# Flight to quality, flight to liquidity, and the pricing of risk

Dimitri Vayanos

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**Abstract**

We propose a dynamic equilibrium model of a multi-asset market with stochastic volatility and transaction costs. Our key assumption is that investors are fund managers, subject to withdrawals when fund performance falls below a threshold. This generates a preference for liquidity that is time-varying and increasing with volatility. We show that during volatile times, assets' liquidity premia increase, investors become more risk averse, assets become more negatively correlated with volatility, assets' pairwise correlations can increase, and illiquid assets' market betas increase. Moreover, an unconditional CAPM can understate the risk of illiquid assets because these assets become riskier when investors are the most risk averse.

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# 1 Introduction

Financial assets differ in their liquidity (i.e., the ease at which they can be transacted), and an important question is how liquidity affects asset prices. One way to get at this question is to measure the price differential (liquidity premium) between two assets of different liquidity but very similar other characteristics. Many examples of such asset pairs come from the government bond market. Consider, for instance, a just-issued (on-the-run) thirty-year US Treasury bond, and a thirty-year bond issued three months ago (off-the-run). The two bonds are very similar in terms of cash flows, but the on-the-run bond is significantly more liquid than its off-the-run counterpart.[1]

The analysis of liquid/illiquid asset pairs reveals that liquidity premia vary considerably over time, widening dramatically during extreme market episodes. A term often used to characterize these episodes is flight to liquidity, suggestive of the notion that investors experience a sudden and strong preference for holding liquid assets. An example of such an episode is the 1998 financial crisis. During the crisis, the yield spread between off- and on-the-run thirty-year US Treasury bonds increased to 24 basis points (bp), from a pre-crisis value of 4bp. Similar increases occurred in other on-the-run spreads, as well as in other spreads for which liquidity plays some role.[2]

An important factor driving the variation in liquidity premia seems to be the extent of uncertainty in the market. For example, during the 1998 crisis there was a dramatic increase in uncertainty, as measured by the implied volatilities of key financial indices.[3] A link between liquidity premia and volatility has also been empirically documented for less extreme times.[4]

In this paper, we propose an equilibrium model in which (i) assets differ in their liquidity, and (ii) the extent of uncertainty, represented by the volatility of asset payoffs, is stochastic. Our model generates liquidity premia that are time-varying and increasing with volatility. Thus, times of high volatility are associated with a flight to liquidity.

[1]For example, Sundaresan (1997, pp.16-18) reports that trading volume for on-the-run bonds is about ten times as large as for their off-the-run counterparts. See also Boudoukh and Whitelaw (1991), who document the on-the-run phenomenon in Japan, and Warga (1992), Krishnamurthy (2002), and Goldreich, Hanke, and Nath (2003), who study the yield spread between off- and on-the-run bonds.

[2]For example, the on-the-run five-year spread increased from 7bp to 28bp, the ten-year spread from 6bp to 16bp, and the ten-year spread in the UK from 6bp to 18bp. The spread between AA-rated corporate bonds (for which default plays only a minor role) and government bonds increased from 80bp to 150bp in the US, and from 60bp to 120bp in the UK. For more information, and a comprehensive description of the crisis, see the report by the Committee on the Global Financial System (CGFS (1999)).

[3]According to CGFS (1999), the implied volatility of the S&P500 index increased from 23% to 43%, that of the FTSE100 index from 27% to 48%, that of the three-month US eurodollar rate from 8% to 33%, and that of the thirty-year US T-bond rate from 7% to 14%.

[4]For example, Kamara (1994) considers the yield spread between T-notes and T-bills with matched maturities. He shows that the spread (which is mainly due to the notes' lower liquidity) is correlated with interest-rate volatility. (See also Amihud and Mendelson (1991) and Strebulaev (2003).) Longstaff (2002) considers the yield spread between the securities of Refcorp, a US government agency, and Treasury securities. He shows that the spread (which is mainly due to the lower liquidity of the Refcorp securities) is correlated with a consumer confidence index.

In our model, times of high volatility are associated with several other interesting phenomena, often mentioned in the context of financial crises. We show that during volatile times, investors' effective risk aversion increases. Thus, there is a flight to quality, in the sense that the risk premium investors require per unit of volatility increases. We also show that assets become more negatively correlated with volatility, and can also become more correlated with each other. Moreover, illiquid assets become riskier, in the sense that their market betas increase. Some of these phenomena have been documented anecdotally or empirically, and some are new and testable predictions of our model.[5] To our knowledge, our model is the first to show that all these phenomena are related, and are also related to flight to liquidity.

Finally, our model has the implication that one cost of illiquidity is to make an asset riskier, and more sensitive to the volatility. We also show that an unconditional CAPM can understate the risk of illiquid assets because of this risk's time-varying nature: illiquid assets become riskier in volatile times, when investors are the most risk averse. This has implications for evaluating the performance of strategies investing in illiquid assets: if performance is evaluated under an unconditional CAPM, the strategies could appear better than they actually are.[6]

We consider a continuous-time, infinite-horizon economy with one riskless and multiple risky assets. The riskless rate is exogenous and constant over time. The risky assets' dividend processes are exogenous, and are characterized by a common volatility parameter which evolves according to a square-root process. The volatility parameter is the key state variable in our model.

The risky assets differ in their liquidity. To model liquidity, we assume that each asset carries an exogenous transaction cost. These costs differ across assets, and could arise for reasons outside the model such as asymmetric information, market-maker inventory costs, search, etc. We assume that transaction costs are constant over time. Thus, the time-varying liquidity premia arise not because of the transaction costs, but because of the investors' willingness to bear these costs.

Our key assumption is in the modelling of investors. We assume that investors are fund managers, managing wealth on behalf of the individuals who own it. Managers receive an exogenous fee which depends on the amount of wealth under management. They are facing, however, the probability that the individuals investing in the fund might withdraw their wealth at any time. Withdrawals occur both for random reasons, and when a fund's performance falls below an ex-

[5] Section 5 presents the empirical evidence. Anecdotal evidence for increased risk aversion during volatile times can be found in the CGFS (1999) report. In page 6, the report describes investment decisions at the onset of the 1998 crisis as follows: "More fundamentally, investment decisions reflected some combination of an upward revision to uncertainty surrounding the expected future prices of financial instruments ... and a reduced tolerance for bearing risk." The dramatic increases in the yields of speculative-grade corporate bonds and emerging market bonds during the crisis can also be viewed as evidence for increased risk aversion. Flight-to-quality phenomena are hard to rationalize within a frictionless representative-investor model, because risk aversion is driven by preferences and should be stable over short time intervals.

[6] Strategies long on illiquid assets were followed by many hedge funds prior to the 1998 crisis.

ogenous threshold. Managers choose a risky-asset portfolio to maximize the expected utility they derive from their fee, taking into account the probability of withdrawals.

Modelling investors as fund managers subject to performance-based withdrawals generates a natural link between volatility and preference for liquidity. Indeed, during volatile times, the probability that performance falls below the threshold increases, and withdrawals become more likely. Thus, managers are less willing to hold illiquid assets during those times. The notion that withdrawals from a fund are based on the fund's performance is closely related to that of the limits of arbitrage (e.g., Shleifer and Vishny (1997)). It also related to the notion of leverage, whereby a trader's levered position must be liquidated when its value falls below a threshold.

We first solve the model in the benchmark case where withdrawals do not depend on performance. In that case, we can determine the equilibrium price process of the risky assets in closed form. (Thus, one contribution of this paper is to derive a simple, closed-form solution for a multi-asset dynamic equilibrium model with stochastic volatility and transaction costs.) We show that the assets' market betas, their pairwise correlations, and the correlations between the assets and the volatility are all constant over time. Moreover, assets are priced by an unconditional two-factor CAPM adjusted for transaction costs, where the two factors are the market and the volatility.

When withdrawals depend on performance, the equilibrium price process can be determined up to a system of ordinary differential equations. This system can be studied both numerically, and through closed-form solutions in the special case where the time-variation in volatility is small.

Because managers are less willing to hold illiquid assets during volatile times, liquidity premia are increasing with volatility. We show that the effect of volatility on liquidity premia can also be convex. The intuition is that when volatility is low, managers are not concerned with withdrawals because the event that performance falls below the threshold requires a movement of several standard deviations. Thus, liquidity premia are very small, and almost insensitive to volatility. When volatility increases, however, the probability of withdrawals starts increasing rapidly, and so do the liquidity premia.[7]

Interestingly, volatility can have a convex effect not only on the liquidity premia, but also on the risk premia. This is because risk premia are affected by the managers' concern with withdrawals. Indeed, withdrawals are personally costly to the managers (because the managers' fee is reduced), and holding a riskier portfolio makes them more likely by increasing the probability that performance falls below the threshold. When volatility is low, managers are not concerned with withdrawals. Thus, the component of the risk premium that corresponds to withdrawals is very small, and almost insensitive to volatility. That component starts, however, increasing rapidly

[7] Because the probability is bounded above, it must eventually become concave in the volatility. In Section 4 we argue, however, that the convexity region is the more relevant.

when volatility increases. Because of the convexity, the managers' effective risk aversion is higher during volatile times, and so are the risk premia per unit of volatility.

The convexity in the risk and liquidity premia generates time-variation in asset correlations and betas. Indeed, because of the convexity, the volatility shock becomes increasingly important, relative to the other shocks, in driving asset returns as volatility increases. Thus, during volatile times, assets move more closely with volatility, and the correlation between returns and volatility decreases (becoming more negative). Moreover, the pairwise correlation between assets can increase, because assets move more closely with volatility and thus there is greater scope for common variation. Finally, because of the convexity in the liquidity premia, illiquid assets become increasingly sensitive to volatility, relative to other assets, as volatility increases. Thus, during volatile times, the negative effect of volatility on the average asset is reflected more strongly on illiquid assets, giving rise to an increased market beta.

In equilibrium, assets are priced by a conditional two-factor CAPM, but not by its unconditional version. The unconditional CAPM can understate, in particular, the risk of illiquid assets because these assets become riskier when investors are the most risk averse.

Our model has a rich set of empirical implications. Some of the implications are sketched above, and concern (i) the time-variation of expected returns, betas, correlations, and liquidity premia, with aggregate volatility, and (ii) the effect of liquidity on cross-sectional expected returns. An additional set of implications concern the role of a volatility or a liquidity factor in explaining returns (i.e., liquidity as a risk factor rather than as an asset characteristic). Our model provides a theoretical foundation to the recent empirical studies that show the presence of such a factor.[8] It also provides insights into the factor's risk premium and the assets' loadings along the factor. In Section 5, we present the empirical implications of our model, and link our results to the empirical findings.

This paper is closely related to the theoretical literature studying the effects of transaction costs on equilibrium asset prices.[9] In most of that literature, liquidity premia are constant over time. This is because transaction costs are constant, and so are investors' expected horizons (which determine the investors' willingness to bear the transaction costs). Acharya and Pedersen (2003) assume that transaction costs are time-varying and investors have one-period horizons, and they show the existence of a liquidity factor. Our work is complementary to theirs, in that we are

[8]See, for example, Amihud (2002), Acharya and Pedersen (2003), Pastor and Stambaugh (2003), and Sadka (2003), for a liquidity factor, and Ang, Hodrick, Xing, and Zhang (2003) for a volatility factor.

[9]See, for example, Amihud and Mendelson (1986), Constantinides (1986), Heaton and Lucas (1996), Vayanos (1998), Vayanos and Vila (1999), Huang (2002), Acharya and Pedersen (2003), and Lo, Mamaysky, and Wang (2003). In some of these papers (e.g., Constantinides (1986) and Vayanos (1998)), there is a cross-sectional relationship between liquidity premia and asset volatility. (See also Duffie, Garleanu, and Pedersen (2003), for a similar result in a setting where transaction costs derive from search.) Volatility, however, is assumed constant over time.

assuming constant transaction costs but time-varying horizons (horizons are determined by the probability of withdrawals, which depends on the volatility).

This paper is also related to the literature on delegated portfolio management, and especially to papers emphasizing the relationship between a fund's performance and inflows. Starks (1987), Chevalier and Ellison (1997), and Basak, Pavlova, and Shapiro (2003) examine how the performance-flow relationship affects a fund manager's behavior in partial equilibrium settings. Shleifer and Vishny (1997) show that professional arbitrageurs who are subject to a performance-flow relationship might be unable to bring prices close to fundamentals. This is because when prices diverge from fundamentals, arbitrageurs realize capital losses, and this leads to an outflow of funds from the arbitrage sector.[10] In this paper, we examine the implications of the performance-flow relationship for the pricing of liquidity.

Kyle and Xiong (2001) derive some of the same phenomena as in this paper through a wealth effect. They show that subsequent to a noise shock that depresses prices, arbitrageurs' wealth decreases, risk aversion increases, and volatility and asset correlations increase. Danielsson and Zigrand (2003) derive the increase in risk aversion through value-at-risk (VaR) constraints: subsequent to a noise shock, some traders hit their VaR constraints, and the market's effective risk aversion increases. In those papers, assets do not carry any transaction costs. Also, the time-variation is generated through noise shocks rather than through exogenous changes in uncertainty.

Scholes (2000) points out that liquid assets have an option-type feature because they give their owner the option to convert them easily into cash if needed. Thus, liquid assets are more valuable during volatile times, a result that we show formally in this paper.[11] In fact, liquid assets in our model are similar to out-of-the-money digital options. This is because substituting a liquid asset for an illiquid one saves the manager the transaction cost of selling the illiquid asset when performance falls below the threshold, but makes no difference otherwise.

The rest of this paper is organized as follows. Section 2 presents the model. Section 3 considers the case where withdrawals do not depend on performance, and Section 4 allows withdrawals to be performance-based. Section 5 presents the model's empirical implications. Section 6 concludes, and all proofs are in the Appendix.

[10]The literature has pointed out several other general-equilibrium implications of delegated portfolio management. Scharfstein and Stein (1990) show that the fund managers' desire to signal themselves as high-ability can lead them to herd on the same asset. Allen and Gorton (1993), Dow and Gorton (1997), and Dasgupta and Prat (2003) show that the same signalling concern can lead managers to engage in asset churning. Brennan (1993) and Cuoco and Kaniel (2001) show that benchmarking managers on a subset of available assets can affect these assets' risk premia relative to the other assets.

[11]Scholes' argument is exposed in more detail in Dunbar (2000, pp.196-197). See also Longstaff (1995), who uses option-pricing theory to derive an upper bound on the cost of illiquidity.

# 2 Model

We consider a continuous-time, infinite-horizon economy. The riskless rate is exogenous and equal to $r$. There are $N$ risky assets, whose dividends exhibit stochastic volatility. Volatility is characterized by a parameter $v_t$, which is common to all assets, and evolves according to a square-root process

$$dv_t = \gamma(\overline{v} - v_t)dt + \sigma\sqrt{v_t}dB_t^v. \tag{1}$$

The dividend flow $\delta_{nt}$ of asset $n = 1, .., N$, evolves according to the process

$$d\delta_{nt} = \kappa(\overline{\delta} - \delta_{nt})dt + \sqrt{v_t}\left(\phi_n dB_t + \psi_n \sigma dB_t^v + dB_{nt}\right), \tag{2}$$

reverting to a long-run mean $\overline{\delta}$ at rate $\kappa$, and subject to three independent Brownian shocks $dB_t$, $dB_t^v$, and $dB_{nt}$. The shock $dB_t^v$ is that of the volatility process, and represents systematic volatility risk. The shock $dB_t$ represents residual systematic risk, and $dB_{nt}$ represents idiosyncratic risk. The effects of the systematic shocks $dB_t$ and $dB_t^v$ are amplified by the coefficients $\phi_n$ and $\psi_n$, which are specific to asset $n$.[12] The effects of all shocks are also amplified by the square root of the volatility parameter $v_t$. For brevity, we refer to $v_t$ simply as the volatility from now on. The assumption that systematic volatility affects also the idiosyncratic shocks is for simplicity, and plays only a small role in our analysis.

Asset $n$ is in supply of $S_n$ shares. Trading these shares is subject to an exogenous cost $\epsilon_n$ per share, paid by both the buyer and the seller. Thus, asset $n$ can be bought at the price $p_{nt} + \epsilon_n$, and sold at $p_{nt} - \epsilon_n$, where $p_{nt}$ denotes the average of the two prices. We refer to $p_{nt}$ as asset $n$'s price, and evaluate an investment in the asset at that price. Transaction costs can differ across assets, and we refer to assets with low (high) transaction costs as liquid (illiquid). While transaction costs are taken as exogenous, they could arise for reasons outside the model such as asymmetric information, market-maker inventory costs, search, etc.

Our modelling of investors is based on the notion that these are fund managers, subject to withdrawals that depend on the fund's performance. We attempt to capture this notion in the simplest possible manner, to keep our dynamic asset pricing model manageable.

We denote a fund's size by $W_t$, and assume that the manager has full discretion over the allocation of $W_t$ across assets. The manager cares about fund performance through an incentive fee that depends on fund size. This fee is exogenous, and accrues to the manager at the rate $aW_t$, where $a$ is a positive constant. The manager is infinitely lived and has preferences over

[12]We multiply $\psi_n$ by $\sigma$ to ensure that the effects of $dB_t^v$ on the volatility and the dividend flow are comparable. We could, of course, eliminate $\sigma$ by redefining $\psi_n$.

intertemporal consumption. Consumption is derived from the fee, and the manager can save or borrow using the riskless asset. The manager's preferences are of the CARA type

$$-E\int_0^\infty \exp(-\alpha c_t - \beta t)dt, \tag{3}$$

where $c_t$ denotes the consumption rate. There is a continuum of managers with mass one.

