Strategic trading in a dynamic noisy market

Dimitri Vayanos

This is an electronic version of an Article published in the Journal of finance 56 (1) pp. 131-171 © 2001 Blackwell Publishing.
http://www.blackwellpublishing.com/journals/JOF

Strategic Trading in a Dynamic Noisy Market

Dimitri Vayanos*

March 10, 2003

ABSTRACT

This paper studies a dynamic model of a financial market with a strategic trader. In each period the strategic trader receives a privately observed endowment in the stock. He trades with competitive market makers to share risk. Noise traders are present in the market. After receiving a stock endowment, the strategic trader is shown to reduce his risk exposure either by selling at a decreasing rate over time, or by selling and then buying back some of the shares sold. When the time between trades is small, the strategic trader reveals the information regarding his endowment very quickly.

*MIT and NBER. I thank Anat Admati, Denis Gromb, Pete Kyle, Andy Lo, Lee-Bath Nelson, Paul Pfleiderer, Jose Scheinkman, Matt Spiegel, René Stulz, Rangarajan Sundaram, Jean Tirole, Jiang Wang, Ingrid Werner, Jeff Zwiebel, seminar participants at Chicago, Harvard, LSE, MIT, UCSD, and participants at the Western Finance Association, NBER market microstructure, and European Econometric Society conferences for very helpful comments. I am especially grateful to Avanidhar Subrahmanyam for many valuable comments and suggestions. I thank Muhamet Yildiz for excellent research assistance.

Large traders, such as dealers, mutual funds, and pension funds, play an important role in financial markets.[1] Many empirical studies show that these agents' trades have a significant price impact.[2] Recent studies also show that large traders execute their trades slowly, over several days, presumably to reduce their price impact.[3] These studies raise a number of theoretical questions. For instance, what dynamic strategies should large traders employ to minimize their price impact? In particular, how quickly should they execute their trades? How quickly does the price adjust to reflect the presence of a large trader? Given the importance of large traders, an analysis of their dynamic strategies is relevant for understanding the daily and weekly behavior of returns, volume, and bid-ask spreads. It is also relevant for comparing trading mechanisms, in terms of liquidity provided to large traders, and information revealed in prices.

An analysis of large traders' dynamic strategies requires an assumption about their motives to trade. Trading motives are generally divided into "informational" motives, arising from private information about asset payoffs, and "allocational" motives, such as risk-sharing, portfolio rebalancing, and liquidity. In a seminal paper, Kyle (1985) studies the dynamic strategy of a large trader with informational motives (an "insider"). The insider is risk-neutral and trades with risk-neutral market makers. Market makers agree to trade because they cannot distinguish the insider from noise traders who are also present in the market. Kyle shows that the insider reveals his information slowly, until the time when it is publicly announced.[4]

If most large trades were motivated by information, large traders would significantly outperform the market. However, many empirical studies show that large traders do not significantly outperform, and may even underperform, the market. Moerover, this performance is in spite of high portfolio turnover.[5] Therefore, allocational motives must be important.

The dynamic strategies of large traders with allocational motives have received comparatively less attention. The problem has some similarities with the case of informational motives. For instance, a large trader who wants to sell in order to hedge a risky position, has private information that he will sell and that the price will fall. This is similar to an insider who has private information that the price will fall because of a negative earnings announcement. The crucial difference, however, is that the time of the earnings announcement is exogenous, while the time at which the large trader sells is endogenous. Therefore, the large trader's speed of trade execution cannot be deduced from the insider's, since the latter depends crucially on the time of the earnings announcement. Notice that in the presence of allocational motives, the speed of trade execution determines two important aspects of the trading process. First, the speed of price adjustment, as in the presence of informational motives, and second, the allocative efficiency. If, for instance, the large trader sells slowly, optimal risk-sharing is slowly achieved, and the trading process is not very efficient.

In this paper we study the dynamic strategy of a large trader who trades to share risk.[6] We consider a discrete-time, infinite-horizon, stationary economy with a consumption good and two investment opportunities. The first is a riskless bond and the second is a risky stock that pays a random dividend in each period. The large trader is risk-averse and trades with competitive market makers. We introduce a risk-sharing motive through a privately observed stock endowment that the large trader receives in each period.[7] For simplicity, we eliminate informational motives by assuming that dividend information is public. To obtain the price impact in the absence of informational motives, we assume that market makers are risk-averse.[8] In addition to the large trader and the market makers, small "noise" traders are present in the market. Since the large trader has a risk-sharing motive, trade can take place even without noise. We introduce noise because it is realistic to assume that the large trader can conceal his trades to some extent.

Our model is similar to Kyle (1985) in that a large trader trades with competitive market makers and noise traders. The main differences between the two models are the following. First, in Kyle's model the large trader trades to exploit private information about asset payoffs. By contrast, in our model, information about asset payoffs is public and the large trader trades to share risk. Second, in Kyle's model the large trader and the market makers are risk-neutral, while in our model they are risk-averse. We assume risk aversion so that there is scope for risk-sharing. Third, Kyle's model is non-stationary, since the large trader receives private information only at the beginning of the trading session. By contrast, our model is stationary, since the large trader receives a stock endowment in each period. We assume stationarity so that the model is tractable even when traders are risk-averse.

Our main results concern the dynamics of stock holdings and prices. We show that after an endowment shock, the large trader's stock holdings converge to a long-run limit, determined by optimal risk-sharing between the large trader and the market makers. Moreover, there are two patterns of convergence to the long-run limit: stock holdings can either decrease over time, or they can decrease and then increase.

The first convergence pattern is very intuitive. Stock holdings decrease over time as the large trader sells to the market makers the fraction of his endowment corresponding to optimal risk-sharing. We show that the first convergence pattern also has the following properties. First, the trading rate, defined as trade size (of the large trader's trades) over the size of subsequent trades, decreases over time. In particular, trade size decreases over time. Second, the price impact, defined as price change over trade size, decreases over time. The trading rate decreases over time as the information regarding the large trader's endowment gets reflected in the price. When the information is not reflected in the price, the large trader has an incentive to sell quickly in order to "frontrun" on this information.

The second convergence pattern is somewhat surprising. The large trader sells to the market makers the fraction of his endowment corresponding to optimal risk-sharing. He then engages in a "round-trip transaction", selling some shares only to buy them back later. This pattern occurs when there is enough noise, and when the large trader is not very risk-averse relative to the market makers. The large trader engages in the round-trip transaction because the market makers misinterpret the sale prior to that transaction, i.e. the sale that led to optimal risk-sharing. Indeed, the market makers attribute the sale to the small traders, who account for most order flow, and expect the large trader to absorb a fraction of the sale. The large trader knows that he initiated the sale and that he will not absorb back a fraction. Therefore, he has private information that the price will fall. He can trade on that information, selling some shares and buying them back when the price falls. The second convergence pattern suggests that large traders' strategies can be complicated and reminiscent of "market manipulation".

The dynamics of stock holdings and prices take a very simple form in the "continuous-time" case, where the time between trades, h, is small. They consist of two phases: a first short phase, whose length goes to 0 when h goes to 0, and a second long phase. In the first phase, the large trader sells a fraction of his endowment (bounded away from 0) and the information regarding the endowment gets fully reflected in the price. In the second phase, optimal risk-sharing is achieved. The trading rate is thus very high in the first phase and lower in the second phase. An important property of the dynamics for small h, is that the large trader reveals his information "quickly", i.e. within a time that goes to 0 when h goes to 0. This is in contrast to the slow revelation of information in Kyle's (1985) model.[9]

The dynamics of stock holdings and prices depend on the noise and on traders' risk aversion. We show that, as the noise increases, the large trader sells faster. However, the market makers' and the large trader's risk aversion have an ambiguous effect on the speed of trade.

Our results have several empirical implications. In Section VI we derive these implications, and relate our results to the empirical literature on large trades. Moreover, in Section VII we calibrate the model and study how quickly the large trader trades for realistic parameter values.

There is a small literature on the dynamic behavior of large traders with allocational motives. Admati and Pfleiderer (1988) and Foster and Vishwanathan (1990) allow large traders to optimize the timing of a single trade. They study whether such timing decisions can explain the daily and weekly behavior of returns, volume, and bid-ask spreads. Seppi (1990) compares the upstairs market, where large traders execute their trades as single blocks, to the downstairs market, where they can trade over time. He derives conditions under which large traders with allocational motives prefer the upstairs market. Almgren and Chriss (1999) and Bertsimas and Lo (1998) study the dynamic strategies of large traders who have a fixed time horizon to complete a trade and face an exogenous price reaction function.

Vayanos (1999) studies the dynamic strategies of large traders who trade to share risk. The main difference with this paper is that there is no noise. The no noise assumption is somewhat unrealistic, since it implies that large traders cannot conceal their trades. In this paper we make the more realistic assumption that there is noise and show that in its presence, large traders' strategies are very different. Indeed, in Vayanos (1999) large traders' stock holdings decrease after an endowment shock and the trading rate is constant. By contrast, in this paper stock holdings either decrease, in which case the trading rate also decreases, or they decrease and then increase. Cao and Lyons (1999) study the dynamic strategies of large dealers who share risk in an inter-dealer market. The main differences with this paper is that there is no noise and that dealers can trade only for two periods.

The remainder of this paper is organized as follows. In Section I we present the model. In Section II we determine a Nash equilibrium of the game between the market makers and the large trader. In Section III we study the dynamics of stock holdings and prices after the large trader receives an endowment shock, and in Section IV we study the dynamics in the continuous-time case. In Section V we examine how the dynamics depend on the noise and on traders' risk aversion. In Section VI we derive the empirical implications of our results, and in Section VII we calibrate the model. Finally, in Section VIII we conclude. All proofs are in the Appendix.

I. The Model

Time is continuous and goes from $-\infty$ to ∞. Trade takes place at times ℓh, where $\ell = .., -1, 0, 1, ..,$ and $h > 0$ is the time between trades. We refer to time ℓh as period ℓ. There is a consumption good and two investment opportunities. The first investment opportunity is a riskless bond with an exogenous, continuously compounded rate of return r. One unit of the consumption good invested in the bond in period $\ell - 1$ returns e^{rh} units in period ℓ. The second investment opportunity is a risky stock that pays a dividend $d_\ell h$ in period ℓ. We set

$$d_\ell = d_{\ell-1} + \delta_\ell, \tag{1}$$

and assume that the dividend shocks δ_ℓ are independent of each other and are normal with mean 0 and variance $\sigma^2 h$. All traders learn δ_ℓ in period ℓ, i.e. dividend information is public.

There are three types of traders: a large trader, market makers, and small "noise" traders. The large trader is infinitely-lived and consumes $c_\ell h$ in period ℓ.[10] His utility over consumption is exponential with coefficient of absolute risk aversion (CARA) α and discount rate β, i.e.

$$-h \sum_{\ell=\ell_0}^{\infty} exp(-\alpha c_\ell - \beta(\ell - \ell_0)h). \tag{2}$$

In period ℓ the large trader receives an endowment of ϵ_ℓ shares of the stock and

$$-d_\ell \frac{h}{1 - e^{-rh}} \epsilon_\ell$$

units of the consumption good. The consumption good endowment is the negative of the present value of expected dividends, $d_\ell h/(1 - e^{-rh})$, times the stock endowment, ϵ_ℓ.[11] The stock endowments ϵ_ℓ are independent of each other and of the dividend shocks, and are normal with mean 0 and variance $\sigma_\epsilon^2 h$. They are private information to the large trader.

Market makers are infinitely-lived and form a continuum with measure 1. A market maker consumes $\overline{c}_\ell h$ in period ℓ. His utility over consumption is exponential with CARA $\overline{\alpha}$ and discount rate $\overline{\beta}$, i.e.

$$-h \sum_{\ell=\ell_0}^{\infty} exp(-\overline{\alpha}\overline{c}_\ell - \overline{\beta}(\ell - \ell_0)h). \tag{3}$$

The assumption that market makers form a continuum with measure 1 means the following. First, a market maker maximizes equation (3) taking price as given. Market makers are thus competitive. Second, the market makers' aggregate demand is derived by assigning each market maker an index m in $[0, 1]$ and integrating market makers' demands over m.

Aggregate demand is thus the average of market makers' demands. The same is true for the market makers' aggregate stock holdings, endowment, etc.[12]

Small traders' behavior is not derived from utility maximization. Small traders simply sell u_ℓ shares in period ℓ. (Small traders buy if u_ℓ is negative.) The sell orders u_ℓ are independent of each other, of the dividend shocks, and of the large trader's endowments. They are normal with mean 0 and variance $\sigma_u^2 h$.

The sequence of events in period ℓ is as follows. First, the large trader receives his endowments. Second, the market makers and the large trader learn the dividend shock. Third, trade in the stock takes place. Fourth, the stock pays the dividend and fifth, the market makers and the large trader consume. Trade is organized as follows. The large trader and the small traders submit market orders and, simultaneously, the market makers submit demand functions. The market-clearing price is then determined and all trades take place at this price.

II. Equilibrium Determination

In this section we determine a stationary Nash equilibrium of the game between the market makers and the large trader. (A Nash equilibrium is stationary if traders' equilibrium strategies are independent of time, holding the state variables constant.) In Section II.A we conjecture traders' equilibrium strategies. These strategies form a Nash equilibrium if it is optimal for a trader to follow his strategy when other traders follow theirs. In Sections II.B and II.C we study traders' optimization problems, and show that determining a stationary Nash equilibrium reduces to solving a system of non-linear equations. In Section II.D we solve the system for small order flow. We also derive an equation that we use in Sections III and IV to explain the intuition behind our results.[13]

A. Strategies

We first introduce some notation. We denote by e_ℓ the large trader's stock holdings after trade in period ℓ, and by s_ℓ the market makers' expectation of e_ℓ, conditional on information available after trade in period ℓ. We divide the large trader's stock holdings before trade in period ℓ, $e_{\ell-1} + \epsilon_\ell$, into expected stock holdings, defined as the market makers' expectation, $s_{\ell-1}$, and into unexpected stock holdings. We denote by $\overline{e}_\ell$ the market makers' aggregate stock holdings after trade at period ℓ. When all market makers follow their equilibrium strategy, $\overline{e}_\ell$ are also the stock holdings of each market maker. To study the market makers' optimization problem, we will allow a market maker to deviate from his equilibrium strategy, but assume that the other market makers follow theirs. Since market makers form a continuum, the stock holdings of each non deviating market maker will be $\overline{e}_\ell$. We will denote the stock holdings of the deviating market maker by $\overline{e}_\ell + \Delta\overline{e}_\ell$.

The conjectured period ℓ equilibrium sell order for the large trader is

$$x_\ell = a_e(e_{\ell-1} + \epsilon_\ell - s_{\ell-1}) + a_s s_{\ell-1} - a_{\overline{e}}\overline{e}_{\ell-1}. \tag{4}$$

(If x_ℓ is negative, it is a buy order.) The sell order, x_ℓ, is linear in the large trader's unexpected stock holdings, $e_{\ell-1} + \epsilon_\ell - s_{\ell-1}$, the large trader's expected stock holdings, $s_{\ell-1}$, and the market makers' stock holdings, $\overline{e}_{\ell-1}$, all evaluated before trade in period ℓ.[14] We expect the coefficients a_e, a_s, and $a_{\overline{e}}$ to be positive, i.e. the large trader sells when his stock holdings are high and the market makers' stock holdings are low. We allow a_e and a_s to be different, i.e. the large trader can sell at different rates out of unexpected and expected stock holdings. We will show that the coefficients a_e, a_s, and $a_{\overline{e}}$ are indeed positive and $a_e > a_s$.

The conjectured period ℓ equilibrium demand for a market maker is

$$\overline{x}_\ell(p_\ell) = B\left(\frac{h}{1-e^{-rh}}d_\ell - p_\ell\right) - \overline{A}_{\overline{e}}\overline{e}_{\ell-1} - \overline{A}_s s_{\ell-1}. \tag{5}$$

Demand, $\overline{x}_\ell(p_\ell)$, is linear in the dividend rate, d_ℓ, the price, p_ℓ, the market makers' stock holdings before trade in period ℓ, $\overline{e}_{\ell-1}$, and the market makers' expectation of the large trader's stock holdings before trade in period ℓ, $s_{\ell-1}$. Notice that a unit increase in the dividend rate leaves demand unaffected only if the price increases by $h/(1-e^{-rh})$. This is because $h/(1-e^{-rh})$ is equal to the increase in the present value of expected dividends. Notice also that traders' conjectured strategies are stationary, since the coefficients a_e, a_s, $a_{\overline{e}}$, $\overline{A}_{\overline{e}}$, $\overline{A}_s$, and B, are independent of ℓ.

The price in period ℓ is given by the market-clearing condition

$$\overline{x}_\ell(p_\ell) = x_\ell + u_\ell. \tag{6}$$

Using equation (5), we get

$$p_\ell = \frac{h}{1-e^{-rh}}d_\ell - \frac{\overline{A}_{\overline{e}}}{B}\overline{e}_{\ell-1} - \frac{\overline{A}_s}{B}s_{\ell-1} - \frac{1}{B}(x_\ell + u_\ell). \tag{7}$$

B. The Market Makers' Optimization Problem

A market maker maximizes the expectation of equation (3) assuming that the other market makers and the large trader follow their equilibrium strategies. We formulate the market maker's problem as a dynamic programming problem. The "state" in period ℓ is evaluated after trade takes place and before the stock pays the dividend. There are five state variables: the market maker's consumption good holdings that we denote by $\overline{M}_\ell$, the dividend rate d_ℓ, the market maker's stock holdings $\overline{e}_\ell + \Delta\overline{e}_\ell$, the other market makers' stock holdings $\overline{e}_\ell$, and the market makers' expectation s_ℓ, of the large trader's stock holdings. There are two control variables chosen between the state in period $\ell-1$ and the state in period ℓ: the consumption $\overline{c}_{\ell-1}$, and the demand $\overline{x}_\ell(p_\ell)$.

