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ON THE CLOSURE IN THE EMERY TOPOLOGY OF SEMIMARTINGALE WEALTH-PROCESS SETS

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Abstract. A wealth-process set is abstractly defined to consist of nonnegative càdlàg processes containing a strictly positive semimartingale and satisfying an intuitive re-balancing property. Under the condition of absence of arbitrage of the first kind, it is established that all wealth processes are semimartingales, and that the closure of the wealth-process set in the Emery topology contains all "optimal" wealth processes.

Introduction

In financial modeling, it is customary to start by describing a set of wealth processes that can be achieved in some elementary way. Concrete examples include:

- wealth processes arising from finite combinations of buy-and-hold strategies;
- wealth processes resulting from taking positions on a finite number of investment assets, when there is an infinite number of such assets available in the market. This is the case in the theoretical modeling of bond markets, where there exist zero-coupon bonds with a continuum of maturities—see, for example, [3] and [7]. Another case is the approximation of "large" financial markets, as is discussed in [6].

Although such initial descriptions of available wealth processes are natural and unquestionable, the thus-constructed classes are typically insufficient for analysis. Indeed, important problems like portfolio optimization and hedging of contingent claims might fail to have solutions within the class of wealth processes, if the latter is lacking any reasonable closedness property. Therefore, the need arises to pass to the closure, in some appropriate sense, of these elementary wealth-process sets. Such passage is a rather subtle issue: although the closure should be large enough to ensure that all "interesting" (or "optimal") elements are there, the need to keep a tight financial interpretation of the resulting enlarged wealth-process set dictates that fine topologies are required.

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In the literature, a balance between the aforementioned opposing forces has to be resolved individually for each problem-at-hand. For example, when wealth processes are defined using simple integrands (i.e., finite combinations of buy-and-hold strategies) against a finite-dimensional semimartingale integrator, the class of all stochastic integrals using general predictable integrands turns out to be the appropriate enlargement—indeed, this has been demonstrated in a number of papers, with [8], [19], and [20] being the ones related to questions of market viability and optimization that are close to the spirit of the present discussion. In fact, the class of stochastic integrals using general predictable integrands coincides with the closure of the set of all simple integrals in the so-called *Emery* (or *semimartingale*) *topology*, introduced in [11]. An enlargement of the initial wealth-process set using limits of semimartingales in the Emery topology is also utilized in [6] and [7], when approximating stochastic integrals with respect to an infinite-dimensional integrator via stochastic integrals with integrands having only a finite number of nonzero coordinates.

The Emery topology is extremely strong and, at the same time, very natural when dealing with semimartingales. The purpose of this paper is to show, in an abstract and general setting, that it is the closure of wealth-process sets in the Emery topology that is indeed appropriate if one wants to ensure that "optimal" elements are contained in the enlarged class of wealth processes. For the sake of generality, admissible wealth-process sets are defined in an abstract way, asking that they consist of nonnegative adapted càdlàg processes containing one strictly positive semimartingale (which can be, for example, the outcome of investing in a locally riskless asset) and satisfying an intuitive re-balancing property, called *fork-convexity* in [27]. It is first established that, under the mild condition of absence of arbitrage of the first kind in the market, all wealth processes are semimartingales—because of this fact, taking the closure of the wealth-process set in the Emery topology becomes both relevant and possible. Following this preliminary result, the main message of the paper is the following: the closure of wealth-process sets in the Emery topology is rich enough in order to allow for solutions to expected utility maximization problems. More precisely, even though an optimal wealth process might not exist in the original wealth-process set, one can find a sequence of "nearly-optimal" wealth processes that has a limit in the Emery topology, and the latter limit is indeed optimal in the enlarged wealth-process set.

The results of this paper serve as a guideline in efficiently defining enlargements of wealth-process sets, after an elementary and acceptable initial description has been carried out. The fineness of the Emery topology on semimartingales ensures that the resulting enlarged wealth-process set will be quite close to the original one. It is exactly the general and abstract nature of the definition of wealth-process sets that makes the hereby presented results valuable. Needless to say, when faced with a specific application one should aim for more "intrinsic" and elegant descriptions of the closure of elementary wealth-process sets in the Emery topology.

The structure of the paper is simple. Section 1 contains all the set-up, discussion and results. All proofs are deferred to Section 2.

1. Results

1.1. **Preliminaries.** Throughout, $T \in (0, \infty)$ will be denoting a fixed financial planning horizon. We shall be working on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where $\mathbf{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is a filtration satisfying the usual hypotheses of right-continuity and saturation by $\mathbb{P}$-null sets of $\mathcal{F}$. Without loss of generality, we assume that $\mathcal{F}_0$ is trivial modulo $\mathbb{P}$ and that $\mathcal{F} = \mathcal{F}_T$. Random variables are identified modulo $\mathbb{P}$-a.s. equality. Stochastic processes that are indistinguishable modulo $\mathbb{P}$ are also identified. A càdlàg (right-continuous with left limits) stochastic process X will be called *nonnegative* (resp., *strictly positive*) if $\mathbb{P}\left[\inf_{t \in [0,T]} X_t \geq 0\right] = 1$ (resp., if $\mathbb{P}\left[\inf_{t \in [0,T]} X_t > 0\right] = 1$).

The class of semimartingales on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ is denoted by $\mathcal{S}$. If $X \in \mathcal{S}$ and η is a predictable and X-integrable process, $\eta \cdot X$ denotes the stochastic integral of η with respect to X—by convention, $(\eta \cdot X)_0 = \eta_0 X_0$. Let $\mathcal{P}_1$ be the set of predictable processes η with $|\eta| \leq 1$. For $X \in \mathcal{S}$, define

$$\lceil X \rceil_{\mathcal{S}} := \sup_{\eta \in \mathcal{P}_1} \mathbb{E}\left[1 \wedge \left(\sup_{t \in [0,T]} |(\eta \cdot X)_t|\right)\right],$$

where "$\mathbb{E}$" is used to denote expectation under $\mathbb{P}$ and "$\wedge$" is used to denote the minimum operation. The metric $\mathcal{S} \times \mathcal{S} \ni (X, X') \mapsto \lceil X - X' \rceil_{\mathcal{S}}$ induces the *Emery topology* on $\mathcal{S}$, introduced in [11]. Whenever $\lim_{n \to \infty} \lceil X^n - X \rceil_{\mathcal{S}} = 0$, we write $\mathcal{S}\text{-}\lim_{n \to \infty} X^n = X$. Convergence in the Emery topology is extremely strong; for example, it implies uniform convergence in probability and (as Proposition 2.10 later in the text shows) convergence of quadratic variations.

1.2. **Financial set-up.** The first line of business is to model the class of wealth processes available to an investor with (normalized) unit initial capital. The wealth-process set will be defined in a rather abstract and general-encompassing way: any reasonable class of (potentially, constrained) nonnegative wealth processes resulting from frictionless trading that has appeared in the literature falls within its scope.

Definition 1.1. A set $\mathcal{X}$ of stochastic processes will be called a *wealth-process set* if:

(1) Each $X \in \mathcal{X}$ is a nonnegative càdlàg process with $X_0 = 1$.
(2) There exists a strictly positive semimartingale in $\mathcal{X}$.
(3) $\mathcal{X}$ is *fork-convex*: for any $s \in [0,T]$, $X \in \mathcal{X}$, any strictly positive processes $X' \in \mathcal{X}$ and $X'' \in \mathcal{X}$, and any $[0,1]$-valued $\mathcal{F}_s$-measurable random variable α, the process

$$[0,T] \ni t \mapsto X_t \mathbb{I}_{\{t<s\}} + \left(\alpha \left(X_s / X'_s\right) X'_t + (1-\alpha) \left(X_s / X''_s\right) X''_t\right) \mathbb{I}_{\{s \leq t\}} \tag{1.1}$$

is also an element of $\mathcal{X}$.

In Definition 1.1 of a wealth-process set, fork-convexity corresponds to the possibility of re-balancing. In fact, (1.1) exactly describes the wealth generated when a financial agent invests according to X up to time s, and then reinvests a fraction α of the money in the wealth process described by X' and the remaining fraction $(1-\alpha)$ in the wealth process described by X''. On the other hand, condition (2) is always true when a locally riskless investment opportunity exists leading to a wealth process that is adapted, right-continuous and nondecreasing.

Definition 1.2. Let $\mathcal{X}$ be a wealth-process set. For $x \in (0, \infty)$, define $\mathcal{X}(x) := \{xX \mid X \in \mathcal{X}\}$. We say that there are opportunities for arbitrage of the first kind in the market if there exists an $\mathcal{F}_T$-measurable random variable ξ such that:

- $\mathbb{P}[\xi \geq 0] = 1$ and $\mathbb{P}[\xi > 0] > 0$;
- for all $x \in (0, \infty)$ there exists $X \in \mathcal{X}(x)$, which may depend on x, with $\mathbb{P}[X_T \geq \xi] = 1$.

If there are *no* opportunities for arbitrage of the first kind, we shall say that condition NA_1 holds.

In the context of Definition 1.2, $\mathcal{X}(x)$ represents all wealth processes that are available to an investor with initial capital $x \in (0, \infty)$. Keeping this in mind, the definition of arbitrage of the first kind is very natural: regardless of how minuscule the initial capital is, an investor is able to choose a wealth process that will result in an outcome which dominates ξ, the latter being a nonnegative random variable which is strictly positive on an event of strictly positive probability.

1.3. **Results.** We are ready to present the findings of the paper; proofs are deferred to Section 2.

We start with a result stating that condition NA_1 already enforces a semimartingale structure on wealth-process sets. Similar results, in the case where the wealth-process set is defined as nonnegative *simple* stochastic integrals (using linear combinations of buy-and-hold strategies) against a càdlàg adapted process have been established in [8, Section 7], [18], and [2].

Theorem 1.3. *Let $\mathcal{X}$ be a wealth-process set, and assume condition* NA_1. *Then, every process in $\mathcal{X}$ is a semimartingale.*

In view of Theorem 1.3, whenever $\mathcal{X}$ is a wealth-process set such that condition NA_1 is valid, we define $\overline{\mathcal{X}}$ as the closure of $\mathcal{X}$ in the Emery topology. It follows that $\overline{\mathcal{X}}$ is also a wealth-process set that is further closed in the Emery topology. Indeed, the only fact that is not trivial is that $\overline{\mathcal{X}}$ is fork-convex. Fix $s \in [0, T]$, $X \in \overline{\mathcal{X}}$, any strictly positive processes $X' \in \overline{\mathcal{X}}$ and $X'' \in \overline{\mathcal{X}}$, and any $[0, 1]$-valued $\mathcal{F}_s$-measurable random variable α. Pick $\mathcal{X}$-valued sequences $(X^n)_{n \in \mathbb{N}}$, $((X')^n)_{n \in \mathbb{N}}$ and $((X'')^n)_{n \in \mathbb{N}}$ such that $\mathcal{S}\text{-}\lim_{n \to \infty} X^n = X$, $\mathcal{S}\text{-}\lim_{n \to \infty} (X')^n = X'$ and $\mathcal{S}\text{-}\lim_{n \to \infty} (X'')^n = X''$. It can be assumed without loss of generality that the sequences $((X')^n)_{n \in \mathbb{N}}$ and $((X'')^n)_{n \in \mathbb{N}}$ consist of strictly positive wealth processes in $\mathcal{X}$; otherwise, with $\chi \in \mathcal{X}$ being strictly positive, one may replace $(X')^n$ with $(1 - n^{-1})(X')^n + n^{-1}\chi$ and $(X'')^n$ with $(1 - n^{-1})(X'')^n + n^{-1}\chi$ for all $n \in \mathbb{N}$; the previous are strictly positive wealth processes, and $\mathcal{S}\text{-}\lim_{n \to \infty} \big((1 - n^{-1})(X')^n + n^{-1}\chi\big) = X'$ as well as $\mathcal{S}\text{-}\lim_{n \to \infty} \big((1 - n^{-1})(X'')^n + n^{-1}\chi\big) = X''$ still hold. It follows that the process ψ^n, defined via $\psi^n_t := X^n_t \mathbb{I}_{\{t < s\}} + (\alpha\, (X^n_s / (X')^n_s)\, (X')^n_t + (1 - \alpha)\, (X^n_s / (X'')^n_s)\, (X'')^n_t)\, \mathbb{I}_{\{s \leq t\}}$ for $t \in [0, T]$ is an element of $\mathcal{X}$ for all $n \in \mathbb{N}$. Furthermore, it is straightforward from the definition of $\mathcal{S}$-convergence that the sequence $(\psi^n)_{n\in\mathbb{N}}$ converges in the Emery topology to the process in (1.1). This establishes the fork-convexity of $\overline{\mathcal{X}}$.