The individuals investing in a fund can decide to withdraw their wealth. For simplicity, we assume that withdrawals are extreme in that when they occur, a fund's size is reduced to zero and all assets are liquidated. Liquidation can occur after poor performance, and also for random reasons. To motivate our modelling of liquidation, suppose that individuals monitor performance at discrete times $k\Delta t$, where $k \in \mathbb{Z}$ and $\Delta t > 0$. Suppose further that performance at time $t = k\Delta t$ is measured by the change in fund size during the interval $[t - \Delta t, t]$.[13] (Thus, individuals have short memories, focusing on performance only in the most recent period.) Suppose finally that individuals monitor with probability $\hat{\mu}$, and decide to liquidate if performance is below a threshold $-\hat{L}$. Then, liquidation at time $t$ occurs with probability

$$\hat{\mu}\text{Prob}\left(W_t - W_{t-\Delta t} \leq -\hat{L}\right).$$

We next assume that $\Delta t$ is small, and set $\hat{\mu} = \mu\Delta t$ and $\hat{L} = L\sqrt{\Delta t}$. Liquidation then occurs at the rate $\mu\pi_t$, where

$$\pi_t \equiv \lim_{\Delta t \to 0} \text{Prob}\left(W_t - W_{t-\Delta t} \leq -L\sqrt{\Delta t}\right). \tag{4}$$

To this rate, we add a constant $\lambda$, capturing the notion that liquidation can also occur for random reasons. The rate $\lambda$ can be arbitrarily small, and its purpose is to ensure that trade occurs even when $\mu = 0$.

The limit probability $\pi_t$ depends on the liquidation threshold and the instantaneous variance of the fund's portfolio. The latter depends on the fund's investment in the risky assets, and the assets' equilibrium price process. Quite importantly, it also depends on the volatility $v_t$, and it is this link that generates the manager's time-varying preference for liquidity. We compute $\pi_t$ as part of the equilibrium in Section 4.

After liquidation, a manager starts a new fund whose size is an increasing function of the old fund's liquidation value (a low liquidation value might, for example, reduce a manager's reputation, and ability to raise the new fund). For simplicity, we assume that the manager starts the new fund immediately after liquidation, and the new fund's size is exactly equal to the old fund's liquidation

[13] Note that we are measuring performance by the change in fund size, rather than the fund's return. This is consistent with the assumption of CARA preferences, since under these preferences the investment in the risky assets is independent of fund size.

value.[14]

We finally assume that in addition to the liquidating withdrawal, the individuals investing in a fund perform a continuous withdrawal at the rate $(r - a)W_t$. This assumption is for simplicity, to ensure that the total outflow while a fund is in operation (i.e., the manager's fee plus the continuous withdrawal) is equal to $rW_t$, the annuity value of the fund's size.

Some of our assumptions are admittedly quite strong. For example, we are assuming that the individuals investing in a fund base their liquidation decision only on the fund's very recent performance, rather than on a longer-term measure. Moreover, a fund is evaluated in isolation, rather than relative to other funds (which in equilibrium, have the same performance because they are holding the same portfolio). A more satisfactory model of liquidation would be based on an explicit agency problem between rational fund investors and the fund manager. For example, we could assume that managers differ in their ability, and investors decide to liquidate when they are sufficiently convinced that the manager is of low ability.[15] Allowing for manager heterogeneity would, however, complicate the model because managers would hold different portfolios. Furthermore, the inferences on ability might depend on a fund's full history, and this would introduce additional state variables.

The assumption that the size of a new fund depends on the old fund's liquidation value ensures that managers care about the transaction costs incurred at liquidation. An alternative, and perhaps less ad-hoc, way to ensure this is to assume that transaction costs are not verifiable. That is, individuals receive the fund's market value at liquidation (computed using the average of buying and selling prices), minus a penalty that does not depend on the transaction costs that liquidation entails. This seems an accurate description of how mutual funds manage withdrawals: investors are not charged for the specific transaction costs incurred in selling assets to meet a withdrawal, but are sometimes charged a fixed withdrawal penalty.

Our motive in making the above assumptions is not to derive an elaborate model of delegated portfolio management, but one that is simple enough to embed into a dynamic asset pricing setting. At the same time, our results should hold under more general assumptions. Suppose, for example, that managers differ in their ability, and hold different portfolios. Then, differences in relative performance would increase in volatile times, and so would the probability of liquidation. The ideas in our model might also apply outside the fund-management context. Suppose, for example, that investors are holding levered positions. Then, a decrease in the positions' value could trigger

[14]Penalizing the manager for liquidation, by assuming that the new fund's size is lower than the old fund's liquidation value, would strengthen our results.

[15]Heinkel and Stoughton (1994) show that performance-based liquidation can be the outcome of optimal contracting. They consider a two-period model, in which managers' ability is unknown to investors, and managers can exert unobservable effort. In their model, managers are screened not through their choice of first-period contract, but at the end of that period when they can be fired based on the fund's performance.

liquidation of assets. Moreover, investors would have a preference for holding liquid assets in volatile times, when liquidation is more likely.

Finally, our model is formally equivalent to a representative-investor model with "liquidation events." Suppose that a continuum of identical investors manage their own wealth, with preferences given by (3). Suppose also that these investors are subject to liquidation events which arrive at the rate $\lambda + \mu\pi_t$, forcing them to liquidate their portfolio and immediately form a new one. Then, asset prices would be the same is in our fund-manager model, provided that the investors' risk-aversion coefficient is equal to $\alpha a/r$.[16] The representative-investor model requires a smaller set of assumptions than the fund-manager model. (For example, we do not need to introduce the incentive fee, or distinguish between the fund's size and the manager's own wealth.) We use the fund-manager model, however, because of its more transparent interpretation.[17]

# 3 No Performance-Based Liquidation

In this section, we study the equilibrium in the benchmark case where liquidation is not based on the fund's performance, but can occur only for random reasons. Liquidation thus occurs at the constant rate $\lambda$, and the monitoring rate $\mu$ is equal to zero.

## 3.1 Equilibrium

An equilibrium is characterized by price processes for the $N$ risky assets, and portfolio policies by the fund managers. Since managers have CARA preferences, their risky-asset portfolio is independent of fund size. Thus, all managers hold the same portfolio, which by market clearing is the market portfolio, i.e., $S_n$ shares of asset $n = 1, .., N$.

To determine the equilibrium price process, we must consider the managers' optimization problem, and write that the solution to that problem is to hold the market portfolio. A manager is facing two budget constraints: one describing the evolution of the fund's size and one the evolution

[16]The modified risk-aversion coefficient is because the investors consume the annuity value of their wealth, $rW_t$, while the managers' consumption is derived from the incentive fee $aW_t$.

[17]Our model can also be given an interpretation based on value-at-risk (VaR) constraints. Suppose that a trader's position is liquidated if the trader breaches a VaR constraint, and if the trader's superiors happen to monitor the position. Since a VaR constraint is of the form $\pi_t \leq \pi$, the position is liquidated at a rate $F(\pi_t)$, where $F(\pi_t) = 0$ if $\pi_t \leq \pi$, and $F(\pi_t) = \mu$ if $\pi_t > \pi$. Thus, the liquidation rate is a monotone transformation of $\pi_t$. Note that under the VaR interpretation, the assumption that performance is evaluated over a very short time interval becomes quite natural.

of his own wealth. While a fund is in operation, its size $W_t$ evolves according to

$$dW_t = r\left(W_t - \sum_{n=1}^{N} x_{nt}p_{nt}\right)dt + \sum_{n=1}^{N} x_{nt}\left(\delta_{nt}dt + dp_{nt}\right) - rW_t dt - \sum_{n=1}^{N} \epsilon_n \left|\frac{dx_{nt}}{dt}\right|, \tag{5}$$

where $x_{nt}$ denotes the number of shares invested in asset $n$. The first term in equation (5) is the return on the amount invested in the riskless asset, the second is the return on the risky-asset portfolio, the third is the outflow from the fund (due to the manager's fee and the withdrawals), and the fourth are the transaction costs of modifying the risky-asset portfolio.[18] When liquidation occurs, the change $\Delta W_t$ between the size of the new fund and the (pre-liquidation) size of the old fund is

$$\Delta W_t = -\sum_{n=1}^{N} \epsilon_n \left(|x_{nt}^-| + |x_{nt}^+|\right), \tag{6}$$

where $x_{nt}^+$ and $x_{nt}^-$ denote the investment in asset $n$ by the new and the old fund, respectively. The fund size decreases due to the transaction costs incurred in liquidation and in establishing new positions.

The manager's own wealth $w_t$ evolves according to

$$dw_t = \left(rw_t + aW_t - c_t\right)dt, \tag{7}$$

where the first term is the return on wealth (the manager can save or borrow using the riskless asset), the second is the fee, and the third is the consumption. The manager chooses the fund's risky-asset portfolio $(x_{1t}, .., x_{Nt})$ and his own consumption $c_t$ to maximize the utility (3), subject to the constraints (5)-(7) and a no-Ponzi condition stated in the Appendix.

To simplify the analysis of the manager's optimization problem, we make a conjecture as to the form of the equilibrium price process. We conjecture that in equilibrium, the price process of asset $n$ is of the form

$$p_{nt} = \frac{\overline{\delta}}{r} + \frac{\delta_{nt} - \overline{\delta}}{r+\kappa} - q_n(v_t), \tag{8}$$

for some function $q_n(v_t)$. The intuition for equation (8) is very simple. The first two terms represent the present value of expected dividends discounted at the riskless rate. This present value depends on the long-run mean, $\overline{\delta}$, of the dividend flow, and the deviation, $\delta_{nt} - \overline{\delta}$, between the dividend flow and the long-run mean. Since the deviation is temporary, it has a smaller impact on the present value. The third term, $q_n(v_t)$, measures the extent to which the price is discounted relative to the present value. It is a combination of a risk and a liquidity premium, and is a function of the

[18] We are restricting attention to processes $x_{nt}$ that are differentiable almost everywhere. This is without loss since in equilibrium the process $x_{nt}$ is constant (and equal to $S_n$).

volatility $v_t$.

Using the conjectured form of the price process, we can simplify the budget constraint (5). To state the budget constraint in its simpler form, we introduce some notation that we use throughout this paper. For a risky-asset portfolio $x_t \equiv (x_{1t}, .., x_{Nt})$, we denote by

$$\phi(x_t) \equiv \sum_{n=1}^{N} x_{nt}\phi_n \tag{9}$$

and

$$\psi(x_t) \equiv \sum_{n=1}^{N} x_{nt}\psi_n \tag{10}$$

the coefficients characterizing the sensitivity of the portfolio's dividend flow to the systematic shocks $dB_t$ and $dB_t^v$. We denote by

$$q(v_t, x_t) \equiv \sum_{n=1}^{N} x_{nt} q_n(v_t) \tag{11}$$

the portfolio's price discount relative to the present value of expected dividends, and by

$$\epsilon(x_t) \equiv \sum_{n=1}^{N} x_{nt}\epsilon_n \tag{12}$$

the transaction costs of liquidating the portfolio. We also set

$$\|x_t\|^2 \equiv \sum_{n=1}^{N} x_{nt}^2. \tag{13}$$

Finally, we denote the market portfolio $(S_1, .., S_N)$ by $S$, and set $\phi \equiv \phi(S)$, $\psi \equiv \psi(S)$, $\epsilon = \epsilon(S)$, and $q(v_t) \equiv q(v_t, S)$.

**Lemma 1** *The budget constraint (5) is equivalent to*

$$\begin{aligned} dW_t &= \left[ rq(v_t, x_t) - q_v(v_t, x_t)\gamma(\overline{v} - v_t) - \frac{1}{2} q_{vv}(v_t, x_t)\sigma^2 v_t \right] dt \\ &\quad + \sqrt{v_t} \left[ \frac{\phi(x_t)}{r+\kappa} dB_t + \left[ \frac{\psi(x_t)}{r+\kappa} - q_v(v_t, x_t) \right] \sigma dB_t^v + \frac{\sum_{n=1}^{N} x_{nt} dB_{nt}}{r+\kappa} \right] \\ &\quad - \sum_{n=1}^{N} \epsilon_n \left| \frac{dx_{nt}}{dt} \right|. \end{aligned} \tag{14}$$

Lemma 1 shows that the drift of the fund's size is determined by the price discount of the fund's portfolio, and the price discount's own drift. The diffusion is, in turn, determined by the diffusion of the present value of the portfolio's expected dividends, and the diffusion of the portfolio's price discount. Note that neither the drift nor the diffusion involve the assets' dividend flows $\{\delta_{nt}\}_{n=1,..,N}$. Thus, dividend flows do not appear as state variables in the manager's optimization problem, and do not influence the manager's optimal risky-asset portfolio. Because of CARA preferences, the optimal portfolio is also independent of fund size, and can thus only depend on the volatility $v_t$. Writing that the optimal portfolio is independent of $v_t$ (and equal to the market portfolio, i.e., $x_t = S$) is what determines the functions $\{q_n(v_t)\}_{n=1,..,N}$.

To solve the manager's optimization problem, we must consider the maximum utility that the manager can achieve under the equilibrium price process (i.e., the value function). We conjecture that it is of the form

$$V(W_t, w_t, v_t) = -\exp\left[-A\left[W_t + zw_t + Z(v_t)\right]\right], \tag{15}$$

where $A \equiv \alpha a$, $z \equiv r/a$, and $Z(v_t)$ is a function of $v_t$. Utility over the wealth variables (i.e., the fund's size $W_t$ and the manager's own wealth $w_t$) is of the CARA type, as is the utility over consumption. Utility is more sensitive to the manager's own wealth than the fund's size as long as $r > a$, i.e., the manager's fee is lower than the annuity value of the fund's size. Utility also depends on the volatility $v_t$, because $v_t$ affects the manager's investment opportunity set.

The manager's first-order condition takes a simple and intuitive form. Define the instantaneous return on asset $n$ as

$$dR_{nt} \equiv dp_{nt} + \delta_{nt}dt - rp_{nt}dt.$$

This return is per share (i.e., is the dollar return times the share price), and is in excess of the riskless rate. Define also the instantaneous return on the market portfolio as

$$dR_{Mt} \equiv \sum_{n=1}^{N} S_n dR_{nt}.$$

This return is again per share, and the market portfolio is treated as one big share. The manager's first-order condition (i.e., that holding the market portfolio is optimal) is

$$E_t(dR_{nt}) = A\mathrm{Cov}_t(dR_{nt}, dR_{Mt}) + AZ'(v_t)\mathrm{Cov}_t(dR_{nt}, dv_t) + \left[r + 2\lambda\exp(2A\epsilon)\right]\epsilon_n dt.^{19} \tag{16}$$

Equation (16) is a conditional two-factor CAPM, adjusted for transaction costs. Asset $n$'s expected return is determined by the asset's covariance with the market portfolio, the covariance with volatility, and the asset's transaction costs. The covariance with volatility matters because $v_t$ enters into the value function, introducing a hedging demand. Transaction costs increase the asset's return

[19] We derive this first-order condition as part of the proof of Proposition 1.

because the manager must be compensated for incurring these costs every time liquidation occurs. From now on, we refer to the adjustment for transaction costs as the asset's liquidity premium, and to the covariance terms as the asset's risk premium.

The first-order condition (16) can be written as an ordinary differential equation (ODE), involving the functions $\{q_n(v_t)\}_{n=1,..,N}$ and $Z(v_t)$. In the proof of Lemma 1, we show that asset $n$'s return is

$$\begin{aligned} dR_{nt} &= \left[rq_n(v_t) - q_n'(v_t)\gamma(\overline{v} - v_t) - \frac{1}{2}q_n''(v_t)\sigma^2 v_t\right] dt \\ &\quad + \sqrt{v_t}\left[\frac{\phi_n}{r+\kappa}dB_t + \left[\frac{\psi_n}{r+\kappa} - q_n'(v_t)\right]\sigma dB_t^v + \frac{1}{r+\kappa}dB_{nt}\right]. \end{aligned} \tag{17}$$

Multiplying equation (17) by $S_n$, and adding over assets, we find that the return on the market portfolio is

$$\begin{aligned} dR_{Mt} &= \left[rq(v_t) - q'(v_t)\gamma(\overline{v} - v_t) - \frac{1}{2}q''(v_t)\sigma^2 v_t\right] dt \\ &\quad + \sqrt{v_t}\left[\frac{\phi}{r+\kappa}dB_t + \left[\frac{\psi}{r+\kappa} - q'(v_t)\right]\sigma dB_t^v + \frac{\sum_{n=1}^N S_n dB_{nt}}{r+\kappa}\right]. \end{aligned} \tag{18}$$

Using equations (1), (17), and (18), we can compute asset $n$'s expected return, covariance with the market portfolio, and covariance with volatility. Substituting these into equation (16), we find the second-order ODE

$$\begin{aligned} & rq_n(v_t) - q_n'(v_t)\gamma(\overline{v} - v_t) - \frac{1}{2}q_n''(v_t)\sigma^2 v_t \\ & -A\left[\frac{\phi_n\phi}{(r+\kappa)^2} + \left[\frac{\psi_n}{r+\kappa} - q_n'(v_t)\right]\left[\frac{\psi}{r+\kappa} - \left[q'(v_t) - Z'(v_t)\right]\right]\sigma^2 + \frac{S_n}{(r+\kappa)^2}\right]v_t \\ & - \left[r + 2\lambda\exp(2A\epsilon)\right]\epsilon_n = 0. \end{aligned} \tag{19}$$

There are $N$ ODEs (19), one for every risky asset. An additional ODE can be derived by the manager's Bellman equation. Taken together, these constitute a system of $N+1$ ODEs for the functions $\{q_n(v_t)\}_{n=1,..,N}$ and $Z(v_t)$. In the case of no performance-based liquidation, this system has a simple, affine solution.

