $\perp$, is given in 7.14 of Humberstone forthcoming.

9. Rules with a unique occurrence of the connective on the left of a premiss-sequent include those called 'oblique' in Dummett 1991 pp. 257, 297.

10. This is not the only area of logic in which the presence or absence of $\perp$ as a primitive connective has significant consequences. In modal propositional logic it affects the structure of the lattice of all modal logics, as was observed in Makinson 1973; see also Humberstone 1993. Other examples of unexpected effects of changing primitives are referenced from the literature in Humberstone forthcoming Subsect. 7.13.

11. We sketch a proof that for the relation defined we have neither $p \Rightarrow \neg\neg p$ nor even $\neg p \Rightarrow \neg(p\&p)$. Consider a semantics that is just like the classical, except that when $v(\alpha) = 0$ then $v(\neg\alpha)$ may take value 1 or 0, chosen freely according to the choice of formula α (so that the negations of different occurrences of the same formula get the same value, but negations of logically equivalent formulas need not). Clearly, neither of the two sequents above is valid under this semantics, while it is not difficult, though rather tedious, to show that every classically correct elementary rule is correct under it. On the other hand, if we admit empty right hand sides to sequents, then we admit the elementary rules $A, \alpha \Rightarrow \varnothing \,/\, A \Rightarrow \neg\alpha$ and $A \Rightarrow \alpha \,/\, A, \neg\alpha \Rightarrow \varnothing$, which are used in Dummett 2000, Sect. 4.2 page 97, to axiomatize intuitionistic consequence. Thus there are two distinct ways of using elementary rules to get intuitionistic out of classical consequence: (1) from the set of all classically correct elementary rules with set/formula sequents and falsum primitive, and (2) from the set of all classically correct elementary rules but with set/formula-or-empty sequents and negation primitive. The latter appears to be the avenue favoured by Tennant 1997, esp. Sect. 10.3.

12. We recall a well-known derivation of $p \Rightarrow \neg\neg p$ from classically correct elementary rules when $\perp$ is taken as primitive and $\neg$ is defined, which brings out the role played by the introduction and elimination rules for $\rightarrow$. Under the definition, $p \Rightarrow \neg\neg p$ abbreviates $p \Rightarrow (p\rightarrow\perp)\rightarrow\perp$. Now $p, p\rightarrow\perp \Rightarrow \perp$ is an instance of the classically correct zero-premiss elimination rule for $\rightarrow$ with underlying 1-tuple $p, p\rightarrow q \Rightarrow q$. Also, $p, p\rightarrow\perp \Rightarrow \perp \,/\, p \Rightarrow (p\rightarrow\perp)\rightarrow\perp$ is an instance of the classically correct one-premiss introduction rule for $\rightarrow$ with underlying 2-tuple $p, q \Rightarrow r \,/\, p \Rightarrow q\rightarrow r$. Applying the latter to the former gives $p \Rightarrow (p\rightarrow\perp)\rightarrow\perp$ as desired. See Béziau 1999 and Humberstone 2005 for further discussion

of the way in which $\neg$ may inherit properties from the behaviour of $\rightarrow$ when defined in terms of it.

13. See e.g. Restall 2005 for an epistemological defense of multiple conclusions, and Steinberger 2011 for an opposing view.

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