We proceed in giving justice to the claim (made in the Introduction) that $\overline{\mathcal{X}}$ already contains all interesting "optimal" elements, by examining the problem of expected utility maximization. Let $U : (0,\infty) \mapsto \mathbb{R}$ be a strictly increasing, strictly concave and continuously differentiable function, satisfying the Inada conditions $\lim_{x\downarrow 0} U'(x) = \infty$ and $\lim_{x\uparrow\infty} U'(x) = 0$. Also, set $U(0) := \lim_{x\downarrow 0} U(x)$ in order to accommodate possibly zero wealth. With $\mathcal{X}$ being a wealth-process set such that $1 \in \mathcal{X}$, define the *indirect utility function* $u : (0,\infty) \mapsto \mathbb{R} \cup \{\infty\}$ via $u(x) = \sup_{X\in\mathcal{X}(x)} \mathbb{E}\big[U(X_T)\big]$ for $x \in (0,\infty)$. (In order for an expression of the form $\mathbb{E}\big[U(X_T)\big]$, where $X \in \mathcal{X}(x)$ for some $x \in (0,\infty)$, to be well defined, the usual convention $\mathbb{E}\big[U(X_T)\big] = -\infty$ whenever $\mathbb{E}\big[0 \wedge U(X_T)\big] = -\infty$ is used. Also, note that $u \geq U$ follows from $1 \in \mathcal{X}$, which implies that $u(x) > -\infty$ for all $x \in (0,\infty)$.) In accordance with Definition 1.2, set $\overline{\mathcal{X}}(x) := \big\{xX \mid X \in \overline{\mathcal{X}}\big\}$ for $x \in (0,\infty)$. It is not *a priori* clear that $\sup_{X\in\mathcal{X}(x)} \mathbb{E}\left[U(X_T)\right] = \sup_{X\in\overline{\mathcal{X}}(x)} \mathbb{E}\left[U(X_T)\right]$ holds for $x \in (0,\infty)$; however, as Theorem 1.4 states, this is indeed true under assumption NA_1. What *is* clear is that, in general, maximal expected utility will not be achieved by a wealth process in $\mathcal{X}(x)$ for $x \in (0,\infty)$; as it turns out, maximal utility *can* be achieved by a process in $\overline{\mathcal{X}}(x)$, at least under condition NA_1 and the validity of the following:

$$\text{(FIN-DUAL)} \qquad \sup_{x>0} \{u(x) - xy\} < \infty \text{ holds for all } y \in (0,\infty).$$

Furthermore, for all $x \in (0,\infty)$, the optimal wealth process in $\overline{\mathcal{X}}(x)$ along with its expected utility can be approximated arbitrarily by wealth processes in $\mathcal{X}(x)$. The exact statement follows.

Theorem 1.4. *Let $\mathcal{X}$ be a wealth process set with $1 \in \mathcal{X}$, and suppose that condition* NA_1 *is valid.*

(1) $u(x) := \sup_{X\in\mathcal{X}(x)} \mathbb{E}\left[U(X_T)\right] = \sup_{X\in\overline{\mathcal{X}}(x)} \mathbb{E}\left[U(X_T)\right]$ *holds for all* $x \in (0,\infty)$.

(2) Suppose that (FIN-DUAL) *is also valid. Then, for all $x \in (0,\infty)$, there exists $\widehat{X}(x) \in \overline{\mathcal{X}}(x)$ satisfying $\mathbb{E}\big[U(\widehat{X}(x)_T)\big] = u(x) < \infty$; furthermore, there exists an $\mathcal{X}(x)$-valued sequence $(X^n(x))_{n\in\mathbb{N}}$ such that both $\mathcal{S}\text{-}\lim_{n\to\infty} X^n(x) = \widehat{X}(x)$ and $\lim_{n\to\infty} \mathbb{E}\big[U(X^n(x)_T)\big] = \mathbb{E}\big[U(\widehat{X}(x)_T)\big] = u(x)$ hold.*

Remark 1.5. For $U : (0,\infty) \mapsto \mathbb{R}$ as before, define $U(\infty) := \lim_{x\uparrow\infty} U(x)$.

When $U(\infty) = \infty$ and condition NA_1 fails for a wealth-process set $\mathcal{X}$ with $1 \in \mathcal{X}$, it is straightforward that $u(x) = \infty$ holds for all $x \in (0,\infty)$. On the other hand, condition (FIN-DUAL) always implies that u is finitely-valued. It then follows that, when $U(\infty) = \infty$ and $\mathcal{X}$ is a wealth process with $1 \in \mathcal{X}$, (FIN-DUAL) is sufficient to have both statements of Theorem 1.4 valid, since condition NA_1 is indirectly forced.

Note also that when $U(\infty) < \infty$ condition (FIN-DUAL) is always trivially valid; therefore it does not have to be assumed in statement (2) of Theorem 1.4.

Remark 1.6. The proof of the existence of optimal wealth processes in statement (2) of Theorem 1.4 heavily depends on the two seminal papers of Kramkov and Schachermayer [19, 20]. At first sight, the setting of the present paper does not match the one of [19] and [20]—indeed, in the latter papers the wealth-process sets are modeled via outcomes of stochastic integrals with respect to a finite-dimensional semimartingale integrator. However, [19] and [20] contain certain "abstract results" that we shall be eventually able to use in order to show the validity of Theorem 1.4.

In fact, there is an intermediate result used in order to establish Theorem 1.4, which is in some sense more fundamental.

Theorem 1.7. *Let $\mathcal{X}$ be a wealth-process set, and assume condition* NA_1*. Then, for any $\mathbb{Q} \sim \mathbb{P}$ there exists a strictly positive $\widehat{X}^{\mathbb{Q}} \in \overline{\mathcal{X}}$ such that $X / \widehat{X}^{\mathbb{Q}}$ is a $\mathbb{Q}$-supermartingale for all $X \in \overline{\mathcal{X}}$.*

Remark 1.8. Theorem 1.7 is related to the idea of *change of numéraire*—see [9]. Using notation from Theorem 1.7, the probability $\mathbb{Q}$ is an equivalent supermartingale measure in the market where wealth is denominated by $\widehat{X}^{\mathbb{Q}} \in \overline{\mathcal{X}}$. In accordance to the terminology of [23], [1] and [14], one can call $\widehat{X}^{\mathbb{Q}}$ the *numéraire portfolio* in $\mathcal{X}$ under the probability $\mathbb{Q}$.

Remark 1.9. We elaborate on how Theorem 1.4 and Theorem 1.7 are connected. Technicalities aside, the numéraire portfolio $\widehat{X}^{\mathbb{Q}}$ in the notation of Theorem 1.7 corresponds to the optimal wealth process for the expected logarithmic utility maximization problem under the probability $\mathbb{Q}$. (This follows by formally applying first-order conditions for log-optimality and deriving the "numéraire property" of log-optimal portfolios—extensive discussion in the special case of financial models driven by a finite-dimensional semimartingale integrator can be found in [14].) As can be seen from the proof of Theorem 1.4 in Subsection 2.7, any optimal process stemming from utility maximization problems can be regarded as the log-optimal wealth (more precisely, a multiple of the numéraire portfolio in $\mathcal{X}$) under an auxiliary probability measure that is equivalent to $\mathbb{P}$. The idea is certainly not new—for example, in the work of Kramkov and Sîrbu [21, 22], such changes of numéraire and probability are utilized in questions related to sensitivity analysis of the expected utility maximization problem as well as utility indifference prices.

Remark 1.10. Suppose that $\mathcal{X}$ is a wealth-process set such that condition NA_1 holds. In view of Theorem 1.7, condition NA_1 also holds for the wealth-process set $\overline{\mathcal{X}}$. Indeed, the existence of a strictly positive $\widehat{X} \in \overline{\mathcal{X}}$ such that $\mathbb{E}\big[X_T / \widehat{X}_T\big] \leq 1$ holds for all $X \in \overline{\mathcal{X}}$ can be easily seen to imply that no arbitrage of the first kind can exist in the market with wealth-process set $\overline{\mathcal{X}}$.

Remark 1.11. Suppose that $\mathcal{X}$ is the wealth-process set generated by nonnegative stochastic integrals with respect to a finite-dimensional semimartingale integrator. Then, $\mathcal{X}$ is already closed in the Emery topology. (The ideas behind the proof of the last claim are present in Mémin's work [24]—see also [14, discussion after Theorem 4.4], as well as [5].) In this special case, more elaborate versions of Theorem 1.7 appear in [16] and [26]: condition NA_1 implies that for any $\mathbb{Q} \sim \mathbb{P}$ there exists a strictly positive $\widehat{X}^{\mathbb{Q}} \in \mathcal{X}$ such that $X/\widehat{X}^{\mathbb{Q}}$ is a local $\mathbb{Q}$-martingale for all $X \in \mathcal{X}$. Furthermore, the results of [9] imply that for each *maximal* strictly positive wealth process $\widehat{X} \in \mathcal{X}$, there exists $\mathbb{Q} \sim \mathbb{P}$ such that $X/\widehat{X}$ is a local $\mathbb{Q}$-martingale for all $X \in \mathcal{X}$.

Remark 1.12. Theorem 1.7—which is the basis for proving Theorem 1.4—underlies the need for assuming that wealth remains nonnegative; indeed, the concept of numéraire portfolio is only available for collections of nonnegative processes. The supermartingale property of properly discounted processes is not suitable to describe optimality when wealth may become negative. It would be interesting to explore whether a theory parallel to the one presented here can be developed for wealth-process sets when processes are not constrained to remain nonnegative. Naturally, different conditions will be required from a wealth-process set in such case; for example, an additive analogue of the multiplicative fork-convexity property of Definition 1.1 may be more appropriate. Such a project will certainly require different tools than the ones used here and is beyond the scope of this paper.

2. Proofs

2.1. **Some modes of convergence.** Let $\mathbb{L}^0$ be the space of $\mathcal{F}$-measurable $\mathbb{P}$-a.s. finitely-valued random variables. For $g \in \mathbb{L}^0$, define $\lceil g \rceil_{\mathbb{P}} := \mathbb{E}[1 \wedge |g|]$. The metric $(g, g') \mapsto \lceil g - g' \rceil_{\mathbb{P}}$ on $\mathbb{L}^0$ induces the topology of convergence in $\mathbb{P}$-measure. We simply write $\mathbb{P}\text{-}\lim_{n\to\infty} g^n = g$ whenever $\lim_{n\to\infty} \lceil g^n - g \rceil_{\mathbb{P}} = 0$. We use $\mathbb{L}^0_+$ to denote the set of $g \in \mathbb{L}^0$ with $\mathbb{P}[g \geq 0] = 1$.