**Proposition 1** *Suppose that $\mu = 0$ and*

$$r + \gamma > A\frac{\sqrt{\phi^2 + \psi^2\sigma^2 + \|S\|^2} - \psi\sigma}{r+\kappa}\sigma. \tag{20}$$

*Then, there exists an equilibrium in which the functions* $\{q_n(v_t)\}_{n=1,..,N}$ *and* $Z(v_t)$ *are affine, i.e.,*

$$q_n(v_t) = q_{n0} + q_{n1}(v_t - \overline{v}),$$

$$Z(v_t) = Z_0 + Z_1(v_t - \overline{v}).$$

*The coefficients* $q_{n0}$ *and* $q_{n1}$ *are*

$$q_{n0} = \frac{A\overline{v}}{r}\left[\frac{\phi_n\phi + S_n}{(r+\kappa)^2} + \left(\frac{\psi_n}{r+\kappa} - q_{n1}\right)\left(\frac{\psi}{r+\kappa} - f_1\right)\sigma^2\right] + \left[1 + 2\frac{\lambda}{r}\exp(2A\epsilon)\right]\epsilon_n \tag{21}$$

*and*

$$q_{n1} = \frac{\frac{\phi_n\phi+S_n}{(r+\kappa)^2} + \frac{\psi_n}{r+\kappa}\left(\frac{\psi}{r+\kappa} - f_1\right)\sigma^2}{\frac{r+\gamma}{A} + \left(\frac{\psi}{r+\kappa} - f_1\right)\sigma^2}, \tag{22}$$

*where* $f_1$ *is the smaller of the two positive roots of the quadratic equation*

$$(r+\gamma)f_1 - \frac{1}{2}A\left[\frac{\phi^2 + \|S\|^2}{(r+\kappa)^2} + \left(\frac{\psi}{r+\kappa} - f_1\right)^2\sigma^2\right] = 0. \tag{23}$$

*The coefficients* $q_1 \equiv \sum_{n=1}^N S_n q_{n1}$ *and* $Z_1$ *are positive.*

The coefficient $q_{n0}$ is the average value of asset $n$'s price discount $q_n(v_t)$ (since $E(v_t) = \overline{v}$). The discount arises because of asset $n$'s risk premium and liquidity premium. The risk premium is reflected in the first term of equation (21). Asset $n$'s risk is measured by the parameters $\phi_n$ (sensitivity of dividend flow to residual systematic risk), $\psi_n$ (sensitivity of dividend flow to volatility risk), $q_{n1}$ (sensitivity of price discount to volatility risk), and $S_n$ (asset supply), since these determine the asset's covariance with the market portfolio and the volatility. The liquidity premium is reflected in the second term of equation (21). This premium is increasing in asset $n$'s transaction costs $\epsilon_n$, and in the liquidation rate $\lambda$. It is also increasing in the costs $\epsilon$ of liquidating the overall portfolio because high costs make liquidation a bad outcome, increasing the marginal utility associated with it.

The coefficient $q_{n1}$ measures the sensitivity of asset $n$'s price discount to changes in the volatility. Volatility affects the discount because it affects the risk premium. Thus, $q_{n1}$ depends on the characteristics of asset $n$ that determine the risk premium, i.e., $\phi_n$, $\psi_n$, and $S_n$. It also depends on the rate $\gamma$ at which volatility reverts to its long-run mean: a high $\gamma$ means that volatility shocks are transitory and have a small effect on the discount.

In the case of no performance-based liquidation, the liquidation rate is independent of the volatility, and so is the liquidity premium. Thus, the coefficient $q_{n1}$ does not depend on asset's $n$'s transaction costs $\epsilon_n$. In other words, transaction costs do not introduce any variation in an asset's price, and do not affect the asset's risk premium. This will no longer be the case when liquidation is performance-based.

### 3.2 Betas and Correlations

We next compute asset betas and correlations. We show that in the case of no performance-based liquidation, conditional betas and correlations are independent of the volatility, and are thus constant over time. Furthermore, the constant values of these variables coincide with the unconditional betas and correlations. These results provide a simple benchmark for our subsequent analysis: for example, any dependence of the conditional betas on volatility can be only because liquidation is performance-based.

For each asset $n$, we consider the conditional beta w.r.t. the market portfolio:

$$\beta_{nt}^M \equiv \frac{\mathrm{Cov}_t(dR_{nt}, dR_{Mt})}{\mathrm{Var}_t(dR_{Mt})},$$

the conditional beta w.r.t. the volatility:

$$\beta_{nt}^v \equiv \frac{\mathrm{Cov}_t(dR_{nt}, dv_t)}{\mathrm{Var}_t(dv_t)},$$

the conditional correlation with any other asset $m$:

$$\rho_{mnt} \equiv \frac{\mathrm{Cov}_t(dR_{mt}, dR_{nt})}{\sqrt{\mathrm{Var}_t(dR_{mt})\mathrm{Var}_t(dR_{nt})}},$$

and the conditional correlation with the volatility:

$$\rho_{nt}^v \equiv \frac{\mathrm{Cov}_t(dR_{nt}, dv_t)}{\sqrt{\mathrm{Var}_t(dR_{nt})\mathrm{Var}_t(dv_t)}}.$$

We also consider the unconditional counterparts of these betas and correlations, and denote them without the subscript $t$. In the case of no performance-based liquidation, conditional betas and correlations are independent of the volatility, and are thus constant over time.

**Proposition 2** *For $\mu = 0$, conditional betas and correlations are independent of $v_t$.*

The intuition behind the proposition can be seen from equation (17) which determines the return on an asset:

$$dR_{nt} \quad = \quad \left[ rq_n(v_t) - q_n'(v_t)\gamma(\overline{v} - v_t) - \frac{1}{2}q_n''(v_t)\sigma^2 v_t \right] dt$$

$$+\sqrt{v_t}\left[\frac{\phi_n}{r+\kappa}dB_t+\left[\frac{\psi_n}{r+\kappa}-q_n'(v_t)\right]\sigma dB_t^v+\frac{1}{r+\kappa}dB_{nt}\right].$$

The diffusion component of the return is generated by the three shocks $dB_t$, $dB_t^v$, and $dB_{nt}$. Quite crucially, the relative effects of these shocks are independent of the volatility, and are thus constant over time. Indeed, the relative effects of $dB_t$ to $dB_t^v$ to $dB_{nt}$ are

$$\frac{\phi_n}{r+\kappa}\Big/\left[\frac{\psi_n}{r+\kappa}-q_n'(v_t)\right]\sigma\Big/\frac{1}{r+\kappa}, \tag{24}$$

and are constant because the function $q_n(v_t)$ is affine. Since the relative effects are constant and the multiplicative effect of the volatility is in $\sqrt{v_t}$, the conditional covariance matrix of returns is equal to a constant matrix times $v_t$. The same applies to the conditional covariance matrix of returns and volatility because the diffusion component of the volatility is in $\sqrt{v_t}$. Since conditional betas and correlations are homogeneous functions of degree zero in the elements of the latter matrix, they are independent of $v_t$.

Two assumptions of our model are critical for Proposition 2. The first is that the relative effects of the shocks $dB_t$, $dB_t^v$, and $dB_{nt}$ on the dividend flow are independent of the volatility. This ensures that the shocks' relative effects on the present value of expected dividends are constant. The second assumption is that $v_t$ follows a square-root process. This ensures that the diffusion component of the affine price discount $q_n(v_t)$ is in $\sqrt{v_t}$, as are those of the present value of expected dividends, and of the volatility itself. While our assumptions are plausible, they are also special, and one could envision plausible alternatives.[20] An advantage of these assumptions, however, is to provide a simple benchmark for isolating the new effects introduced by performance-based liquidation.

We next consider the unconditional betas and correlations, which are determined by the unconditional covariance matrix of returns and volatility. In continuous time, an unconditional covariance is the expectation of its conditional counterpart. Indeed, since

$$\mathrm{Cov}(dX_t, dY_t) = E(dX_t dY_t) - E(dX_t)E(dY_t),$$

and the second term is negligible relative to the first, we have

$$\mathrm{Cov}(dX_t, dY_t) = E(dX_t dY_t) = E\left[E_t(dX_t dY_t)\right] = E\left[\mathrm{Cov}_t(dX_t, dY_t)\right]. \tag{25}$$

Therefore, the unconditional covariance matrix of returns and volatility is the expectation of the

[20] One alternative is that the systematic volatility $v_t$ does not affect the idiosyncratic shocks. Under this assumption, the pairwise correlations between assets would increase in $v_t$, and so would the correlation between an asset and $v_t$. (Since with high systematic volatility, idiosyncratic shocks would have small relative effects.) Asset betas would, however, be independent of $v_t$, and so would the correlation between $v_t$ and the market portfolio. (Assuming, of course, a large number of assets, so that idiosyncratic shocks have no effect on the market portfolio return.)

conditional covariance matrix. Since the latter is equal to a constant matrix times $v_t$, the former is equal to the same constant matrix times the expectation of $v_t$. Therefore, the unconditional betas and correlations coincide with their conditional counterparts.

**Proposition 3** *For $\mu = 0$, unconditional betas and correlations coincide with their conditional counterparts.*

### 3.3 Expected Returns

In equilibrium, expected returns are determined by a conditional two-factor CAPM, adjusted for transaction costs. This CAPM is given by equation (16), and the two factors are the market and the volatility. In Lemma 2, we restate this CAPM, using the asset betas instead of the covariances.

**Lemma 2** *In equilibrium, conditional expected returns are given by*

$$E_t(dR_{nt}) = \left[\beta_{nt}^M \Lambda_{Mt} + \beta_{nt}^v \Lambda_{vt} + [r + 2\lambda \exp(2A\epsilon)]\, \epsilon_n\right] dt, \tag{26}$$

*where*

$$\Lambda_{Mt} \equiv A \left[\frac{\phi^2}{(r+\kappa)^2} + \left(\frac{\psi}{r+\kappa} - q_1\right)^2 \sigma^2 + \frac{\|S\|^2}{(r+\kappa)^2}\right] v_t,$$

$$\Lambda_{vt} \equiv A Z_1 \sigma^2 v_t.$$

Starting from the conditional CAPM, we can easily derive an unconditional CAPM. Since conditional and unconditional betas coincide, the unconditional CAPM follows immediately by taking expectations in the conditional CAPM equation.

**Proposition 4** *In equilibrium, unconditional expected returns are given by*

$$E(dR_{nt}) = \left[\beta_n^M \Lambda_M + \beta_n^v \Lambda_v + [r + 2\lambda \exp(2A\epsilon)]\, \epsilon_n\right] dt, \tag{27}$$

*where $\Lambda_M \equiv E(\Lambda_{Mt})$ and $\Lambda_v \equiv E(\Lambda_{vt})$.*

Proposition 4 implies that an asset's risk can be characterized completely by two coefficients: the unconditional market and volatility betas. This result provides a benchmark for the analysis of performance-based liquidation, where the characterization of asset risk is more complicated.

The coefficients $\Lambda_M$ and $\Lambda_v$ are the factor risk premia. The volatility risk premium arises because of the managers' hedging demand, i.e., because volatility affects the managers' investment opportunity set. When volatility is zero, all assets have the same return as the riskless asset, and thus the opportunity set is equivalent to that asset. An increase in volatility improves the opportunity set because managers have more options. Therefore, holding wealth constant, managers are better off with high volatility. (Formally, Proposition 1 shows that $Z_1$, the derivative of the function $Z(v_t)$ which characterizes the effect of volatility on the value function, is positive.) This implies that holding the covariance with the market portfolio constant, managers prefer assets that pay off when volatility is low. Thus, assets that are positively correlated with volatility carry a positive premium, and this means that the volatility risk premium is positive.[21]

# 4 Performance-Based Liquidation

In this section, we allow for liquidation to be based on the fund's performance. We assume that the monitoring rate $\mu$ is positive, so that the liquidation rate $\lambda + \mu\pi_t$ depends on the probability $\pi_t$ of poor performance. The probability $\pi_t$ depends on the equilibrium price process, and we compute it below as part of the equilibrium.

## 4.1 Equilibrium

We conjecture that the equilibrium price process takes the same form as in the case of no performance-based liquidation: it is given by equation (8), for a possibly different set of discount functions $\{q_n(v_t)\}_{n=1,..,N}$. Given this process, we can compute the probability $\pi_t$ of poor performance. Recall from equation (4) that $\pi_t$ is the probability that the change $W_t - W_{t-\Delta t}$ in fund size over the small interval $[t - \Delta t, t]$ is below the threshold $-L\sqrt{\Delta t}$. The change in fund size is approximately normal with a variance of order $\Delta t$. Denoting this variance by $z_t \Delta t$, $\pi_t$ is

$$\pi_t = N\left(-\frac{L}{\sqrt{z_t}}\right) \equiv g(z_t), \tag{28}$$

where $N(.)$ is the cumulative distribution function of the normal distribution. From the budget constraint (14), $z_t$ is

$$z_t = \left[\frac{\phi(x_t)^2}{(r+\kappa)^2} + \left[\frac{\psi(x_t)}{r+\kappa} - q_v(v_t, x_t)\right]^2 \sigma^2 + \frac{\|x_t\|^2}{(r+\kappa)^2}\right] v_t. \tag{29}$$

Equations (28) and (29) determine the probability $\pi_t$ of poor performance. This probability depends on the instantaneous variance $z_t$ of the fund's portfolio, through the function $g(z_t)$. In turn, $z_t$ depends on the volatility $v_t$, the fund's investments $x_t$, and the equilibrium discount functions

[21] The result that managers prefer assets that pay off when the investment opportunity set is poor follows because of CARA preferences. It also holds for an investor with CRRA preferences, provided that risk aversion is greater than in the logarithmic case.

$\{q_n(v_t)\}_{n=1,..,N}$. In what follows, we often denote $\pi_t$ by $\pi(v_t, x_t)$ and $z_t$ by $z(v_t, x_t)$, to emphasize the dependence on $v_t$ and $x_t$. We also set $\pi(v_t) \equiv \pi(v_t, S)$ and $z(v_t) \equiv z(v_t, S)$.

Figure 1 plots $\pi_t$ as a function of $z_t$. For small values of $z_t$, $\pi_t$ cannot be distinguished from zero, because the event that performance falls below the threshold requires a movement of several standard deviations. As $z_t$ increases, $\pi_t$ moves away from zero, and increases in an approximately linear manner. For larger values of $z_t$, $\pi_t$ becomes concave. This concavity is necessary because $\pi_t$ must converge to 1/2 when $z_t$ goes to infinity: for large portfolio variance, performance falls below the threshold with almost any negative movement.

The convexity or concavity of $\pi_t$ matters for some of our results. For example, market betas of illiquid assets, as well as asset correlations, increase with volatility only when the latter takes values in the region where $\pi_t$ is convex. This makes it important to determine whether the convexity region is the more relevant.

The convexity region is consistent with the intuitive notion that liquidations are rare during normal times but pick up dramatically in volatile, crisis times. A more detailed argument in support of convexity can be constructed by mapping our model more closely to the real world. A simplifying, and somewhat unrealistic, feature of our model is that liquidation is based only on a fund's very recent ("current") performance. Suppose instead that liquidation is based on a longer-term performance measure, to which current performance contributes only slightly. Then, current performance has a small impact on the liquidation probability, except in two cases. The first is when past performance has already brought the fund close to liquidation. More to our point, the second case is when current volatility is high, because current performance can then contribute significantly to the long-term measure. Consider now the liquidation probability as a function of current volatility (for an average level of past performance). This probability is almost independent of volatility when volatility is low, and increases for larger values, thus being convex.[22]

In what follows, we give greater emphasis to the results obtained in the convexity region (while also noting that some results hold in both regions). This is primarily because the convexity region seems the more relevant. Relevance aside, the results requiring convexity can also be viewed as a set of joint predictions such that if one holds, all others should hold as well.

When liquidation is performance-based, the manager's first-order condition is

$$\begin{aligned} E_t(dR_{nt}) \quad &= \quad A\text{Cov}_t(dR_{nt}, dR_{Mt}) + AZ'(v_t)\text{Cov}_t(dR_{nt}, dv_t) \\ &\quad +\mu\, \pi_{x_n}(v_t, x_t)|_{x_t=S}\, \frac{\exp(2A\epsilon) - 1}{A} dt \end{aligned}$$

[22]A different way to phrase this argument is that when liquidation is based on a long-term performance measure, to which current performance contributes only slightly, it takes an extremely high value of current volatility to get into the concavity region.

$$+\left[r+2[\lambda+\mu\pi(v_t)]\exp(2A\epsilon)\right]\epsilon_n dt. \tag{30}$$

Compared to the case of no performance-based liquidation (equation (16)), there are two differences. First, the liquidation rate used in computing the liquidity premium (i.e., the last term in the RHS) is $\lambda+\mu\pi(v_t)$ instead of $\lambda$. Second, there is a new term (third term in the RHS) involving the derivative of the probability of poor performance w.r.t. the investment in asset $n$. This term is a risk premium that arises because of the manager's concern with liquidation: increasing the investment in asset $n$ affects the risk of the fund's overall portfolio, thus affecting the probability that performance falls below the threshold. We refer to this term as a liquidation risk premium, to distinguish it from the standard risk premium that arises because of the risk aversion in the manager's utility function. (The standard risk premium corresponds to the first two terms in the RHS).