The dynamics of $\overline{M}_\ell$ are given by the budget constraint

$$\overline{M}_\ell = e^{rh}(\overline{M}_{\ell-1} + d_{\ell-1}(\overline{e}_{\ell-1} + \Delta\overline{e}_{\ell-1})h - \overline{c}_{\ell-1}h) - p_\ell\overline{x}_\ell(p_\ell).$$

Since market makers form a continuum, the price p_ℓ in the budget constraint is given by equation (7), which assumes that all market makers follow their equilibrium strategy. Given that the large trader also follows his equilibrium strategy, equation (7) becomes

$$p_\ell = \frac{h}{1-e^{-rh}}d_\ell - \frac{A_{\overline{e}}}{B}\overline{e}_{\ell-1} - \frac{A_s}{B}s_{\ell-1} - \frac{1}{B}(a_e(e_{\ell-1} + \epsilon_\ell - s_{\ell-1}) + u_\ell), \tag{8}$$

where $A_s = \overline{A}_s + a_s$ and $A_{\overline{e}} = \overline{A}_{\overline{e}} - a_{\overline{e}}$. The dynamics of d_ℓ are given by equation (1). The dynamics of $\overline{e}_\ell + \Delta\overline{e}_\ell$ are given by

$$\overline{e}_\ell + \Delta\overline{e}_\ell = \overline{e}_{\ell-1} + \Delta\overline{e}_{\ell-1} + \overline{x}_\ell(p_\ell). \tag{9}$$

The market maker's stock holdings after trade in period ℓ are equal to his stock holdings after trade in period $\ell-1$, plus the shares that he buys in period ℓ. Similarly, the dynamics of $\overline{e}_\ell$ are given by

$$\overline{e}_\ell = \overline{e}_{\ell-1} + x_\ell + u_\ell. \tag{10}$$

The other market makers' stock holdings after trade in period ℓ are equal to their stock holdings after trade in period $\ell-1$, plus the order flow in period ℓ. Given that the large trader follows his equilibrium strategy, equation (10) becomes

$$\overline{e}_\ell = (1-a_{\overline{e}})\overline{e}_{\ell-1} + a_s s_{\ell-1} + a_e(e_{\ell-1} + \epsilon_\ell - s_{\ell-1}) + u_\ell. \tag{11}$$

We finally determine the dynamics of s_ℓ, the market makers' expectation of the large trader's stock holdings e_ℓ. To determine these dynamics, we first determine the dynamics of e_ℓ and then use recursive filtering. The dynamics of e_ℓ are given by

$$e_\ell = (e_{\ell-1} + \epsilon_\ell) - x_\ell. \tag{12}$$

The large trader's stock holdings after trade in period ℓ are equal to his stock holdings after trade in period $\ell-1$, plus his stock endowment in period ℓ, minus his sell order in period ℓ. Given that the large trader follows his equilibrium strategy, equation (12) becomes

$$e_\ell = (e_{\ell-1} + \epsilon_\ell) - a_e(e_{\ell-1} + \epsilon_\ell - s_{\ell-1}) - a_s s_{\ell-1} + a_{\overline{e}}\overline{e}_{\ell-1}. \tag{13}$$

We next use recursive filtering. We denote the information available to the market makers after trade in period ℓ by $\mathcal{I}_\ell$. The information $\mathcal{I}_\ell$ consists of $\mathcal{I}_{\ell-1}$, the dividend rate d_ℓ (which does not contain information on e_ℓ), and the order flow $x_\ell + u_\ell$. To compute s_ℓ, the mean of e_ℓ conditional on $\mathcal{I}_\ell$, we proceed recursively. We first condition on $\mathcal{I}_{\ell-1}$ and then compute the mean of e_ℓ conditional on the order flow $x_\ell + u_\ell$. Since all variables are normal, we can simply regress e_ℓ on $x_\ell + u_\ell$. (The regression is conditional on $\mathcal{I}_{\ell-1}$.) To do the regression, we use equations (4) and (13). We also denote by Σ_e^2 the variance of $e_{\ell-1}$ conditional on $\mathcal{I}_{\ell-1}$.[15] The regression implies that

$$s_\ell = E_{\mathcal{I}_\ell} e_\ell = (1-a_s)s_{\ell-1} + a_{\overline{e}}\overline{e}_{\ell-1} + g(x_\ell + u_\ell - (a_s s_{\ell-1} - a_{\overline{e}}\overline{e}_{\ell-1})), \tag{15}$$

where

$$g = \frac{a_e(1-a_e)(\Sigma_e^2 + \sigma_e^2 h)}{a_e^2(\Sigma_e^2 + \sigma_e^2 h) + \sigma_u^2 h}. \tag{16}$$

The mean of e_ℓ conditional on $\mathcal{I}_\ell$, is the sum of two terms. The first term, $(1-a_s)s_{\ell-1} + a_{\overline{e}}\overline{e}_{\ell-1}$, is the mean of e_ℓ conditional on $\mathcal{I}_{\ell-1}$. The second term corresponds to the market makers' "surprise" in period ℓ. The surprise is proportional to the difference between the order flow $x_\ell + u_\ell$, and the mean of the order flow conditional on $\mathcal{I}_{\ell-1}$, $a_s s_{\ell-1} - a_{\overline{e}}\overline{e}_{\ell-1}$. The coefficient of proportionality g increases in $(\Sigma_e^2 + \sigma_e^2 h)/\sigma_u^2 h$, which is a measure of the relative order flow coming from the large and the small traders. The dynamics of s_ℓ are given by equation (15). Since the large trader follows his equilibrium strategy, equation (15) becomes

$$s_\ell = (1-a_s)s_{\ell-1} + a_{\overline{e}}\overline{e}_{\ell-1} + g(a_e(e_{\ell-1} + \epsilon_\ell - s_{\ell-1}) + u_\ell). \tag{17}$$

We conjecture that the market maker's value function is

$$\overline{V}(\overline{M}_\ell, d_\ell, \Delta\overline{e}_\ell, \overline{e}_\ell, s_\ell) = -exp(-\alpha(\frac{1-e^{-rh}}{h}\overline{M}_\ell + d_\ell(\overline{e}_\ell + \Delta\overline{e}_\ell) + F(\overline{Q}, \begin{pmatrix} \Delta\overline{e}_\ell \\ \overline{e}_\ell \\ s_\ell \end{pmatrix}) + \overline{q})), \tag{18}$$

where $F(\overline{Q}, v) = (1/2)v^t\overline{Q}v$ for a matrix $\overline{Q}$ and a vector v, v^t is the transpose of v, $\overline{Q}$ is a symmetric 3×3 matrix, and $\overline{q}$ is a constant. In Proposition 1 (in the Appendix) we provide sufficient conditions for the demand in equation (5) to solve the market maker's optimization problem and for the function (18) to be the value function. The conditions are on the coefficients $A_{\overline{e}}$, A_s, B, a_e, a_s, $a_{\overline{e}}$, $\overline{Q}$, and $\overline{q}$. We explore the economic intuition behind these conditions in Section II.D.

C. The Large Trader's Optimization Problem

The large trader maximizes the expectation of equation (2) assuming that the market makers follow their equilibrium strategy. We formulate the large trader's problem as a dynamic programming problem. The state in period ℓ is evaluated after trade takes place and before the stock pays the dividend. There are five state variables: the large trader's consumption good holdings that we denote by M_ℓ, the dividend rate d_ℓ, the large trader's stock holdings e_ℓ, the market makers' expectation s_ℓ, of the large trader's stock holdings, and the market makers' stock holdings $\overline{e}_\ell$. There are two control variables chosen between the state in period $\ell - 1$ and the state in period ℓ: the consumption $c_{\ell-1}$, and the market order x_ℓ. The dynamics of M_ℓ are given by the budget constraint

$$M_\ell = e^{rh}(M_{\ell-1} + d_{\ell-1}e_{\ell-1} - c_{\ell-1}h) - d_\ell \frac{h}{1 - e^{-rh}} \epsilon_\ell + p_\ell x_\ell. \tag{19}$$

The price p_ℓ in the budget constraint is given by equation (7). The dynamics of d_ℓ, e_ℓ, s_ℓ, and $\overline{e}_\ell$ are given by equations (1), (12), (15), and (10), respectively.

We conjecture that the large trader's value function is

$$V(M_\ell, d_\ell, e_\ell, s_\ell, \overline{e}_\ell) = -exp(-\alpha(\frac{1 - e^{-rh}}{h} M_\ell + d_\ell e_\ell + F(Q, \begin{pmatrix} e_\ell - s_\ell \\ s_\ell \\ \overline{e}_\ell \end{pmatrix}) + q)). \tag{20}$$

where Q is a symmetric 3×3 matrix and q a constant. In Proposition 2 (in the Appendix) we provide sufficient conditions for the market order in equation (4) to solve the large trader's optimization problem, and for the function (20) to be the value function. The conditions are on the coefficients $A_{\overline{e}}$, A_s, B, a_e, a_s, $a_{\overline{e}}$, Q, and q. We explore the economic intuition behind these conditions in the next section.

D. Equilibrium for Small Order Flow

The optimization problems of Sections II.B and II.C produce a set of sufficient conditions for traders' strategies to form a stationary Nash equilibrium. The conditions are on $A_{\overline{e}}$, A_s, B, a_e, a_s, and $a_{\overline{e}}$, the coefficients of traders' strategies, and $\overline{Q}$, $\overline{q}$, Q, and q, the coefficients of traders' value functions. In the Appendix we combine these conditions with equations (14) and (16) of the market makers' recursive filtering problem, and obtain a system, (S), of 20 non-linear equations. The 20 unknowns are the six coefficients $A_{\overline{e}}$, A_s, B, a_e, a_s, and $a_{\overline{e}}$, the 12 elements of the symmetric 3×3 matrices $\overline{Q}$ and Q, and the two coefficients g and Σ_e^2.

In general, the system (S) is complicated and can only be solved numerically. This is because traders face both "fundamental" risk (dividend risk) and "price" risk (risk coming from the effect of the other traders on price). The large trader, for instance, faces the risk of a negative dividend shock and the risk of selling at the same time as the small traders. The cost of bearing dividend risk is exogenous and easy to compute. However, the cost of bearing price risk depends on traders' strategies and is a non-linear function of $A_{\overline{e}}$, A_s, B, a_e, a_s, $a_{\overline{e}}$, $\overline{Q}$, and Q.

Since dealing with dividend risk is easy and with price risk difficult, it is natural to study the case where price risk is small. The large trader's problem does not change qualitatively for small price risk. Indeed, the determinants of the large trader's strategy are price impact and the risk of holding a large position. Neither goes to 0 as price risk gets small, since there is dividend risk.

Price risk is small when order flow is small, i.e. when the variances σ_e^2 and σ_u^2 of the endowment and the noise shocks are small. To study the system (S) for small order flow, we set $\sigma_u^2 = \overline{\sigma}_u^2 \sigma_e^2$, and assume that σ_e^2 goes to 0 while $\overline{\sigma}_u^2$ stays constant. We also set $\Sigma_e^2 = \overline{\Sigma}_e^2 \sigma_e^2$. For $\sigma_e^2 = 0$ the system (S) simplifies dramatically. In Theorem 1 we show that (S) collapses to a system of four non-linear equations in a_e, a_s, g, and $\overline{\Sigma}_e^2$. This new system, (s), has a solution (obtained in closed-form for small h in Section IV) and thus (S) has a solution. The solution of (S) can be extended for small σ_e^2, by the implicit function theorem.

Theorem 1 *The system (S) has a solution for small σ_e^2. For $\sigma_e^2 = 0$, the coefficients a_e, a_s, g, and $\overline{\Sigma}_e^2$ solve the system (s) of*

$$a_e(1-(1-a_e(1+g))e^{-rh})(1-a_s(1+g))\overline{\alpha}-(1-a_e(1+g))(1-a_s-a_{\overline{e}})(1-e^{-rh})\alpha = 0, \quad (21)$$

$$-a_s \frac{1-(1-a_s(1+g)(2-a_s-a_{\overline{e}}))e^{-rh}}{1-(1-a_s(1+g))(1-a_s-a_{\overline{e}})e^{-rh}}\overline{\alpha} + (\alpha(1-a_s) - \overline{\alpha} a_s)(1-e^{-rh}) = 0, \quad (22)$$

$$g = \frac{a_e(1-a_e)(\overline{\Sigma}_e^2 + h)}{a_e^2(\overline{\Sigma}_e^2 + h) + \overline{\sigma}_u^2 h}, \quad (23)$$

and

$$\overline{\Sigma}_e^2 = \frac{(1-a_e)^2(\overline{\Sigma}_e^2 + h)\overline{\sigma}_u^2 h}{a_e^2(\overline{\Sigma}_e^2 + h) + \overline{\sigma}_u^2 h}. \tag{24}$$

The coefficient $a_{\overline{e}}$ *is given by*

$$a_{\overline{e}}\alpha - a_s\overline{\alpha} = 0. \tag{25}$$

The coefficients $A_{\overline{e}}$, A_s, B, $\overline{Q}$, *and* Q, *are determined (in the Appendix) by solving systems of linear equations.*

When the variance σ_e^2 of the endowment shocks is zero, we can simplify the system (S) and explore the economic intuition behind its equations. We do not consider all the equations of (S), but only two equations that we can derive by combining the equations of (S). (We derive the two equations in the Appendix.) The first, "market maker" equation follows by combining the equations coming out of the market makers' optimization problem. The second, "large trader" equation similarly follows from the large trader's problem. Combining the market maker and large trader equations, we can obtain an equation that we use in Sections III and IV to explain the intuition behind our results.

The market maker equation is

$$-p_\ell + d_\ell h - \frac{\overline{\alpha}\sigma^2 h^2 e^{-rh}}{1-e^{-rh}}\overline{e}_\ell + \overline{E}_\ell p_{\ell+1} e^{-rh} = 0. \tag{26}$$

Equation (26) states that a market maker's marginal benefit of buying Δx shares in period ℓ and selling them in period $\ell+1$, is 0. The marginal benefit is the sum of four terms. The first term represents the price p_ℓ that the market maker pays at period ℓ. The second term represents the dividend $d_\ell h$ that the market maker receives in period ℓ. The third term represents an "inventory" cost that the market maker bears by holding a riskier position between periods ℓ and $\ell+1$. This inventory cost is proportional to the dividend risk σ^2, the market maker's CARA $\overline{\alpha}$, and the market maker's stock holdings $\overline{e}_\ell$. Finally, the fourth term represents the price that the market maker expects to receive in period $\ell+1$, discounted in period ℓ. (We denote by $\overline{E}_\ell$ the expectation w.r.t. the market makers' information, to distinguish it from the expectation w.r.t. the large trader's information.)

The large trader equation is

$$-p_\ell + d_\ell h - \frac{\alpha\sigma^2 h^2 e^{-rh}}{1-e^{-rh}} e_\ell + E_\ell p_{\ell+1} e^{-rh}$$

$$+\frac{1}{B}x_\ell - (g a_s - a_{\overline{e}})(1+g)\frac{A_s - a_s}{B}\sum_{\ell' \geq \ell+1}(1 - a_s(1+g))^{\ell'-\ell-2} E_\ell x_{\ell'} e^{-(\ell'-\ell)rh} = 0. \tag{27}$$

Equation (27) states that the large trader's marginal benefit of buying Δx shares in period ℓ and selling them in period $\ell+1$, is 0. It is the counterpart of equation (26) which sets the marginal benefit of a market maker to 0. The large trader's marginal benefit is the sum of six terms. The first four terms are as for the market maker. The fifth and sixth terms represent the impact of buying and selling shares on prices. The fifth term represents the impact on the period ℓ price. It is the product of x_ℓ, the number of shares that the large trader sells in equilibrium, times $1/B$, the "marginal" price increase that corresponds to the purchase of Δx shares. (Equation (7) implies that a purchase of Δx shares increases the price by $(1/B)\Delta x$.) The sixth term represents the impact on the price in periods $\ell' \geq \ell+1$. For each ℓ', we multiply $E_\ell x_{\ell'}$, the expected number of shares that the large trader sells in equilibrium, by

$$-(ga_s - a_{\overline{e}})(1+g)\frac{A_s - a_s}{B}(1 - a_s(1+g))^{\ell'-\ell-2}, \tag{28}$$

the marginal price increase that corresponds to the purchase of Δx shares in period ℓ and the sale of Δx shares in period $\ell+1$. We then discount in period ℓ, and sum over ℓ'.

Equation (28) implies that the purchase and subsequent sale of Δx shares decreases the price in periods $\ell' \geq \ell+1$ if $ga_s - a_{\overline{e}} > 0$ and increases the price otherwise. We first explain why this zero cumulative trade affects prices, and then why the effect depends on the sign of $ga_s - a_{\overline{e}}$. The purchase and subsequent sale of Δx shares affects prices because it affects the market makers' expectation of the large trader's stock holdings. The reason why it affects the market makers' expectation, is the following. The market makers revise their expectation by comparing order flow to expected order flow. In period ℓ, they compare the purchase of Δx shares to 0. However, in period $\ell+1$, they compare the sale of Δx shares not to 0, but to the expectation that the purchase in period ℓ has created. Suppose, for instance, that after a purchase, the market makers expect a purchase. Then, the sale of Δx shares will send a stronger signal than the purchase, and the market makers will increase their expectation of the large trader's stock holdings. To determine whether the market makers expect a purchase after a purchase, we use equations (15) and (10). These equations imply that a purchase of Δx shares in period ℓ reduces s_ℓ by $g\Delta x$, $\overline{e}_\ell$ by Δx, and expected (sell) order flow in period $\ell+1$, $a_s s_\ell - a_{\overline{e}}\overline{e}_\ell$, by

$$-(a_s \Delta s_\ell - a_{\overline{e}}\Delta \overline{e}_\ell) = (ga_s - a_{\overline{e}})\Delta x.$$

Therefore, the market makers expect a purchase after a purchase if $ga_s - a_{\overline{e}} > 0$. Notice that when g is small, i.e. most order flow comes from the small traders, the market makers expect a sale after a purchase. This is because they attribute the purchase to the small traders, and expect the large trader to satisfy part of the demand, i.e. to sell some shares.

Combining the market maker equation (26) and the large trader equation (27), we can obtain an equation that we use to explain the intuition behind our subsequent results. Substituting the price p_ℓ from equation (26) into equation (27), we get

$$\frac{\sigma^2 h^2 e^{-rh}}{1-e^{-rh}}\left(\overline{\alpha}\overline{e}_\ell - \alpha e_\ell\right) + \left(E_\ell p_{\ell+1} - \overline{E}_\ell p_{\ell+1}\right) e^{-rh}$$

$$+\frac{1}{B}x_\ell - (ga_s - a_{\overline{e}})(1+g)\frac{A_s - a_s}{B}\sum_{\ell' \geq \ell+1}(1-a_s(1+g))^{\ell'-\ell-2}E_\ell x_{\ell'} e^{-(\ell'-\ell)rh} = 0. \quad (29)$$

The large trader's marginal benefit of buying shares in period ℓ and selling them in period $\ell+1$, can be reduced to a sum of four terms. The first term is the "risk-sharing" term. Buying shares benefits the large trader when his stock holdings e_ℓ, adjusted by his CARA α, are smaller than the market makers' stock holdings. The second term is the "frontrunning" term. Suppose that in equilibrium the large trader sells in period $\ell+1$, and that the sale is not expected by the market makers. In period ℓ, the large trader has private information that the price will fall, i.e. $E_\ell p_{\ell+1} - \overline{E}_\ell p_{\ell+1} < 0$. By buying shares, he foregoes "frontrunning" on this information. The third term is the "price impact" term and represents the impact of buying shares on the period ℓ price. The fourth term represents the impact of buying and selling shares on the price in periods $\ell' \geq \ell+1$. Since the impact is through the market makers' expectation, we refer to the fourth term as the "belief manipulation" term.

III. Dynamics of Stock Holdings and Prices

In this section we study the dynamics of stock holdings and prices. To study these dynamics, we first show a result on the coefficients a_e and a_s of the large trader's equilibrium strategy.