For a càdlàg process X, define $X^* := \sup_{t\in[0,\cdot]} |X_t|$; then, define $\lceil X \rceil_{\mathsf{u}\mathbb{P}} := \lceil X^*_T \rceil_{\mathbb{P}}$. The metric $(X, X') \mapsto \lceil X - X' \rceil_{\mathsf{u}\mathbb{P}}$ induces the topology of uniform (on $[0, T]$) convergence in $\mathbb{P}$-measure on the space of càdlàg processes. We write $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty} X^n = X$ when $\lim_{n\to\infty} \lceil X^n - X \rceil_{\mathsf{u}\mathbb{P}} = 0$. With the previous notation, note that $\lceil X \rceil_{\mathcal{S}} = \sup_{\eta\in\mathcal{P}_1} \lceil \eta \cdot X \rceil_{\mathsf{u}\mathbb{P}}$ holds for $X \in \mathcal{S}$—in particular, since considering $\eta \equiv 1$ gives $\lceil X \rceil_{\mathsf{u}\mathbb{P}} \leq \lceil X \rceil_{\mathcal{S}}$ for $X \in \mathcal{S}$, $\mathcal{S}$-convergence implies $\mathsf{u}\mathbb{P}$-convergence.

Lastly, we introduce yet another mode of convergence. Say that a sequence of nonnegative càdlàg processes $(X^n)_{n\in\mathbb{N}}$ *Fatou-converges* to a nonnegative càdlàg process X, and write $\mathsf{F}\text{-}\lim_{n\to\infty} X^n = X$, if there exists a countably dense set $\mathbb{T} \subseteq [0, T]$ with $T \in \mathbb{T}$ such that, $\mathbb{P}$-a.s.,

$$X_t = \liminf_{\mathbb{T}\ni s\downarrow t} \left(\liminf_{n\to\infty} X^n_s \right) = \limsup_{\mathbb{T}\ni s\downarrow t} \left(\limsup_{n\to\infty} X^n_s \right), \text{ for all } t \in [0, T].$$

(For $t = T$ the last equality should be read as $X_T = \liminf_{n\to\infty} X^n_T = \limsup_{n\to\infty} X^n_T$.)

Remark 2.1. Fatou-convergence certainly lacks elegance compared to the previous modes of convergence. However, it proves extremely useful in the theory of Mathematical Finance, as was made clear in [12], [19] and [27], to name a few. The main reason for its usefulness is a "convex compactness" property that allows to obtain existence of optimal wealth processes in the Fatou-closure (the set of all possible limits in the Fatou sense) of a wealth-process set for concave maximization problems. Indeed, as stated in Lemma 2.14 (which follows from [12, Lemma 5.2(1)] and a change-of-numéraire argument), if $\mathcal{X}$ is a wealth-process set such that NA_1 holds, any $\mathcal{X}$-valued sequence $(X^n)_{n \in \mathbb{N}}$ has a sequence of forward convex combinations that is Fatou-convergent. Although convenient, this ability to easily find Fatou-convergent sequences in wealth-process sets has the undesirable implication that the Fatou-closure of a wealth-process set tends to be quite large, making the corresponding limits difficult to justify from a financial viewpoint. In fact, Fatou-closures contain "wealth processes" that fail to be maximal, in the sense that they allow for free disposal of wealth—Subsection 2.6 offers a better understanding of such issues. However, as it turns out, "optimal" elements in the Fatou-closure, which are exactly the numéraires mentioned in Theorem 1.7, can be approximated also in the Emery topology. As already mentioned in Remark 1.11, when $\mathcal{X}$ is the wealth-process set generated by nonnegative stochastic integrals with respect to a finite-dimensional semimartingale integrator, it is established in [9] that all strictly positive maximal processes are actually numéraire portfolios under a suitable equivalent change of probability. However, in the case of possible constraints on investment, it may happen that maximal elements do not correspond to numéraire portfolios—for an example in a one time-period model, see [17, Subsection 1.3].

2.2. **Preliminaries towards proving Theorem 1.3 and Theorem 1.7.** We start with an auxiliary result.

Lemma 2.2. *Suppose that $\mathcal{X}$ is a wealth-process set. Then, condition* NA_1 *holds if and only if* $\lim_{\ell \to \infty} \left(\sup_{(X,t) \in \mathcal{X} \times [0,T]} \mathbb{P}\left[X_t > \ell \right] \right) = 0$, *i.e., when the collection* $\{X_t \,|\, X \in \mathcal{X}, \ t \in [0,T]\}$ *of random variables is bounded in* $\mathbb{P}$*-measure.*

Proof. The proof of the fact that condition NA_1 holds if and only if $\{X_T \,|\, X \in \mathcal{X}\}$ is bounded in $\mathbb{P}$-measure follows *mutatis mutandis* from [18, proof of Proposition 1.1]. It only remains to show that boundedness in $\mathbb{P}$-measure of $\{X_T \,|\, X \in \mathcal{X}\}$ implies the stronger boundedness in $\mathbb{P}$-measure of $\{X_t \,|\, X \in \mathcal{X}, \ t \in [0,T]\}$. Fix some strictly positive $\chi \in \mathcal{X}$, and define $\kappa \in \mathbb{L}^0_+$ via $\kappa := \sup_{t \in [0,T]} \chi_t / \chi_T$. For $(X,t) \in \mathcal{X} \times [0,T]$, the fork-convexity of $\mathcal{X}$ implies that $X_t(\chi_T/\chi_t)$ is equal to X'_T for some $X' \in \mathcal{X}$. It follows that for any $(X,t) \in \mathcal{X} \times [0,T]$ there exists $X' \in \mathcal{X}$ such that $X_t \leq \kappa X'_T$. Since $\{X_T \,|\, X \in \mathcal{X}\}$ is bounded in $\mathbb{P}$-measure and $\kappa \in \mathbb{L}^0_+$, it follows that $\{X_t \,|\, X \in \mathcal{X}, \ t \in [0,T]\}$ is bounded in $\mathbb{P}$-measure as well. □

For a wealth-process set $\mathcal{X}$, let $\overline{\mathcal{X}}^{\mathsf{F}}$ denote the set of all possible limits of Fatou-convergent sequences of $\mathcal{X}$. We state and prove a result that will help establish both Theorem 1.3 and Theorem 1.7. (Note the similarity between the statements of Lemma 2.3 and Theorem 1.7.)

Lemma 2.3. *Suppose that $\mathcal{X}$ is a wealth-process set and that condition* NA_1 *is in force. Then, for all $\mathbb{Q} \sim \mathbb{P}$ there exists a strictly positive $\widehat{X}^{\mathbb{Q}} \in \overline{\mathcal{X}}^{\mathsf{F}}$ with $\widehat{X}^{\mathbb{Q}}_0 \geq 1$, such that $X/\widehat{X}^{\mathbb{Q}}$ is a $\mathbb{Q}$-supermartingale for all $X \in \overline{\mathcal{X}}^{\mathsf{F}}$.*

Proof. We shall give the proof for the case $\mathbb{Q} = \mathbb{P}$ and suppress the superscript "$\mathbb{P}$" from notation; the proof for the general case follows in exactly the same way.

Let $\mathbb{T}$ be a countable dense subset of $[0, T]$ with $\{0, T\} \subseteq \mathbb{T}$. Recalling Lemma 2.2, it follows exactly as in [15, proof of Theorem 2.3] that there exists an $\mathcal{X}$-valued sequence $(X^n)_{n \in \mathbb{N}}$ such that:

(a) $\widetilde{X}_s := \mathbb{P}\text{-}\lim_{n\to\infty} X^n_s$ exists and satisfies $\mathbb{P}\big[\widetilde{X}_s > 0\big] = 1$ for all $s \in \mathbb{T}$; and

(b) for all $X \in \mathcal{X}$, $\big(X_s/\widetilde{X}_s\big)_{s\in\mathbb{T}}$ is a $\mathbb{P}$-supermartingale with respect to the filtration $(\mathcal{F}_s)_{s\in\mathbb{T}}$.

Using a diagonalization argument and passing to a subsequence if necessary, we may strengthen $\widetilde{X}_s = \mathbb{P}\text{-}\lim_{n\to\infty} X^n_s$ for all $s \in \mathbb{T}$ into that $\mathbb{P}\big[\lim_{n\to\infty} X^n_s = \widetilde{X}_s, \text{ for all } s \in \mathbb{T}\big] = 1$. Furthermore, the fact that $X_0 = 1$ for all $X \in \mathcal{X}$ coupled with property (b) above gives that $\mathbb{E}\big[X_s/\widetilde{X}_s\big] \leq 1$ holds for all $X \in \mathcal{X}$ and $s \in \mathbb{T}$.

Fix a strictly positive semimartingale $X \in \mathcal{X}$. Since the process $\big(X_s/\widetilde{X}_s\big)_{s\in\mathbb{T}}$ is a nonnegative $\mathbb{P}$-supermartingale with respect to the filtration $(\mathcal{F}_s)_{s\in\mathbb{T}}$, it follows that $\mathbb{P}\big[\inf_{s\in\mathbb{T}} \widetilde{X}_s > 0\big] = 1$. For each $t \in [0, T]$, define $\widehat{X}_t := \lim_{\mathbb{T}\ni s\downarrow t} \widetilde{X}_s$; the $\mathbb{P}$-a.s. existence of this limit is ensured by the nonnegative supermartingale convergence theorem. (Note that $\mathbb{P}[\widehat{X}_t < \infty] = 1$ holds since Lemma 2.2 implies that the closure in $\mathbb{P}$-measure of $\{X_s \,|\, X \in \mathcal{X}, \ s \in [0, T]\}$, where $\widehat{X}_t$ belongs to, is bounded in $\mathbb{P}$-measure.) Since the filtration $\mathbf{F}$ satisfies the usual hypotheses, it follows that $\widehat{X}$ (viewed as a process) has an adapted càdlàg version, which we shall be using from now on; then, $\mathsf{F}\text{-}\lim_{n\to\infty} X^n = \widehat{X}$. Furthermore, $\mathbb{P}\big[\inf_{s\in\mathbb{T}} \widetilde{X}_s > 0\big] = 1$ implies that $\mathbb{P}\big[\inf_{t\in[0,T]} \widehat{X}_t > 0\big] = 1$, i.e., that $\widehat{X}$ is strictly positive. The fact that $\mathbb{E}\big[X_s/\widetilde{X}_s\big] \leq 1$, for all $s \in \mathbb{T}$ and Fatou's lemma give $\mathbb{E}\big[X_t/\widehat{X}_t\big] \leq 1$ for all $t \in [0, T]$. In particular, $1/\widehat{X}_0 = \mathbb{E}\big[X_0/\widehat{X}_0\big] \leq 1$, i.e., $\widehat{X}_0 \geq 1$.