Using equations (28) and (29), we can write the liquidation risk premium as

$$2\mu g'\left[z(v_t)\right]\frac{\exp(2A\epsilon)-1}{A}\mathrm{Cov}_t(dR_{nt},dR_{Mt}). \tag{31}$$

The liquidation risk premium involves the covariance between asset $n$ and the market portfolio, exactly as the standard risk premium. This might seem surprising, given that liquidation is a tail event. The intuition is that because performance is evaluated over a short interval, asset returns are normal, and thus variance is the only measure of risk.

Substituting equation (31) into (30), we find

$$\begin{aligned} E_t(dR_{nt}) &= A(v_t)\mathrm{Cov}_t(dR_{nt},dR_{Mt})+AZ'(v_t)\mathrm{Cov}_t(dR_{nt},dv_t) \\ &\quad +\left[r+2[\lambda+\mu\pi(v_t)]\exp(2A\epsilon)\right]\epsilon_n dt, \end{aligned} \tag{32}$$

where

$$A(v_t)\equiv A+2\mu g'\left[z(v_t)\right]\frac{\exp(2A\epsilon)-1}{A}. \tag{33}$$

The manager's first-order condition takes the form of a conditional CAPM adjusted for transaction costs, as in the case of no performance-based liquidation. The existence of the liquidation risk premium, however, modifies the coefficient multiplying the covariance with the market portfolio. When liquidation is not performance-based, the covariance is multiplied by $A$, the coefficient of absolute risk aversion of the manager's value function. When liquidation is performance-based, the coefficient becomes $A(v_t)$, and it incorporates both the risk aversion inherent in the manager's preferences, and that arising because of the concern with liquidation. From now on, we refer to $A(v_t)$ as an effective risk-aversion coefficient.

The coefficient $A(v_t)$ increases with volatility in the convexity region of Figure 1.[23] Intuitively, when volatility is low, managers are not concerned with liquidation because the event that performance falls below the threshold requires a movement of several standard deviations. When volatility increases, however, liquidation becomes a concern, and managers choose their portfolios in a more risk-averse fashion.

The result that effective risk aversion can increase with volatility is consistent with anecdotal descriptions of flight-to-quality phenomena, i.e., of sharp increases in the market's risk aversion during periods of turmoil. (See Footnote 5 for references.) Such phenomena are hard to rationalize within a frictionless representative-investor model, because risk aversion is driven by preferences and should be stable over short time intervals. They can, however, be readily generated in our model: risk aversion increases during volatile times because managers become concerned with liquidation.[24]

The first-order condition (32) is equivalent to an ODE, as in the case of no performance-based liquidation. As in that case, there is one ODE for every asset, and one following from the Bellman equation. These $N+1$ ODEs determine the functions $\{q_n(v_t)\}_{n=1,..,N}$ and $Z(v_t)$. In Proposition 9 (stated in the Appendix), we characterize the equilibrium and derive the ODEs.

The ODEs are quite complicated because they involve the cumulative distribution function of the normal distribution. To determine the functions $\{q_n(v_t)\}_{n=1,..,N}$ and $Z(v_t)$, we focus on the special case where the parameter $\sigma$, which determines the volatility of volatility, is small. In this case, volatility stays close to its long-run mean $\overline{v}$, and we can solve the ODEs in closed form by approximating all functions by Taylor expansions. The solutions provide significant insight into the qualitative properties of the equilibrium. To study the quantitative properties, we need to solve the ODEs numerically for general values of $\sigma$. We plan to do so in a companion paper where we will calibrate the model.[25]

For our analysis, it suffices to consider Taylor expansions up to the second order. We thus set

$$q_n(v_t) = q_{n0} + q_{n1}(v_t - \overline{v}) + \frac{1}{2}q_{n2}(v_t - \overline{v})^2, \tag{34}$$

[23] This is because $z(v_t)$ is increasing in $v_t$, and $g'(z_t)$ is increasing in $z_t$ in the convexity region.

[24] At a methodological level, our model shows that asset-pricing formulas derived in the presence of constraints (e.g., liquidation) can be equivalent to formulas without constraints, but with modified preference parameters. At the same time, we show that constraints cannot be ignored, because the modified parameters can have properties that are hard to rationalize based on preferences. (In particular, the effective risk-aversion coefficient $A(v_t)$ depends on the volatility.) Other models with constraints also have implications along those lines. For example, Grossman and Vila (1992) show that a portfolio problem with leverage constraints is equivalent to an unconstrained problem in which the investor's risk-aversion coefficient is different. Basak and Cuoco (1998) show that representative-investor pricing holds even when markets are incomplete, provided that the investor's utility function is constructed using stochastic weights.

[25] The ODEs become simpler to solve if $g(z_t)$ is replaced by a piecewise-linear function (which from Figure 1 is not a bad approximation), and the interest rate $r$ is assumed small relative to the rate $\gamma$ at which volatility reverts to its long-run mean. In that case, the ODEs are of the Ricatti type, and solutions can be derived in terms of the Gamma and Kummer functions.

$$Z(v_t) = Z_0 + Z_1(v_t - \overline{v}) + \frac{1}{2}Z_2(v_t - \overline{v})^2, \tag{35}$$

ignoring terms of order three and higher. The coefficients $q_{ni}$ and $Z_i$, $i = 0, 1, 2$, are functions of $\sigma$, but it suffices to determine only the dominant term, i.e., the term of order zero in $\sigma$.[26]

**Proposition 5** *Up to order zero in $\sigma$, the coefficients $q_{ni}$, $i = 0, 1, 2$, are*

$$q_{n0} = \left[\frac{A}{r} + \frac{\mu n(\ell)\ell}{r\overline{z}}\frac{\exp(2A\epsilon) - 1}{A}\right]\overline{z}_n + \left[1 + 2\frac{\lambda + \mu N(-\ell)}{r}\exp(2A\epsilon)\right]\epsilon_n, \tag{36}$$

$$q_{n1} = \left[\frac{A}{(r+\gamma)\overline{v}} + \frac{\mu n(\ell)\ell(\ell^2 - 1)}{2(r+\gamma)\overline{zv}}\frac{\exp(2A\epsilon) - 1}{A}\right]\overline{z}_n + \frac{\mu n(\ell)\ell}{(r+\gamma)\overline{v}}\exp(2A\epsilon)\epsilon_n, \tag{37}$$

$$q_{n2} = \frac{\mu n(\ell)\ell(\ell^4 - 6\ell^2 + 3)}{4(r+2\gamma)\overline{zv}^2}\frac{\exp(2A\epsilon) - 1}{A}\overline{z}_n + \frac{\mu n(\ell)\ell(\ell^2 - 3)}{2(r+2\gamma)\overline{v}^2}\exp(2A\epsilon)\epsilon_n, \tag{38}$$

*where*

$$\overline{z}_n \equiv \frac{\phi_n\phi + S_n}{(r+\kappa)^2}\overline{v},$$

$\overline{z} \equiv \sum_{n=1}^N S_n\overline{z}_n$, *$\ell \equiv L/\sqrt{\overline{z}}$, and $n(.)$ is the density function of the normal distribution.*

The main difference relative to the case of no performance-based liquidation is in the coefficients $q_{n1}$ and $q_{n2}$. The coefficient $q_{n1}$ measures the average sensitivity of asset $n$'s price discount to changes in the volatility (since $E\left[q_n'(v_t)\right] = q_{n1} + q_{n2}E(v_t - \overline{v}) = q_{n1}$). One reason why volatility affects the discount is because it affects asset $n$'s risk premium. This effect is as in the case of no performance-based liquidation, with the difference that the risk premium now consists of the standard and the liquidation risk premium. The effect of volatility on the risk premium corresponds to the first term in equation (37): the two terms in the bracket capture the two components of the risk premium, and both are multiplied by the coefficient $\overline{z}_n$ which is the instantaneous covariance between asset $n$ and the market portfolio.

A second reason why volatility affects the discount is because it increases the liquidity premium. This is a key new effect introduced by performance-based liquidation. The intuition is that at times of high volatility, there is an increased probability that performance falls below the threshold. Thus, liquidation is more likely, and this reduces a manager's willingness to invest in illiquid (i.e., high transaction cost) assets. The effect of volatility on the liquidity premium corresponds to the second term in equation (37). This term is increasing in asset $n$'s transaction costs $\epsilon_n$, meaning that volatility has a greater impact on the discount of illiquid assets.

The coefficient $q_{n2}$ measures the convexity of the price discount, i.e., how the price discount's

[26] Since the Taylor expansions are up to the second order, and $v_t - \overline{v}$ is of the same order as $\sigma$, the coefficients $q_{n0}$ and $Z_0$ can be determined up to order two in $\sigma$, $q_{n1}$ and $Z_1$ up to order one, and $q_{n2}$ and $Z_2$ up to order zero.

sensitivity to volatility changes with volatility. In the case of no performance-based liquidation, $q_{n2}$ is zero because the price discount is affine. Performance-based liquidation can introduce convexity through the liquidation risk premium and the liquidity premium. Intuitively, both premia are very small, and almost insensitive to volatility, when volatility is low and liquidation is not a concern. For larger values of volatility, however, liquidation becomes a concern, and the two premia start increasing rapidly.

More precisely, the liquidity premium is convex in the volatility when the probability of poor performance is convex, and the liquidation risk premium is convex when the marginal increase in that probability, by taking more risk, is convex. Recall from Figure 1 that the probability of poor performance is convex when the risk of the fund's portfolio is small relative to the liquidation threshold. The exact condition is that $\ell$, the number of standard deviations required to reach the threshold, exceeds $\sqrt{3} \approx 1.73$. Under this condition, the second term in equation (38) is positive. The marginal increase in the probability of poor performance is convex under the more restrictive condition $\ell > 2.34$, which ensures that the first term in equation (38) is positive.[27]

Since the liquidity premium depends on the volatility, it accounts for a fraction of asset-price movements, thus adding to the risk in an asset's return. This risk is priced because it is systematic. Thus, transaction costs have not only a direct effect on returns through the liquidity premium, but also an indirect effect through the risk premium. In other words, assuming that transaction costs affect returns only through the liquidity premium (as is implicitly done in several empirical studies) would understate their impact. We return to this point in Section 5, where we present the model's empirical implications.

## 4.2 Betas and Correlations

When liquidation is performance-based, conditional betas and correlations depend on the volatility, and are thus time-varying. In this section, we characterize the dependence on volatility for small $\sigma$. We show, in particular, that convexity of the assets' price discounts, together with the assumption that volatility affects asset returns negatively, imply several interesting and testable properties: the correlation between an asset and the volatility becomes more negative when volatility increases, the pairwise correlation between similar assets increases with volatility, and the market betas of illiquid assets increase with volatility.

**Proposition 6** *For small $\sigma$:*

[27] Convexity of the liquidity premium requires the probability of poor performance to be convex in the volatility. In Figure 1, however, that probability is treated as a function of the instantaneous variance of the fund's portfolio. We do not emphasize the difference because for small $\sigma$, convexity under one variable implies convexity under the other, and they both hold when $\ell > \sqrt{3}$.

- *The conditional correlation between asset $n$ and the volatility decreases with volatility if $q_{n2} > 0$.*

- *The conditional correlation between assets $m$ and $n$ increases with volatility if*

$$\sum_{\substack{i,j=m,n\\ i\neq j}} q_{i2}(\phi_j^2+1)\left[-\chi_j - (\chi_j\phi_i - \chi_i\phi_j)\,\phi_i\right] > 0, \tag{39}$$

  *where*

$$\chi_n \equiv \frac{\psi_n}{r+\kappa} - q_{n1}.$$

- *The conditional market betas of illiquid assets increase with volatility (i.e., $\frac{\partial^2 \beta^M_{nt}}{\partial\epsilon_n \partial v_t} > 0$) if*

$$\frac{q_2}{r+\gamma} - \frac{\chi(\ell^2-3)}{2(r+2\gamma)\overline{v}} > 0, \tag{40}$$

  *where $q_2 \equiv \sum_{n=1}^N S_n q_{n2}$ and $\chi \equiv \sum_{n=1}^N S_n \chi_n$.*

To explain the intuition behind the proposition, we consider the diffusion component of asset returns. From equations (24) and (34), the relative effects of the shocks $dB_t$ to $dB_t^v$ to $dB_{nt}$ in generating asset $n$'s diffusion component are

$$\frac{\phi_n}{r+\kappa} \Bigg/ \left[\frac{\psi_n}{r+\kappa} - \left[q_{n1} + q_{n2}(v_t - \overline{v})\right]\right]\sigma \Bigg/ \frac{1}{r+\kappa}.$$

When liquidation is performance-based, the relative effects depend on the volatility because the price discount is not affine. To illustrate the role of volatility, we focus on the case where the average effect of the volatility shock $dB_t^v$ is negative. This amounts to assuming that

$$\chi_n \equiv \frac{\psi_n}{r+\kappa} - q_{n1} < 0,$$

i.e., any positive effect of volatility on asset $n$'s dividend flow is dominated by the negative effect on the price discount.

Suppose now that asset $n$'s price discount is convex ($q_{n2} > 0$). Then, the negative effect of volatility on asset $n$'s return becomes even more negative at times of high volatility. At those times, the volatility shock plays a greater role relative to the other shocks in driving asset $n$'s return, and the asset moves more closely with volatility. Thus, the correlation between the asset and the volatility becomes more negative (i.e., decreases), as the first result of the proposition shows.

The second result concerns the pairwise correlation between assets. One would expect this correlation to increase with volatility when (i) volatility affects asset returns negatively ($\chi_n < 0$), and (ii) price discounts are convex ($q_{n2} > 0$): these conditions ensure that at times of high volatility assets move more closely with volatility, and thus there is more scope for common variation. Conditions (i) and (ii) are, however, not fully sufficient. Suppose, for example, that asset $m$ is almost insensitive to volatility, i.e., $\chi_m$ and $q_{m2}$ are close to zero, and $\phi_m = \phi_n$. Then, the volatility shock does not affect the covariance between assets $m$ and $n$, but its relative impact on asset $n$'s variance increases with volatility. Thus, the correlation decreases with volatility (as can also be confirmed from equation (39)). Conditions (i) and (ii) ensure that the correlation increases with volatility when, for example, $\chi_m/\phi_m \approx \chi_n/\phi_m$, i.e., assets $m$ and $n$ are similar in terms of the relative effects that the two systematic shocks have on their returns.

The third result of the proposition is that the market betas of illiquid assets increase with volatility when (i′) volatility is negatively correlated with the return on the market portfolio ($\chi < 0$), and (ii′) the market portfolio's price discount is convex ($q_2 > 0$).[28] To explain the intuition, we examine how the effect of volatility on an asset's market beta depends on the characteristics of the asset's price discount.

First, the effect depends on the discount's convexity. Suppose, for example, that asset $n$'s convexity exceeds that of the average asset. Then, as volatility increases, asset $n$'s negative sensitivity to volatility increases faster than that of the average asset. Thus, at times of high volatility, the negative effect of volatility on the average asset ($\chi < 0$) is reflected more strongly on asset $n$, giving rise to an increased market beta.

Second, the effect depends on the discount's average sensitivity to volatility. When the market portfolio's price discount is convex ($q_2 > 0$), the fraction of market movements driven by volatility increases with volatility. Then, if asset $n$'s sensitivity to volatility exceeds that of the average asset, asset $n$ exhibits greater common variation with the market at times of high volatility, and has an increased market beta.

Illiquid assets have exactly the properties that lead to an increased market beta at times of high volatility. Recall that liquidity premia are increasing in volatility, and are convex when $\ell > \sqrt{3}$, a condition which is satisfied when $q_2 > 0$. Since illiquid assets carry greater liquidity premia than liquid assets, their price discounts are more sensitive to volatility, and more convex than those of liquid assets. Thus, the market betas of illiquid assets increase with volatility.

[28] Conditions (i′) and (ii′) imply that the cross-partial derivative $\frac{\partial^2 \beta^M_{nt}}{\partial \epsilon_n \partial v_t}$ is positive. The reason why this ensures that the partial derivative $\frac{\partial \beta^M_{nt}}{\partial v_t}$ is positive for high-transaction-cost assets is because of the adding-up constraint $\sum_{n=1}^N S_n \beta^M_{nt} = 1$: this constraint implies that $\sum_{n=1}^N S_n \frac{\partial \beta^M_{nt}}{\partial v_t} = 0$.

### 4.3 Expected Returns

In equilibrium, expected returns are determined by a conditional two-factor CAPM, adjusted for transaction costs. This is as in the case of no performance-based liquidation. The complication relative to that case arises in deriving an unconditional CAPM. When liquidation is not performance-based, conditional betas are constant over time and equal to the unconditional betas. Thus, an unconditional CAPM follows immediately by taking expectations in the conditional CAPM equation. When liquidation is performance-based, this no longer applies because the conditional betas depend on the volatility.