A. The Coefficients a_e and a_s

The large trader's sell order in period ℓ is given by equation (4) that we reproduce below

$$x_\ell = a_e(e_{\ell-1} + \epsilon_\ell - s_{\ell-1}) + a_s s_{\ell-1} - a_{\overline{e}}\overline{e}_{\ell-1}.$$

The coefficient a_s is the rate at which the large trader sells out of his expected stock holdings (conditional on the market makers' information) $s_{\ell-1}$. The coefficient a_e is similarly the rate at which the large trader sells out of his unexpected stock holdings $e_{\ell-1} + \epsilon_\ell - s_{\ell-1}$. In Proposition 3 we compare the coefficients a_e and a_s for small order flow. Our numerical solutions confirm Proposition 3 for large order flow.

Proposition 3 *For small σ_e^2, $a_e > a_s$.*

Proposition 3 states that the large trader sells at a higher rate out of unexpected than out of expected stock holdings. To explain the intuition behind this result, we consider two extreme cases. First, the case where the large trader's stock holdings are expected, i.e. where they are equal to the market makers' expectation $s_{\ell-1}$. Second, the case where the large trader's stock holdings are unexpected, i.e. where the market makers' expectation is 0. We also suppose that the large trader's stock holdings are high relative to the market makers', so that the large trader sells over time and the price falls. When stock holdings are unexpected, the large trader has private information that the price will fall. By selling slowly, he bears the opportunity cost of not "frontrunning" on this information. By contrast, when stock holdings are expected, the large trader has no private information and bears no such cost. Therefore, the large trader sells more slowly when stock holdings are expected. In terms of equation (29), the "frontrunning" term is 0 when stock holdings are expected and negative when they are unexpected. Therefore, the marginal benefit of buying in period ℓ and selling in period $\ell+1$, i.e. the marginal benefit of selling more slowly, is higher when stock holdings are expected.

B. Dynamics

The dynamics of stock holdings and prices are, a priori, complicated, because they are generated by multiple endowment and noise shocks. We can, however, simplify the dynamics, using the linearity of the model. Indeed, because of linearity, the dynamics are simply the sum over all shocks of the dynamics generated by each shock.[16] We will study the dynamics generated by an endowment shock. The dynamics generated by a noise shock have a similar flavor. We normalize the endowment shock to 1, and assume that it comes in period ℓ, i.e. we set $\epsilon_\ell = 1$. To isolate the dynamics generated by this shock, we set all other endowment and noise shocks to 0. In Proposition 4 we determine the dynamics of the large trader's stock holdings.

Proposition 4 *The large trader's stock holdings before trade in period $\ell' \geq \ell$ are*

$$e_{\ell'-1} = \frac{a_{\overline{e}}}{a_s + a_{\overline{e}}} + \frac{a_e(ga_s - a_{\overline{e}})}{(a_e(1+g) - a_s - a_{\overline{e}})(a_s + a_{\overline{e}})}(1 - a_s - a_{\overline{e}})^{\ell'-\ell}$$

$$+\frac{a_e - a_s}{a_e(1+g) - a_s - a_{\overline{e}}}(1 - a_e(1+g))^{\ell'-\ell}. \tag{30}$$

Proposition 4 implies that stock holdings converge to the long-run limit $a_{\overline{e}}/(a_s + a_{\overline{e}})$, and that convergence takes place at a combination of the rates $(1 - a_s - a_{\overline{e}})^{\ell'-\ell}$ and $(1 - a_e(1+g))^{\ell'-\ell}$. To prove Proposition 4, we determine the joint dynamics of the large trader's stock holdings, e_ℓ, the market makers' expectation of the large trader's stock holdings, s_ℓ, and the market makers' stock holdings, $\overline{e}_\ell$. Equations (13), (17), and (11), give e_ℓ, s_ℓ, and $\overline{e}_\ell$ as linear functions of $e_{\ell-1}$, $s_{\ell-1}$, and $\overline{e}_{\ell-1}$. The 3×3 matrix corresponding to these equations has three eigenvalues. The first eigenvalue is 1, and corresponds to the long-run limit. The other two eigenvalues are $1 - a_s - a_{\overline{e}}$ and $1 - a_e(1+g)$ and correspond to the rates of convergence.

The dynamics of prices can be deduced from the dynamics of e_ℓ, s_ℓ, and $\overline{e}_\ell$. Equation (8) implies that the price p_ℓ is a linear function of e_ℓ, s_ℓ, and $\overline{e}_\ell$. Therefore, the price converges to a long-run limit, and convergence takes place at a combination of the rates $(1 - a_s - a_{\overline{e}})^{\ell'-\ell}$ and $(1 - a_e(1+g))^{\ell'-\ell}$.

We next study the long-run limits of stock holdings and prices, and then the convergence to these limits. The long-run limit of the large trader's stock holdings is $a_{\overline{e}}/(a_s + a_{\overline{e}})$. Since the endowment shock is equal to 1, the long-run limit of the market makers' stock holdings is $a_s/(a_s + a_{\overline{e}})$. Therefore, in the long run, the endowment shock is divided between the large trader and the market makers according to the ratio $a_{\overline{e}}/a_s$. This ratio is in fact the optimal risk-sharing rule. Theorem 1 implies that in the absence of price risk, the risk-sharing rule coincides with the standard risk-sharing rule $\overline{\alpha}/\alpha$.[17] The long-run limit of the price can be deduced from the long-run limits of stock holdings and the market makers' expectation of the large trader's stock holdings. In the proof of Proposition 4, we show that the latter is $a_{\overline{e}}/(a_s + a_{\overline{e}})$. This means that, in the long run, the market makers form a correct expectation of the large trader's stock holdings. (As all results in this section, this result concerns the dynamics generated by one endowment shock. Because of the subsequent endowment and noise shocks, the market makers never learn the large trader's stock holdings.)

In Proposition 5 we show some results on the convergence of stock holdings and prices to their long-run limits. To state the proposition, we define two variables, the trading rate and the price impact. The trading rate is the ratio of the large trader's sell order at a given period, to the sum of his sell orders at that and subsequent periods. The price impact is the ratio of minus the price change at a given period, to the large trader's sell order at that period. Formally, the trading rate in period ℓ' is $x_{\ell'}/\sum_{\ell''\geq\ell'} x_{\ell''}$, and the price impact is $(p_{\ell'-1} - p_{\ell'})/x_{\ell'}$.

Proposition 5 *If $ga_s - a_{\overline{e}} \geq 0$, the large trader's stock holdings decrease over time. The trading rate and the price impact also decrease over time. If $ga_s - a_{\overline{e}} < 0$, the large trader's stock holdings decrease and then increase over time.*

Proposition 5 implies that there are two patterns of convergence to the long-run limit: the large trader's stock holdings can decrease over time, or they can decrease and then increase. Moreover, when stock holdings decrease over time, the trading rate and the price impact also decrease.[18]

The first convergence pattern is very intuitive. Stock holdings decrease over time, as the large trader sells to the market makers the fraction of his endowment corresponding to optimal risk-sharing. The trading rate decreases over time because, first, the large trader sells at a higher rate out of unexpected than out of expected stock holdings (Proposition 3) and, second, the fraction of stock holdings that are expected increases, as the market makers' expectation becomes more accurate. The price impact decreases over time because the fraction of the large trader's sell order that is unexpected, and that causes the price to change, decreases. Figure 1 illustrates the first convergence pattern.[19]

The second convergence pattern is somewhat surprising. The large trader sells to the market makers the fraction of his endowment corresponding to optimal risk-sharing. He then engages in a "round-trip transaction", selling some shares only to buy them back later. This pattern occurs when g is small, i.e. most order flow comes from the small traders, and when $a_{\overline{e}}/a_s$ is large, i.e. the optimal risk-sharing rule assigns many shares to the large trader. The large trader engages in the round-trip transaction because the market makers misinterpret the sale prior to that transaction, i.e. the sale that led to optimal risk-sharing. Indeed, the market makers attribute the sale to the small traders, who account for most order flow, and expect the large trader to absorb a fraction of the sale. The large trader knows that he initiated the sale and that he will not absorb back a fraction. Therefore, he has private information that the price will fall. He can trade on that information, selling some shares and buying them back when the price falls. Figure 2 illustrates the second convergence pattern.

IV. Dynamics in the Continuous-Time Case

In this section we study the dynamics of stock holdings and prices in the "continuous-time" case, where the time between trades h, is small. We consider the continuous-time case for two reasons. First, because in real-world financial markets the time between trades can be very small. Second, because when both the order flow and the time between trades are small, we can determine the Nash equilibrium in closed-form. In Section IV.A we determine the closed-form solution and in Section IV.B we use the solution to study the dynamics of stock holdings and prices.

A. The Closed-Form Solution

We obtain the closed-form solution when both the order flow and the time between trades are small. More precisely, we consider the Taylor expansion in h of the coefficients $A_{\overline{e}}$, A_s, B, a_e, a_s, $a_{\overline{e}}$, $\overline{Q}$, Q, g, and Σ_e^2. For small σ_e^2, i.e. for small order flow, we determine the order of the dominant term in the Taylor expansion, i.e. we determine whether the dominant term is of order h, or $\sqrt{h}$, or 1, etc. For $\sigma_e^2 = 0$, we determine the dominant term in closed-form. In Proposition 6 we present the results for the coefficients a_e, a_s, $a_{\overline{e}}$, and g. We omit the other coefficients, since we do not use them in what follows.

Proposition 6 *For small σ_e^2,*

$$a_e = \phi_e\sqrt{h} + o(\sqrt{h}), \qquad a_s = \phi_s h + o(h), \qquad a_{\overline{e}} = \phi_{\overline{e}} h + o(h), \qquad \textit{and} \qquad g = g_0 + o(1).$$

For $\sigma_e^2 = 0$,

$$g_0 = \sqrt{1 + \frac{1}{\overline{\sigma}_u^2}} - 1,$$

$$\phi_e = \sqrt{\frac{\alpha r}{\overline{\alpha}(1 + g_0)}},$$

ϕ_s is the positive root of

$$2(\phi_s)^2(1 + g_0)\frac{\overline{\alpha}}{\alpha} - \phi_s r(2 + g_0) - r^2 = 0,$$

and $\phi_{\overline{e}} = \phi_s\overline{\alpha}/\alpha$.

Proposition 6 implies that for small order flow, a_e is of order $\sqrt{h}$, a_s and $a_{\overline{e}}$ of order h, and g of order 1. Our numerical solutions confirm these results for large order flow. In Section IV.B we use these results to study the dynamics of stock holdings and prices. Proposition 6 has also implications on how the coefficients a_e, a_s, $a_{\overline{e}}$, and g depend on the exogenous parameters $\overline{\sigma}_u^2$, α, and $\overline{\alpha}$, for $\sigma_e^2 = 0$. We explore these implications in Section V, where we perform comparative statics.

The result that a_e is of order $\sqrt{h}$ and a_s of order h, is stronger than the result $a_e > a_s$ of Proposition 3. For small h, the large trader sells at a much higher rate, and not simply at a higher rate, out of unexpected than out of expected stock holdings. This result turns out to be crucial for the dynamics of stock holdings and prices, so we explain the intuition behind it in the rest of this section.

We suppose that before trade in period ℓ, the large trader's stock holdings are equal to 1, and the market makers' stock holdings to 0. The large trader thus sells over time. Equation (29) states that the marginal benefit of buying in period ℓ and selling in period $\ell+1$, i.e. the marginal benefit of selling more slowly, is the sum of the "risk-sharing" term, the "frontrunning" term, the "price impact" term, and the "belief manipulation" term. The risk-sharing term is negative, since the large trader bears more dividend risk between periods ℓ and $\ell+1$. Since the variance of the dividend shock $\delta_{\ell+1}$ is $\sigma^2 h$, the risk-sharing term is of order h. The price impact term is positive, since the price in period ℓ increases. It is of the same order as the trade x_ℓ, of the large trader. The belief manipulation term can be negative or positive, depending on whether the market makers expect a purchase or a sale after a purchase. For simplicity, we assume that it is 0. When stock holdings are expected, i.e. when $s_{\ell-1} = 1$, the frontrunning term is 0. Therefore, the price impact benefit of selling more slowly, has to equal the risk-sharing cost. This means that the trade x_ℓ is of order h. Equation (4) implies that $x_\ell = a_s$. Therefore, a_s is of order h.

When stock holdings are unexpected, i.e. when $s_{\ell-1} = 0$, the frontrunning term is negative. This is because the period ℓ price does not reflect the large trader's future sales, and is higher than the period $\ell+1$ price. Substituting the price $p_{\ell+1}$ from equation (7), we can write the frontrunning term as

$$-\frac{1}{B}\left(E_\ell x_{\ell+1} - \overline{E}_\ell x_{\ell+1}\right)e^{-rh}.$$

The term in parentheses is the "unexpected" period $\ell+1$ sell order, i.e. the sell order that can be predicted by the large trader and not by the market makers. The frontrunning term is simply the price impact of the unexpected sell order, discounted in period ℓ. Notice that the price impact term, $(1/B)x_\ell$, is the "negative" of the frontrunning term, since it is the negative of the price impact of x_ℓ, the unexpected period ℓ sell order. The sum of the two terms is of order

$$x_\ell - \left(E_\ell x_{\ell+1} - \overline{E}_\ell x_{\ell+1}\right)e^{-rh}.$$

This, rather than x_ℓ, has to be of the same order as the risk-sharing term, i.e. of order h. Equations (4), (13), and (17) imply that

$$x_\ell - \left(E_\ell x_{\ell+1} - \overline{E}_\ell x_{\ell+1}\right) e^{-rh} = a_e - a_e(e_\ell - s_\ell)e^{-rh} = a_e(1 - (1 - a_e(1+g))e^{-rh}).$$

Therefore, a_e is of order $\sqrt{h}$.

B. Dynamics

As in Section III.B, we study the dynamics generated by one endowment shock. We first proceed heuristically. To complement the heuristic analysis, we then study the dynamics in the limit when h goes to 0. In Proposition 6 we showed that a_e, the selling rate out of unexpected stock holdings, is of order $\sqrt{h}$. Therefore, unexpected stock holdings get close to 0, their long-run limit, within a number of periods of order $1/\sqrt{h}$, i.e. within a time of order $h \times (1/\sqrt{h}) = \sqrt{h}$. Within that time, total (expected plus unexpected) stock holdings get close to expected stock holdings. This happens both because total stock holdings decrease, as the large trader sells a fraction of his endowment, and because expected stock holdings increase, as the market makers' expectation becomes more accurate. Once total stock holdings get close to expected stock holdings, they evolve more slowly. Indeed, in Proposition 6 we showed that a_s, the selling rate out of expected stock holdings, is of order h. Therefore, expected (and total) stock holdings get close to their long-run limit, within a number of periods of order $1/h$, i.e. within a time of order $h \times (1/h) = 1$.

The above heuristic analysis implies that for small h, the dynamics consist of two phases. The first phase is short, with length of order $\sqrt{h}$. In this phase, the large trader sells a fraction of his endowment, and the market makers form a correct expectation of the large trader's stock holdings. The second phase is long. In this phase, optimal risk-sharing is achieved. Figure 3 illustrates the dynamics of the large trader's stock holdings for small h.

To complement the heuristic analysis, we study the dynamics in the limit when h goes to 0. We assume that the endowment shock comes at time t, and we determine the limit when h goes to 0, of the large trader's stock holdings at time $t' > t$. We fix times t and t', rather than periods ℓ and ℓ', because the time between two periods goes to 0 when h goes to 0. We determine the limit of the large trader's stock holdings in Proposition 7.

Proposition 7 *The limit when h goes to 0, of the large trader's stock holdings at time $t' > t$ is*

$$\frac{\phi_{\overline{e}}}{\phi_s + \phi_{\overline{e}}} + \frac{g_0\phi_s - \phi_{\overline{e}}}{(1+g_0)(\phi_s + \phi_{\overline{e}})} e^{-(\phi_s + \phi_{\overline{e}})(t'-t)}. \tag{31}$$

Proposition 7 implies that in the limit when h goes to 0, stock holdings behave as follows. At time t they are equal to 1, the endowment shock. Immediately after time t they drop discontinuously to $g_0/(1+g_0)$. They then converge to the long-run limit $\phi_{\overline{e}}/(\phi_s+\phi_{\overline{e}})$ at the rate $e^{-(\phi_s+\phi_{\overline{e}})(t'-t)}$. The discontinuous drop corresponds to the first phase of the dynamics for small h, and the exponential rate of convergence corresponds to the second phase. We should emphasize that the discontinuous drop does not correspond to a single block trade, but to many small trades. Each of these trades is of order $\sqrt{h}$, and the trades are completed within a time of order $\sqrt{h}$.

An important property of the dynamics for small h, is that the large trader reveals his information "quickly", i.e. within a time that goes to 0 when h goes to 0. Indeed, the market makers form a correct expectation of the large trader's stock holdings within a time of order $\sqrt{h}$. The "fast" revelation of information is in contrast to the "slow" revelation of information (within a time that does not go to 0 when h goes to 0) in Kyle's (1985) insider trading model. One reason why the results are different may have to do with traders' impatience. Our large trader is risk-averse and bears a cost when holding a risky position. He is thus impatient to reduce his position. By contrast, Kyle's insider is risk-neutral and is not impatient to establish a position, as long as he does so before his information is publicly announced.[20] It is, however, unlikely that the difference in results is due to impatience. Indeed, we get fast revelation of information for all parameter values, including when the risk aversion of the large trader is small.[21]

A second reason why the results are different may have to do with stationarity. Our model is stationary, since the large trader receives a stock endowment in each period. By contrast, Kyle's model is non-stationary, since the insider receives private information only at the beginning of the trading session. Non-stationarity implies that the fast revelation of information cannot be an equilibrium. Indeed, if the insider reveals his information quickly, market depth will increase over time and the insider will prefer to trade more slowly. To determine whether the difference in results is due to stationarity, one can examine a stationary model with a risk-neutral insider who receives private information over time. Such a model is of independent interest. For instance, a result that the insider reveals his information quickly would suggest that markets are informationally efficient even in the presence of informational monopolists.[22]

V. Comparative Statics

In this section we study how the dynamics of the large trader's stock holdings after an endowment shock depend on the exogenous parameters $\overline{\sigma}_u^2$, α, and $\overline{\alpha}$. The parameter $\overline{\sigma}_u^2 = \sigma_u^2/\sigma_e^2$ is a measure of the relative order flow coming from the small traders and the large trader. We refer to it as noise. The parameters α and $\overline{\alpha}$ are the coefficients of absolute risk aversion of the large trader and the market makers. We refer to them as the large trader's and the market makers' risk aversion.

It is easiest to perform the comparative statics for small order flow and time between trades, i.e. for small σ_e^2 and h. Indeed, in the limit when h goes to 0, the dynamics of the large trader's stock holdings simplify to the ones given by Proposition 7. Moreover, for $\sigma_e^2 = 0$, the parameters g_0, ϕ_s, and $\phi_{\overline{e}}$, are given in closed form by Proposition 6. We perform the comparative statics for small σ_e^2 and h (more precisely, in the limit when σ_e^2 and h go to 0) in Corollaries 1 and 2. Corollary 1 concerns the comparative statics w.r.t. the noise, and Corollary 2 those w.r.t. traders' risk aversion. After stating the corollaries, we explain how they follow from Propositions 6 and 7. Finally, we show that the comparative statics for general values of σ_e^2 and h are consistent with the corollaries.