It only remains to show that $X/\widehat{X}$ is a $\mathbb{P}$-supermartingale for all $X \in \overline{\mathcal{X}}^{\mathsf{F}}$. In view of the conditional version of Fatou's lemma, it suffices to show that $X/\widehat{X}$ is a $\mathbb{P}$-supermartingale for all $X \in \mathcal{X}$. Initially fix X being strictly positive. Let $t \in [0, T]$, $s \in [0, t]$ and $A \in \mathcal{F}_s$. Consider two $\mathbb{T}$-valued sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ such that $\downarrow \lim_{n\to\infty} s_n = s$, $\downarrow \lim_{n\to\infty} t_n = t$, and $s_n \leq t_n$ for all $n \in \mathbb{N}$. Since $A \in \mathcal{F}_{s_n}$ for all $n \in \mathbb{N}$, property (b) above gives

$$\mathbb{E}\left[\frac{\widetilde{X}_{s_n} X_{t_n}}{X_{s_n} \widetilde{X}_{t_n}} \mathbb{I}_A\right] \leq \mathbb{P}[A]$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$ and using Fatou's lemma, we obtain

$$\mathbb{E}\left[\frac{\widehat{X}_s X_t}{X_s \widehat{X}_t} \mathbb{I}_A\right] \leq \mathbb{P}[A].$$

As $t \in [0, T]$, $s \in [0, t]$ and $A \in \mathcal{F}_s$ are arbitrary, the last inequality shows that $X/\widehat{X}$ is a $\mathbb{P}$-supermartingale. The final step is to remove the assumption that X is strictly positive. Pick any $X \in \mathcal{X}$ and a strictly positive $X' \in \mathcal{X}$. For all $n \in \mathbb{N}$, define the strictly positive process $X^n := (1 - n^{-1})X + n^{-1}X'$, which is a wealth process in $\mathcal{X}$. It follows that $X^n / \widehat{X}$ is a nonnegative $\mathbb{P}$-supermartingale for all $n \in \mathbb{N}$. Using the conditional version of Fatou's lemma, it follows that $X/\widehat{X}$ is a nonnegative $\mathbb{P}$-supermartingale, which concludes the argument. □

2.3. **Proof of Theorem 1.3.** Fix a strictly positive semimartingale $X' \in \mathcal{X}$ and (in view of Lemma 2.3) a strictly positive $\widehat{X} \in \overline{\mathcal{X}}^{\mathsf{F}}$ such that $X/\widehat{X}$ is a $\mathbb{P}$-supermartingale for all $X \in \mathcal{X}$. Pick any $X \in \mathcal{X}$ and write $X = (X/\widehat{X})(\widehat{X}/X')X'$. The process $X/\widehat{X}$ is a càdlàg supermartingale, therefore a semimartingale. As $X' \in \mathcal{S}$, $X \in \mathcal{S}$ will follow as soon as $(\widehat{X}/X') \in \mathcal{S}$ is established. The last follows upon noticing that $\widehat{X}/X' = 1/(X'/\widehat{X})$ and using Itô's formula with the function $(0, \infty) \ni x \mapsto 1/x \in (0, \infty)$ on the strictly positive semimartingale $X'/\widehat{X}$.

2.4. **Convergence in the Emery topology.** Below, we collect the essential results regarding convergence in the Emery topology that shall be needed for the proof of Theorem 1.7. We provide full details for the convenience of the reader; however, versions of some of them have appeared previously—for example, see the original paper [11].

Convention 2.4. In several occasions until the end of Subsection 2.5, we define stopping times as first passage times of processes in certain sets. On the event that the process never enters the specific set up to time T, the stopping time is defined by convention equal to ∞.

The first result contains a convenient necessary and sufficient condition for $\mathcal{S}$-convergence.

Lemma 2.5. *Let $(X^n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{S}$. Then, $\mathcal{S}\text{-}\lim_{n \to \infty} X^n = 0$ holds if and only if for all $\mathcal{P}_1$-valued sequences $(\eta^n)_{n \in \mathbb{N}}$, $\mathbb{P}\text{-}\lim_{n \to \infty} (\eta^n \cdot X^n)_T = 0$ holds.*

Proof. By definition, $\mathcal{S}\text{-}\lim_{n \to \infty} X^n = 0$ implies $\mathbb{P}\text{-}\lim_{n \to \infty} (\eta^n \cdot X^n)_T = 0$ whenever $(\eta^n)_{n \in \mathbb{N}}$ is a $\mathcal{P}_1$-valued sequence. Now, assume the latter condition and, by way of contradiction, that $\mathcal{S}\text{-}\lim_{n \to \infty} X^n = 0$ fails. Passing to a subsequence if necessary, one can find $\epsilon > 0$ and a $\mathcal{P}_1$-valued sequence $(\theta^n)_{n \in \mathbb{N}}$ such that $\mathbb{P}\left[(\theta^n \cdot X^n)^*_T > \epsilon\right] > \epsilon$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define the stopping time $\tau^n := \inf\{t \in [0, T] \,|\, |\theta^n \cdot X^n|_t > \epsilon\}$. With $\eta^n := \theta^n \mathbb{I}_{[\![0, \tau^n \wedge T]\!]}$, $(\eta^n)_{n \in \mathbb{N}}$ is $\mathcal{P}_1$-valued sequence, and $\mathbb{P}\text{-}\lim_{n \to \infty} (\eta^n \cdot X^n)_T = 0$ fails. We reached a contradiction, which means that $\mathcal{S}\text{-}\lim_{n \to \infty} X^n = 0$ holds. □

We introduce some notation that will be used in all that follows. For $X \in \mathcal{S}$, X_- denotes its left-continuous version, with the understanding that $X_{0-} = 0$. We define $\Delta X := X - X_-$. The quadratic covariation process between $X \in \mathcal{S}$ and $Y \in \mathcal{S}$ is $[X, Y] := XY - X_- \cdot Y - Y_- \cdot X$. (Note that $[X, Y]_0 = X_0 Y_0$.) Furthermore, $\mathsf{Var}(X)$ denotes the first-variation process of $X \in \mathcal{S}$.

Remark 2.6. During the remainder of Subsection 2.4, some proofs make use of the following *double subsequence* trick. Suppose that *any* subsequence of a given a sequence of random variables has a further subsequence that converges in $\mathbb{P}$-measure to zero. As convergence in $\mathbb{P}$-measure comes from a metric topology, it follows that the whole sequence has to converge to zero in $\mathbb{P}$-measure.

The next result discusses sufficient conditions for $\mathcal{S}$-convergence that will be used in the main text.

Proposition 2.7. *If $(X^n)_{n\in\mathbb{N}}$ is a sequence of semimartingales, $\mathcal{S}\text{-}\lim_{n\to\infty} X^n = 0$ holds in all of the following three cases:*

- $\lim_{n\to\infty} \mathbb{P}[(X^n)^*_T > 0] = 0$.
- *Each X^n is a process of finite variation, and $\mathbb{P}\text{-}\lim_{n\to\infty} \mathsf{Var}(X^n)_T = 0$.*
- *Each X^n is a local martingale with $|\Delta X^n| \leq C$, where $C \in \mathbb{R}_+$ does not depend on $n \in \mathbb{N}$, and $\mathbb{P}\text{-}\lim_{n\to\infty}[X^n, X^n]_T = 0$.*

Proof. We treat each case separately below.

First, assume that $\lim_{n\to\infty} \mathbb{P}[(X^n)^*_T > 0] = 0$. On the event $\{(X^n)^*_T = 0\}$ we have $\eta^n \cdot X^n = 0$ for all $\eta^n \in \mathcal{P}_1$ in view of [25, Chapter IV, Theorem 26]. Then, the result follows from Lemma 2.5.

Now, assume that each X^n is a process of finite variation, and $\mathbb{P}\text{-}\lim_{n\to\infty} \mathsf{Var}(X^n)_T = 0$ holds. For $\eta^n \in \mathcal{P}_1$ we have $|(\eta^n \cdot X^n)_T| \leq \mathsf{Var}(X^n)_T$—then, Lemma 2.5 allows to conclude.

Finally, assume that each X^n is a local martingale with $|\Delta X^n| \leq C$ for $C \in \mathbb{R}_+$, and that $\mathbb{P}\text{-}\lim_{n\to\infty}[X^n, X^n]_T = 0$. Let $(\eta^n)_{n\in\mathbb{N}}$ be a $\mathcal{P}_1$-valued sequence and set $M^n = \eta^n \cdot X^n$ for $n \in \mathbb{N}$. We need to show that $\mathbb{P}\text{-}\lim_{n\to\infty} M^n_T = 0$. Note that $|\Delta M^n| = |\eta^n \Delta X^n| \leq C$ and $[M^n, M^n] = |\eta^n|^2 \cdot [X^n, X^n] \leq [X^n, X^n]$ so that $\mathbb{P}\text{-}\lim_{n\to\infty}[M^n, M^n]_T = 0$. Let $(M^{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(M^n)_{n\in\mathbb{N}}$ such that $\mathbb{P}\left[[M^{n_k}, M^{n_k}]_T > 1/2^k\right] \leq 1/2^k$ holds for all $k \in \mathbb{N}$; then, by the first Borel-Cantelli lemma it follows that $A := \sum_{k\in\mathbb{N}}[M^{n_k}, M^{n_k}]$ is a finite nondecreasing adapted process. For $m \in \mathbb{N}$, define $\tau_m := \inf\{t \in [0,T] \,|\, A_t \geq m\}$. Then, $[M^{n_k}, M^{n_k}]_{\tau_m} \leq A_{\tau_m -} + (\Delta M^{n_k})^2_{\tau_m} \leq m + C^2$ holds for all $k \in \mathbb{N}$ and $m \in \mathbb{N}$. Therefore, using the well-known $\mathbb{L}^2$-isometry for square-integrable martingales and the dominated convergence theorem, we obtain

$$\lim_{k\to\infty} \mathbb{E}\left[|M^{n_k}_{\tau_m\wedge T}|^2\right] = \lim_{k\to\infty} \mathbb{E}\left[[M^{n_k}, M^{n_k}]_{\tau_m\wedge T}\right] = 0.$$

This implies that $\mathbb{P}\text{-}\lim_{k\to\infty} M^{n_k}_{\tau_m\wedge T} = 0$ and, in turn, that $\mathbb{P}\text{-}\lim_{k\to\infty}\left(M^{n_k}_T \mathbb{I}_{\{\tau_m=\infty\}}\right) = 0$. The fact that $\mathbb{P}\left[\bigcup_{m\in\mathbb{N}}\{\tau_m = \infty\}\right] = 1$ implies that $\mathbb{P}\text{-}\lim_{k\to\infty} M^{n_k}_T = 0$. Up to now we have shown that there exists a subsequence of $(M^n_T)_{n\in\mathbb{N}}$ that converges in $\mathbb{P}$-measure to zero. The same argument shows that *any* subsequence of $(M^n_T)_{n\in\mathbb{N}}$ has a further subsequence that converges in $\mathbb{P}$-measure to zero. By the double subsequence trick mentioned in Remark 2.6, it follows that $\mathbb{P}\text{-}\lim_{n\to\infty} M^n_T = 0$, which concludes the argument. □

Remark 2.8. Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of local martingales such that $\mathbb{P}\text{-}\lim_{n\to\infty}[X^n, X^n]_T = 0$ holds. In the case where there does *not* exist any $C \in \mathbb{R}_+$ with $|\Delta X^n| \leq C$ holding for all $n \in \mathbb{N}$, $\mathcal{S}\text{-}\lim_{n\to\infty} X^n = 0$ may fail. For example, consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that affords a collection $\{\tau_n \,|\, n \in \mathbb{N}\}$ of independent (under $\mathbb{P}$) random variables such that $\mathbb{P}[\tau_n > t] = \exp(-t/n)$ for $t \in \mathbb{R}_+$. Define $(\mathcal{F}_t)_{t\in[0,T]}$ as (the restriction on $[0,T]$ of) the usual augmentation of the smallest filtration that makes all random times in the collection $\{\tau_n \,|\, n \in \mathbb{N}\}$ stopping times. Then, for each $n \in \mathbb{N}$, define a martingale X^n via the formula $X^n_t = n\mathbb{I}_{\{\tau^n \leq t\}} - \tau_n \wedge t$ for $t \in [0,T]$. (It is straightforward to check that each X^n, $n \in \mathbb{N}$, is a martingale in its own filtration; then, the independence of the random variables in $\{\tau_n \,|\, n \in \mathbb{N}\}$ implies that X^n is also a martingale in the larger filtration $(\mathcal{F}_t)_{t\in[0,T]}$, for all $n \in \mathbb{N}$.) In this case, $[X^n, X^n]_T = n^2\mathbb{I}_{\{\tau^n \leq T\}}$ for all $n \in \mathbb{N}$; as $\lim_{n\to\infty} \mathbb{P}[\tau^n \leq T] = 0$, $\mathbb{P}\text{-}\lim_{n\to\infty}[X^n, X^n]_T = 0$ holds. However, $\mathbb{P}\text{-}\lim_{n\to\infty} X^n_T = -T$, which of course implies that $\mathcal{S}\text{-}\lim_{n\to\infty} X^n = 0$ fails.