To characterize unconditional expected returns, we take expectations in equation (32).[29] Using equation (25) and the fact that

$$E(XY) = E(X)E(Y) + \mathrm{Cov}(X,Y),$$

we find

$$\begin{aligned} E(dR_{nt}) \quad = \quad & E\left[A(v_t)\right]\mathrm{Cov}(dR_{nt}, dR_{Mt}) + E\left[AZ'(v_t)\right]\mathrm{Cov}(dR_{nt}, dv_t) \\ & + \mathrm{Cov}\left[A(v_t), \mathrm{Cov}_t(dR_{nt}, dR_{Mt})\right] + \mathrm{Cov}\left[AZ'(v_t), \mathrm{Cov}_t(dR_{nt}, dv_t)\right] + L\epsilon_n dt, \end{aligned} \tag{41}$$

where

$$L \equiv \left[r + 2[\lambda + \mu E\left[\pi(v_t)\right]]\exp(2A\epsilon)\right].$$

The first two terms in the RHS involve an asset's unconditional covariances with the two factors. These terms represent an adjustment for risk that is fully captured by the unconditional CAPM, because the covariances are linear functions of the unconditional betas. The unconditional CAPM might, however, fail to capture the risk adjustment represented by the third and fourth terms.

Consider, for example, the third term, which involves the covariance between the effective risk-aversion coefficient $A(v_t)$ and an asset's conditional covariance with the market. To illustrate the role of this term, consider two assets $n_1$ and $n_2$ that have the same unconditional covariance with the market. Suppose, however, that the conditional covariance of asset $n_1$ rises significantly in states where managers are more risk averse, while that of asset $n_2$ varies less strongly. Then, asset $n_1$ has a higher unconditional expected return than asset $n_2$ because of a very high conditional expected return in the high risk-aversion states.

Summarizing, an asset's risk is characterized not only by the unconditional betas, but also

[29] Our approach is similar in spirit to Jagannathan and Wang (1996). Notice that we are using the conditional CAPM equation in its covariance form, rather than the beta form. This simplifies the analysis because the expectation of a conditional covariance is an unconditional covariance (in continuous time), while the same is not true for the beta (unless beta is constant over time).

by the extent to which the conditional covariances covary with managers' risk aversion. This characterization is more complicated than in the case of no performance-based liquidation: in that case, the coefficients $A(v_t)$ and $Z'(v_t)$ are constants, and the third and fourth terms in equation (41) are zero.

Despite the more complicated characterization, the unconditional CAPM can still hold, if the third and fourth terms in equation (41) are linear functions of the unconditional betas. In Proposition 7 we show, however, that these terms involve cross-sectional variation which is not captured by the unconditional betas.

**Proposition 7** *For small $\sigma$, equation (41) becomes approximately*

$$E(dR_{nt}) = \left[\beta_n^M \Lambda_M + \beta_n^v \Lambda_v + K\epsilon_n + L\epsilon_n\right] dt, \tag{42}$$

*where*

$$K \equiv -\left[\frac{\mu n(\ell)\ell(\ell^2-3)\chi}{2\overline{z}\overline{v}}\frac{\exp(2A\epsilon)-1}{A} + AZ_2\right]\frac{\mu n(\ell)\ell(\ell^2-3)\sigma^4}{4\gamma(r+2\gamma)}\exp(2A\epsilon), \tag{43}$$

$$\Lambda_v \equiv \left[AZ_1 + \frac{A(2Z_2+Z_3\overline{v})\sigma^2}{4\gamma} - \frac{\mu n(\ell)\ell(\ell^2-3)q_2\sigma^2}{4\gamma\overline{z}}\frac{\exp(2A\epsilon)-1}{A}\right]\sigma^2\overline{v}, \tag{44}$$

*and $\Lambda_M$ is determined in the Appendix.*

Proposition 7 shows that the adjustment for risk operates not only through the unconditional betas $\beta_n^M$ and $\beta_n^v$, but also through the term $K\epsilon_n$. Thus, the unconditional CAPM does not adjust correctly for risk, failing to capture the risk associated with illiquidity.

To explain the intuition, suppose that the probability of poor performance is convex. Convexity ensures that both the market betas of illiquid assets, and the managers' effective risk-aversion coefficient, increase with volatility. Thus, at times of high volatility, illiquid assets covary with the market more strongly than other assets, and this is also when managers are the most risk averse. As a result, illiquid assets carry a risk premium not fully captured by their unconditional betas. Interestingly, the same risk premium applies when the probability of poor performance is concave: at times of high volatility, illiquid assets covary with the market less than other assets, and this is also when managers are the least risk averse. Thus, in both cases, the unconditional CAPM can understate the risk of illiquid assets.[30]

[30] The reason why the CAPM does not always understate this risk (i.e., $K$ is not always positive) is because of the hedging demand. Recall from Section 3.3 that the hedging demand is positive for assets that covary negatively with volatility. During volatile times, illiquid assets covary with volatility more negatively than other assets. If during those times, the coefficient $AZ'(v_t)$ (which determines the strength of the hedging demand) is larger, the conditional hedging demand is very strong for illiquid assets, and this can make $K$ negative. The effect of the hedging demand is, however, small when $A$ is small. (Formally, for small $A$, the first term in brackets in equation (43) dominates the term $AZ_2$. This makes $K$ positive, provided that $\chi < 0$, i.e., volatility impacts negatively the return on the market portfolio.)

Equation (42) illustrates the direct and indirect effects of transaction costs. The direct effect is through the liquidity premium, and corresponds to the term $L\epsilon_n$. The indirect effect is through the risk premium, and arises because transaction costs make an asset more sensitive to the volatility, and thus riskier. Part of the indirect effect is through the unconditional betas. Because transaction costs make an asset more (negatively) sensitive to volatility, they decrease the volatility beta (making it more negative). They also increase the market beta, as long as volatility impacts negatively the return on the market portfolio ($\chi < 0$).

**Proposition 8** *For small $\sigma$:*

- *The unconditional market beta of asset $n$ increases with $\epsilon_n$ if $\chi < 0$.*
- *The unconditional volatility beta of asset $n$ decreases with $\epsilon_n$.*

The effect of transaction costs on the unconditional betas represents the part of the indirect effect that is captured by the unconditional CAPM. The indirect effect is, however, not fully captured by the CAPM, and the part that the CAPM does not capture corresponds to the term $K\epsilon_n$.

Ignoring the transaction costs' indirect effect can understate the full cost of asset illiquidity. The indirect effect is absent from most theoretical studies of transaction costs, because these consider liquidity premia that are constant over time.[31] The indirect effect is also not accounted for in empirical studies because these measure the impact of transaction costs controlling for risk (a point to which we return in the next section). Our model explains why the indirect effect can arise, and allows us to compare it to the direct effect. A precise comparison requires, of course, a calibration of the model, which we plan to do in a companion paper.

A final implication of Proposition 7 concerns the volatility risk premium $\Lambda_v$. Recall that in the case of no performance-based liquidation, the volatility risk premium is determined by the hedging demand, and is positive. The effect of the hedging demand corresponds to the first two terms in equation (44). (These terms involve $Z_1$, $Z_2$, and $Z_3$, the first, second, and third derivatives of the function $Z(v_t)$ which characterizes the effect of volatility on the value function.) Equation (44) contains, however, an additional term which is not related to the hedging demand, and which is generally negative.[32] This term arises because of the time-variation in the conditional market betas. Recall from Section 4.2 that the market beta of an asset which is very (negatively) sensitive to volatility increases with volatility. Thus, the unconditional market beta of such an asset understates

[31] See, for example, Amihud and Mendelson (1986), Constantinides (1986), Vayanos (1998), Vayanos and Vila (1999), Huang (2002), and Lo, Mamaysky, and Wang (2003). An exception is Acharya and Pedersen (2003), who assume time-varying transaction costs.

[32] The third term in equation (44) is negative when $q_2 > 0$ (since then $\ell > \sqrt{3}$) or when $\ell < \sqrt{3}$ (since then $q_2 < 0$).

the asset's risk (because times of high volatility are also times of high risk aversion), and the unconditional CAPM attributes incorrectly the extra risk to the volatility beta. This can give rise to a negative volatility risk premium.[33] In the next section, we explain why this point is relevant for the empirical studies that consider a volatility or a liquidity factor.

So far, we have focused on the expected returns on individual assets. Our model has also implications for the expected return on the market portfolio. Indeed, substituting $R_{Mt}$ for $R_{nt}$ in equation (41), we find

$$\begin{aligned} E(dR_{Mt}) &= E\left[A(v_t)\right]\mathrm{Var}(dR_{Mt}) + E\left[AZ'(v_t)\right]\mathrm{Cov}(dR_{Mt}, dv_t) \\ &\quad +\mathrm{Cov}\left[A(v_t), \mathrm{Var}_t(dR_{Mt})\right] + \mathrm{Cov}\left[AZ'(v_t), \mathrm{Cov}_t(dR_{Mt}, dv_t)\right] + L\epsilon_M dt. \end{aligned} \quad (45)$$

Using equation (45), we can explore how performance-based liquidation affects the market portfolio's expected return. First, because the investors' risk-aversion coefficient increases from $A$ to $A(v_t)$, the expected return becomes more sensitive to the market portfolio's unconditional variance (first term in the RHS). Second, if the probability of poor performance is convex, the investors' risk-aversion coefficient $A(v_t)$ covaries positively with the market portfolio's conditional variance. This raises further the market portfolio's expected return, holding the unconditional variance constant (third term in the RHS). Thus, performance-based liquidation can have implications for the equity-premium puzzle, whereby the risk premium on the stock market is too high relative to what is implied by investors' risk aversion (characterized by $A$ in our model). We plan to explore these implications in a companion paper where we will calibrate the model.

# 5 Empirical Implications

In this section, we present the empirical implications of our model, and link our results to existing empirical findings.

## 5.1 Time-Variation with Aggregate Volatility

Our model predicts that liquidity premia, correlations, betas, and expected returns, should vary over time with aggregate volatility.

**Liquidity Premia:** Proposition 5 shows that liquidity premia increase with volatility. Empirical support for this result comes from Kamara (1994), who shows that the yield spread between T-notes and (more liquid) T-bills with matched maturities is correlated with interest-rate volatility.

[33] The third term in equation (44) dominates the first two terms when $A$ is small.

**Correlations:** Proposition 6 shows that the correlation between an asset and volatility decreases (i.e., becomes more negative) as volatility increases.[34] While a negative correlation between aggregate stock returns and volatility has been documented empirically (e.g., French, Schwert, and Stambaugh (1987), Campbell and Hentschel (1992)), we are not aware of any study that examines how this correlation depends on the volatility.

Proposition 6 also shows that the pairwise correlation between similar assets (in terms of relative loadings on the market and volatility portfolios) increases with volatility. Indirect support for this result comes from studies documenting an increase in pairwise correlations during market downturns (when volatility is also higher). See, for example, Longin and Solnik (2001) for international stock indices, and Ang and Chen (2002) for US stocks.

**Betas:** Proposition 6 shows that the market betas of illiquid assets increase with volatility. To our knowledge, this prediction of the model has not been tested empirically.

**Expected Returns:** Equation (32) implies that the market risk premium per unit of volatility increases with volatility.[35] Many empirical studies have examined the more basic question of whether the market risk premium, per se, increases with volatility (see Lettau and Lydvigson (2003) for a recent survey), but the overall evidence is inconclusive.

## 5.2 Liquidity and Expected Returns

In our model, transaction costs make an asset riskier, and more sensitive to the volatility. Thus, they affect the asset's expected return not only through the liquidity premium (direct effect), but also through the risk premium (indirect effect). Formally, Proposition 7 shows that the expected return of asset $n$ is

$$E(dR_{nt}) = \left[\beta_n^M \Lambda_M + \beta_n^v \Lambda_v + (K+L)\epsilon_n\right] dt, \tag{46}$$

where $\beta_n^M$ and $\beta_n^v$ are the asset's market and volatility betas, and $\epsilon_n$ are the transaction costs. Transaction costs affect the expected return through the liquidity premium $L\epsilon_n$, through the market and volatility betas, and through the term $K\epsilon_n$ which is the part of the risk premium not captured by the CAPM. In particular, illiquid assets have higher market betas, as shown in Proposition 8.

Accounting for the indirect effect of transaction costs is relevant for the empirical studies that

[34] For all implications listed in the remainder of this subsection, we require that (i) volatility is negatively correlated with asset returns, and (ii) assets' price discounts are convex.

[35] Writing equation (32) as

$$E_t(dR_{nt}) = \left[\beta_{nt}^M \Lambda_{Mt} + \beta_{nt}^v \Lambda_{vt} + [r + 2[\lambda + \mu\pi(v_t)]\exp(2A\epsilon)]\,\epsilon_n\right] dt,$$

the market risk premium is $\Lambda_{Mt} = A(v_t)\mathrm{Var}_t(dR_{Mt})$. When price discounts are convex, the effective risk-aversion coefficient $A(v_t)$ is increasing in $v_t$, and this implies that for small $\sigma$, $\Lambda_{Mt}/v_t$ is also increasing.

measure the impact of illiquidity on stocks' expected returns (e.g., Amihud and Mendelson (1986), Brennan and Subrahmanyam (1996), Eleswarapu (1997), and Chalmers and Kadlec (1998)). These studies regress returns on a transaction-cost measure (e.g., bid-ask spread, market depth), and the market beta (as to control for risk). Our model implies that such a regression (which corresponds to equation (46) without the volatility beta) can understate the full effect of illiquidity. Indeed, illiquidity increases the market beta, and controlling for the market beta suppresses this effect.

### 5.3 Liquidity Factor

According to equation (46), the cross-section of expected returns can be explained by the market beta, the volatility beta, and the transaction costs. The presence of a volatility factor is consistent with Ang, Hodrick, Xing, and Zhang (2003), who show that such a factor affects stocks' expected returns. It is also broadly consistent with Amihud (2002), Acharya and Pedersen (2003), Pastor and Stambaugh (2003), and Sadka (2003), who show the same result for a liquidity factor. Indeed, these papers construct the liquidity factor through some measure of price impact. Since in our model, price impact increases with volatility, the liquidity factor corresponds exactly to (the opposite of) the volatility factor.[36] In generating a volatility or a liquidity factor, our model provides a theoretical foundation to the above empirical findings. It also has implications for the factor's risk premium, and the assets' loadings along the factor.

**Factor Risk Premium:** Ang, Hodrick, Xing, and Zhang (2003) find a negative volatility risk premium. Likewise, Acharya and Pedersen (2003), Pastor and Stambaugh (2003), and Sadka (2003) find a positive liquidity risk premium (and thus a negative premium on the illiquidity factor). Our model explains what can drive such a premium. We show that the volatility risk premium in the conditional CAPM is positive because of the hedging demand. However, the premium can become negative in the unconditional CAPM (Proposition 7) because of the time-variation in the conditional market betas.

**Factor Loadings:** Amihud (2002), Acharya and Pedersen (2003), and Pastor and Stambaugh (2003) show that illiquid assets (i.e., assets whose own price-impact measure is high) are more sensitive to the liquidity factor (i.e., to changes in the aggregate price-impact measure). Thus, the notion of liquidity as an asset characteristic is related to that as a priced factor. This is consistent with our model: in Proposition 8, we show that illiquid assets have low (i.e., more negative) volatility betas. Acharya and Pedersen's finding that illiquid assets have high market betas is also

[36] If, for example, we introduce exogenous supply shocks in our model, these will have a higher price impact during volatile times. Of course, we do not mean to imply that the actual liquidity factor is just volatility. However, there is some correlation between the two factors. For example, the correlation between Pastor-Stambaugh's or Sadka's liquidity factors and the volatility is around -40%.

consistent with Proposition 8.

## 6 Conclusion

In this paper, we propose a theory of time-varying liquidity premia, and explore its asset-pricing implications. Our theory is based on the notion that investors are fund managers, subject to withdrawals that depend on the fund's performance. During volatile times, the probability that performance falls below an exogenous threshold increases, and withdrawals become more likely. This reduces the managers' willingness to hold illiquid assets, and raises the liquidity premia.

Our theory generates a rich set of empirical implications. These concern (i) the time-variation of expected returns, betas, correlations, and liquidity premia, with aggregate volatility, and (ii) the role of liquidity, both as an asset characteristic and as a risk factor, in explaining cross-sectional expected returns. At a more theoretical level, a contribution of this paper is to solve analytically a continuous-time equilibrium model with multiple assets, stochastic volatility, and transaction costs.

This research can be extended in a number of directions. An obvious extension is to test the model's implications empirically. One can also calibrate the model, and evaluate the quantitative importance of the various effects. Such a calibration would reveal, for example, the relative importance of the direct effect of transaction costs (through the liquidity premium) relative to the indirect effect (through the risk premium).

On the theoretical front, one extension is to endogenize the assets' transaction costs, focusing especially on their time-variation. (A time-variation in transaction costs has, in fact, been documented in several recent studies.[37]) If, for example, transaction costs increase during volatile times, the convexity in liquidity premia would be stronger, reinforcing many of our results.

Another extension is to consider a liquidation criterion that is based on a longer-term measure of performance. Under such a criterion, past performance would be a state variable: for example, after poor past performance, managers would become more concerned with liquidation and have an increased preference for liquid assets. Thus, liquidity premia would increase in down markets, and the same might apply to assets' pairwise correlations, and the market betas of illiquid assets. Deriving such results would broaden the empirical implications of our theory.

Finally, it might be interesting to introduce heterogeneity across investors, assuming, for example, constrained investors (the fund managers) and unconstrained ones. This would allow for an analysis of portfolio policies (which in our model is straightforward because all investors hold the

[37]See, for example, Chordia, Roll, and Subrahmanyam (2001), Hasbrouck and Seppi (2001), Huberman and Halka (2001), and Jones (2002). Acharya and Pedersen (2003) incorporate exogenous time-varying transaction costs into a multi-asset equilibrium model.

market portfolio). In particular, we could examine how the constrained investors should manage the liquidity of their portfolios, given that illiquid assets become very risky in volatile times, when constraints are the most severe.