Corollary 1 *If the noise is small, the large trader's stock holdings decrease over time, while if the noise is large, they decrease and then increase over time. Moreover, as the noise increases, the large trader sells faster.*

Corollary 2 *If the large trader's risk aversion is large or the market makers' risk aversion small, the large trader's stock holdings decrease over time. In the opposite case, they decrease and then increase over time. Both the large trader's and the market makers' risk aversion have an ambiguous effect on the speed at which the large trader sells.*

To prove the corollaries, we first note that from Proposition 7 the large trader's stock holdings drop discontinuously from 1 to $g_0/(1+g_0)$, and then converge to the long-run limit $\phi_{\overline{e}}/(\phi_s + \phi_{\overline{e}})$ at the rate $e^{-(\phi_s+\phi_{\overline{e}})(t'-t)}$. We then use Proposition 6 to study how the long-run limit, the discontinuous drop, and the rate of convergence depend on the noise and on traders' risk aversion. Proposition 6 implies that the long-run limit is $\overline{\alpha}/(\alpha+\overline{\alpha})$. The long-run limit thus decreases in the large trader's risk aversion, increases in the market makers' risk aversion, and is independent of the noise. The intuition is that the long-run limit is determined by optimal risk-sharing. Proposition 6 also implies that the discontinuous drop

$1/(1+g_0)$, in the large trader's stock holdings increases in the noise and is independent of the large trader's and the market makers' risk aversion. The intuition is that the discontinuous drop represents the number of shares the large trader can sell before the market makers form a correct expectation of his stock holdings. This number of shares depends only on the noise. Finally, Proposition 6 implies that the rate of convergence $e^{-(\phi_s+\phi_{\overline{\epsilon}})(t'-t)}$ becomes faster (i.e. $\phi_s + \phi_{\overline{\epsilon}}$ increases) as the noise increases, the large trader's risk aversion increases, and the market makers' risk aversion decreases.[23] The intuition is that the large trader trades quickly if he is impatient to reduce his position and if the price impact is small. The large trader's impatience increases in his risk aversion, and the price impact decreases in the noise and increases in the market makers' risk aversion.

Using the above results, it is straightforward to prove the corollaries. The first result of Corollary 1 follows by noting that the long-run limit of stock holdings is independent of the noise, while the discontinuous drop increases in the noise. Therefore, stock holdings drop above their long-run limit if the noise is small, and below otherwise. For the second result of the corollary, suppose that the noise is small, in which case stock holdings drop above their long-run limit. As the noise increases, the discontinuous drop increases, and the subsequent rate of convergence to the long-run limit becomes faster. Therefore, the large trader sells faster. Moreover, as the noise gets large, the large trader sells immediately, since stock holdings drop discontinuously below their long-run limit.

To prove Corollary 2, we note that the large trader's and the market makers' risk aversion have opposite effects on the long-run limit of stock holdings, the discontinuous drop, and the rate of convergence. Therefore, we can focus on the large trader's risk aversion. The first result of the corollary follows by noting that the long-run limit of stock holdings increases as α decreases, while the discontinuous drop is independent of α. Therefore, stock holdings drop above their long-run limit if α is large, and below otherwise. For the second result of the corollary, note that as α decreases, the rate of convergence to the long-run limit (that applies subsequent to the discontinuous drop) becomes slower, but the long-run limit increases and becomes easier to reach. In other words, the large trader sells more slowly, but has a smaller quantity to sell. To show that either effect can dominate, consider three cases: $\alpha = \infty$, i.e. infinitely risk-averse large trader, α large but not ∞, and α small. If $\alpha = \infty$, the large trader sells immediately, since he bears an infinite cost of holding a risky position. If α is large but not ∞, the large trader does not sell immediately, since stock holdings drop above their long-run limit and then converge slowly to that limit. Finally, if

α is small, the large trader sells immediately, since stock holdings drop below their long-run limit.

Figure 4 illustrates the effects of the noise on the dynamics of the large trader's stock holdings and Figure 5 illustrates the effects of the large trader's risk aversion. These figures are drawn for general values of σ_e^2 and h, and are consistent with the corollaries.

VI. Empirical Implications

In this section we derive the empirical implications of our results. We also relate our results to the empirical literature on the price impact and the execution of large trades.

We focus on the results concerning the dynamics of stock holdings and prices. These results have implications for time-series data on institutional trades. We should note that the dynamics in our model are generated by multiple endowment and noise shocks. The data are also likely to be generated by multiple shocks, where shocks can, for instance, be interpreted as changes in institutions' desired portfolios. Indeed, institutions' desired portfolios change frequently, in response to inflows and outflows of funds, changing market conditions, etc. To study the dynamics in our model, we considered the dynamics generated by each shock in isolation. Therefore, one way to bring our model to the data is to filter the data, and produce sequences of trades corresponding to each shock. Our results have empirical implications for these sequences.

In Proposition 5 we show that, after an endowment shock, the large trader's stock holdings can decrease over time, or they can decrease and then increase. Moreover, when stock holdings decrease over time, the trading rate and the price impact also decrease. These results imply that in a sequence of institutional trades, in which all trades are in the same direction, the following should be true. First, trading rate, defined as trade size over size of subsequent trades, decreases over time. In particular, trade size decreases over time, i.e. the first trades are the largest. Second, price impact, defined as price change over trade size, decreases over time.

In Corollaries 1 and 2 we show that stock holdings decrease and then increase, when the noise is large, the large trader not very risk-averse, and the market makers very risk-averse. Therefore, a sequence of institutional trades in which not all trades are in the same direction is more likely to be observed (i) for stocks with a high ratio of non-institutional to institutional trades, i.e. for stocks where the noise is large, (ii) for large, and thus not very risk-averse, institutions, and (iii) for stocks with few, weakly capitalized, and thus very risk-averse, market makers.

In Corollaries 1 and 2 we show that, as the noise increases, the large trader sells faster. We also show that the large trader's and the market makers' risk aversion have an ambiguous effect on the speed of trade. Therefore, a sequence of institutional trades should be completed faster for stocks with a high ratio of non-institutional to institutional trades. However, the size of the institution and the number of market makers should have an ambiguous effect on the length of a trading sequence.

Most empirical studies of large trades, isolate individual trades and measure their price impact. (See, for instance Kraus and Stoll (1972), Scholes (1972), Holthausen, Leftwich, and Mayers (1987, 1990), Hausman, Lo, and McKinlay (1992), Chan and Lakonishok (1993), and Keim and Madhavan (1996).) Chan and Lakonishok (1995) recognize that individual trades may be part of a sequence or, in their terminology, a "package". They show that the price impact of trades is larger when these are analyzed in terms of packages rather than individually. They determine the distribution of package lengths and show that 53 percent of packages take four or more days to be completed. They also show that, quite surprisingly, packages in small stocks are completed faster than packages in large stocks.

The above relation between package length and stock size, although surprising, is not inconsistent with our results. Indeed, small stocks have few market makers and our results imply that the length of a package does not necessarily increase as the number of market makers decreases. On the other hand, our results imply a strong negative relation, across stocks, between package length and the ratio of non-institutional to institutional trades. Such a relation is not examined in Chan and Lakonishok (1995), but can be tested in future research. The other implications of our results concern "finer" properties of packages, such as how trade size and price impact vary over the course of the package and when all trades in the package are in the same direction. These properties are not examined in Chan and Lakonishok, but can also be tested in future research.

VII. Calibration

In this section we calibrate the model, and study how quickly the large trader trades after receiving an endowment shock. We measure the speed of trade by the average time of trade execution (ATTE). This is the weighted sum of the times at which trades are executed, with weights equal to the trades divided by the total trade. In Figure 6 we plot the ATTE as a function of the noise. (The noise is $\overline{\sigma}_u^2 = \sigma_u^2/\sigma_e^2$ and measures the relative order flow coming from the small traders and the large trader.) The values of the other parameters are the same as for Figures 1, 2, and 4.[24] The parameter $\overline{\alpha}/\alpha$ is equal to 1/9. The large trader is thus nine times as risk-averse as the market makers as a group. We assume that the market makers are not very risk-averse to capture the real-world feature that market makers can unload, to outside investors and over time, trades they receive from the large trader. (In our model market makers cannot unload trades since there are no outside investors.) The annual interest rate r is equal to 5 percent. The time between trading periods h is equal to 1/5000'th of a year. Since there are approximately 250 trading days per year, there are $(250 \times h)^{-1} = 20$ trading periods per day.[25] The parameter $\overline{\alpha}\sigma\sigma_e/r$ is equal to 0.1. This parameter determines the ratio of price risk (price variability due to trades) to dividend risk (price variability due to dividends). If $\overline{\alpha}\sigma\sigma_e/r = 0.1$, the ratio is approximately 10 percent. Finally, the endowment shock of the large trader is equal to 1. The size of the endowment shock does not affect the ATTE, because of the linearity of the model.

Figure 6 shows that when the noise is greater than four, the ATTE is approximately five trading days. When the noise is smaller than four, however, the ATTE increases substantially. For instance, when the noise is two, the ATTE is 53 trading days. Such an ATTE seems somewhat long (although there is anecdotal evidence that it can take a very long time to get out of an extremely large position.) On the other hand, an ATTE of five trading days seems quite consistent with the findings of Chan and Lakonishok (1995).

VIII. Conclusion

This paper studies a dynamic model of a financial market with a large trader. In each period the large trader receives a privately observed endowment in the stock. He trades with competitive market makers to share risk. Small noise traders are present in the market. After receiving a stock endowment, the large trader is shown to reduce his risk exposure either by selling at a decreasing rate over time, or by selling and then buying back some of the shares sold. He follows the second strategy when there is enough noise and when he is not very risk-averse relative to the market makers. When the time between trades is small, the large trader reveals the information regarding his endowment very quickly. Finally, the large trader sells faster as the noise increases, but the market makers' and the large trader's risk aversion have an ambiguous effect on the speed of trade. Our results have several empirical implications that can be tested in future research.

Future research can also address some theoretical issues. First, in this paper we obtain analytical results only for small order flow, i.e. only for small endowment and noise shocks. For large order flow, we can only obtain numerical results (which are consistent with the analytical results). Since many institutions need to execute large trades on a frequent basis, it would be desirable to gain more insight on the equilibrium for large order flow. In the context of our model, this can be done by studying in more depth the non-linear system (S).

Second, in our model, trading needs are "smooth". Indeed, normality implies that the endowment shocks are never equal to 0, and are small when the time between trades is small. However, in the real-world trading needs are frequently "lumpy". Many institutions do not trade for a period of time and then need to execute a large trade. It would be interesting to know whether our results are robust to endowment shocks not being normal.

Finally, it would be interesting to know why our large trader reveals his information quickly while Kyle's insider reveals his information slowly. In Section IV.B we suggested that this might be because our model is stationary while Kyle's model is not. To determine whether the stationarity is indeed the reason, one can examine a stationary model with a risk-neutral insider who receives private information over time. Such a model is of independent interest. For instance, a result that the insider reveals his information quickly would suggest that markets are informationally efficient even in the presence of informational monopolists.

Appendix A

A. The Market Makers' Optimization Problem

In this section we state concisely the market makers' optimization problem, and then state Proposition 1. The proofs of this proposition, of Propositions 2, 3, 5, 6, and of Theorem 1, are not included in the printed version, in order to save space. They are available from the author upon request and on the Journal of Finance web site `http://www.afajof.org/`.

We define the matrix $\overline{N}$ by

$$\overline{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - a_{\overline{e}} & a_s \\ 0 & a_{\overline{e}} & 1 - a_s \end{pmatrix}, \tag{A1}$$

and the vector $\overline{n}$ by

$$\overline{n} = \begin{pmatrix} 0 \\ 1 \\ g \end{pmatrix}. \tag{A2}$$

We also denote by ζ_ℓ the unexpected order flow (conditional on the market makers' information) i.e.

$$\zeta_\ell = a_e(e_{\ell-1} + \epsilon_\ell - s_{\ell-1}) + u_\ell.$$

The market maker's optimization problem, $(\overline{P})$, is

$$\sup_{\overline{c}_\ell, \overline{x}_\ell(p_\ell)} -\overline{E}_{\ell_0}(h \sum_{\ell=\ell_0}^{\infty} exp(-\overline{\alpha}\,\overline{c}_\ell - \overline{\beta}(\ell - \ell_0)h))$$

subject to

$$d_\ell = d_{\ell-1} + \delta_\ell,$$

$$\overline{M}_\ell = e^{rh}(\overline{M}_{\ell-1} + d_{\ell-1}(\overline{e}_{\ell-1} + \Delta\overline{e}_{\ell-1})h - \overline{c}_{\ell-1}h) - p_\ell \overline{x}_\ell(p_\ell), \tag{A3}$$

$$p_\ell = \frac{h}{1 - e^{-rh}} d_\ell - \frac{A_{\overline{e}}}{B}\overline{e}_{\ell-1} - \frac{A_s}{B} s_{\ell-1} - \frac{1}{B}\zeta_\ell, \tag{A4}$$

$$\begin{pmatrix} \Delta\overline{e}_\ell \\ \overline{e}_\ell \\ s_\ell \end{pmatrix} = \overline{N} \begin{pmatrix} \Delta\overline{e}_{\ell-1} + \overline{x}_\ell(p_\ell) - (\zeta_\ell + a_s s_{\ell-1} - a_{\overline{e}}\overline{e}_{\ell-1}) \\ \overline{e}_{\ell-1} \\ s_{\ell-1} \end{pmatrix} + \overline{n}\zeta_\ell, \tag{A5}$$

and the transversality condition[26]

$$\lim_{\ell\to\infty} \overline{E}_{\ell_0} \overline{V}(\overline{M}_\ell, d_\ell, \Delta\overline{e}_\ell, \overline{e}_\ell, s_\ell) exp(-\overline{\beta}(\ell - \ell_0)h) = 0. \tag{A6}$$

To state Proposition 1 we define the following. First, the symmetric 4×4 matrix $\overline{Q}'$ by the condition that

$$F(\overline{Q}', \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}) = \frac{1-e^{-rh}}{h} \left(\frac{A_{\overline{e}}}{B} z_2 + \frac{A_s}{B} z_3 + \frac{1}{B} z_4 \right) (z_4 + a_s z_3 - a_{\overline{e}} z_2 - z_1)$$

$$+F(\overline{Q}, \overline{N} \begin{pmatrix} 0 \\ z_2 \\ z_3 \end{pmatrix} + \overline{n} z_4) \tag{A7}$$

for all (z_1, z_2, z_3, z_4). Second, the scalar $\overline{R}$ by

$$\overline{R} = 1 + \overline{\alpha} \left(a_e^2 (\Sigma_e^2 + \sigma_e^2 h) + \sigma_u^2 h \right) \overline{Q}'_{4,4}. \tag{A8}$$

Third, the symmetric 3×3 matrix $\overline{R}'$ by

$$\overline{R}' = \overline{\alpha} \left(a_e^2 (\Sigma_e^2 + \sigma_e^2 h) + \sigma_u^2 h \right) \overline{R}^{-1} \overline{Q}'_{\{1,2,3\},\{4\}} \overline{Q}'_{\{4\},\{1,2,3\}}. \tag{A9}$$

Finally, the symmetric 3×3 matrix $\overline{P}$ by

$$\overline{P} = (\overline{Q}'_{\{1,2,3\}} - \overline{R}' - \overline{\alpha} \sigma^2 h \Gamma) e^{-rh}. \tag{A10}$$

In equation (A9), $\overline{Q}'_{I,J}$ denotes the matrix formed by the rows of $\overline{Q}'$ whose indices belong to the set I and by the columns whose indices belong to the set J. In equation (A10), $\overline{Q}'_I$ denotes the matrix $\overline{Q}'_{I,I}$, and Γ is a symmetric 3×3 matrix whose elements are 1 for $(i,j) = (1,1), (1,2), (2,1), (2,2)$ and 0 otherwise.