The two last results of Subsection 2.4 concern stability of $\mathcal{S}$-convergence.

Lemma 2.9. *Let $\mathcal{S}\text{-}\lim_{n\to\infty} X^n = X$ and (Y^n) be a sequence of adapted càdlàg processes such that $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty} Y^n = Y$. Then, $\mathcal{S}\text{-}\lim_{n\to\infty}(Y^n_- \cdot X^n) = Y_- \cdot X$.*

Proof. Upon writing $Y^n_- \cdot X^n - Y_- \cdot X = Y_- \cdot (X^n - X) + (Y^n - Y)_- \cdot X + (Y^n - Y)_- \cdot (X^n - X)$, it suffices to treat three special cases: (i) when $Y^n = Y$ for all $n \in \mathbb{N}$ and $\mathcal{S}\text{-}\lim_{n\to\infty} X^n = 0$ (ii) when $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty} Y^n = 0$ and $X^n = X$ for all $n \in \mathbb{N}$, and (iii) when $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty} Y^n = 0$ and $\mathcal{S}\text{-}\lim_{n\to\infty} X^n = 0$ both hold.

First, assume case (i): $Y^n = Y$ for all $n \in \mathbb{N}$ and $\mathcal{S}\text{-}\lim_{n\to\infty} X^n = 0$. For $k \in \mathbb{N}$, define $\tau_k := \inf\{t \in [0,T] \,|\, |Y_t| > k\}$. Let $(\eta^n)_{n\in\mathbb{N}}$ be a $\mathcal{P}_1$-valued sequence and set $\theta^{k,n} := \eta^n (Y_-/k)\mathbb{I}_{[\![0,\tau_k\wedge T]\!]}$. Noting that $(\theta^{k,n})_{n\in\mathbb{N}}$ is a $\mathcal{P}_1$-valued sequence, it follows that $\mathbb{P}\text{-}\lim_{n\to\infty} (\eta^n \cdot (Y_- \cdot X^n))_{\tau_k\wedge T} = k\mathbb{P}\text{-}\lim_{n\to\infty} \left(\theta^{k,n} \cdot X^n\right)_T = 0$. Therefore, $\mathbb{P}\text{-}\lim_{n\to\infty} (\eta^n \cdot (Y_- \cdot X^n))_T \,\mathbb{I}_{\{\tau_k=\infty\}} = 0$. Since it holds that $\mathbb{P}\left[\bigcup_{k\in\mathbb{N}} \{\tau_k = \infty\}\right] = 1$, we obtain $\mathbb{P}\text{-}\lim_{n\to\infty} (\eta^n \cdot (Y_- \cdot X^n))_T = 0$. As the $\mathcal{P}_1$-valued sequence $(\eta^n)_{n\in\mathbb{N}}$ was arbitrary, Lemma 2.5 implies that $\mathcal{S}\text{-}\lim_{n\to\infty}(Y_- \cdot X^n) = 0$.

Now, assume case (ii): $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty} Y^n = 0$ and $X^n = X$ for all $n \in \mathbb{N}$. For an arbitrary $\mathcal{P}_1$-valued sequence $(\eta^n)_{n\in\mathbb{N}}$, we shall show that $\mathbb{P}\text{-}\lim_{n\to\infty} \left(\eta^n \cdot (Y^n_- \cdot X)\right)_T = 0$. Pick a subsequence $(Y^{n_k})_{k\in\mathbb{N}}$ such that $\xi := \sum_{k\in\mathbb{N}} |Y^{n_k}|$ is a real-valued càdlàg process. The facts that $\mathbb{P}\text{-}\lim_{k\to\infty}(\eta^{n_k} Y^{n_k}_-)^* = 0$, ξ_- is X-integrable (since ξ_- is locally bounded) and $|\eta^{n_k} Y^{n_k}_-| \leq \xi_-$ for all $k \in \mathbb{N}$, coupled with the dominated convergence theorem for stochastic integrals, imply that $\mathbb{P}\text{-}\lim_{k\to\infty} \left((\eta^{n_k} Y^{n_k}_-) \cdot X\right)_T = 0$, i.e., $\mathbb{P}\text{-}\lim_{k\to\infty} \left(\eta^{n_k} \cdot (Y^{n_k}_- \cdot X)\right)_T = 0$. Up to now we have shown that there exists a subsequence of $\left(\left(\eta^n \cdot (Y^n_- \cdot X)\right)_T\right)_{n\in\mathbb{N}}$ that converges in $\mathbb{P}$-measure to zero. The same argument shows that *any* subsequence of $\left(\left(\eta^n \cdot (Y^n_- \cdot X)\right)_T\right)_{n\in\mathbb{N}}$ has a further subsequence that converges in $\mathbb{P}$-measure to zero. The double subsequence trick of Remark 2.6 allows to conclude that $\mathbb{P}\text{-}\lim_{n\to\infty} \left(\eta^n \cdot (Y^n_- \cdot X)\right)_T = 0$. As the sequence $(\eta^n)_{n\in\mathbb{N}}$ was arbitrary, Lemma 2.5 implies that $\mathcal{S}\text{-}\lim_{n\to\infty} Y^n \cdot X = 0$.

Finally, assume case (iii): $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty} Y^n = 0$ and $\mathcal{S}\text{-}\lim_{n\to\infty} X^n = 0$ for all $n \in \mathbb{N}$. In view of Lemma 2.5, we only need to show that $\mathbb{P}\text{-}\lim_{n\to\infty} \left(\eta^n \cdot (Y^n_- \cdot X^n)\right)_T = 0$ for an arbitrary $\mathcal{P}_1$-valued sequence $(\eta^n)_{n\in\mathbb{N}}$. Similar to case (ii), pick a subsequence $(Y^{n_k})_{k\in\mathbb{N}}$ such that

$\xi := \sum_{k\in\mathbb{N}} |Y^{n_k}|$ is a real-valued càdlàg process. For $m \in \mathbb{N}$, define $\tau_m := \inf\{t \in [0,T] \,|\, |\xi_t| > m\}$. For $m \in \mathbb{N}$ and $k \in \mathbb{N}$, set $\theta^{m,k} := \eta^{n_k}(Y^{n_k}_-/m)\mathbb{I}_{[\![0,\tau_m\wedge T]\!]}$. As $(\theta^{m,k})_{k\in\mathbb{N}}$ is $\mathcal{P}_1$-valued, we have $\mathbb{P}\text{-}\lim_{k\to\infty}\left(\eta^{n_k}\cdot(Y^{n_k}_-\cdot X^{n_k})\right)_{\tau_m\wedge T} = m\mathbb{P}\text{-}\lim_{k\to\infty}\left(\theta^{m,k}\cdot X^{n_k}\right)_T = 0$. Therefore, for all $m \in \mathbb{N}$, $\mathbb{P}\text{-}\lim_{k\to\infty}\left(\eta^{n_k}\cdot(Y^{n_k}_-\cdot X^{n_k})\right)_T \mathbb{I}_{\{\tau_m=\infty\}} = 0$ holds. Since $\mathbb{P}\left[\bigcup_{m\in\mathbb{N}}\{\tau_m=\infty\}\right] = 1$, we obtain that $\mathbb{P}\text{-}\lim_{k\to\infty}\left(\eta^{n_k}\cdot(Y_-\cdot X^{n_k})\right)_T = 0$. We have shown that there exists a subsequence of $\left(\left(\eta^n\cdot(Y^n_-\cdot X^n)\right)_T\right)_{n\in\mathbb{N}}$ that converges in $\mathbb{P}$-measure to zero. The same argument shows that *any* subsequence of $\left(\left(\eta^n\cdot(Y^n_-\cdot X^n)\right)_T\right)_{n\in\mathbb{N}}$ has a further subsequence that converges in $\mathbb{P}$-measure to zero. By the double subsequence trick of Remark 2.6, it follows that $\mathbb{P}\text{-}\lim_{n\to\infty}\left(\eta^n\cdot(Y^n_-\cdot X^n)\right)_T = 0$. Then, another invocation of Lemma 2.5 implies that $\mathcal{S}\text{-}\lim_{n\to\infty}\left(Y^n_-\cdot X^n\right) = 0$. □

Proposition 2.10. *Let $\mathcal{S}\text{-}\lim_{n\to\infty} X^n = X$ and $\mathcal{S}\text{-}\lim_{n\to\infty} Y^n = Y$. Then, it further holds that $\mathcal{S}\text{-}\lim_{n\to\infty}[X^n, Y^n] = [X,Y]$ and $\mathcal{S}\text{-}\lim_{n\to\infty}(X^nY^n) = XY$.*

Proof. We shall establish below that $\mathcal{S}\text{-}\lim_{n\to\infty}[X^n,Y^n] = [X,Y]$; then, $\mathcal{S}\text{-}\lim_{n\to\infty}(X^nY^n) = XY$ follows from Lemma 2.9 and a use of the integration-by-parts formula.

Using the identity $4[X^n,Y^n] = [X^n+Y^n, X^n+Y^n] - [X^n-Y^n, X^n-Y^n]$, it follows that it suffices to show that $\mathcal{S}\text{-}\lim_{n\to\infty} X^n = X$ implies $\mathcal{S}\text{-}\lim_{n\to\infty}[X^n,X^n] = [X,X]$. Furthermore, since quadratic variation processes of semimartingales are of finite variation, the estimate

$$\begin{aligned}\mathsf{Var}\left([X^n,X^n]-[X,X]\right)_T &= \mathsf{Var}\left([X^n-X,X^n-X]+2[X,X^n-X]\right)_T \\ &\leq [X^n-X,X^n-X]_T + 2\sqrt{[X,X]_T}\sqrt{[X^n-X,X^n-X]_T}\end{aligned}$$

implies that we only have to establish that, whenever $\mathcal{S}\text{-}\lim_{n\to\infty} X^n = 0$, $\mathcal{S}\text{-}\lim_{n\to\infty}[X^n,X^n] = 0$ holds. In view of Proposition 2.7, $\mathcal{S}\text{-}\lim_{n\to\infty}[X^n,X^n] = 0$ is equivalent to $\mathbb{P}\text{-}\lim_{n\to\infty}[X^n,X^n]_T = 0$. Using $[X^n,X^n] = |X^n|^2 - 2X^n_-\cdot X^n$ as well as that $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty} X^n = 0$ and $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty}(X^n_-\cdot X^n) = 0$, the latter holding in view of Lemma 2.9, we obtain the result. □

2.5. **Proof of Theorem 1.7.** In the course of the proof of Theorem 1.7, we shall actually assume that $\mathbb{Q} = \mathbb{P}$ and use "$\mathbb{P}$" in what follows for notational simplicity. Of course, this does not entail any loss of generality whatsoever. (Note that the Emery topology depends on the probability measure only through its equivalence class.)