# Appendix

**Proof of Lemma 1:** The budget constraint (5) is equivalent to

$$dW_t = \sum_{n=1}^{N} x_{nt}\left(dp_{nt} + \delta_{nt}dt - rp_{nt}dt\right) - \sum_{n=1}^{N} \epsilon_n \left|\frac{dx_{nt}}{dt}\right|. \tag{47}$$

Equations (1), (2), (8), and Ito's lemma, imply that

$$\begin{aligned} dp_{nt} + \delta_{nt}dt - rp_{nt}dt &= \left[rq_n(v_t) - q_n'(v_t)\gamma(\overline{v} - v_t) - \frac{1}{2}q_n''(v_t)\sigma^2 v_t\right]dt \\ &\sqrt{v_t}\left[\frac{\phi_n}{r+\kappa}dB_t + \left[\frac{\psi_n}{r+\kappa} - q_n'(v_t)\right]\sigma dB_t^v + \frac{1}{r+\kappa}dB_{nt}\right]. \end{aligned}$$

Substituting into equation (47), and using equations (9)-(11), we find equation (14). ∎

**Proof of Proposition 1:** We proceed in two steps. First, we solve the manager's optimization problem, and show that optimality of the market portfolio is equivalent to the system of $N+1$ ODEs. Second, we show that the solution of that system is as in the proposition.

**Step 1:** For brevity, we consider only buy-and-hold policies (i.e., policies where the manager buys $x_n$ shares of asset $n = 1,..,N$, holds them until liquidation, buys the same number of shares again, and so on), and show that optimality of the market portfolio within that set of policies implies the system of ODEs. A proof that (i) considers the larger set of policies which are differentiable almost everywhere, and (ii) shows that optimality is equivalent to the ODE system, can be constructed using the calculus-of-variation method in Vayanos (1998).

For a buy-and-hold policy $x \equiv (x_1,..,x_N)$, the budget constraint (14) becomes

$$\begin{aligned} dW_t &= \left[rq_n(v_t,x) - q_v(v_t,x)\gamma(\overline{v} - v_t) - \frac{1}{2}q_{vv}(v_t,x)\sigma^2 v_t\right]dt \\ &+\sqrt{v_t}\left[\frac{\phi(x)}{r+\kappa}dB_t + \left[\frac{\psi(x)}{r+\kappa} - q_v(v_t,x)\right]\sigma dB_t^v + \frac{\sum_{n=1}^N x_n dB_{nt}}{r+\kappa}\right]. \end{aligned} \tag{48}$$

Consider the optimization problem where the manager follows a given buy-and-hold policy $x$, and maximizes over consumption. To solve this problem, we use dynamic programming and conjecture that the manager's value function is of the form

$$V(W_t, w_t, v_t) = -\exp\left[-A\left[W_t + zw_t + Z(v_t,x)\right]\right],$$

for some function $Z(v_t, x)$. The Bellman equation is

$$
\begin{aligned}
\max_{c_t} \quad & [-\exp(-\alpha c_t) + D^c_{Wwv} V(W_t, w_t, v_t) - \beta V(W_t, w_t, v_t) \\
& + \lambda [V(W_t - 2\epsilon(x), w_t, v_t) - V(W_t, w_t, v_t)]] = 0, \qquad (49)
\end{aligned}
$$

where the last term represents the effect of liquidation (equation (6) implies that the size of the new fund is equal to the size of the old fund minus $2\epsilon(x)$), and

$$
\begin{aligned}
D^c_{Wwv} V \equiv \; & V_W \left[ rq(v_t, x) - q_v(v_t, x)\gamma(\overline{v} - v_t) - \frac{1}{2} q_{vv}(v_t, x)\sigma^2 v_t \right] \\
& + V_w (rw_t + aW_t - c_t) + V_v \gamma(\overline{v} - v_t) \\
& + \frac{1}{2} V_{WW} \left[ \frac{\phi(x)^2}{(r+\kappa)^2} + \left[ \frac{\psi(x)}{r+\kappa} - q_v(v_t, x) \right]^2 \sigma^2 + \frac{\|x\|^2}{(r+\kappa)^2} \right] v_t \\
& + V_{Wv} \left[ \frac{\psi(x)}{r+\kappa} - q_v(v_t, x) \right] \sigma^2 v_t + \frac{1}{2} V_{vv} \sigma^2 v_t.
\end{aligned}
$$

The first-order-condition w.r.t. $c_t$ is

$$
\alpha \exp(-\alpha c_t) = V_w.
$$

Plugging $c_t$ back into the Bellman equation, substituting for the partial derivatives of $V$, and dividing by $AV$ throughout, we find

$$
\begin{aligned}
& r [q(v_t, x) - Z(v_t, x)] - [q_v(v_t, x) - Z_v(v_t, x)] \gamma(\overline{v} - v_t) - \frac{1}{2} [q_{vv}(v_t, x) - Z_{vv}(v_t, x)] \sigma^2 v_t \\
& - \frac{1}{2} A \left[ \frac{\phi(x)^2}{(r+\kappa)^2} + \left[ \frac{\psi(x)}{r+\kappa} - [q_v(v_t, x) - Z_v(v_t, x)] \right]^2 \sigma^2 + \frac{\|x\|^2}{(r+\kappa)^2} \right] v_t \\
& + \frac{\beta - r}{A} + \frac{r \log r}{A} - \lambda \frac{\exp[2A\epsilon(x)] - 1}{A} = 0. \qquad (50)
\end{aligned}
$$

In equilibrium, the manager must be holding the market portfolio, i.e., $x = S$. Noting that $\phi(S) = \phi$, $\psi(S) = \psi$, $\epsilon(S) = \epsilon$, $q(v_t, S) = q(v_t)$, and $Z(v_t, S) = Z(v_t)$, we can write equation (50) as

$$
\begin{aligned}
& r [q(v_t) - Z(v_t)] - [q'(v_t) - Z'(v_t)] \gamma(\overline{v} - v_t) - \frac{1}{2} [q''(v_t) - Z''(v_t)] \sigma^2 v_t \\
& - \frac{1}{2} A \left[ \frac{\phi^2}{(r+\kappa)^2} + \left[ \frac{\psi}{r+\kappa} - [q'(v_t) - Z'(v_t)] \right]^2 \sigma^2 + \frac{\|S\|^2}{(r+\kappa)^2} \right] v_t \\
& + \frac{\beta - r}{A} + \frac{r \log r}{A} - \lambda \frac{\exp(2A\epsilon) - 1}{A} = 0. \qquad (51)
\end{aligned}
$$

Equation (51) is the $N+1$st ODE of the system. The first $N$ ODEs will derive from the optimality of the market portfolio. Consider a manager who forms a new risky-asset portfolio after liquidation. Buying $x_n$, rather than $S_n$, shares of asset $n$, changes the transaction costs by $\epsilon_n(x_n - S_n)$, thus changing $W_t$ by the opposite of that amount. The manager's utility changes by

$$-\frac{\partial}{\partial x_n} \exp\left[-A\left[W_t - \epsilon_n(x_n - S_n) + zw_t + Z(v_t, S^{-n})\right]\right],$$

where $S^{-n} \equiv (S_1, .., S_{n-1}, x_n, S_{n+1}, .., S_N)$. Since buying the market portfolio is optimal, the above derivative is zero, and thus

$$Z_{x_n}(v_t, x)|_{x=S} = \epsilon_n. \tag{52}$$

Equation (52) implies that

$$Z_{vx_n}(v_t, x)|_{x=S} = Z_{x_n v}(v_t, x)|_{x=S} = \frac{\partial \epsilon_n}{\partial v} = 0, \tag{53}$$

and similarly

$$Z_{vvx_n}(v_t, x)|_{x=S} = 0. \tag{54}$$

Differentiating equation (50) w.r.t. $x_n$ at $x = S$, and using equations (9)-(13) and (52)-(54), we find the ODE (19). Thus, optimality of the market portfolio implies the system of ODEs.

To derive the ODEs, we used the Bellman equation of the manager's optimization problem. For the Bellman equation to ensure optimality, the set of consumption policies must be restricted in a way that Ponzi schemes are ruled out (i.e., the manager must be prevented from borrowing indefinitely). Since Ponzi schemes arise because of the infinite horizon, a simple way to rule them out is to consider a finite-horizon economy and require that the infinite-horizon equilibrium is the limit of a finite-horizon one as the horizon goes to infinity. Considering this limit will also provide initial conditions for the ODEs.

More specifically, we consider an economy ending at time $T$, with asset $n$ paying a liquidating dividend

$$\frac{\bar{\delta}}{r} + \frac{\delta_{nT} - \bar{\delta}}{r + \kappa},$$

and the manager receiving a utility

$$-\exp\left[-A\left(W_T + zw_T\right)\right].$$

We look for a finite-horizon equilibrium in which the price process is of the form

$$p_{nt} = \frac{\bar{\delta}}{r} + \frac{\delta_{nt} - \bar{\delta}}{r + \kappa} - q_n(v_t, t)$$

and the manager's value function is

$$V(W_t, w_t, v_t, t) = -\exp\left[-A\left[W_t + zw_t + Z(v_t, x, t)\right]\right].$$

With finite horizon, the budget constraint (48) is replaced by

$$\begin{aligned} dW_t &= \sum_{n=1}^{N} x_n \left[ r q_n(v_t, t) - [q_n]_v(v_t, t)\gamma(\overline{v} - v_t) - \frac{1}{2}[q_n]_{vv}(v_t, t)\sigma^2 v_t - [q_n]_t(v_t, t) \right] dt \\ &\quad + \sum_{n=1}^{N} x_n \sqrt{v_t} \left[ \frac{\phi_n}{r+\kappa} dB_t + \left[ \frac{\psi_n}{r+\kappa} - [q_n]_v(v_t, t) \right] \sigma dB_t^v + \frac{1}{r+\kappa} dB_{nt} \right], \end{aligned}$$

and the Bellman equation (49) by

$$\begin{aligned} \max_{c_t} \quad & \left[ -\exp(-\alpha c_t) + D^c_{Wwv} V(W_t, w_t, v_t, t) + V_t(W_t, w_t, v_t, t) - \beta V(W_t, w_t, v_t, t) \right. \\ & \left. + \lambda \left[ V(W_t - 2\epsilon(x), w_t, v_t, t) - V(W_t, w_t, v_t, t) \right] \right] = 0. \end{aligned}$$

Proceeding as in the infinite-horizon case, we find the equations

$$\begin{aligned} & r\left[q(v_t, t) - Z(v_t, t)\right] - \left[q_v(v_t, t) - Z_v(v_t, t)\right]\gamma(\overline{v} - v_t) - \frac{1}{2}\left[q_{vv}(v_t, t) - Z_{vv}(v_t, t)\right]\sigma^2 v_t \\ & -\frac{1}{2} A \left[ \frac{\phi^2}{(r+\kappa)^2} + \left[ \frac{\psi}{r+\kappa} - \left[q_v(v_t, t) - Z_v(v_t, t)\right] \right]^2 \sigma^2 + \frac{\|S\|^2}{(r+\kappa)^2} \right] v_t \\ & + \frac{\beta - r}{A} + \frac{r \log r}{A} - \lambda \frac{\exp(2A\epsilon) - 1}{A} - \left[q_t(v_t, t) - Z_t(v_t, t)\right] = 0 \end{aligned} \tag{55}$$

and

$$\begin{aligned} & r q_n(v_t, t) - [q_n]_v(v_t, t)\gamma(\overline{v} - v_t) - \frac{1}{2}[q_n]_{vv}(v_t, t)\sigma^2 v_t \\ & -A \left[ \frac{\phi_n \phi}{(r+\kappa)^2} + \left[ \frac{\psi_n}{r+\kappa} - [q_n]_v(v_t, t) \right] \left[ \frac{\psi}{r+\kappa} - \left[q_v(v_t, t) - Z_v(v_t, t)\right] \right] \sigma^2 + \frac{S_n}{(r+\kappa)^2} \right] v_t \\ & - \left[r + 2\lambda \exp(2A\epsilon)\right]\epsilon_n - [q_n]_t(v_t, t) = 0. \end{aligned} \tag{56}$$

These equations are PDEs for the functions $q_n(v_t, t)$ and $Z(v_t, t)$, and must be solved with the terminal conditions $q_n(v_T, T) = Z(v_T, T) = 0$. We require that the solution to the ODEs (51) and (19) is the limit of the PDE solution as $T$ goes to infinity and $t$ is held constant. This ensures that the infinite-horizon equilibrium is the limit of the finite-horizon one.

**Step 2:** Setting $f(v_t) \equiv q(v_t) - Z(v_t)$, we can write the ODEs (51) and (19) as

$$rf(v_t) - f'(v_t)\gamma(\overline{v} - v_t) - \frac{1}{2} f''(v_t)\sigma^2 v_t$$

$$
\begin{aligned}
&-\frac{1}{2}A\left[\frac{\phi^2}{(r+\kappa)^2}+\left[\frac{\psi}{r+\kappa}-f'(v_t)\right]^2\sigma^2+\frac{\|S\|^2}{(r+\kappa)^2}\right]v_t \\
&+\frac{\beta-r}{A}+\frac{r\log r}{A}-\lambda\frac{\exp(2A\epsilon)-1}{A}=0
\end{aligned}
\tag{57}
$$

and

$$
\begin{aligned}
&rq_n(v_t)-q_n'(v_t)\gamma(\overline{v}-v_t)-\frac{1}{2}q_n''(v_t)\sigma^2 v_t \\
&-A\left[\frac{\phi_n\phi}{(r+\kappa)^2}+\left[\frac{\psi_n}{r+\kappa}-q_n'(v_t)\right]\left[\frac{\psi}{r+\kappa}-f'(v_t)\right]\sigma^2+\frac{S_n}{(r+\kappa)^2}\right]v_t \\
&-\left[r+2\lambda\exp(2A\epsilon)\right]\epsilon_n=0,
\end{aligned}
\tag{58}
$$

respectively. The ODE (57) involves only the function $f(v_t)$, and has an affine solution

$$
f(v_t)=f_0+f_1(v_t-\overline{v}),
$$

provided that $f_1$ satisfies equation (23), and

$$
f_0=\frac{A\overline{v}}{2r}\left[\frac{\phi^2+\|S\|^2}{(r+\kappa)^2}+\left(\frac{\psi}{r+\kappa}-f_1\right)^2\sigma^2\right]-\frac{\beta-r}{rA}-\frac{\log r}{A}+\frac{\lambda}{r}\frac{\exp(2A\epsilon)-1}{A}=0. \tag{59}
$$

The function $q_n(v_t)$ satisfying the ODE (58) is also affine, i.e.,

$$
q_n(v_t)=q_{n0}+q_{n1}(v_t-\overline{v}),
$$

provided that $q_{n0}$ and $q_{n1}$ satisfy equations (21) and (22), respectively.

Equations (59), (21), and (22), determine $f_0$, $q_{n0}$, and $q_{n1}$, uniquely, as functions of $f_1$. Equation (23) is quadratic in $f_1$ and has two real roots if and only if

$$
\left(r+\gamma+A\frac{\psi\sigma^2}{r+\kappa}\right)^2>A^2\frac{\phi^2+\psi^2\sigma^2+\|S\|^2}{(r+\kappa)^2}\sigma^2. \tag{60}
$$

Equation (60) can hold only if

$$
r+\gamma+A\frac{\psi\sigma^2}{r+\kappa}>0, \tag{61}
$$

and is thus equivalent to equation (20). Since the LHS of equation (23) goes to $-\infty$ when $f_1$ goes to $\pm\infty$, it is negative outside the roots and positive within them. Since, in addition, the LHS is negative for $f_1=0$, and has a positive derivative (from equation (61)), the two roots are positive.

Given that $f_1$ can take two values, there are two affine solutions to the ODEs (51) and (19). To determine which solution to select (and to explain why we consider only affine solutions), we solve the PDEs (55) and (56) with the terminal conditions $q_n(v_T,T) = Z(v_T,T) = 0$. It is easy to check that the solution is affine:

$$f(v_t,t) \equiv q(v_t,t) - Z(v_t,t) = f_0(t) + f_1(t)(v_t - \overline{v}),$$

$$q_n(v_t,t) = q_{n0}(t) + q_{n1}(t)(v_t - \overline{v}),$$

where the functions $f_0(t)$, $f_1(t)$, $q_{n0}(t)$, and $q_{n1}(t)$, satisfy a system of ODEs. The terminal conditions for the ODEs follow from $q_n(v_T,T) = Z(v_T,T) = 0$ and are $f_0(T) = f_1(T) = q_{n0}(T) = q_{n1}(T) = 0$. The ODE for the function $f_1(t)$ is

$$f_1'(t) = (r+\gamma)f_1(t) - \frac{1}{2}A\left[\frac{\phi^2 + \|S\|^2}{(r+\kappa)^2} + \left(\frac{\psi}{r+\kappa} - f_1(t)\right)^2 \sigma^2\right]. \tag{62}$$

To determine the behavior of $f_1(t)$, we recall that the two roots of equation (23) are positive, and the LHS of that equation is negative outside the roots and positive within them. Thus, the function $f_1(t)$ (which starts from zero at $t = T$) increases as $t$ decreases, stays below the smaller root, and converges to that root as $T$ goes to infinity. This means that in the limit we obtain an affine solution, and furthermore, the one corresponding to the smaller root. Note that if condition (20) does not hold, the LHS of equation (23) is negative and bounded away from zero, and thus $f_1(t)$ converges to infinity.