Proposition 1 *The demand in equation (5) solves ($\overline{P}$) and the function (18) is the value function, if the following conditions hold. First, the "optimality conditions"*

$$\frac{1-e^{-rh}}{h} \frac{A_{\overline{e}}}{B} + (1 - a_{\overline{e}}) \overline{Q}_{1,2} + a_{\overline{e}} \overline{Q}_{1,3} = 0, \tag{A11}$$

$$\frac{1-e^{-rh}}{h} \frac{A_s}{B} + a_s \overline{Q}_{1,2} + (1 - a_s) \overline{Q}_{1,3} = 0, \tag{A12}$$

$$\frac{1-e^{-rh}}{h} \frac{1}{B} + \overline{Q}_{1,2} + g \overline{Q}_{1,3} = 0, \tag{A13}$$

and $\overline{Q}_{1,1} < 0$. Second, the "Bellman conditions"

$$\overline{Q} = \overline{P} \tag{A14}$$

and

$$\overline{q} = \frac{log(\overline{R}) e^{-rh}}{2\overline{\alpha}(1-e^{-rh})} + \frac{(\overline{\beta} e^{-rh} - r)h}{\overline{\alpha}(1-e^{-rh})} - \frac{1}{\overline{\alpha}} log(\frac{h}{e^{rh}-1}). \tag{A15}$$

B. The Large Trader's Optimization Problem

In this section we state concisely the large trader's optimization problem, and then state Proposition 2. We define the matrix N by

$$N = \begin{pmatrix} 1 - a_e(1+g) & 0 & 0 \\ g a_e & 1 - a_s & a_{\overline{e}} \\ a_e & a_s & 1 - a_{\overline{e}} \end{pmatrix}, \tag{A16}$$

the vector n by

$$n = \begin{pmatrix} -g \\ g \\ 1 \end{pmatrix}, \tag{A17}$$

and the vector $\hat{n}$ by

$$\hat{n} = \begin{pmatrix} -(1+g) \\ g \\ 1 \end{pmatrix}. \tag{A18}$$

We also introduce Δx_ℓ, the difference between the large trader's market order and the candidate market order, i.e.

$$\Delta x_\ell = x_\ell - \left(a_e(e_{\ell-1} + \epsilon_\ell - s_{\ell-1}) + a_s s_{\ell-1} - a_{\overline{e}}\overline{e}_{\ell-1}\right). \tag{A19}$$

We will write p_ℓ and the dynamics of e_ℓ, s_ℓ, and $\overline{e}_\ell$ in terms of Δx_ℓ instead of x_ℓ. The large trader's optimization problem, (P), is

$$\sup_{c_\ell, x_\ell} -E_{\ell_0}\left(h \sum_{\ell=\ell_0}^{\infty} exp(-\alpha c_\ell - \beta(\ell - \ell_0)h)\right)$$

subject to

$$d_\ell = d_{\ell-1} + \delta_\ell,$$

$$M_\ell = e^{rh}(M_{\ell-1} + d_{\ell-1}e_{\ell-1} - c_{\ell-1}h) - d_\ell \frac{h}{1 - e^{-rh}}\epsilon_\ell + p_\ell x_\ell,$$

$$p_\ell = \frac{h}{1 - e^{-rh}} d_\ell - \frac{A_{\overline{e}}}{B}\overline{e}_{\ell-1} - \frac{A_s}{B}s_{\ell-1} - \frac{1}{B}(a_e(e_{\ell-1} + \epsilon_\ell - s_{\ell-1}) + u_\ell + \Delta x_\ell), \tag{A20}$$

$$\begin{pmatrix} e_\ell - s_\ell \\ s_\ell \\ \overline{e}_\ell \end{pmatrix} = N \begin{pmatrix} e_{\ell-1} + \epsilon_\ell - s_{\ell-1} \\ s_{\ell-1} \\ \overline{e}_{\ell-1} \end{pmatrix} + n u_\ell + \hat{n}\Delta x_\ell, \tag{A21}$$

and the transversality condition

$$\lim_{\ell \to \infty} E_{\ell_0} V(M_\ell, d_\ell, e_\ell, s_\ell, \overline{e}_\ell) exp(-\overline{\beta}(\ell - \ell_0)h) = 0. \tag{A22}$$

To state Proposition 2 we define the following. First, the 1×4 vector Q_u by

$$Q_u = -\frac{1-e^{-rh}}{h}\left(\frac{a_e}{B}, \frac{a_s}{B}, -\frac{a_{\overline{e}}}{B}, \frac{1}{B}\right) + (n^t Q N, n^t Q \hat{n}). \tag{A23}$$

Second, the scalar R_u and the symmetric 4×4 matrix R'_u by

$$R_u = 1 + \alpha\sigma_u^2 h n^t Q n, \tag{A24}$$

and

$$R'_u = \alpha\sigma_u^2 h R_u^{-1} Q_u^t Q_u, \tag{A25}$$

respectively. Third, the symmetric 3×3 matrix Q' by

$$Q' = -\frac{1-e^{-rh}}{h}\left[\begin{pmatrix} \frac{a_e}{B} \\ \frac{A_s}{B} \\ \frac{A_{\overline{e}}}{B} \end{pmatrix}(a_e, a_s, -a_{\overline{e}}) + \begin{pmatrix} a_e \\ a_s \\ -a_{\overline{e}} \end{pmatrix}\left(\frac{a_e}{B}, \frac{A_s}{B}, \frac{A_{\overline{e}}}{B}\right)\right] + N^t Q N - (R'_u)_{\{1,2,3\}}. \tag{A26}$$

Fourth, the scalar R and the symmetric 3×3 matrix R' by

$$R = 1 + \alpha\sigma_e^2 h Q'_{1,1}, \tag{A27}$$

and

$$R' = \alpha\sigma_e^2 h R^{-1} Q'_{\{1,2,3\},\{1\}} Q'_{\{1\},\{1,2,3\}}, \tag{A28}$$

respectively. Finally, the symmetric 3×3 matrix P by

$$P = (Q' - R' - \alpha\sigma^2 h \Gamma)e^{-rh}, \tag{A29}$$

where Γ is as in equation (A10).

Proposition 2 *The market order in equation (4) solves* (P) *and the function (20) is the value function, if the following conditions hold. First, the optimality conditions*

$$-\frac{1-e^{-rh}}{h}\left(\frac{2a_e}{B}, \frac{A_s + a_s}{B}, \frac{A_{\overline{e}} - a_{\overline{e}}}{B}\right) + \hat{n}^t Q N - (R'_u)_{\{4\},\{1,2,3\}} = 0 \tag{A30}$$

and

$$-\frac{1-e^{-rh}}{h}\frac{2}{B} + \hat{n}^t Q \hat{n} - (R'_u)_{4,4} < 0. \tag{A31}$$

Second, the Bellman conditions

$$Q = P \tag{A32}$$

and

$$q = \frac{log(R_u R)e^{-rh}}{2\alpha(1-e^{-rh})} + \frac{(\beta e^{-rh} - r)h}{\alpha(1-e^{-rh})} - \frac{1}{\alpha}log(\frac{h}{e^{rh}-1}). \tag{A33}$$

C. The System (S)

The system (S) consists of the following equations. First, the three optimality conditions (A11), (A12), (A13), and the six Bellman conditions (A14) of the market makers' optimization problem. Second, the three optimality conditions (A30) and the six Bellman conditions (A32) of the large trader's problem. Finally, the two equations (14) and (16) of the market makers' recursive filtering problem. Having solved this system, we should check that equations $\overline{Q}_{1,1} < 0$ and (A31) are satisfied. To study the small order flow case, we use $\overline{\sigma}_u^2$ and $\overline{\Sigma}_e^2$ instead of σ_u^2 and Σ_e^2. We also replace equations (16) and (14) by (23) and (24).

D. Proofs of Propositions 4 and 7

Proof of Proposition 4: We first show that the dynamics are the sum over endowment and noise shocks of the dynamics generated by each shock. We then determine the dynamics generated by an endowment shock.

We define the vectors $v_{\ell-1}$ and b by

$$v_{\ell-1} = \begin{pmatrix} e_{\ell-1} + \epsilon_\ell - s_{\ell-1} \\ s_{\ell-1} \\ \overline{e}_{\ell-1} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{A34}$$

Equation (A21) implies that in equilibrium

$$v_\ell = N v_{\ell-1} + n u_\ell + b \epsilon_{\ell+1}.$$

Iterating this equation between periods m_0 and m', we get

$$v_{m'} = N^{m'-m_0} v_{m_0} + \sum_{m=m_0+1}^{m'} N^{m'-m} n u_m + \sum_{m=m_0+1}^{m'} N^{m'-m} b \epsilon_{m+1}. \tag{A35}$$

The first term in equation (A35) corresponds to the dynamics generated by the endowment and noise shocks prior to period m_0. The second and third terms correspond to the dynamics generated by the shocks between periods m_0 and m'.

To isolate the dynamics generated by a unit endowment shock in period ℓ, we use equation (A35) and set $m' = \ell' - 1$, $v_{m_0} = 0$, $u_m = 0$ for all m, $\epsilon_m = 0$ for all $m \neq \ell$, and $\epsilon_\ell = 1$. We get

$$v_{\ell'-1} = N^{\ell'-\ell} b. \tag{A36}$$

To determine $v_{\ell'-1}$, we need to determine the eigenvalues and eigenvectors of the matrix N, and write b as a linear combination of the eigenvectors. The eigenvalues of N are 1, $1-a_s-a_{\overline{e}}$, and $1-a_e(1+g)$, and the corresponding eigenvectors are

$$w_1 = \begin{pmatrix} 0 \\ \frac{a_{\overline{e}}}{a_s+a_{\overline{e}}} \\ \frac{a_s}{a_s+a_{\overline{e}}} \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad \text{and} \qquad w_3 = \begin{pmatrix} 1 \\ \frac{a_{\overline{e}}-ga_e}{a_e(1+g)-a_s-a_{\overline{e}}} \\ \frac{a_s-a_e}{a_e(1+g)-a_s-a_{\overline{e}}} \end{pmatrix}.$$

We can write b as

$$b = w_1 + \frac{a_e(ga_s-a_{\overline{e}})}{(a_e(1+g)-a_s-a_{\overline{e}})(a_s+a_{\overline{e}})} w_2 + w_3.$$

Equation (A36) implies that

$$v_{\ell'-1} = w_1 + (1-a_s-a_{\overline{e}})^{\ell'-\ell} \frac{a_e(ga_s-a_{\overline{e}})}{(a_e(1+g)-a_s-a_{\overline{e}})(a_s+a_{\overline{e}})} w_2 + (1-a_e(1+g))^{\ell'-\ell} w_3. \quad \text{(A37)}$$

Equation (30) follows from equation (A37), by noting that $e_{\ell'-1}$ is the sum of the first and second components of $v_{\ell'-1}$. The first component of $v_{\ell'-1}$ is $e_{\ell'-1}-s_{\ell'-1}$ and goes to 0. Therefore, $s_{\ell'-1}$ has the same limit as $e_{\ell'-1}$. Q.E.D.

Proof of Proposition 7: We use equation (30) of Proposition 4. We set $\ell'-\ell = (t'-t)/h$, and replace a_e, a_s, $a_{\overline{e}}$, and g, by the Taylor expansions of Proposition 6. Noting that

$$\lim_{h\to 0}(1-\phi_s h-\phi_{\overline{e}} h)^{\frac{t'-t}{h}} = e^{-(\phi_s+\phi_{\overline{e}})(t'-t)},$$

and

$$\lim_{h\to 0}\left(1-\phi_e(1+g_0)\sqrt{h}\right)^{\frac{t'-t}{h}} = 0,$$

we get equation (31). Q.E.D.

References

Admati, Anat R., 1991, The informational role of prices: A review essay, *Journal of Monetary Economics* 28, 347-361.

Admati, Anat R., and Paul Pfleiderer, 1988, A theory of intraday patterns: Volume and price variability, *Review of Financial Studies* 1, 3-40.

Almgren, Robert, and Neil Chriss, 1999, Optimal liquidation, Working paper, University of Chicago.

Back, Kerry, 1992, Insider trading in continuous time, *Review of Financial Studies* 5, 387-410.

Back, Kerry, H. Henry Cao, and Gregory A. Willard, 2000, Imperfect competition among informed traders, *Journal of Finance* forthcoming.

Baruch, Schmuel, 1999, Insider trading and risk aversion, Working paper, University of Utah.

Bertsimas, Dimitris, and Andrew W. Lo, 1998, Optimal control of execution costs, *Journal of Financial Markets* 1, 1-50.

Cao, H. Henry, and Richard K. Lyons, 1999, Inventory information, Working paper, UC Berkeley.

Chan, Louis K.C., and Josef Lakonishok, 1993, Institutional trades and intraday stock price behavior, *Journal of Financial Economics* 33, 173-199.

Chan, Louis K.C., and Josef Lakonishok, 1995, The behavior of stock prices around institutional trades, *Journal of Finance* 50, 1147-1174.

Chau, Minh, 2000, Dynamic trading and market making with inventory costs and private information, Working paper, ESSEC.

Chevalier, Judith, and Glenn Ellison, 1999, Are some mutual fund managers better than others? Cross-sectional patterns in behavior and performance, *Journal of Finance* 54, 875-899.

Fama, Eugene F., 1991, Efficient capital markets: II, *Journal of Finance* 46, 1575-1617.

Foster, F. Douglas, and S. Vishwanathan, 1990, A theory of interday variations in volume, variance, and trading costs in securities markets, *Review of Financial Studies* 3, 593-624.

Foster, F. Douglas, and S. Vishwanathan, 1996, Strategic trading when agents forecast the forecasts of others, *Journal of Finance* 51, 1437-1478.

Gennotte, Gerard, and Albert Kyle, 1993, Intertemporal insider trading with a smooth order flow, Working paper, Duke University.

Hausman, Jerry A., Andrew W. Lo, and A. Craig McKinlay, 1992, An ordered probit analysis of transaction stock prices, *Journal of Financial Economics* 31, 319-379.

Holden, Craig W., and Avanidhar Subrahmanyam, 1992, Long-lived private information and imperfect competition, *Journal of Finance* 47, 247-270.

Holden, Craig W., and Avanidhar Subrahmanyam, 1994, Risk aversion, imperfect competition and long-lived information, *Economic Letters* 44, 181-190.

Holthausen, Robert, Richard Leftwich, and David Mayers, 1987, The effect of large block transactions on security prices: A cross-sectional analysis, *Journal of Financial Economics* 19, 237-267.

Holthausen, Robert, Richard Leftwich, and David Mayers, 1990, Large block transactions, the speed of response, and temporary and permanent stock-price effects, *Journal of Financial Economics* 26, 71-95.

Keim, Donald B., and Ananth Madhavan, 1996, The upstairs market for large-block transactions: Analysis and measurement of price effects, *Review of Financial Studies* 9, 1-36.

Kraus, Alan, and Hans R. Stoll, 1972, Price impacts of block trading on the New York Stock Exchange, *Journal of Finance* 27, 569-588.

Kyle, Albert, 1985, Continuous auctions and insider trading, *Econometrica* 53, 1315-1336.

Merton, Robert C., 1969, Lifetime portfolio selection under uncertainty: The continuous-time case, *Review of Economics and Statistics* 51, 247-257.

O'Hara, Maureen, 1995, *Market Microstructure Theory* (Blackwell Publishers, Cambridge, MA.)

Schwartz, Robert A., and James E. Shapiro, 1992, The challenge of institutionalization for the equity markets, in Anthony Saunders, ed.: *Recent Developments in Finance* (New York University Salomon Center, New York, NY, and Business One Irwin, Homewood, IL.)

Scholes, Myron, 1972, The market for securities: Substitution versus price pressure and the effects of information on share prices, *Journal of Business* 45, 179-211.

Seppi, Duane J., 1990, Equilibrium block trading and asymmetric information, *Journal of Finance* 45, 73-94.

Vayanos, Dimitri, 1999, Strategic trading and welfare in a dynamic market, *Review of Economic Studies* 66, 219-254.

Wang, Jiang, 1994, A model of competitive stock trading volume, *Journal of Political Economy* 102, 127-168.

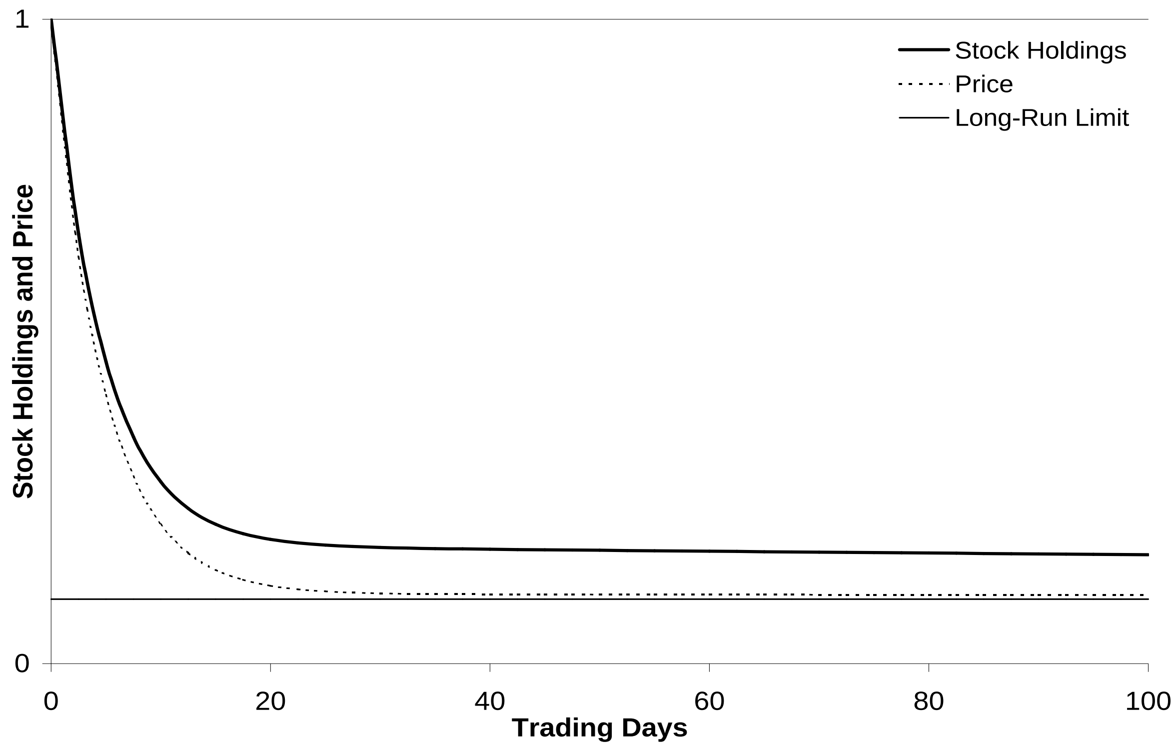

Figure 1. The first convergence pattern. The thick solid line represents the large trader's stock holdings, the dotted line the price, and the thin solid line the long-run limit of stock holdings and price. (Price is normalized so that it has the same initial value and long-run limit as stock holdings.) The large trader's stock holdings decrease over time. The trading rate, which is the ratio of the slope of stock holdings to the difference between stock holdings and the long-run limit, decreases over time. The price impact, which is the ratio of the slope of price to the slope of stock holdings, also decreases over time. This figure is drawn for the following parameter values. The parameter $\overline{\sigma}_u^2 = \sigma_u^2/\sigma_e^2$ is equal to 2, which means that the small traders produce twice as much order flow as the large trader. The parameter $\overline{\alpha}/\alpha$ is equal to 1/9, which means that the large trader is nine times as risk-averse as the market makers as a group. The annual interest rate r, is equal to 5 percent. The time between trading periods h, is equal to 1/5000'th of a year. Since there are approximately 250 trading days per year, our choice of h implies 20 trading periods per day. Finally, the parameter $\overline{\alpha}\sigma\sigma_e/r$ is equal to 0.1. This parameter determines the ratio of price risk (price variability due to trades) to dividend risk (price variability due to dividends). Our choice of $\overline{\alpha}\sigma\sigma_e/r$ implies a ratio of approximately 10 percent.