Suppose that $\mathcal{X}$ is a wealth-process set and that condition NA_1 is valid. Keeping the notation of Lemma 2.3, consider the strictly positive $\widehat{X} \equiv \widehat{X}^{\mathbb{P}} \in \overline{\mathcal{X}}^{\mathsf{F}}$ with $\widehat{X}_0 \geq 1$ and such that $X/\widehat{X}$ is a $\mathbb{P}$-supermartingale for all $X \in \overline{\mathcal{X}}^{\mathsf{F}} \supseteq \mathcal{X}$. Pick an $\mathcal{X}$-valued sequence $(X^n)_{n\in\mathbb{N}}$ such that $\mathsf{F}\text{-}\lim_{n\to\infty} X^n = \widehat{X}$; in particular, $\mathbb{P}\text{-}\lim_{n\to\infty} X^n_T = \widehat{X}_T$. Define $Z^n := X^n/\widehat{X}$, which is a nonnegative $\mathbb{P}$-supermartingale with $Z^n_0 \leq 1$ for all $n \in \mathbb{N}$. The convergence $\mathbb{P}\text{-}\lim_{n\to\infty} X^n_T = \widehat{X}_T$ translates to $\mathbb{P}\text{-}\lim_{n\to\infty} Z^n_T = 1$. If one can show that $\mathcal{S}\text{-}\lim_{n\to\infty} Z^n = 1$, an application of Proposition 2.10 shows that $\mathcal{S}\text{-}\lim_{n\to\infty} X^n = \widehat{X}$, which will complete the argument. Therefore, we shall prove below that if a sequence $(Z^n)_{n\in\mathbb{N}}$ of nonnegative $\mathbb{P}$-supermartingales with $Z^n_0 \leq 1$ for all $n \in \mathbb{N}$ satisfies

$\mathbb{P}\text{-}\lim_{n\to\infty} Z^n_T = 1$, then $\mathcal{S}\text{-}\lim_{n\to\infty} Z^n = 1$. We prepare the ground with the following result, which establishes $\mathsf{u}\mathbb{P}$-convergence. In the course of the proofs below, Convention 2.4 will be used.

Lemma 2.11. *Suppose that $(Z^n)_{n\in\mathbb{N}}$ is a sequence of nonnegative $\mathbb{P}$-supermartingales such that $Z^n_0 \leq 1$ for all $n \in \mathbb{N}$, as well as $\mathbb{P}\text{-}\lim_{n\to\infty} Z^n_T = 1$. Then, in fact, $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty} Z^n = 1$.*

Proof. Since $\mathbb{E}[Z^n_T] \leq 1$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} \mathbb{E}[Z^n_T] = 1$ holds by Fatou's lemma. Then, [10, Theorem 5.5.2] implies the uniform integrability of $(Z^n_T)_{n\in\mathbb{N}}$; therefore, $\lim_{n\to\infty} \mathbb{E}\left[|Z^n_T - 1|\right] = 0$.

We shall now show that $\mathbb{P}\text{-}\lim_{n\to\infty} \sup_{t\in[0,T]} Z^n_t = 1$. Fix $\epsilon \in (0,\infty)$ and define the stopping time $\tau^n := \inf\{t \in [0,T] \,|\, Z^n_t > 1+\epsilon\}$ for all $n \in \mathbb{N}$. Showing that $\lim_{n\to\infty} \mathbb{P}[\tau^n = \infty] = 1$ will imply that $\mathbb{P}\text{-}\lim_{n\to\infty} \sup_{t\in[0,T]} Z^n_t = 1$, since $\epsilon \in (0,\infty)$ is arbitrary. Suppose on the contrary (passing to a subsequence if necessary) that $\lim_{n\to\infty} \mathbb{P}[\tau^n = \infty] = 1 - p$, where $p > 0$. Then, since $\left|\mathbb{E}\left[Z^n_T \mathbb{I}_{\{\tau^n = \infty\}}\right] - \mathbb{P}[\tau^n = \infty]\right| = \left|\mathbb{E}\left[(Z^n_T - 1)\mathbb{I}_{\{\tau^n=\infty\}}\right]\right| \leq \mathbb{E}[|Z^n_T - 1|]$, and the last quantity converges to zero as $n \to \infty$, we obtain $\lim_{n\to\infty} \mathbb{E}\left[Z^n_T \mathbb{I}_{\{\tau^n=\infty\}}\right] = 1 - p$. In turn, this implies

$$
\begin{aligned}
1 \;\geq\; \limsup_{n\to\infty} \mathbb{E}[Z^n_0] \;\geq\; \limsup_{n\to\infty} \mathbb{E}[Z^n_{\tau^n\wedge T}] \;&\geq\; \liminf_{n\to\infty} \mathbb{E}\left[Z^n_{\tau^n} \mathbb{I}_{\{\tau^n \leq T\}}\right] + \lim_{n\to\infty} \mathbb{E}\left[Z^n_T \mathbb{I}_{\{\tau^n=\infty\}}\right] \\
&\geq\; (1+\epsilon)p + (1-p) \;=\; 1 + \epsilon p,
\end{aligned}
$$

which contradicts the fact that $p > 0$. Thus, $\mathbb{P}\text{-}\lim_{n\to\infty} \sup_{t\in[0,T]} Z^n_t = 1$ has been shown.

We shall now establish that $\mathbb{P}\text{-}\lim_{n\to\infty} \inf_{t\in[0,T]} Z^n_t = 1$. Fix $\epsilon \in (0,\infty)$, and for each $n \in \mathbb{N}$ redefine $\tau^n := \inf\{t \in [0,T] \,|\, Z^n_t < 1-\epsilon\}$—we only need to show that $\lim_{n\to\infty} \mathbb{P}[\tau^n = \infty] = 1$. The nonnegative supermartingale property of Z^n gives that, on $\{\tau^n \leq T\}$, where in particular $Z_{\tau^n} \leq 1-\epsilon$ holds, we have $\mathbb{P}[Z^n_T > 1-\epsilon^2 \,|\, \mathcal{F}_{\tau^n}] \leq (1-\epsilon)/(1-\epsilon^2) = 1/(1+\epsilon)$ for all $n \in \mathbb{N}$. Then,

$$
\mathbb{P}[Z^n_T > 1-\epsilon^2] = \mathbb{E}\left[\mathbb{P}[Z^n_T > 1-\epsilon^2 \,|\, \mathcal{F}_{\tau^n}]\right] \;\leq\; \mathbb{P}[\tau^n = \infty] + \mathbb{P}[\tau^n \leq T]\frac{1}{1+\epsilon}.
$$

Using $\mathbb{P}[\tau^n = \infty] = 1 - \mathbb{P}[\tau^n \leq T]$, rearranging and taking the inferior limit as $n \to \infty$, we obtain $\liminf_{n\to\infty} \mathbb{P}[\tau^n = \infty] \geq \left(1+\epsilon^{-1}\right) \liminf_{n\to\infty} \mathbb{P}[Z^n_T > 1-\epsilon^2] - \epsilon^{-1} = 1$, which shows that $\mathbb{P}\text{-}\lim_{n\to\infty} \inf_{t\in[0,T]} Z^n_t = 1$. Together with $\mathbb{P}\text{-}\lim_{n\to\infty} \sup_{t\in[0,T]} Z^n_t = 1$ that was proved above, the proof of Lemma 2.11 is complete. □

Theorem 1.7 immediately follows from Proposition 2.7, Proposition 2.10, and the following result.

Lemma 2.12. *Under the assumptions of Lemma 2.11, one can write $Z^n = 1 + A^n - B^n + L^n$ for each $n \in \mathbb{N}$, where:*

- *Each A^n is a semimartingale, and $\lim_{n\to\infty} \mathbb{P}[(A^n)^*_T > 0] = 0$.*
- *Each B^n is a predictable, nonnegative and nondecreasing process, and $\mathbb{P}\text{-}\lim_{n\to\infty} B^n_T = 0$.*
- *Each L^n is a local martingale with $|\Delta L^n| \leq 4$ and $\mathbb{P}\text{-}\lim_{n\to\infty}[L^n, L^n]_T = 0$.*

Proof. For $n \in \mathbb{N}$, define the stopping time $\tau^n := \inf\{t \in [0,T] \,|\, Z^n_t > 2\}$. Furthermore, for $n \in \mathbb{N}$ define processes ζ^n and A^n via $\zeta^n_t = Z^n_{t\wedge\tau^n} - \Delta Z^n_{\tau^n} \mathbb{I}_{\{\tau^n \leq t\}}$ and $A^n_t = (Z^n_t - Z^n_{\tau^n-})\mathbb{I}_{\{\tau^n \leq t\}}$ for

$t \in [0,T]$. In other words, ζ^n is the process Z^n stopped *just before* time τ^n, while A^n is defined so that $Z^n = A^n + \zeta^n$. Since $\Delta Z^n_{\tau^n} \geq 0$, ζ^n is a supermartingale and $0 \leq \zeta^n \leq 2$ holds for all $n \in \mathbb{N}$. Now, $\lim_{n\to\infty} \mathbb{P}[\tau^n = \infty] = 1$ holds in view of Lemma 2.11; therefore, $\lim_{n\to\infty} \mathbb{P}[(A^n)^*_T > 0] = 0$, as required. Since $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty} Z^n = 1$ and $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty} A^n = 0$, we obtain $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty} \zeta^n = 1$. For each $n \in \mathbb{N}$, write $\zeta^n = -B^n + M^n$ for the Doob-Meyer decomposition of ζ^n, where B^n is predictable, nonnegative and nondecreasing process and such that $B^n_0 = 0$, and M^n is a nonnegative local martingale with $M^n_0 = \zeta^n_0 = Z^n_0 \leq 1$. Since $M^n \geq \zeta^n$ and $\mathbb{P}\text{-}\lim_{n\to\infty} \zeta^n_T = 1$, it necessarily holds that $\mathbb{P}\text{-}\lim_{n\to\infty} M^n_T = 1$; otherwise $\limsup_{n\to\infty} \mathbb{E}[M^n_T] > 1$, which is impossible in view of the fact that $M^n_0 \leq 1$ and M^n is a nonnegative local $\mathbb{P}$-martingale for all $n \in \mathbb{N}$. Using $\mathbb{P}\text{-}\lim_{n\to\infty} \zeta^n_T = 1$ and $\mathbb{P}\text{-}\lim_{n\to\infty} M^n_T = 1$, we obtain $\mathbb{P}\text{-}\lim_{n\to\infty} B^n_T = 0$, which completes the requirements for the sequence $(B^n)_{n\in\mathbb{N}}$.