We finally show that $q_1$ and $Z_1$ are positive. Multiplying equation (22) by $S_n$, and summing over $n$, we find

$$q_1 = \frac{\frac{\phi^2+\|S\|^2}{(r+\kappa)^2} + \frac{\psi}{r+\kappa}\left(\frac{\psi}{r+\kappa} - f_1\right)\sigma^2}{\frac{r+\gamma}{A} + \left(\frac{\psi}{r+\kappa} - f_1\right)\sigma^2}. \tag{63}$$

Since $f_1$ is a root of equation (23), we can write $q_1$ as

$$q_1 = f_1 + \frac{\frac{r+\gamma}{A} f_1}{\frac{r+\gamma}{A} + \left(\frac{\psi}{r+\kappa} - f_1\right)\sigma^2}. \tag{64}$$

Since $f_1$ is the smaller root, the derivative of the LHS of equation (23) at $f_1$ is positive. Therefore,

$$\frac{r+\gamma}{A} + \left(\frac{\psi}{r+\kappa} - f_1\right)\sigma^2 > 0.$$

Since, in addition $f_1 > 0$, equation (64) implies that $q_1 > 0$ and $Z_1 = q_1 - f_1 > 0$. ∎

**Proof of Proposition 2:** The proof follows from the argument immediately after the proposition. We can also compute the conditional betas and correlations directly, and show that when the functions $q_n(v_t)$ are affine, these are constant. We carry these computations below because we use the results in subsequent proofs.

From equations (1), (17), and (18), the market beta of asset $n$ is

$$\beta_{nt}^M = \frac{\frac{\phi_n \phi}{(r+\kappa)^2} + \left[\frac{\psi_n}{r+\kappa} - q_n'(v_t)\right]\left[\frac{\psi}{r+\kappa} - q'(v_t)\right]\sigma^2 + \frac{S_n}{(r+\kappa)^2}}{\frac{\phi^2}{(r+\kappa)^2} + \left[\frac{\psi}{r+\kappa} - q'(v_t)\right]^2 \sigma^2 + \frac{\|S\|^2}{(r+\kappa)^2}} \tag{65}$$

and the volatility beta is

$$\beta_{nt}^v = \frac{\psi_n}{r+\kappa} - q_n'(v_t). \tag{66}$$

From equations (1), (17), and (18), the correlation between assets $n$ and $m$ is

$$\rho_{mnt} = \frac{\frac{\phi_m \phi_n}{(r+\kappa)^2} + \left[\frac{\psi_m}{r+\kappa} - q_m'(v_t)\right]\left[\frac{\psi_n}{r+\kappa} - q_n'(v_t)\right]\sigma^2}{\sqrt{\prod_{i=m,n}\left[\frac{\phi_i^2}{(r+\kappa)^2} + \left[\frac{\psi_i}{r+\kappa} - q_i'(v_t)\right]^2\sigma^2 + \frac{1}{(r+\kappa)^2}\right]}} \tag{67}$$

and the correlation between asset $n$ and the volatility is

$$\rho_{nt}^v = \frac{\left[\frac{\psi_n}{r+\kappa} - q_n'(v_t)\right]\sigma}{\sqrt{\frac{\phi_n^2}{(r+\kappa)^2} + \left[\frac{\psi_n}{r+\kappa} - q_n'(v_t)\right]^2\sigma^2 + \frac{1}{(r+\kappa)^2}}}. \tag{68}$$

∎

**Proof of Proposition 3:** The proof follows from the argument preceeding the proposition. ∎

**Proof of Lemma 2:** Using the asset betas, we can write equation (16) as

$$E_t(dR_{nt}) = A\beta_{nt}^M \mathrm{Var}_t(dR_{Mt}) + AZ'(v_t)\beta_{nt}^v \mathrm{Var}_t(dv_t) + [r + 2\lambda \exp(2A\epsilon)]\,\epsilon_n dt. \tag{69}$$

Equation (18) implies that

$$\mathrm{Var}_t(dR_{Mt}) = \left[\frac{\phi^2}{(r+\kappa)^2} + \left[\frac{\psi}{r+\kappa} - q'(v_t)\right]^2\sigma^2 + \frac{\|S\|^2}{(r+\kappa)^2}\right] v_t dt,$$

and equation (1) implies that

$$\mathrm{Var}_t(dv_t) = \sigma^2 v_t dt.$$

Plugging into equation (69), and noting that $q'(v_t) = q_1$ and $Z'(v_t) = Z_1$, we find equation (26). ∎

**Proof of Proposition 4:** The proposition follows immediately by taking expectations in equation (26), and using the fact that $\beta_{nt}^M = \beta_n^M$ and $\beta_{nt}^v = \beta_n^v$. ∎

We next state and prove Proposition 9.

**Proposition 9** *In equilibrium, the functions $\{q_n(v_t)\}_{n=1,..,N}$ and $Z(v_t)$ must satisfy the equations*

$$
\begin{aligned}
& r\left[q(v_t)-Z(v_t)\right]-\left[q'(v_t)-Z'(v_t)\right]\gamma(\overline{v}-v_t)-\frac{1}{2}\left[q''(v_t)-Z''(v_t)\right]\sigma^2 v_t \\
& -\frac{1}{2}A\left[\frac{\phi^2}{(r+\kappa)^2}+\left[\frac{\psi}{r+\kappa}-\left[q'(v_t)-Z'(v_t)\right]\right]^2\sigma^2+\frac{\|S\|^2}{(r+\kappa)^2}\right]v_t \\
& +\frac{\beta-r}{A}+\frac{r\log r}{A}-\left[\lambda+\mu\pi(v_t)\right]\frac{\exp(2A\epsilon)-1}{A}=0,
\end{aligned}
\tag{70}
$$

*and*

$$
\begin{aligned}
& rq_n(v_t)-q_n'(v_t)\gamma(\overline{v}-v_t)-\frac{1}{2}q_n''(v_t)\sigma^2 v_t \\
& -A\left[\frac{\phi_n\phi}{(r+\kappa)^2}+\left[\frac{\psi_n}{r+\kappa}-q_n'(v_t)\right]\left[\frac{\psi}{r+\kappa}-\left[q'(v_t)-Z'(v_t)\right]\right]\sigma^2+\frac{S_n}{(r+\kappa)^2}\right]v_t \\
& -\mu\pi_{x_n}(v_t,x)|_{x=S}\frac{\exp(2A\epsilon)-1}{A}-\left[r+2\left[\lambda+\mu\pi(v_t)\right]\exp(2A\epsilon)\right]\epsilon_n=0.
\end{aligned}
\tag{71}
$$

**Proof:** We proceed as in the proof of Proposition 1. The only change introduced by performance-based liquidation is that equation (50) holds with $\lambda+\mu\pi(v_t,x)$ instead of $\lambda$. It now is

$$
\begin{aligned}
& r\left[q(v_t,x)-Z(v_t,x)\right]-\left[q_v(v_t,x)-Z_v(v_t,x)\right]\gamma(\overline{v}-v_t)-\frac{1}{2}\left[q_{vv}(v_t,x)-Z_{vv}(v_t,x)\right]\sigma^2 v_t \\
& -\frac{1}{2}A\left[\frac{\phi(x)^2}{(r+\kappa)^2}+\left[\frac{\psi(x)}{r+\kappa}-\left[q_v(v_t,x)-Z_v(v_t,x)\right]\right]^2\sigma^2+\frac{\|x\|^2}{(r+\kappa)^2}\right]v_t \\
& +\frac{\beta-r}{A}+\frac{r\log r}{A}-\left[\lambda+\mu\pi(v_t,x)\right]\frac{\exp[2A\epsilon(x)]-1}{A}=0.
\end{aligned}
\tag{72}
$$

Setting $x=S$ in equation (72), we find equation (70). Taking the derivative of equation (72) w.r.t. $x_n$ for $x=S$, and using equation (52), we find equation (71).

Equations (70) and (71) constitute a system of $N+1$ ODEs in the functions $\{q_n(v_t)\}_{n=1,..,N}$ and $Z(v_t)$. The functions $\pi(v_t)$ and $\pi_{x_n}(v_t,x)|_{x=S}$ can be expressed in terms of $\{q_n(v_t)\}_{n=1,..,N}$ and $Z(v_t)$, using equations (28) and (29). These equations imply that $\pi(v_t)=g[z(v_t)]$,

$$
z(v_t)=\left[\frac{\phi^2}{(r+\kappa)^2}+\left[\frac{\psi}{r+\kappa}-q'(v_t)\right]^2\sigma^2+\frac{\|S\|^2}{(r+\kappa)^2}\right]v_t,
\tag{73}
$$

and

$$
\pi_{x_n}(v_t,x)|_{x=S}=2g'[z(v_t)]z_n(v_t),
\tag{74}
$$

where

$$z_n(v_t) \equiv \left[ \frac{\phi_n \phi}{(r+\kappa)^2} + \left[ \frac{\psi_n}{r+\kappa} - q_n'(v_t) \right] \left[ \frac{\psi}{r+\kappa} - q'(v_t) \right] \sigma^2 + \frac{S_n}{(r+\kappa)^2} \right] v_t. \tag{75}$$

The initial conditions to the ODEs (70) and (71) follow from the behavior of $v_t$ at infinity. When $v_t$ goes to infinity, $\pi(v_t)$ converges to 1/2. The solution to the ODEs must then converge to the affine solution that these ODEs have when the function $\pi(v_t)$ is replaced by 1/2 and $\pi_{x_n}(v_t, x)|_{x=S}$ by zero. ∎

**Proof of Proposition 5:** We first determine the Taylor expansions of $z(v_t)$, $\pi(v_t)$, and $\pi_{x_n}(v_t, x)|_{x=S}$, keeping terms up to order two in $v_t - \overline{v}$, but only up to order zero in $\sigma$. Equations (73) and (75) imply that the Taylor expansions of $z(v_t)$ and $z_n(v_t)$ are

$$z(v_t) = \overline{z} + \overline{z}\frac{v_t - \overline{v}}{\overline{v}}, \tag{76}$$

$$z_n(v_t) = \overline{z}_n + \overline{z}_n \frac{v_t - \overline{v}}{\overline{v}}.$$

Since $\pi(v_t) = g[z(v_t)]$, the Taylor expansion of $\pi(v_t)$ is

$$\pi(v_t) = g(\overline{z}) + g'(\overline{z})\overline{z}\frac{v_t - \overline{v}}{\overline{v}} + \frac{1}{2}g''(\overline{z})\overline{z}^2 \left(\frac{v_t - \overline{v}}{\overline{v}}\right)^2.$$

Equation (74) implies that the Taylor expansion of $\pi_{x_n}(v_t, x)|_{x=S}$ is

$$\begin{aligned}
\pi_{x_n}(v_t, x)|_{x=S} &= 2\left[\overline{z}_n + \overline{z}_n \frac{v_t - \overline{v}}{\overline{v}}\right]\left[g'(\overline{z}) + g''(\overline{z})\overline{z}\frac{v_t - \overline{v}}{\overline{v}} + \frac{1}{2}g'''(\overline{z})\overline{z}^2\left(\frac{v_t - \overline{v}}{\overline{v}}\right)^2\right] \\
&= 2g'(\overline{z})\overline{z}_n + 2\left[g'(\overline{z}) + g''(\overline{z})\overline{z}\right]\overline{z}_n \frac{v_t - \overline{v}}{\overline{v}} + \left[2g''(\overline{z})\overline{z} + g'''(\overline{z})\overline{z}^2\right]\overline{z}_n \left(\frac{v_t - \overline{v}}{\overline{v}}\right)^2.
\end{aligned}$$

The first three derivatives of the function $g(z_t) = N(-L/\sqrt{z_t})$ are

$$g'(z_t) = n\left(\frac{L}{\sqrt{z_t}}\right)\frac{L}{2z_t^{\frac{3}{2}}},$$

$$g''(z_t) = n\left(\frac{L}{\sqrt{z_t}}\right)\frac{L}{4z_t^{\frac{5}{2}}}\left(\frac{L^2}{z_t} - 3\right), \tag{77}$$

$$g'''(z_t) = n\left(\frac{L}{\sqrt{z_t}}\right)\frac{L}{8z_t^{\frac{7}{2}}}\left(\frac{L^4}{z_t^2} - \frac{10L^2}{z_t} + 15\right).$$

Plugging into the Taylor expansions of $\pi(v_t)$ and $\pi_{x_n}(v_t, x)|_{x=S}$, and replacing $L/\sqrt{\overline{z}}$ by $\ell$, we find

$$\pi(v_t) = N(-\ell) + \frac{n(\ell)\ell}{2}\frac{v_t - \overline{v}}{\overline{v}} + \frac{n(\ell)\ell(\ell^2 - 3)}{8}\left(\frac{v_t - \overline{v}}{\overline{v}}\right)^2, \tag{78}$$

$$\pi_{x_n}(v_t,x)|_{x=S} = \left[\frac{n(\ell)\ell}{\overline{z}} + \frac{n(\ell)\ell(\ell^2-1)}{2\overline{z}}\frac{v_t-\overline{v}}{\overline{v}} + \frac{n(\ell)\ell(\ell^4-6\ell^2+3)}{8\overline{z}}\left(\frac{v_t-\overline{v}}{\overline{v}}\right)^2\right]\overline{z}_n. \tag{79}$$

Using equations (34), (35), (78), and (79), we next substitute $q_n(v_t)$, $Z(v_t)$, $\pi(v_t)$, and $\pi_{x_n}(v_t,x)|_{x=S}$, into equation (71). Keeping terms up to order zero in $\sigma$, we find an equation involving terms of order zero, one, and two, in $v_t-\overline{v}$. Setting these terms to zero, we find equations (36)-(38). ∎

**Proof of Proposition 6:** Differentiating equation (68) w.r.t. $v_t$, we find

$$\frac{\partial \rho^v_{nt}}{\partial v_t} = -\frac{q''_n(v_t)\sigma}{U_n} + \frac{q''_n(v_t)X_n^2\sigma^3}{U_n^3}, \tag{80}$$

where

$$U_n \equiv \sqrt{\frac{\phi_n^2}{(r+\kappa)^2} + \left[\frac{\psi_n}{r+\kappa} - q'_n(v_t)\right]^2\sigma^2 + \frac{1}{(r+\kappa)^2}}$$

and

$$X_n \equiv \frac{\psi_n}{r+\kappa} - q'_n(v_t).$$

For small $\sigma$, $v_t$ is close to $\overline{v}$, and $q'_n(v_t)$ and $q''_n(v_t)$ are close to $q_{n1}$ and $q_{n2}$, respectively. The highest-order term in equation (80) is

$$-\frac{q_{n2}(r+\kappa)\sigma}{\sqrt{\phi_n^2+1}}.$$

Therefore, $\rho^v_{nt}$ decreases with volatility if $q_{n2}>0$.

Differentiating equation (67) w.r.t. $v_t$, we find

$$\frac{\partial \rho_{mnt}}{\partial v_t} = -\frac{q''_m(v_t)X_n + q''_n(v_t)X_m}{U_mU_n}\sigma^2 + \left[\frac{\phi_m\phi_n}{(r+\kappa)^2} + X_mX_n\sigma^2\right]\left[\frac{q''_m(v_t)X_m}{U_m^3U_n} + \frac{q''_n(v_t)X_n}{U_mU_n^3}\right]\sigma^2. \tag{81}$$

For small $\sigma$, the highest-order term in equation (81) is

$$\frac{-(q_{m2}\chi_n + q_{n2}\chi_m)(\phi_m^2+1)(\phi_n^2+1) + \phi_m\phi_n\left[q_{m2}\chi_m(\phi_n^2+1) + q_{n2}\chi_n(\phi_m^2+1)\right]}{(\phi_m^2+1)^{\frac{3}{2}}(\phi_n^2+1)^{\frac{3}{2}}}(r+\kappa)^2\sigma^2.$$

Rearranging the numerator, we find that $\rho^v_{mnt}$ increases with volatility if equation (39) holds.

Differentiating equation (65) w.r.t. $v_t$, we find

$$\frac{\partial \beta^M_{nt}}{\partial v_t} = -\frac{q''_n(v_t)X + q''(v_t)X_n}{U^2}\sigma^2 + \left[\frac{\phi_n\phi}{(r+\kappa)^2} + X_nX\sigma^2 + \frac{S_n}{(r+\kappa)^2}\right]\frac{2q''(v_t)X}{U^4}\sigma^2,$$

where

$$U \equiv \sqrt{\frac{\phi^2}{(r+\kappa)^2} + \left[\frac{\psi}{r+\kappa} - q'(v_t)\right]^2\sigma^2 + \frac{\|S\|^2}{(r+\kappa)^2}}$$

and $X \equiv \sum_{n=1}^{N} S_n X_n$. Differentiating again w.r.t. $\epsilon_n$, and noting that only the function $q_n(v_t)$ is affected,[38] we find

$$\begin{aligned}
\frac{\partial^2 \beta^M_{nt}}{\partial \epsilon_n \partial v_t} &= -\left[\frac{\partial q''_n(v_t)}{\partial \epsilon_n} X + q''(v_t)\frac{\partial X_n}{\partial \epsilon_n}\left(1 - \frac{2X^2\sigma^2}{U^2}\right)\right]\frac{\sigma^2}{U^2} \\
&= -\left[\frac{\partial q''_n(v_t)}{\partial \epsilon_n} X - q''(v_t)\frac{\partial q'_n(v_t)}{\partial \epsilon_n}\left(1 - \frac{2X^2\sigma^2}{U^2}\right)\right]\frac{\sigma^2}{U^2}.
\end{aligned} \tag{82}$$

For small $\sigma$, the highest-order term in equation (82) is

$$-\left[\frac{\partial q_{n2}}{\partial \epsilon_n}\chi - q_2 \frac{\partial q_{n1}}{\partial \epsilon_n}\right]\frac{(r+\kappa)^2\sigma^2}{\phi^2 + \|S\|^2}.$$

Substituting $q_{n1}$ and $q_{n2}$ from equations (37) and (38), we find that $\partial^2\beta^M_{nt}/\partial\epsilon_n\partial v_t > 0$ if equation (40) holds. ∎

**Proof of Proposition 7:** Setting

$$C^M_n \equiv \frac{\text{Cov}(dR_{nt}, dR_{Mt})}{dt},$$

$$C^v_n \equiv \frac{\text{Cov}(dR_{nt}, dv_t)}{dt},$$

$$C^A_n \equiv \frac{\text{Cov}\left[A(v_t), \text{Cov}_t(dR_{nt}, dR_{Mt})\right]}{dt},$$

$$C^Z_n \equiv \frac{\text{Cov}\left[Z'(v_t), \text{Cov}_t(dR_{nt}, dv_t)\right]}{dt},$$

we can write equation (41) as

$$E(dR_{nt}) = \left[E\left[A(v_t)\right]C^M_n + AE\left[Z'(v_t)\right]C^v_n + C^A_n + AC^Z_n + L\epsilon_n\right]dt. \tag{83}$$

We will show that this equation is the same as (42), plus terms of higher order in $\sigma$.