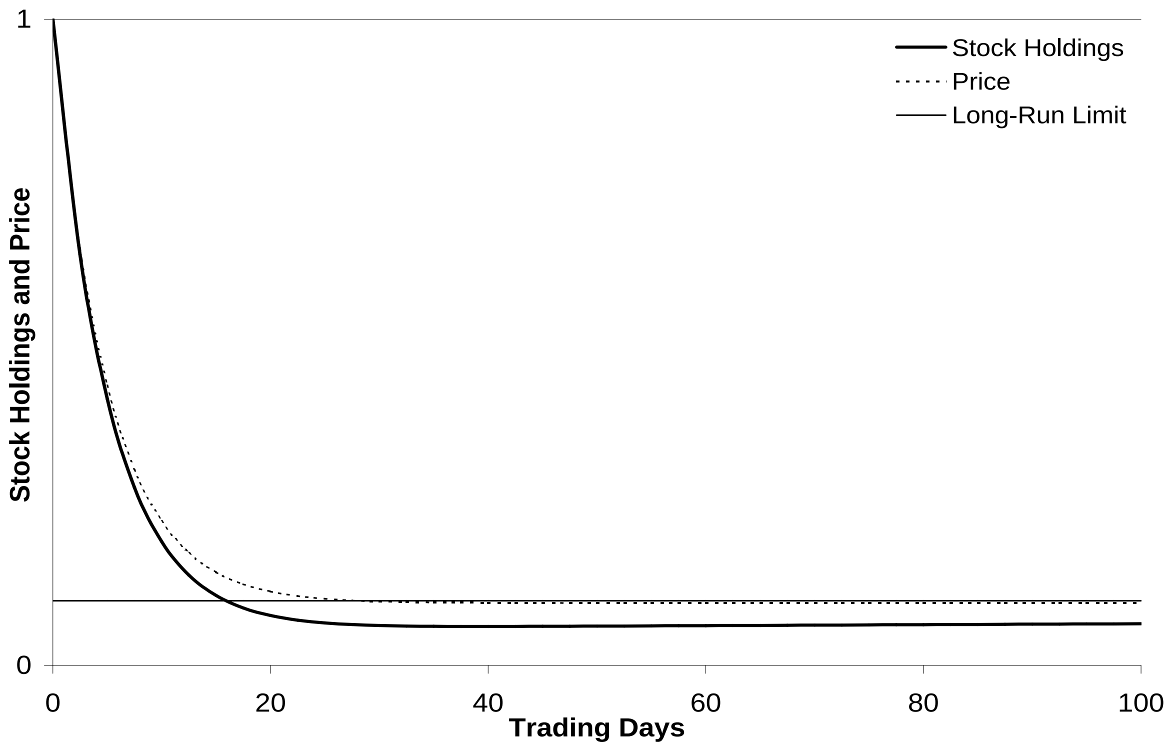

Figure 2. The second convergence pattern. The thick solid line represents the large trader's stock holdings, the dotted line the price, and the thin solid line the long-run limit of stock holdings and price. The large trader's stock holdings decrease and then increase over time. First, they decrease to their long-run limit. Then, they decrease further, and increase back to their long-run limit, as the large trader engages in a "round-trip transaction". Notice that the round-trip transaction is profitable, since the average price of sales exceeds the average price of purchases. The parameter values for this figure are $\overline{\sigma}_u^2 = 8$ (the small traders produce eight times as much order flow as the large trader), $\overline{\alpha}/\alpha = 1/9$ (the large trader is nine times as risk-averse as the market makers as a group), $r = 5$ percent, $h = 1/5000$'th of a year (20 trading periods per day), and $\overline{\alpha}\sigma\sigma_e/r = 0.1$.

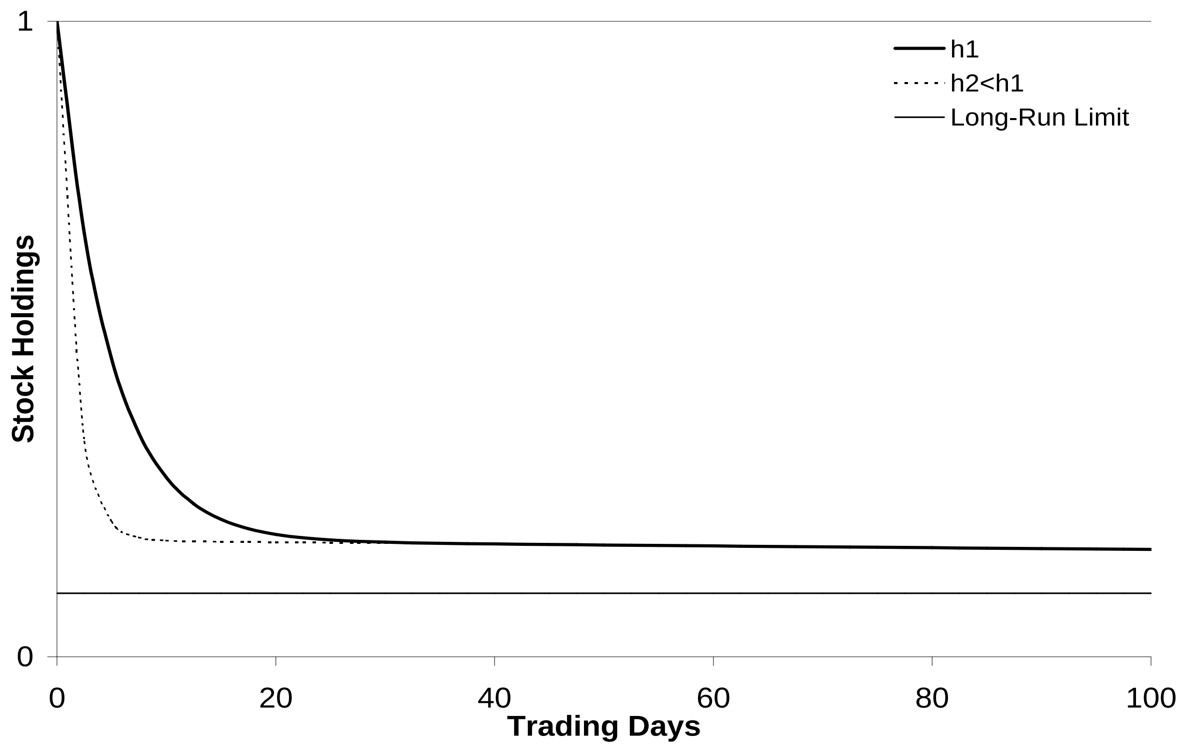

Figure 3. The dynamics of the large trader's stock holdings for small time between trades h. The thick solid line represents stock holdings for h_1, the dotted line stock holdings for $h_2 < h_1$, and the thin solid line the long-run limit of stock holdings. The dynamics consist of two phases. In the first, short phase, stock holdings decrease quickly, while in the second, long phase, they evolve more slowly. Notice that the first phase is shorter for h_2 than for h_1. The parameter values for this figure are $\overline{\sigma}_u^2 = 2$ (the small traders produce twice as much order flow as the large trader), $\overline{\alpha}/\alpha = 1/9$ (the large trader is nine times as risk-averse as the market makers as a group), $r = 5$ percent, $h_1 = 1/5000$ and $h_2 = 1/50000$'th of a year (20 or 200 trading periods per day), and $\overline{\alpha}\sigma\sigma_e/r = 0.1$.

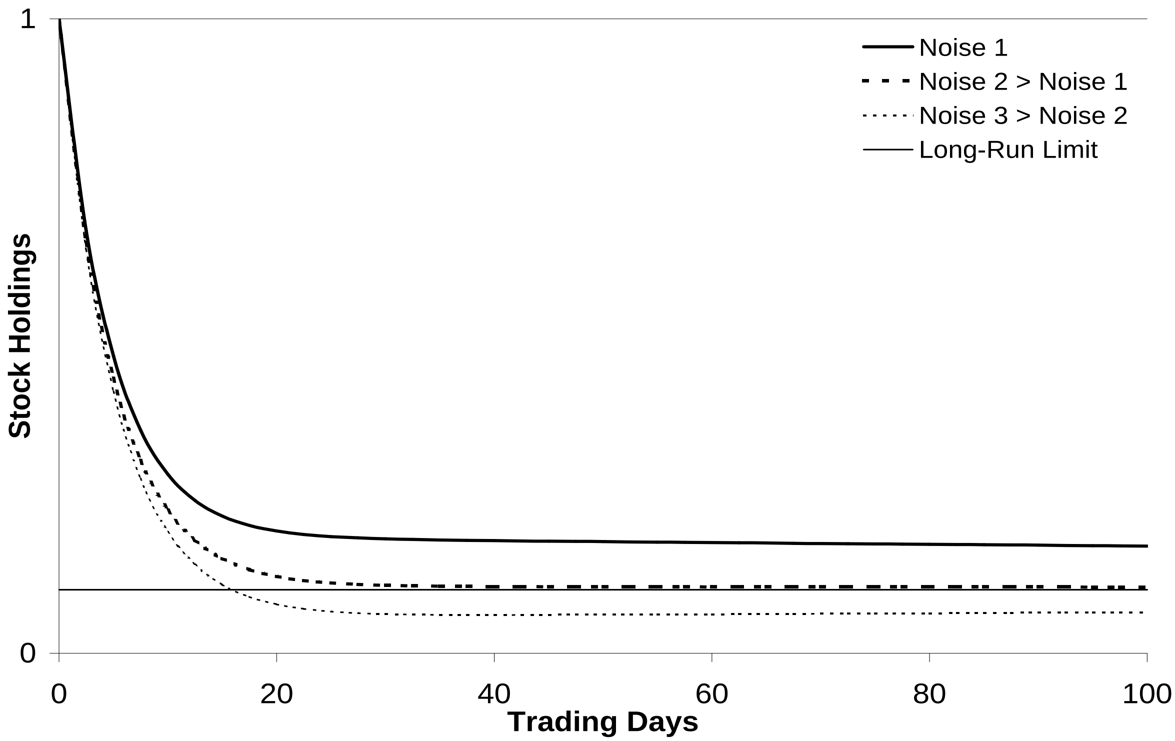

Figure 4. The effects of the noise $\overline{\sigma}_u^2$, on the dynamics of the large trader's stock holdings. The thick solid line represents stock holdings for $(\overline{\sigma}_u^2)_1$, the thick dotted line stock holdings for $(\overline{\sigma}_u^2)_2 > (\overline{\sigma}_u^2)_1$, the thin dotted line stock holdings for $(\overline{\sigma}_u^2)_3 > (\overline{\sigma}_u^2)_2$, and the thin solid line the long-run limit of stock holdings. As the noise increases, the initial drop in stock holdings increases, and the large trader sells faster. For $(\overline{\sigma}_u^2)_3$, stock holdings drop below their long-run limit, and thus decrease and then increase over time. The parameter values for this figure are $(\overline{\sigma}_u^2)_1 = 2$, $(\overline{\sigma}_u^2)_2 = 4$, and $(\overline{\sigma}_u^2)_3 = 8$ (the small traders produce two, four, or eight times as much order flow as the large trader), $\overline{\alpha}/\alpha = 1/9$ (the large trader is nine times as risk-averse as the market makers as a group), $r = 5$ percent, $h = 1/5000$'th of a year (20 trading periods per day), and $\overline{\alpha}\sigma\sigma_e/r = 0.1$.

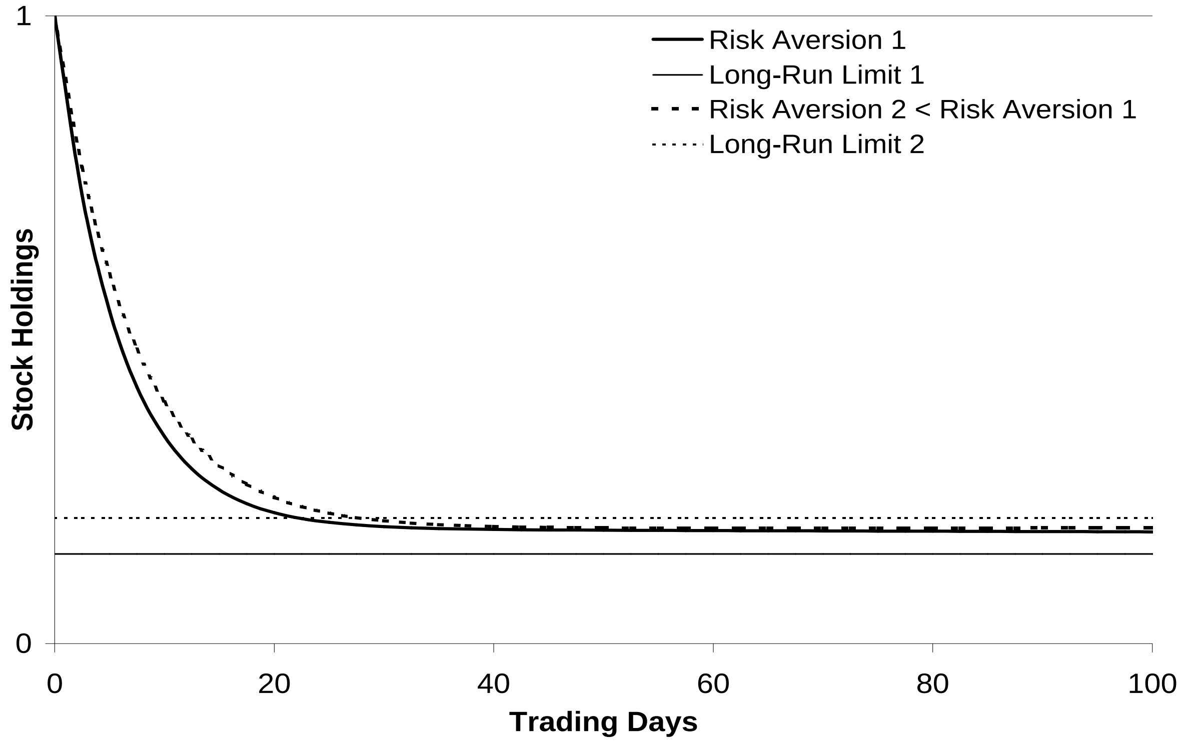

Figure 5. The effects of the large trader's risk aversion α, on the dynamics of the large trader's stock holdings. The thick solid line represents stock holdings for α_1, the thick dotted line stock holdings for $\alpha_2 < \alpha_1$, the thin solid line the long-run limit of stock holdings for α_1, and the thin dotted line the long-run limit for α_2. As α decreases, the long-run limit of stock holdings increases, while the initial drop stays roughly the same. For α_2, stock holdings drop below their long-run limit, and thus decrease and then increase over time. Since for α_2 stock holdings drop below their long-run limit, the large trader sells faster for α_2 than for α_1. The parameter values for this figure are $\overline{\sigma}_u^2 = 2$ (the small traders produce twice as much order flow as the large trader), $\overline{\alpha}/\alpha_1 = 1/6$ and $\overline{\alpha}/\alpha_2 = 1/4$ (the large trader is six or four times as risk-averse as the market makers as a group), $r = 5$ percent, $h = 1/5000$'th of a year (20 trading periods per day), and $\overline{\alpha}\sigma\sigma_e/r = 0.1$.

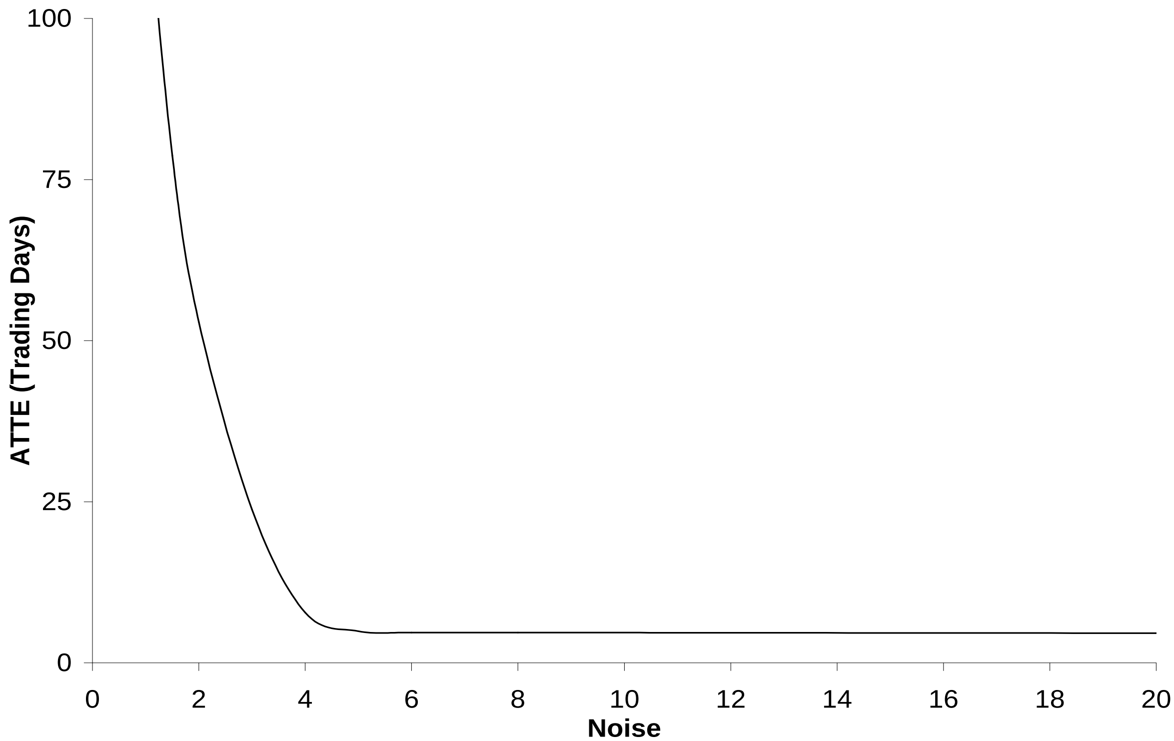

Figure 6. The average time of trade execution as a function of the noise. The parameter values for this figure are $\overline{\alpha}/\alpha = 1/9$ (the large trader is nine times as risk-averse as the market makers as a group), $r = 5$ percent, $h = 1/5000$'th of a year (20 trading periods per day), and $\overline{\alpha}\sigma\sigma_e/r = 0.1$.

Notes

[1]Schwartz and Shapiro (1992) estimate that in 1989 about 70 percent of the trading volume in the NYSE was accounted for by member firms and institutional investors.

[2]Holthausen, Leftwich, and Mayers (1990) examine the 50 largest buy and sell trades for 109 randomly selected NYSE firms in 1983. They find a price impact of around 1 percent. Keim and Madhavan (1996) examine block trades for small NYSE, AMEX, and Nasdaq firms, from 1985 to 1992, and find a much larger price impact of around 8 percent. See also Kraus and Stoll (1972), Scholes (1972), Holthausen, Leftwich, and Mayers (1987), Hausman, Lo, and McKinlay (1992), and Chan and Lakonishok (1993).

[3]Chan and Lakonishok (1995) examine the trades of 37 large investment management firms from 1986 to 1988. They find that individual trades are generally part of a sequence or, in their terminology, a "package", and that 53 percent of packages take four or more days to be completed.

[4]Kyle's paper has been extended in a number of directions. Holden and Subrahmanyam (1992) assume many insiders who have the same information. Foster and Vishwanathan (1996) and Back, Cao, and Willard (2000) assume many insiders who have different pieces of information. Back (1992) assumes general distributions for asset payoffs. Gennotte and Kyle (1993) assume an infinite horizon. Holden and Subrahmanyam (1994) and Baruch (1999) assume risk-averse insiders. See also the survey by Admati (1991) and the book by O'Hara (1995).

[5]For evidence on the performance of mutual and pension funds, see the survey by Fama (1991). For evidence on turnover, see, for instance, Chevalier and Ellison (1999). They report that the growth, and growth and income funds in the 1994 Morningstar CD turn over 76.8 percent of their portfolios every year, but underperform the market by 0.5 percent.

[6]We choose risk-sharing over portfolio rebalancing and liquidity because it is easier to model. Portfolio rebalancing requires wealth effects and a different utility function than the exponential. Liquidity requires borrowing constraints.

[7]The stock endowment can be interpreted as a futures position or as a position in a correlated stock.

[8]The assumption that market makers are risk-averse was standard in the early, "inventory-based", market-microstructure literature. For a survey of that literature and for more recent references, see O'Hara (1995).

[9]We explain why the results are different in Section IV.B.

[10]The time between trades h, is smaller than a day. Therefore, a criticism of the utility in equation (2) is that it connects intraday trading decisions and consumption/savings decisions. To address this criticism, we could assume that consumption takes place once every given number of periods. (Traders are evaluated every quarter.) This assumption would complicate the model but would probably not change the results.