Continuing, a use of Lemma 2.11 with $(M^n)_{n\in\mathbb{N}}$ in place of $(Z^n)_{n\in\mathbb{N}}$, gives $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty} M^n = 1$. We define L^n in the obvious way: $L^n = M^n - 1$; it remains to show that the requirements for the sequence $(L^n)_{n\in\mathbb{N}}$ are fulfilled. Firstly, note that $0 \leq \zeta^n \leq 2$ implies that $|\Delta\zeta^n| \leq 2$; therefore, $0 \leq \Delta B^n \leq 2$, since $\Delta B^n_\tau = -\mathbb{E}[\Delta\zeta^n_\tau \,|\, \mathcal{F}_{\tau-}] + [\Delta M^n_\tau \,|\, \mathcal{F}_{\tau-}] = -\mathbb{E}[\Delta\zeta^n_\tau \,|\, \mathcal{F}_{\tau-}]$ holds for all predictable times τ. This implies that $|\Delta L^n| = |\Delta M^n| \leq |\Delta\zeta^n| + \Delta B^n \leq 4$. It only remains to show that $\mathbb{P}\text{-}\lim_{n\to\infty}[L^n, L^n]_T = 0$. Fix $\epsilon \in (0,\infty)$ and redefine, for each $n \in \mathbb{N}$, the stopping time $\tau^n := \inf\{t \in [0,T] \,|\, M^n_t > 1/\epsilon\}$. Since $M^n_0 \leq 1$ and each M^n is a nonnegative local $\mathbb{P}$-martingale, we obtain that $\mathbb{P}[\tau^n = \infty] \geq 1-\epsilon$. Also, note that $\sup_{t\in[0,T]} |L_{\tau^n\wedge t}| \leq 1 + \sup_{t\in[0,T]} M_{\tau^n\wedge t} \leq 5 + 1/\epsilon$ for all $n \in \mathbb{N}$. Coupled with the fact that $\mathbb{P}\text{-}\lim_{n\to\infty} M^n_{\tau^n\wedge T} = 1$ (recall that $\mathsf{u}\mathbb{P}\text{-}\lim_{n\to\infty} M^n = 1$) and the L^2-isometry for square-integrable martingales, we obtain

$$\lim_{n\to\infty} \mathbb{E}\left[[L^n, L^n]_{\tau^n\wedge T}\right] = \lim_{n\to\infty} \mathbb{E}\left[|L^n_{\tau^n\wedge T}|^2\right] = \lim_{n\to\infty} \mathbb{E}\left[|M^n_{\tau^n\wedge T} - 1|^2\right] = 0.$$

It follows that $\limsup_{n\to\infty} \mathbb{P}\left[[L^n, L^n]_T > \epsilon\right] \leq \epsilon$ holds for all $\epsilon \in (0,\infty)$. Therefore, we obtain that $\mathbb{P}\text{-}\lim_{n\to\infty}[L^n, L^n]_T = 0$, which completes the proof. □

2.6. **Preliminaries towards proving Theorem 1.4.** Consider a wealth-process set $\mathcal{X}$. Define $\mathcal{X}^\circ$, the *process-polar* of $\mathcal{X}$, as the set of all nonnegative càdlàg adapted processes Y such that $Y_0 \leq 1$ and YX is a $\mathbb{P}$-supermartingale for all $X \in \mathcal{X}$. Similarly, define $\mathcal{X}^{\circ\circ}$, the *process-bipolar* of $\mathcal{X}$, as the set of all nonnegative càdlàg adapted processes X such that $X_0 \leq 1$ and YX is a $\mathbb{P}$-supermartingale for all $Y \in \mathcal{X}^\circ$. (The terminology of the process-polar and the process-bipolar was introduced in [27].)

By definition, it is clear that $\mathcal{X} \subseteq \mathcal{X}^{\circ\circ}$ holds for any wealth-process set $\mathcal{X}$—actually, one can provide a very concrete description of the structure of $\mathcal{X}^{\circ\circ}$. Suppose that $\mathcal{X}$ is a wealth-process set and that condition NA_1 holds—in particular, by Theorem 1.3, $\mathcal{X} \subseteq \mathcal{S}$. In [27], and using the terminology of that paper, it is shown that $\mathcal{X}^{\circ\circ}$ is the smallest set of nonnegative càdlàg adapted processes that includes $\mathcal{X}$ and is fork-convex, process-solid and Fatou-closed. The following statement repeats this structural result for the process-bipolar, in a slightly altered way to be useful later in the paper.

Theorem 2.13 (Žitković [27])**.** *Let $\mathcal{X}$ be a wealth-process set such that* NA_1 *holds. Then,* $X \in \mathcal{X}^{\circ\circ}$ *if and only if there exists an $\mathcal{X}$-valued sequence $(X^n)_{n\in\mathbb{N}}$ and a sequence $(A^n)_{n\in\mathbb{N}}$ of nondecreasing adapted càdlàg processes with $0 \leq A^n \leq 1$ for each $n \in \mathbb{N}$ such that $\mathsf{F}\text{-}\lim_{n\to\infty} X^n(1 - A^n) = X$.*

If follows from Theorem 2.13 above that, if condition NA_1 is valid for a wealth-process set $\mathcal{X}$, the set inclusions $\mathcal{X} \subseteq \overline{\mathcal{X}} \subseteq \overline{\mathcal{X}}^{\mathsf{F}} \subseteq \mathcal{X}^{\circ\circ}$ hold.

The following result regarding "forward convex convergence" will be used twice in the sequel.

Lemma 2.14. *Let $\mathcal{X}$ be a wealth-process set such that* NA_1 *holds. Consider any $\mathcal{X}$-valued sequence $(X^n)_{n\in\mathbb{N}}$. Then, there exists an $\mathcal{X}$-valued sequence $(\chi^n)_{n\in\mathbb{N}}$, with each χ^n belonging in the convex hull of $\{X^k \mid k \geq n\}$, as well as some $\chi \in \overline{\mathcal{X}}^{\mathsf{F}} \subseteq \mathcal{X}^{\circ\circ}$ such that $\mathsf{F}\text{-}\lim_{n\to\infty} \chi^n = \chi$.*

Proof. In the notation of Theorem 1.7, consider the strictly positive process $\widehat{X} \equiv \widehat{X}^{\mathbb{P}} \in \overline{\mathcal{X}}$ and define $\widetilde{\mathcal{X}} := \{X/\widehat{X} \mid X \in \mathcal{X}\}$. It is straightforward to check that $\widetilde{\mathcal{X}}$ is also a wealth-process set in the sense of Definition 1.1. All elements of $\widetilde{\mathcal{X}}$ are nonnegative càdlàg $\mathbb{P}$-supermartingales starting from unit value. For the given $\mathcal{X}$-valued sequence $(X^n)_{n\in\mathbb{N}}$, consider the $\widetilde{X}$-valued sequence $(\widetilde{X}^n)_{n\in\mathbb{N}}$ defined via $\widetilde{X}^n := X^n/\widehat{X}$ for all $n \in \mathbb{N}$. Then, [12, Lemma 5.2(1)] implies that there exists an $\mathcal{X}$-valued sequence $(\widetilde{\chi}^n)_{n\in\mathbb{N}}$, with each $\widetilde{\chi}^n$ being in the convex hull of $\{\widetilde{X}^k \mid k \geq n\}$, as well as some nonnegative càdlàg $\mathbb{P}$-supermartingale $\widetilde{\chi}$ such that $\mathsf{F}\text{-}\lim_{n\to\infty} \widetilde{\chi}^n = \widetilde{\chi}$. Defining $\chi^n := \widehat{X}\widetilde{\chi}^n$ for all $n \in \mathbb{N}$ and $\chi := \widehat{X}\widetilde{\chi}$, the statement of Lemma 2.14 immediately follows. □

We pause for an interesting remark that will be soon useful. Assuming condition NA_1 on a wealth-process set $\mathcal{X}$, note that $\widehat{Y} := 1/\widehat{X}^{\mathbb{P}}$ (in the notation of Theorem 1.7) is a strictly positive process in $\mathcal{X}^{\circ}$—in fact, it is easy to show that the converse also holds: existence of a strictly positive process in $\mathcal{X}^{\circ}$ implies condition NA_1.

Proposition 2.15 that follows (a static version of "bipolarity", a topic taken up in a general $\mathbb{L}^0_+$ setting in [4]) is exactly the result that will allow us to use the abstract formulation of results on expected utility maximization from [19] and [20].

Proposition 2.15. *Suppose that $\mathcal{X}$ is a wealth-process set and that condition* NA_1 *is in force. Define $\mathcal{C} := \{X_T \mid X \in \mathcal{X}^{\circ\circ}\}$ and $\mathcal{D} := \{Y_T \mid Y \in \mathcal{X}^{\circ}\}$. Then, we have the following:*

- *for $g \in \mathbb{L}^0_+$, $g \in \mathcal{C}$ holds if and only if $\mathbb{E}[hg] \leq 1$ holds for all $h \in \mathcal{D}$;*
- *for $h \in \mathbb{L}^0_+$, $h \in \mathcal{D}$ holds if and only if $\mathbb{E}[hg] \leq 1$ holds for all $g \in \mathcal{C}$.*

Proof. If $g \in \mathcal{C}$ and $h \in \mathcal{D}$, $\mathbb{E}[hg] \leq 1$ trivially holds.

Let $g \in \mathbb{L}^0_+$ be such that $\sup_{h\in\mathcal{D}} \mathbb{E}[hg] \leq 1$. We shall show that there exists $X \in \mathcal{X}^{\circ\circ}$ such that $X_T = g$. As mentioned before the statement of Proposition 2.15, under condition NA_1

there exists a strictly positive $\widehat{Y} \in \mathcal{X}^{\circ}$; replacing, in obvious notation, $\mathcal{X}$ and $\mathcal{X}^{\circ\circ}$ by $\widehat{Y}\mathcal{X}$ and $\widehat{Y}\mathcal{X}^{\circ\circ}$ and $\mathcal{X}^{\circ}$ by $(1/\widehat{Y})\mathcal{X}^{\circ}$, we may (and shall) assume that $1 \in \mathcal{X}^{\circ}$. Let $\mathcal{X}^{\circ}_{++}$ be the set of all strictly positive processes in $\mathcal{X}^{\circ}$. For all $t \in [0,T]$, define the (*a priori*, possibly infinite-valued) $\mathcal{F}_t$-measurable random variable $X^0_t := \operatorname{ess\,sup}_{Y \in \mathcal{X}^{\circ}_{++}} \mathbb{E}\left[(Y_T/Y_t)g \,|\, \mathcal{F}_t\right]$. As $\mathcal{X}^{\circ}_{++}$ is easily seen to be fork-convex, the class of random variables $\left\{\mathbb{E}\left[(Y_T/Y_t)g \,|\, \mathcal{F}_t\right] \,|\, Y \in \mathcal{X}^{\circ}_{++}\right\}$ is upwards directed. (For the definition of upwards directed collections of random variables and their connection with the notion of essential supremum, see [13, Theorem A.32 in Appendix A.5].) Furthermore, the fork-convexity of $\mathcal{X}^{\circ}_{++}$ combined with the fact that $1 \in \mathcal{X}^{\circ}_{++}$ implies that $(Y_T/Y_t) \in \mathcal{D}$ holds for all $Y \in \mathcal{X}^{\circ}_{++}$ and $t \in [0,T]$; therefore, $\mathbb{E}\left[\mathbb{E}\left[(Y_T/Y_t)g \,|\, \mathcal{F}_t\right]\right] = \mathbb{E}\left[(Y_T/Y_t)g\right] \leq 1$ holds for all $Y \in \mathcal{X}^{\circ}_{++}$. It follows that $\mathbb{E}[X^0_t] \leq 1$ for all $t \in [0,T]$; in particular, $X^0_t \in \mathbb{L}^0_+$ for all $t \in [0,T]$. It is straightforward to check that YX^0 is a nonnegative supermartingale for all $Y \in \mathcal{X}^{\circ}_{++}$. In particular, there exists a càdlàg process X that coincides with the right-continuous version of X^0 (for the terminal value, this means $X_T = X^0_T = g$); then, the conditional version of Fatou's lemma implies again that YX is a a nonnegative supermartingale for all $Y \in \mathcal{X}^{\circ}_{++}$. For any fixed $Y \in \mathcal{X}^{\circ}$, $Y^n := \left(n^{-1} + (1-n^{-1})Y\right) \in \mathcal{X}^{\circ}_{++}$ for all $n \in \mathbb{N}$. Therefore, $Y^n X$ is a supermartingale for all $n \in \mathbb{N}$; sending $n \to \infty$ and using the conditional version of Fatou's lemma, we conclude that YX is a supermartingale for all $Y \in \mathcal{X}^{\circ}$. Also, $X_0 \leq \liminf_{t \downarrow 0} \mathbb{E}\left[X^0_t\right] \leq 1$. By the definition of the process-bipolar, it follows that $X \in \mathcal{X}^{\circ\circ}$; since $X_T = g$, we conclude.