We first compute $C^A_n$ and $C^Z_n$. Equations (17) and (18) imply that

$$\text{Cov}_t(dR_{nt}, dR_{Mt}) = \left[\frac{\phi_n\phi}{(r+\kappa)^2} + X_n X\sigma^2 + \frac{S_n}{(r+\kappa)^2}\right]v_t dt. \tag{84}$$

Using the Taylor expansions of $q_n(v_t)$ and $q(v_t)$, we can write $X_n X$ as

$$X_n X = \chi_n\chi - (\chi_n q_2 + \chi q_{n2})(v_t - \overline{v}) + O\left[(v_t - \overline{v})^2\right], \tag{85}$$

[38] The function $q(v_t)$ (associated to the market portfolio) is affected if the differentiation w.r.t. $\epsilon_n$ is interpreted as a change in an asset's transaction costs, holding the other assets' costs constant. If, however, the differentiation is interpreted as a cross-sectional comparison between assets that differ only in transaction costs, $q(v_t)$ is not affected.

where $O(x)$ means "of order $x$ or higher." Combining equations (84) and (85), and noting that for a function $F(v_t)$,

$$\text{Cov}\left[F(v_t),(v_t-\overline{v})^k\right]=E\left[\left[F(v_t)-E\left[F(v_t)\right]\right](v_t-\overline{v})^k\right]\approx F'(\overline{v})E\left[(v_t-\overline{v})^{k+1}\right]=O(\sigma^{k+1}), \tag{86}$$

we find

$$\begin{aligned} C_n^A &= \left[\frac{\phi_n\phi}{(r+\kappa)^2}+\chi_n\chi\sigma^2+\frac{S_n}{(r+\kappa)^2}\right]\text{Cov}\left[A(v_t),v_t\right] \\ &\quad -\left(\chi_n q_2+\chi q_{n2}\right)\sigma^2\text{Cov}\left[A(v_t),(v_t-\overline{v})v_t\right]+O(\sigma^5). \end{aligned} \tag{87}$$

Since

$$\text{Cov}\left[A(v_t),(v_t-\overline{v})v_t\right]=\text{Cov}\left[A(v_t),(v_t-\overline{v})\overline{v}\right]+\text{Cov}\left[A(v_t),(v_t-\overline{v})^2\right]=\overline{v}\text{Cov}\left[A(v_t),v_t\right]+O(\sigma^3),$$

we can write equation (87) as

$$C_n^A=\left[\frac{\phi_n\phi}{(r+\kappa)^2}+\chi_n\chi\sigma^2+\frac{S_n}{(r+\kappa)^2}-\left(\chi_n q_2+\chi q_{n2}\right)\sigma^2\overline{v}\right]\text{Cov}\left[A(v_t),v_t\right]+O(\sigma^5). \tag{88}$$

To compute $C_n^Z$, we proceed similarly. Equations (1) and (17) imply that

$$\text{Cov}_t(dR_{nt},dv_t)=X_n\sigma^2 v_t dt. \tag{89}$$

Using the Taylor expansion of $q_n(v_t)$, we can write $X_n$ as

$$X_n=\chi_n-q_{n2}(v_t-\overline{v})+O\left[(v_t-\overline{v})^2\right]. \tag{90}$$

Combining equations (89) and (90), and using equation (86), we find

$$\begin{aligned} C_n^Z &= \chi_n\sigma^2\text{Cov}\left[Z'(v_t),v_t\right]-q_{n2}\sigma^2\text{Cov}\left[Z'(v_t),(v_t-\overline{v})v_t\right]+O(\sigma^5) \\ &= \left[\chi_n-q_{n2}\overline{v}\right]\sigma^2\text{Cov}\left[Z'(v_t),v_t\right]+O(\sigma^5). \end{aligned} \tag{91}$$

We next determine whether $C_n^A$ and $C_n^Z$ can be expressed in terms of $C_n^M$ and $C_n^v$. Equations (25), (84), and (85) imply that

$$\begin{aligned} C_n^M &= \frac{E\left[\text{Cov}_t(dR_{nt},dR_{Mt})\right]}{dt} \\ &= \left[\frac{\phi_n\phi}{(r+\kappa)^2}+\chi_n\chi\sigma^2+\frac{S_n}{(r+\kappa)^2}\right]E(v_t)-\left(\chi_n q_2+\chi q_{n2}\right)\sigma^2 E\left[(v_t-\overline{v})v_t\right]+O(\sigma^4) \end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{\phi_n \phi}{(r+\kappa)^2} + \chi_n \chi \sigma^2 + \frac{S_n}{(r+\kappa)^2}\right] \overline{v} - (\chi_n q_2 + \chi q_{n2})\, \sigma^2 E\left[(v_t - \overline{v})^2\right] + O(\sigma^4) \\
&= \left[\frac{\phi_n \phi}{(r+\kappa)^2} + \chi_n \chi \sigma^2 + \frac{S_n}{(r+\kappa)^2}\right] \overline{v} + O(\sigma^4), \qquad (92)
\end{aligned}
$$

where the term $O(\sigma^4)$ follows because

$$E\left[(v_t - \overline{v})^k\right] = O(\sigma^k).$$

Similarly, equations (25), (89), and (90) imply that

$$
\begin{aligned}
C_n^v &= \frac{E\left[\mathrm{Cov}_t(dR_{nt}, dv_t)\right]}{dt} \\
&= \chi_n \sigma^2 E(v_t) - q_{n2}\sigma^2 E\left[(v_t - \overline{v})v_t\right] + O(\sigma^4) \\
&= \chi_n \sigma^2 E(v_t) - q_{n2}\sigma^2 E\left[(v_t - \overline{v})^2\right] + O(\sigma^4) \\
&= \chi_n \sigma^2 \overline{v} + O(\sigma^4). \qquad (93)
\end{aligned}
$$

Using equations (92) and (93), we can write equations (88) and (91) as

$$C_n^A = \left[\frac{C_n^M}{\overline{v}} - q_2 C_n^v - \chi q_{n2}\sigma^2 \overline{v}\right] \mathrm{Cov}\left[A(v_t), v_t\right] + O(\sigma^5) \qquad (94)$$

and

$$C_n^Z = \left[\frac{C_n^v}{\overline{v}} - q_{n2}\sigma^2 \overline{v}\right] \mathrm{Cov}\left[Z'(v_t), v_t\right] + O(\sigma^5), \qquad (95)$$

respectively. Furthermore, using equation (38), and noting that

$$\overline{z}_n \equiv \frac{\phi_n \phi + S_n}{(r+\kappa)^2}\overline{v} = C_n^M + o(\sigma),$$

we can write $q_{n2}$ as

$$q_{n2} = \frac{\mu n(\ell)\ell(\ell^4 - 6\ell^2 + 3)}{4(r+2\gamma)\overline{z}\overline{v}^2} \frac{\exp(2A\epsilon) - 1}{A} C_n^M + \frac{\mu n(\ell)\ell(\ell^2 - 3)}{2(r+2\gamma)\overline{v}^2} \exp(2A\epsilon)\epsilon_n + O(\sigma). \qquad (96)$$

Plugging equations (94), (95), and (96) into equation (83), we find

$$E(dR_{nt}) = \left[\Gamma_M C_n^M + \Gamma_v C_n^v + K\epsilon_n + L\epsilon_n\right] dt + O(\sigma^5), \qquad (97)$$

where

$$\Gamma_M \equiv E\left[A(v_t)\right] + \frac{\mathrm{Cov}\left[A(v_t), v_t\right]}{\overline{v}}$$

$$- \left[\chi \text{Cov}\left[A(v_t), v_t\right] + A\text{Cov}\left[Z'(v_t), v_t\right]\right] \frac{\mu n(\ell)\ell(\ell^4 - 6\ell^2 + 3)\sigma^2}{4(r + 2\gamma)\overline{zv}} \frac{\exp(2A\epsilon) - 1}{A}, \quad (98)$$

$$\Gamma_v \equiv AE\left[Z'(v_t)\right] + \frac{A\text{Cov}\left[Z'(v_t), v_t\right]}{\overline{v}} - q_2 \text{Cov}\left[A(v_t), v_t\right], \quad (99)$$

$$K \equiv - \left[\chi \text{Cov}\left[A(v_t), v_t\right] + A\text{Cov}\left[Z'(v_t), v_t\right]\right] \frac{\mu n(\ell)\ell(\ell^2 - 3)\sigma^2}{2(r + 2\gamma)\overline{v}} \exp(2A\epsilon). \quad (100)$$

To compute $\text{Cov}\left[A(v_t), v_t\right]$, we note that

$$\begin{aligned}
\text{Cov}\left[A(v_t), v_t\right] &= \text{Cov}\left[A(v_t), (v_t - \overline{v})\right] \\
&\approx A'(\overline{v})E\left[(v_t - \overline{v})^2\right] \\
&= 2\mu g''(\overline{z})z'(\overline{v})\frac{\exp(2A\epsilon) - 1}{A}\text{Var}(v_t) \\
&= \frac{\mu n(\ell)\ell(\ell^2 - 3)}{2\overline{zv}} \frac{\exp(2A\epsilon) - 1}{A}\text{Var}(v_t) \\
&= \frac{\mu n(\ell)\ell(\ell^2 - 3)\sigma^2}{4\gamma\overline{z}} \frac{\exp(2A\epsilon) - 1}{A}, \quad (101)
\end{aligned}$$

where the second step follows from equation (86), the third from equation (33), the fourth from equations (76) and (77), and the fifth because $v_t$ follows a square-root process. We similarly have

$$\text{Cov}\left[Z'(v_t), v_t\right] = \frac{Z_2 \sigma^2 \overline{v}}{2\gamma}, \quad (102)$$

and

$$\begin{aligned}
E\left[Z'(v_t)\right] &= Z_1 + \frac{1}{2}Z_3 E\left[(v_t - \overline{v})^2\right] \\
&= Z_1 + \frac{Z_3 \sigma^2 \overline{v}}{4\gamma}. \quad (103)
\end{aligned}$$

Plugging equations (101) and (102) into (100), we find equation (43). Plugging equations (101)-(103) into (99), and noting that $\Lambda_v = \Gamma_v \text{Var}(dv_t)/dt = \Gamma_v \sigma^2 \overline{v}$, we find equation (44). ∎

**Proof of Proposition 8:** The unconditional market beta of asset $n$ is

$$\begin{aligned}
\beta_n^M &= \frac{\text{Cov}(dR_{nt}, dR_{Mt})}{\text{Var}(dR_{Mt})} \\
&= \frac{E\left[\text{Cov}_t(dR_{nt}, dR_{Mt})\right]}{E\left[\text{Var}_t(dR_{Mt})\right]} \\
&= \frac{E\left[\left[\frac{\phi_n \phi}{(r+\kappa)^2} + \left[\frac{\psi_n}{r+\kappa} - q_n'(v_t)\right]\left[\frac{\psi}{r+\kappa} - q'(v_t)\right]\sigma^2 + \frac{S_n}{(r+\kappa)^2}\right] v_t\right]}{E\left[\left[\frac{\phi^2}{(r+\kappa)^2} + \left[\frac{\psi}{r+\kappa} - q'(v_t)\right]^2 \sigma^2 + \frac{\|S\|^2}{(r+\kappa)^2}\right] v_t\right]}.
\end{aligned}$$

Since $\epsilon_n$ affects only the function $q_n(v_t)$, $\partial\beta_n^M/\partial\epsilon_n$ has the same sign as

$$-E\left[\frac{\partial q_n'(v_t)}{\partial\epsilon_n}\left[\frac{\psi}{r+\kappa} - q'(v_t)\right] v_t\right]. \tag{104}$$

For small $\sigma$, the highest-order term in equation (104) is

$$-\frac{\partial q_{n1}}{\partial\epsilon_n}\chi\overline{v},$$

and is positive if $\chi < 0$ (since $\partial q_{n1}/\partial\epsilon_n > 0$ from equation (37)).

Proceeding similarly, we find that the unconditional volatility beta of asset $n$ is

$$\beta_n^v = \frac{E\left[\left[\frac{\psi_n}{r+\kappa} - q_n'(v_t)\right]\sigma^2 v_t\right]}{E\left[\sigma^2 v_t\right]},$$

and $\partial\beta_n^v/\partial\epsilon_n$ has the same sign as

$$-E\left[\frac{\partial q_n'(v_t)}{\partial\epsilon_n} v_t\right]. \tag{105}$$

For small $\sigma$, the highest-order term in equation (105) is

$$-\frac{\partial q_{n1}}{\partial\epsilon_n}\overline{v},$$

and is negative. ∎

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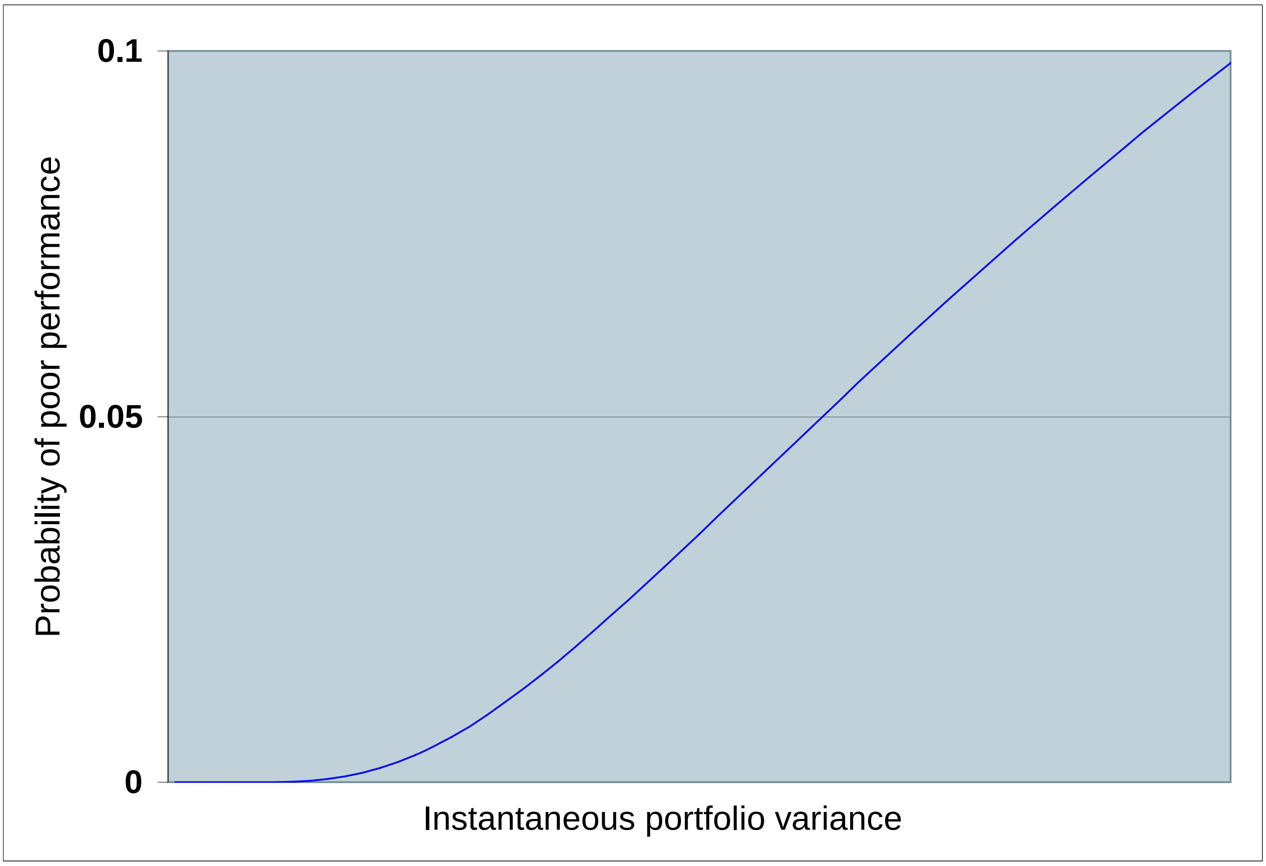


Figure 1: The probability of poor performance as a function of the instantaneous portfolio variance.