[11]We introduce the consumption good endowment for tractability. In our model, the large trader faces "fundamental" risk (the risk of a negative dividend shock), "price" risk (the risk of selling at the same time as the small traders), and "endowment" risk (the risk of receiving a small stock endowment). The latter risk depends on the large trader's marginal valuation of a share. Therefore, it increases in the dividend level and decreases in the large trader's stock holdings. The large trader can hedge this risk by increasing his stock holdings, i.e. buying shares from the market makers. Moreover, he needs to buy more shares when dividends are high. The consumption good endowment makes the hedging demand independent of the dividend level, thus simplifying the model. Indeed, the consumption good endowment represents a cost that the large trader pays in order to receive the stock endowment. The risk of receiving a small stock endowment is accompanied by the risk of paying a small cost. The specification of the consumption good endowment ensures that the sum of the two risks is independent of the dividend level. The model without the consumption good endowment was considered in an earlier version and produces similar results. Moreover, the model with the consumption good endowment is, in a sense, more realistic, since it captures the idea that the stock endowment does not come for "free". If, for instance, the stock endowment is interpreted as a position in a correlated stock, the consumption good endowment captures the cost of establishing the position.

[12]Notice that the large trader is treated as an "atom", while a market maker is treated as part of a continuum. It is in this sense that the large trader is large, and thus strategic, while a market maker is small (infinitesimal), and thus competitive. Notice also that a continuum of market makers is equivalent to a single competitive market maker. We assume

a continuum because this clarifies the exposition.

[13]In this paper we determine one stationary Nash equilibrium, and thus show existence of Nash equilibrium. We do not show uniqueness.

[14]The large trader's sell order depends on the market makers' expectation $s_{\ell-1}$, and the market makers' stock holdings $\overline{e}_{\ell-1}$. The large trader does not observe these variables directly. However, he can infer them from the prices up to period $\ell - 1$. Indeed, equations (7), (10), and (15), imply that s_ℓ and $\overline{e}_\ell$ can be obtained recursively from $s_{\ell-1}$, $\overline{e}_{\ell-1}$, d_ℓ, and p_ℓ.

[15] We thus assume that the variance of $e_{\ell-1}$ conditional on $\mathcal{I}_{\ell-1}$ is independent of ℓ. This is a condition for stationarity and is satisfied if and only if Σ_e^2 is equal to the unique positive root of equation

$$\Sigma_e^2 = \frac{(1-a_e)^2(\Sigma_e^2 + \sigma_e^2 h)\sigma_u^2 h}{a_e^2(\Sigma_e^2 + \sigma_e^2 h) + \sigma_u^2 h}. \tag{14}$$

Indeed, the regression of e_ℓ on $x_\ell + u_\ell$ implies that the variance of e_ℓ conditional on $\mathcal{I}_\ell$ is equal to the RHS of equation (14). Notice that in our stationary equilibrium, Σ_e^2 is endogenous since it is given by equation (14). The endogeneity of Σ_e^2 raises an important issue. Suppose that an exogenous parameter, say σ_u^2, changes from $(\sigma_u^2)_1$ to $(\sigma_u^2)_2$. Then Σ_e^2 must also change, say from $(\Sigma_e^2)_1$ to $(\Sigma_e^2)_2$, since in our stationary equilibrium it is endogenous. Suppose now that the economy starts at a finite time instead of $-\infty$. Then the initial value of Σ_e^2 is an exogenous parameter. Moreover, a ceteris paribus comparison requires holding this initial value constant and equal to $(\Sigma_e^2)_1$ when σ_u^2 changes to $(\sigma_u^2)_2$. In this case, however, we obtain a non-stationary equilibrium and not the stationary equilibrium for $(\sigma_u^2)_2$. Focusing on the stationary equilibrium is valid, if it is the limit of the non-stationary equilibrium when ℓ goes to ∞. Indeed, in the limit, Σ_e^2 is endogenous and equal to $(\Sigma_e^2)_2$, and is independent of the initial value $(\Sigma_e^2)_1$. We studied the non-stationary equilibrium assuming that the initial value $(\Sigma_e^2)_1$ is close to the stationary equilibrium value $(\Sigma_e^2)_2$ and showed (numerically) that convergence indeed obtains. This "local convergence" result ensures that comparing stationary equilibria is valid, when the change in σ_u^2 is small. Indeed, the stationary equilibrium value of Σ_e^2 is continuous in σ_u^2, and thus $(\Sigma_e^2)_1$ is close to $(\Sigma_e^2)_2$ when $(\sigma_u^2)_1$ is close to $(\sigma_u^2)_2$. By starting the economy at $-\infty$ we side-step the above issue. We however implicitly assume that the economy converges to our stationary equilibrium.

[16]This is shown formally in the proof of Proposition 4.

[17]The risk-sharing rule is different in the presence of price risk. The large trader, who is subject to endowment shocks, is more reluctant to take risk than the market makers.

[18]When stock holdings decrease and then increase over time, the trading rate and the price impact have a complicated behavior, because their denominators can become 0.

[19]The parameter values for which this and the subsequent figures are drawn are in the figures' legends. For now, we focus on the qualitative aspects of the figures. We discuss the parameter values and the axis markings in Section VII where we calibrate the model.

[20]Holden and Subrahmanyam (1994) and Baruch (1999) have shown that risk aversion makes the insider impatient. The insider prefers to establish a position early, before the noise traders introduce too much price volatility.

[21]It is worth noting that for a fixed h (i.e. not in the limit when h goes to 0) the large trader reveals his information more slowly when he is less risk-averse. This can be seen from Figure 5, where the first phase of the dynamics is longer when the large trader is less risk-averse. However, in the limit when h goes to 0, the large trader reveals his information immediately, independently of his risk aversion.

[22]Gennotte and Kyle (1993) and Chau (2000) study infinite-horizon models with insiders. However, Gennotte and Kyle do not report any results on the speed of information revelation, and Chau does not consider the continuous-time case.

[23]The proof is available from the author upon request. To show that $\phi_s + \phi_{\overline{e}}$ increases in α and decreases in $\overline{\alpha}$, we need to make the plausible assumption that $\overline{\alpha} < 2\alpha$, i.e. the market makers as a group are less than twice as risk-averse as the large trader.

[24]We focus on the effects of the noise, because it is the parameter that affects the ATTE the most.

[25]It is worth emphasizing that h affects the speed of trade only because it affects the length of the first phase. (This length is of order $\sqrt{h}$.) Indeed, when h decreases, the number of trading periods increases, but the quantity traded per period decreases. In the first phase the first effect dominates, because the number of trading periods in a given time interval is of order $1/h$, while the quantity traded per period is of order $\sqrt{h}$. In the second

phase, however, the two effects cancel, because the quantity traded per period is of order h.

[26]The transversality condition (A6) is standard for optimal consumption-investment problems. See, for instance, Merton (1969) and Wang (1994).

Appendix B

A. Proof of Proposition 1

We first show that the function (18) solves the Bellman equation

$$\overline{V}(\overline{M}_{\ell-1}, d_{\ell-1}, \Delta\overline{e}_{\ell-1}, \overline{e}_{\ell-1}, s_{\ell-1}) =$$

$$\sup_{\overline{c}_{\ell-1}, \overline{x}_{\ell}(p_\ell)} \left\{ -exp(-\overline{\alpha}\,\overline{c}_{\ell-1})h + \overline{E}_{\ell-1}\overline{V}(\overline{M}_{\ell}, d_\ell, \Delta\overline{e}_{\ell}, \overline{e}_{\ell}, s_\ell) exp(-\overline{\beta}h) \right\}, \tag{B1}$$

for the demand

$$\overline{x}_\ell(p_\ell) = B\left(\frac{h}{1-e^{-rh}} d_\ell - p_\ell\right) - \overline{A}_{\overline{e}}\overline{e}_{\ell-1} - \overline{A}_s s_{\ell-1} - \Delta\overline{e}_{\ell-1} \tag{B2}$$

and the optimal consumption. We then show that the demand in equation (B2) and the optimal consumption satisfy the transversality condition (A6). These results will imply that the demand in equation (B2) solves $(\overline{P})$ and the function (18) is the value function. The demand in equation (B2) is equal to the demand in equation (5) minus $\Delta\overline{e}_{\ell-1}$, and produces the trade $x_\ell + u_\ell - \Delta\overline{e}_{\ell-1}$. The two demands are equal along the optimal path. Indeed, equation (9) implies that if $\overline{x}_\ell(p_\ell) = x_\ell + u_\ell - \Delta\overline{e}_{\ell-1}$ then $\Delta\overline{e}_\ell = 0$.

Bellman Equation

We proceed in 3 steps. First, we show that the optimality conditions are sufficient for the demand in equation (B2) to maximize the RHS of the Bellman equation (B1). (This is why we refer to these conditions as "optimality conditions".) Second, we compute the expectation of the RHS conditional on period $\ell - 1$ information. Finally, we show that the Bellman conditions are sufficient for the function (18) to satisfy the Bellman equation. (This is why we refer to these conditions as "Bellman conditions".)

Step 1: Optimal Demand

We define the vector $\overline{v}_{\ell-1}$ by

$$\overline{v}_{\ell-1} = \begin{pmatrix} \Delta\overline{e}_{\ell-1} \\ \overline{e}_{\ell-1} \\ s_{\ell-1} \end{pmatrix}.$$

A market maker chooses his demand $\overline{x}_\ell(p_\ell)$ to maximize the expectation of the period ℓ value function w.r.t. ζ_ℓ. Using the budget constraint (A3) and equation (A5), we can write this expectation as

$$-E_{\zeta_\ell} exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(\overline{M}_{\ell-1}+d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})h-\overline{c}_{\ell-1}h)-\frac{1-e^{-rh}}{h}p_\ell\overline{x}_\ell(p_\ell)$$

$$+d_\ell(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1}+\overline{x}_\ell(p_\ell))+F(\overline{Q},\overline{N}(\overline{v}_{\ell-1}+\begin{pmatrix}\overline{x}_\ell(p_\ell)-(\zeta_\ell+a_s s_{\ell-1}-a_{\overline{e}}\overline{e}_{\ell-1})\\0\\0\end{pmatrix})+\overline{n}\zeta_\ell)+\overline{q})), \tag{B3}$$

where p_ℓ is given by equation (A4). The market maker can condition his trade $\overline{x}_\ell$ on ζ_ℓ, since he can infer ζ_ℓ from the price. Therefore, his problem is the same as choosing a trade $\overline{x}_\ell$ to maximize equation (B3) without the expectation sign. The first-order condition is

$$-\frac{1-e^{-rh}}{h}p_\ell+d_\ell+(1,0,0)\overline{Q}(\overline{N}(\overline{v}_{\ell-1}+\begin{pmatrix}\overline{x}_\ell-(\zeta_\ell+a_s s_{\ell-1}-a_{\overline{e}}\overline{e}_{\ell-1})\\0\\0\end{pmatrix})+\overline{n}\zeta_\ell)=0. \tag{B4}$$

The first-order condition determines a maximum since $\overline{Q}_{1,1}<0$. Denoting by $\overline{G}$ the row vector formed by the LHS of equations (A11), (A12), and (A13), we can write the first-order condition as

$$\overline{G}\begin{pmatrix}\overline{e}_{\ell-1}\\s_{\ell-1}\\\zeta_\ell\end{pmatrix}+\overline{Q}_{1,1}(\overline{x}_\ell-(\zeta_\ell+a_s s_{\ell-1}-a_{\overline{e}}\overline{e}_{\ell-1}-\Delta\overline{e}_{\ell-1}))=0.$$

Therefore, the optimal trade is

$$\overline{x}_\ell=\zeta_\ell+a_s s_{\ell-1}-a_{\overline{e}}\overline{e}_{\ell-1}-\Delta\overline{e}_{\ell-1}. \tag{B5}$$

The demand in equation (B2) is optimal since it produces the optimal trade. Substituting p_ℓ from equation (A4), and using the definition of $\overline{Q}'$, we can write equation (B3), evaluated for the optimal trade, as

$$-E_{\zeta_\ell} exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(\overline{M}_{\ell-1}+d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})h-\overline{c}_{\ell-1}h)$$

$$+d_\ell(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})+F(\overline{Q}',\begin{pmatrix}\overline{v}_{\ell-1}\\\zeta_\ell\end{pmatrix})+\overline{q})). \tag{B6}$$

Step 2: Computing the Expectation

We have to compute the expectation of equation (B6) conditional on period $\ell-1$ information, i.e. w.r.t. δ_ℓ and ζ_ℓ. Computing the expectation w.r.t. δ_ℓ is straightforward. We get

$$-E_{\zeta_\ell} exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(\overline{M}_{\ell-1}-\overline{c}_{\ell-1}h)+e^{rh}d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})$$

$$-\frac{1}{2}\overline{\alpha}\sigma^2 h(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})^2+F(\overline{Q}',\begin{pmatrix}\overline{v}_{\ell-1}\\ \zeta_\ell\end{pmatrix})+\overline{q})). \tag{B7}$$

To compute the expectation w.r.t. ζ_ℓ, we use the formula

$$E(exp(-\overline{\alpha}(ax+\frac{1}{2}bx^2)))=exp(-\overline{\alpha}(-\frac{1}{2}\overline{\alpha}\Sigma^2(1+\overline{\alpha}\Sigma^2 b)^{-1}a^2+\frac{1}{2\overline{\alpha}}log(1+\overline{\alpha}\Sigma^2 b))), \tag{B8}$$

where x is normal with mean 0 and variance Σ^2, and a and b are constants. (Equation (B8) gives simply the moment generating function of the normal distribution for $b=0$. We can always assume $b=0$ by also assuming that x is normal with mean 0 and variance $\Sigma^2(1+\overline{\alpha}\Sigma^2 b)^{-1}$.)

We set $x=\zeta_\ell$, $\Sigma^2=a_e^2(\Sigma_e^2+\sigma_e^2 h)+\sigma_u^2 h$, $a=\overline{Q}'_{\{4\},\{1,2,3\}}\overline{v}_{\ell-1}$, and $b=\overline{Q}'_{4,4}$. Using the definitions of $\overline{R}$ and $\overline{R}'$, we can write equation (B7) as

$$-exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(\overline{M}_{\ell-1}-\overline{c}_{\ell-1}h)+e^{rh}d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})$$

$$-\frac{1}{2}\overline{\alpha}\sigma^2 h(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})^2+F(\overline{Q}'_{\{1,2,3\}}-\overline{R}',\overline{v}_{\ell-1})+\frac{1}{2\overline{\alpha}}log(\overline{R})+\overline{q})).$$

Finally, using the definition of $\overline{P}$, we can rewrite this equation as

$$-exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(\overline{M}_{\ell-1}-\overline{c}_{\ell-1}h)+e^{rh}d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})+e^{rh}F(\overline{P},\overline{v}_{\ell-1})+\frac{1}{2\overline{\alpha}}log(\overline{R})+\overline{q})).$$

Step 3: Bellman Equation

To compute the RHS of the Bellman equation, we have to maximize w.r.t. $\overline{c}_{\ell-1}$

$$-exp(-\overline{\alpha}\overline{c}_{\ell-1})h-exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(\overline{M}_{\ell-1}-\overline{c}_{\ell-1}h)+e^{rh}d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})$$

$$+e^{rh}F(\overline{P},\overline{v}_{\ell-1})+\frac{1}{2\overline{\alpha}}log(\overline{R})+\overline{q})-\overline{\beta}h). \tag{B9}$$

Simple calculations show that the maximum is

$$-exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}\overline{M}_{\ell-1}+d_{\ell-1}(\overline{e}_{\ell-1}+\Delta\overline{e}_{\ell-1})+F(\overline{P},\overline{v}_{\ell-1})$$

$$+\frac{1}{2\overline{\alpha}}log(\overline{R})e^{-rh}+\overline{q}e^{-rh}+\frac{(\overline{\beta}e^{-rh}-r)h}{\overline{\alpha}}-\frac{1}{\overline{\alpha}}(1-e^{-rh})log(\frac{h}{e^{rh}-1}))). \tag{B10}$$

This is equal to the period $\ell-1$ value function from equations (A14) and (A15).

Transversality Condition

It is easy to check that, by substituting the optimal $\overline{c}_{\ell-1}$ in the second term of equation (B9), we find equation (B10) times e^{-rh}. Therefore, the expectation of the period ℓ value function in period $\ell-1$, is the period $\ell-1$ value function times e^{-rh}. Recursive use of this equation implies equation (A6). Q.E.D.

B. Proof of Proposition 2

We show that the function (20) solves the Bellman equation for the market order in equation (4) and the optimal consumption. The proof that the market order in equation (4) and the optimal consumption satisfy the transversality condition (A22) is as in Section A.

We proceed in 4 steps. First, we compute the expectation of the RHS of the Bellman equation w.r.t. u_ℓ. Second, we use the optimality conditions to show that the market order in equation (4) maximizes the RHS. Third, we compute the expectation of the RHS w.r.t. the remaining information revealed in period ℓ , i.e. δ_ℓ and ϵ_ℓ. Finally, we use the Bellman conditions to show that the function (20) satisfies the Bellman equation. Notice that we take expectations w.r.t. u_ℓ before determining the optimal market order. This is because the large trader does not know u_ℓ and, unlike the market makers, cannot condition his order on price.

Step 1: Expectation w.r.t. u_ℓ

We have to compute the expectation of the period ℓ value function w.r.t. u_ℓ. Using the budget constraint (19), equations (12) and (A21), and the vector $v_{\ell-1}$ defined by equation (A34), we have to compute

$$-E_{u_\ell} exp(-\alpha(\frac{1-e^{-rh}}{h}e^{rh}(M_{\ell-1}+d_{\ell-1}e_{\ell-1}h-c_{\ell-1}h)+\frac{1-e^{-rh}}{h}p_\ell x_\ell$$

$$+d_\ell(e_{\ell-1}-x_\ell)+F(Q,Nv_{\ell-1}+nu_\ell+\hat{n}\Delta x_\ell)+q)),$$

where p_ℓ and x_ℓ are given by equations (A20) and (A19), respectively. The term inside the exponential is a quadratic function of u_ℓ. The coefficient of $u_\ell^2/2$ is n^tQn, and equations (A19) and (A20) imply that the coefficient of u_ℓ is

$$Q_u \begin{pmatrix} v_{\ell-1} \\ \Delta x_\ell \end{pmatrix}.$$

To compute the expectation, we use equation (B8) and set $x = u_\ell$, $\Sigma^2 = \sigma_u^2 h$, a the coefficient of u_ℓ, and b the coefficient of $u_\ell^2/2$. The expectation is

$$-exp(-\alpha(\frac{1-e^{-rh}}{h}e^{rh}(M_{\ell-1}+d_{\ell-1}e_{\ell-1}h-c_{\ell-1}h)+\frac{1-e^{-rh}}{h}p_\ell x_\ell+d_\ell(e_{\ell-1}-x_\ell)$$

$$+F(Q,Nv_{\ell-1}+\hat{n}\Delta x_\ell)-F(R'_u,\begin{pmatrix} v_{\ell-1} \\ \Delta x_\ell \end{pmatrix})+\frac{1}{2\alpha}log(R_u)+q)), \tag{B11}$$

where p_ℓ is evaluated for $u_\ell = 0$.