In a completely similar way, it can be shown that if $h \in \mathbb{L}^0_+$ is such that $\sup_{g \in \mathcal{C}} \mathbb{E}[hg] \leq 1$, then there exists $Y \in \mathcal{Y}$ such that $Y_T = h$. One needs to use the fork-convexity of $\mathcal{X}^{\circ\circ}$ as well as the fact that $\mathcal{X}^{\circ}$ is the set of all càdlàg adapted processes Y with $Y_0 \leq 1$ and such that YX is a nonnegative supermartingale for all $X \in \mathcal{X}^{\circ\circ}$. Indeed, this last fact follows from the filtered bipolar theorem and Lemma 1 (with $\mathcal{G} = \mathcal{F}_0$) in [27], since the process-bipolar of $\mathcal{X}^{\circ}$ coincides with $\mathcal{X}^{\circ}$ itself. □

2.7. **Proof of Theorem 1.4.** We retain all notation from Subsection 2.6. In accordance with the definition of u from Subsection 1.3, for $x \in (0,\infty)$ define $\mathcal{X}^{\circ\circ}(x) := \{xX \,|\, X \in \mathcal{X}^{\circ\circ}\}$ and $u^{\circ\circ}(x) = \sup_{X \in \mathcal{X}^{\circ\circ}(x)} \mathbb{E}\big[U(X_T)\big]$. The first thing to settle is that the functions u and $u^{\circ\circ}$ coincide.

Lemma 2.16. *Let $\mathcal{X}$ be a wealth process set with $1 \in \mathcal{X}$, such that* NA_1 *holds. Then, $u = u^{\circ\circ}$.*

Proof. Of course, $u \leq u^{\circ\circ}$ always holds; by way of contradiction, assume that $u(x) < u^{\circ\circ}(x)$ for some $x \in (0,\infty)$. Pick $X \in \mathcal{X}^{\circ\circ}(x)$ such that $\mathbb{E}\big[U(X_T)\big] > u(x)$. Recalling Theorem 2.13, consider an $\mathcal{X}(x)$-valued sequence $(X^n)_{n\in\mathbb{N}}$ and a sequence $(A^n)_{n\in\mathbb{N}}$ of nondecreasing adapted càdlàg processes with $0 \leq A^n \leq 1$ for each $n \in \mathbb{N}$ such that $\mathsf{F}\text{-}\lim_{n\to\infty} X^n(1-A^n) = X$. By Lemma 2.14, there exists an $\mathcal{X}(x)$-valued sequence $(\chi^n)_{n\in\mathbb{N}}$, with each χ^n being in the convex hull of $\left\{X^k \,|\, k \geq n\right\}$, as well as some $\chi \in \mathcal{X}^{\circ\circ}(x)$ such that $\mathsf{F}\text{-}\lim_{n\to\infty} \chi^n = \chi$. It is clear that $X_T \leq \chi_T$ holds—therefore, $\mathbb{E}\big[U(\chi_T)\big] \geq \mathbb{E}\big[U(X_T)\big] > u(x)$. It follows that we may (and shall) assume that there exists $X \in \mathcal{X}^{\circ\circ}(x)$ such that $\mathbb{E}\big[U(X_T)\big] > u(x)$, as well as an $\mathcal{X}(x)$-valued sequence

$(X^n)_{n\in\mathbb{N}}$ such that $\mathsf{F}\text{-}\lim_{n\to\infty} X^n = X$. For $k \in \mathbb{N}$, define the process $\widetilde{X}^k := (1/k)x + (1-1/k)X$; then $\widetilde{X}^k \in \mathcal{X}^{\circ\circ}(x)$ for all $k \in \mathbb{N}$. Since $\mathbb{E}\big[0 \wedge U(X_T)\big] > -\infty$, the monotone convergence theorem implies that there exists $K \in \mathbb{N}$ such that, with $\psi := \widetilde{X}^K$, $\mathbb{E}\big[U(\psi_T)\big] > u(x)$ holds. Now, for all $n \in \mathbb{N}$, define $\psi^n := (1/K)x + (1-1/K)X^n$, so that $\psi^n \in \mathcal{X}(x)$. Note that $\mathsf{F}\text{-}\lim_{n\to\infty} \psi^n = \psi$; in particular, $\mathbb{P}\big[\lim_{n\to\infty} \psi^n_T = \psi_T\big] = 1$. Since $\psi^n_T \geq x/K$ holds for all $n \in \mathbb{N}$, using Fatou's lemma we obtain $\liminf_{n\to\infty} \mathbb{E}\left[U(\psi^n_T)\right] \geq \mathbb{E}\big[U(\psi_T)\big] > u(x)$, which contradicts the definition of u. □

According to Lemma 2.16, (FIN-DUAL) holds with $u^{\circ\circ}$ replacing u there. Fix $x \in (0,\infty)$. In view of Proposition 2.15, under the assumptions of Theorem 1.7 one can use the abstract results of the utility maximization problem in [19] and the results of [20] on the existence of the optimal wealth process, to show the existence of a strictly positive $\widehat{X} \in \mathcal{X}^{\circ\circ}$ with $\widehat{X}_0 = 1$ such that $\mathbb{E}\big[U(x\widehat{X}_T)\big] = u^{\circ\circ}(x) = u(x) < \infty$, as well as the existence of a strictly positive $\widehat{Y} \in \mathcal{X}^{\circ}$ such that $\widehat{Y}_0 = 1$ and $\widehat{Y}\widehat{X}$ is a uniformly integrable martingale under $\mathbb{P}$. Define a probability $\mathbb{Q} \sim \mathbb{P}$ via $\mathrm{d}\mathbb{Q} = (\widehat{Y}_T\widehat{X}_T)\,\mathrm{d}\mathbb{P}$. Pick $X \in \mathcal{X}$, $t \in [0,T]$ and $s \in [0,t]$. With "$\mathbb{E}_{\mathbb{Q}}$" denoting expectation under $\mathbb{Q}$,

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{X_t}{\widehat{X}_t} \;\Big|\; \mathcal{F}_s\right] = \frac{1}{\widehat{Y}_s\widehat{X}_s}\mathbb{E}\left[\widehat{Y}_t\widehat{X}_t\frac{X_t}{\widehat{X}_t} \;\Big|\; \mathcal{F}_s\right] = \frac{1}{\widehat{Y}_s\widehat{X}_s}\mathbb{E}\left[\widehat{Y}_t X_t \,|\, \mathcal{F}_s\right] \leq \frac{1}{\widehat{Y}_s\widehat{X}_s}\widehat{Y}_s X_s = \frac{X_s}{\widehat{X}_s},$$

i.e., $X/\widehat{X}$ is a $\mathbb{Q}$-supermartingale. Using the conditional version of Fatou's lemma, we can further deduce that $X/\widehat{X}$ is a $\mathbb{Q}$-supermartingale for all $X \in \overline{\mathcal{X}}^{\mathsf{F}}$.

We shall show now that $\widehat{X} \in \overline{\mathcal{X}}$, which will establish both statement (1) of Theorem 1.4 and the part of statement (2) of Theorem 1.4 regarding existence of optimal wealth processes. First, we show that $\widehat{X} \in \overline{\mathcal{X}}^{\mathsf{F}}$. Since $\widehat{X} \in \mathcal{X}^{\circ\circ}$, in view of Theorem 2.13 consider an $\mathcal{X}$-valued sequence $(X^n)_{n\in\mathbb{N}}$ and a sequence of nondecreasing adapted càdlàg processes with $0 \leq A^n \leq 1$ for each $n \in \mathbb{N}$ such that $\mathsf{F}\text{-}\lim_{n\to\infty} X^n(1-A^n) = \widehat{X}$. By Lemma 2.14, consider an $\mathcal{X}$-valued sequence $(\chi^n)_{n\in\mathbb{N}}$, with each χ^n being in the convex hull of $\big\{X^k \,|\, k \geq n\big\}$, as well as some $\chi \in \overline{\mathcal{X}}^{\mathsf{F}}$ such that $\mathsf{F}\text{-}\lim_{n\to\infty} \chi^n = \chi$. From the two limiting relationships, one can deduce that $\widehat{X} \leq \chi$. According to the preceding paragraph, $\chi/\widehat{X}$ is a nonnegative $\mathbb{Q}$-supermartingale with $\chi_0/\widehat{X}_0 \leq 1$. This last fact combined with $\chi/\widehat{X} \geq 1$ is easily seen to imply that $\chi = \widehat{X}$—in other words, that $\widehat{X} \in \overline{\mathcal{X}}^{\mathsf{F}}$. In order to actually show that $\widehat{X} \in \overline{\mathcal{X}}$, note that $\big(\chi^n/\widehat{X}\big)_{n\in\mathbb{N}}$ is a sequence of nonnegative $\mathbb{Q}$-supermartingales with $\chi^n_0/\widehat{X}_0 = 1$ and $\mathbb{Q}\big[\lim_{n\to\infty}\big(\chi^n_T/\widehat{X}_T\big) = 1\big] = 1$. Recalling the arguments of Subsection 2.5, we deduce that $\mathcal{S}\text{-}\lim_{n\to\infty} \chi^n = \widehat{X}$, which implies that $\widehat{X} \in \overline{\mathcal{X}}$.

We now move to establish the existence of an approximating sequence as required in statement (2) of Theorem 1.4, which will complete the proof. Fix $x \in (0,\infty)$. Let $\widehat{X} \equiv \widehat{X}(x)$ be the optimizer in $\overline{\mathcal{X}}(x)$ of the utility maximization problem. We know that there exists an $\mathcal{X}(x)$-valued sequence $(\widetilde{X}^k)_{k\in\mathbb{N}}$ such that $\mathcal{S}\text{-}\lim_{k\to\infty} \widetilde{X}^k = \widehat{X}$. However, it might not hold that $\lim_{k\to\infty}\mathbb{E}[U(\widetilde{X}^k_T)] = u(x)$. To circumvent this issue, set $\widehat{X}^n := (1/n)x + (1-1/n)\widehat{X}$ for $n \in \mathbb{N}$. Note that $\mathcal{S}\text{-}\lim_{n\to\infty} \widehat{X}^n = \widehat{X}$ and $\lim_{n\to\infty}\mathbb{E}[U(\widehat{X}^n_T)] = \mathbb{E}[U(\widehat{X}_T)]$ hold. For each $n \in \mathbb{N}$, pick $k_n \in \mathbb{N}$ such that, with $X^n := (1/n)x + (1-1/n)\widetilde{X}^{k_n}$, we have $\left\lceil X^n - \widehat{X}^n \right\rceil_{\mathcal{S}} \leq n^{-1}$ and $\mathbb{E}[U(X^n_T)] \geq \mathbb{E}[U(\widehat{X}^n_T)] - n^{-1}$, the latter being feasible in view of Fatou's lemma. As $\mathcal{S}\text{-}\lim_{n\to\infty} \widehat{X}^n = \widehat{X}$ and $\lim_{n\to\infty} \mathbb{E}[U(\widehat{X}^n_T)] = \mathbb{E}[U(\widehat{X}_T)]$, we conclude that the sequence $(X^n)_{n\in\mathbb{N}}$ satisfies the requirements of statement (2) of Theorem 1.4.

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