Step 2: Optimal Market Order

The large trader chooses Δx_ℓ to maximize equation (B11). Since $dp_\ell/d\Delta x_\ell = -1/B$ and $dx_\ell/d\Delta x_\ell = 1$, the first-order condition is

$$\frac{1-e^{-rh}}{h}\left(p_\ell - \frac{1}{B}x_\ell\right) - d_\ell + \hat{n}^t Q\left(Nv_{\ell-1}+\hat{n}\Delta x_\ell\right) - (R'_u)_{\{4\},\{1,2,3,4\}}\begin{pmatrix} v_{\ell-1} \\ \Delta x_\ell \end{pmatrix} = 0. \tag{B12}$$

The first-order condition determines a maximum because of equation (A31). Denoting by G the LHS of equation (A30) and by $\mathcal{G}$ the LHS of equation (A31), we can write the first-order condition as

$$Gv_{\ell-1} + \mathcal{G}\Delta x_\ell = 0.$$

Therefore, $\Delta x_\ell = 0$, i.e. the market order in equation (4) is optimal. Substituting p_ℓ and x_ℓ from equations (A20) and (A19), and using the definition of Q', we can write equation (B11), evaluated for $\Delta x_\ell = 0$, as

$$-exp(-\alpha(\frac{1-e^{-rh}}{h}e^{rh}(M_{\ell-1}+d_{\ell-1}e_{\ell-1}h-c_{\ell-1}h)+d_\ell e_{\ell-1}+F(Q',v_{\ell-1})+\frac{1}{2\alpha}log(R_u)+q)). \tag{B13}$$

Step 3: Expectation w.r.t. δ_ℓ and ϵ_ℓ

We have to compute the expectation of equation (B13) w.r.t. δ_ℓ and ϵ_ℓ. Computing the expectation w.r.t. δ_ℓ is straightforward. To compute the expectation w.r.t. ϵ_ℓ, we use equation (B8) and set $x = \epsilon_\ell$, $\Sigma^2 = \sigma_e^2 h$,

$$a = Q'_{\{1\},\{1,2,3\}}\begin{pmatrix} e_{\ell-1}-s_{\ell-1} \\ s_{\ell-1} \\ \overline{e}_{\ell-1} \end{pmatrix},$$

and $b = Q'_{1,1}$. Proceeding as in Section A, and using the definitions of R, R', and P, we get

$$-exp(-\overline{\alpha}(\frac{1-e^{-rh}}{h}e^{rh}(M_{\ell-1}-c_{\ell-1}h)+e^{rh}d_{\ell-1}e_{\ell-1}$$

$$+e^{rh}F(P,\begin{pmatrix} e_{\ell-1}-s_{\ell-1} \\ s_{\ell-1} \\ \overline{e}_{\ell-1} \end{pmatrix})+\frac{1}{2\alpha}log(R_uR)+q)).$$

Step 4: Bellman Equation

We proceed as in Section A. Q.E.D.

C. Proof of Theorem 1

We prove Theorem 1 in Sections C.1, C.2, and C.3. In Section C.1 we replace (S) by an equivalent system, (S'), which is easier to solve. In Section C.2 we show that for $\sigma_e^2 = 0$, (S') collapses to (s), and that (s) has a solution. In Section C.3 we extend the solution of (S') for small σ_e^2.

C.1. The Equivalent System

To form the system (S'), we replace the Bellman conditions (A32) of the large trader's optimization problem, by a new set of conditions, the "envelope conditions". Under both the Bellman and the envelope conditions, the matrix Q can be interpreted as a matrix of marginal benefits. The coefficient $Q_{1,2}$, for instance, is the marginal benefit of increasing $e_\ell - s_\ell$, the "first" state variable, when s_ℓ, the "second" state variable is 1 and the other state variables are 0. The Bellman conditions compute this marginal benefit under the assumption that the large trader changes his strategy in response to the change in $e_\ell - s_\ell$, while the envelope conditions assume that the large trader does not change his strategy. The Bellman and the envelope conditions are of course equivalent, when the large trader's strategy is optimal, i.e. when the optimality conditions hold. We use the envelope conditions instead of the Bellman conditions because they are much easier to solve.

To state the envelope conditions, we define the matrix N_e by

$$N_e = \begin{pmatrix} 1 & a_s(1+g) & -a_{\overline{e}}(1+g) \\ 0 & 1-a_s(1+g) & a_{\overline{e}}(1+g) \\ 0 & 0 & 1 \end{pmatrix}$$

and the matrix $\hat{Q}'$ by

$$\hat{Q}' = -\frac{1-e^{-rh}}{h}\begin{pmatrix} 0 \\ \frac{A_s-a_s}{B} \\ \frac{A_{\overline{e}}+a_{\overline{e}}}{B} \end{pmatrix}(a_e, a_s, -a_{\overline{e}}) + N_e^t Q N - (R'_u)_{\{1,2,3\}} + \begin{pmatrix} a_e \\ a_s \\ -a_{\overline{e}} \end{pmatrix}(R'_u)_{\{4\},\{1,2,3\}}. \tag{B14}$$

We also define the scalar $\hat{R}$ and the matrices $\hat{R}'$ and $\hat{P}$ by proceeding as in Section B and using $\hat{Q}'$ instead of Q'. The envelope conditions are $Q = \hat{P}$. Notice that the matrices $\hat{Q}'$ and $\hat{P}$ are not symmetric a priori. Therefore, the system $(S)'$ consists of 23 equations (since there are nine envelope conditions) and 23 unknowns (since the matrix Q is not symmetric a priori). We first show that the solution of (S') produces a symmetric matrix Q. We then show that the solution of (S') satisfies the Bellman conditions, and is thus the solution of (S).

The Matrix Q is Symmetric

We use the vector $v_{\ell-1}$ defined by equation (A34). We define the vector a by

$$a = \begin{pmatrix} a_e \\ a_s \\ -a_{\overline{e}} \end{pmatrix}. \tag{B15}$$

Finally, we denote by p_ℓ and x_ℓ those given by equations (A20) and (A19) for $u_\ell = \Delta x_\ell = 0$.

We will show that $Q = Q^t$. Using the envelope conditions and the fact that $\hat{R}'$ is symmetric, we get

$$Q - Q^t = (\hat{Q}' - (\hat{Q}')^t)e^{-rh}.$$

Using the definition of $\hat{Q}'$ and noting that $a^t v_{\ell-1} = x_\ell$, we get

$$\hat{Q}' v_{\ell-1} = -\frac{1-e^{-rh}}{h}\begin{pmatrix} 0 \\ \frac{A_s-a_s}{B} \\ \frac{A_{\overline{e}}+a_{\overline{e}}}{B} \end{pmatrix} x_\ell + N_e^t Q N v_{\ell-1} - (R'_u)_{\{1,2,3\}} v_{\ell-1} + a(R'_u)_{\{4\},\{1,2,3\}} v_{\ell-1}. \tag{B16}$$

Equations (A20) and (A19) imply that

$$-\frac{A_s-a_s}{B}s_{\ell-1} - \frac{A_{\overline{e}}+a_{\overline{e}}}{B}\overline{e}_{\ell-1} = p_\ell + \frac{x_\ell}{B} - \frac{h}{1-e^{-rh}}d_\ell.$$

Using this fact we get

$$(\hat{Q}')^t v_{\ell-1} = a\left(\frac{1-e^{-rh}}{h}\left(p_\ell + \frac{1}{B}x_\ell\right) - d_\ell\right) + N^t Q^t N_e v_{\ell-1} - (R'_u)^t_{\{1,2,3\}} v_{\ell-1} + (R'_u)_{\{1,2,3\},\{4\}} x_\ell. \tag{B17}$$

Since the first-order condition (B12) holds for $\Delta x_\ell = 0$, we have

$$\frac{1-e^{-rh}}{h}\left(p_\ell - \frac{1}{B}x_\ell\right) - d_\ell + \hat{n}^t Q N v_{\ell-1} - (R'_u)_{\{4\},\{1,2,3\}} v_{\ell-1} = 0. \tag{B18}$$

We subtract equation (B17) from equation (B16), add the transpose of equation (A30) times x_ℓ, and subtract equation (B18) times a. Noting that the matrix R'_u is symmetric, and that

$$N_e + \hat{n}a^t = N, \tag{B19}$$

we get

$$(\hat{Q}' - (\hat{Q}')^t) v_{\ell-1} = N^t (Q - Q^t) N v_{\ell-1}.$$

Since this holds for all $v_{\ell-1}$, we get

$$Q - Q^t = (\hat{Q}' - (\hat{Q}')^t) e^{-rh} = N^t (Q - Q^t) N e^{-rh}.$$

It is easy to check that this equation produces a system of three linear equations in $Q_{1,2} - Q_{2,1}$, $Q_{1,3} - Q_{3,1}$, and $Q_{2,3} - Q_{3,2}$. Moreover, the solution of this system is zero provided that $1 - a_e(1+g)$ and $1 - a_s - a_{\overline{e}} \in [0,1)$. In Section C.2 we will show that the solution of (S') indeed satisfies $1 - a_e(1+g)$ and $1 - a_s - a_{\overline{e}} \in [0,1)$.

The Bellman Conditions Hold

We only need to show that $Q' = \hat{Q}'$. We subtract equation (B14) from equation (A26), and add the vector a times equation (A30). Using equation (B19), we get $Q' = \hat{Q}'$.

C.2. The Solution for $\sigma_e^2 = 0$

We first assume that (s) has a solution a_s, a_e, g, and $\overline{\Sigma}_e^2$, such that $1-a_e(1+g)$, $1-a_s(1+g)$, and $1 - a_s - a_{\overline{e}} \in (0,1)$. (We define $a_{\overline{e}}$ by equation (25).) Starting from this solution, we construct a solution of (S'). We then show that (s) has a solution with the above properties.

The Solution of (S')

We proceed in three steps. First, we use the equations of the market makers' optimization problem to solve for $A_{\overline{e}}$, A_s, B, and $\overline{Q}$. Second, we use the envelope conditions of the large trader's problem to solve for Q. Finally, we show that the optimality conditions of the large trader's problem are satisfied.

Step 1: The Market Makers' Problem

We first use the Bellman conditions to solve for $\overline{Q}$, as a function of $A_{\overline{e}}$, A_s, and B. We then plug $\overline{Q}_{1,2}$ and $\overline{Q}_{1,3}$ into the optimality conditions, and solve for $A_{\overline{e}}$, A_s, and B.

For $\sigma_e^2 = 0$, $\overline{R}' = 0$. Therefore, the Bellman conditions (A14) become

$$\overline{Q} = (\overline{Q}'_{\{1,2,3\}} - \overline{\alpha}\sigma^2 h\Gamma)e^{-rh}.$$

The equation for $\overline{Q}_{1,1}$ is $\overline{Q}_{1,1} = -\overline{\alpha}\sigma^2 he^{-rh} < 0$. The equations for $\overline{Q}_{1,2}$ and $\overline{Q}_{1,3}$ are

$$\overline{Q}_{1,2} = -\frac{1-e^{-rh}}{h}\frac{A_{\overline{e}}}{B}e^{-rh} - \overline{\alpha}\sigma^2 he^{-rh} \tag{B20}$$

and

$$\overline{Q}_{1,3} = -\frac{1-e^{-rh}}{h}\frac{A_s}{B}e^{-rh}, \tag{B21}$$

respectively. The equations for $\overline{Q}_{2,2}$, $\overline{Q}_{2,3}$, and $\overline{Q}_{3,3}$ form a system of three linear equations. We omit the solution of this system, since we do not use it in what follows.

Plugging $\overline{Q}_{1,2}$ and $\overline{Q}_{1,3}$ into the optimality conditions (A11) and (A12), we get a system of two linear equations in $A_{\overline{e}}/B$ and A_s/B. Solving this system, we get

$$\frac{A_s}{B} = \frac{a_s\overline{\alpha}\sigma^2 h^2 e^{-rh}}{(1-e^{-rh})^2 D_1}, \tag{B22}$$

where

$$D_1 = 1 - (1 - a_s - a_{\overline{e}})e^{-rh}.$$

We omit $A_{\overline{e}}/B$ since we do not use it in what follows. To determine $1/B$, we multiply the optimality condition (A13) by a_s, and subtract it from the optimality condition (A12). Plugging $\overline{Q}_{1,3}$ in the resulting equation, we get

$$\frac{A_s - a_s}{B} = (1 - a_s(1+g))\frac{A_s}{B}e^{-rh}. \tag{B23}$$

Combining equations (B22) and (B23), we get

$$\frac{1}{B} = \frac{D_2\overline{\alpha}\sigma^2 h^2 e^{-rh}}{(1-e^{-rh})^2 D_1}, \tag{B24}$$

where

$$D_2 = 1 - (1 - a_s(1+g))e^{-rh}.$$

Step 2: The Envelope Conditions

For $\sigma_e^2 = 0$, R'_u and R' are equal to 0. Therefore, the envelope conditions become

$$Q = (\hat{Q} - \alpha\sigma^2 h\Gamma)e^{-rh} = \left(-\frac{1-e^{-rh}}{h}\begin{pmatrix} 0 \\ \frac{A_s - a_s}{B} \\ \frac{A_{\overline{e}} + a_{\overline{e}}}{B} \end{pmatrix}(a_e, a_s, -a_{\overline{e}}) + N_e^t Q N - \alpha\sigma^2 h\Gamma\right)e^{-rh}. \tag{B25}$$

Equation (B25) produces a system of nine linear equations in the elements of the matrix Q. We will "break" this system into three subsystems of three equations each. To obtain the first subsystem, we multiply equation (B25) from the left by the vector $(-1,1,0)$. We get

$$(-1,1,0)Q = (-\frac{1-e^{-rh}}{h}\frac{A_s-a_s}{B}(a_e,a_s,-a_{\overline{e}}) + (1-a_s(1+g))(-1,1,0)QN)e^{-rh}, \quad \text{(B26)}$$

i.e. a system in $Q_{2,1}-Q_{1,1}$, $Q_{2,2}-Q_{1,2}$, and $Q_{2,3}-Q_{1,3}$. The solution of this system is

$$Q_{2,1}-Q_{1,1} = -\frac{1-e^{-rh}}{h}\frac{A_s-a_s}{B}\frac{a_e(1-(1-a_s(1+g))^2e^{-rh})e^{-rh}}{D_3D_4}, \quad \text{(B27)}$$

$$Q_{2,2}-Q_{1,2} = -\frac{1-e^{-rh}}{h}\frac{A_s-a_s}{B}\frac{a_se^{-rh}}{D_3}, \quad \text{(B28)}$$

and

$$Q_{2,3}-Q_{1,3} = \frac{1-e^{-rh}}{h}\frac{A_s-a_s}{B}\frac{a_{\overline{e}}e^{-rh}}{D_3}, \quad \text{(B29)}$$

where

$$D_3 = 1-(1-a_s(1+g))(1-a_s-a_{\overline{e}})e^{-rh}$$

and

$$D_4 = 1-(1-a_s(1+g))(1-a_e(1+g))e^{-rh}.$$

To obtain the second and third subsystems, we multiply equation (B25) from the left by the vectors $(1,0,0)$ and $(0,0,1)$, respectively. The second subsystem is in $Q_{1,1}$, $Q_{1,2}$, and $Q_{1,3}$. The third subsystem is in $Q_{3,1}$, $Q_{3,2}$, and $Q_{3,3}$, and in $Q_{2,1}-Q_{1,1}$, $Q_{2,2}-Q_{1,2}$, and $Q_{2,3}-Q_{1,3}$ that we already have determined. We omit the solutions of the second and third subsystems, since we do not use them in what follows.

Step 3: The Optimality Conditions

We proceed in three steps. First, we show that the equations of (S') imply the market maker and large trader equations (26) and (27). Second, we show that the three optimality conditions (A30) are satisfied. Finally, we show that equation (A31) is satisfied.

Step 3.1: The Market Maker and Large Trader Equations

We first derive the market maker equation (26). Plugging the optimal trade of equation (B5) into the first-order condition (B4), and using equation (A5), we get

$$-\frac{1-e^{-rh}}{h}p_\ell + d_\ell + (1,0,0)\overline{Q}\begin{pmatrix}0\\ \overline{e}_\ell\\ s_\ell\end{pmatrix} = -\frac{1-e^{-rh}}{h}p_\ell + d_\ell + \left(\overline{Q}_{1,2}\overline{e}_\ell + \overline{Q}_{1,3}s_\ell\right) = 0. \quad \text{(B30)}$$

Substituting $\overline{Q}_{1,2}$ and $\overline{Q}_{1,3}$ from equations (B20) and (B21) into equation (B30), we get

$$-\frac{1-e^{-rh}}{h}p_\ell + d_\ell - \overline{\alpha}\sigma^2 h e^{-rh}\overline{e}_\ell - \frac{1-e^{-rh}}{h}\left(\frac{A_{\overline{e}}}{B}\overline{e}_\ell + \frac{A_s}{B}s_\ell\right)e^{-rh} = 0. \tag{B31}$$

Equation (A4) implies that

$$\overline{E}_\ell p_{\ell+1} = \overline{E}_\ell\left(\frac{h}{1-e^{-rh}}d_{\ell+1} - \frac{A_{\overline{e}}}{B}\overline{e}_\ell - \frac{A_s}{B}s_\ell - \frac{1}{B}\zeta_{\ell+1}\right) = \frac{h}{1-e^{-rh}}d_\ell - \frac{A_{\overline{e}}}{B}\overline{e}_\ell - \frac{A_s}{B}s_\ell. \tag{B32}$$

Combining equations (B31) and (B32), we get equation (26).

We next derive the large trader equation (27). We use the vectors $v_{\ell-1}$ and a defined by equations (A34) and (B15), respectively. We denote by p_ℓ, x_ℓ, and $(e_\ell, s_\ell, \overline{e}_\ell)$ those given by equations (A20), (A19), and (A21) for $u_\ell = \Delta x_\ell = 0$. Finally. we denote by $p_{\ell+1}$ and $x_{\ell+1}$ those given by equations (A20) and (A19) for $u_{\ell+1} = \Delta x_{\ell+1} = 0$.

For $\sigma_e^2 = 0$, $R'_u = 0$. Since the first-order condition (B12) holds for $\Delta x_\ell = 0$, we have

$$\frac{1-e^{-rh}}{h}\left(p_\ell - \frac{1}{B}x_\ell\right) - d_\ell + \hat{n}^t Q N v_{\ell-1} = 0. \tag{B33}$$

Equation (B25) implies that

$$\hat{n}^t Q N v_{\ell-1} = \hat{n}^t(-\frac{1-e^{-rh}}{h}\begin{pmatrix} 0 \\ \frac{A_s - a_s}{B} \\ \frac{A_{\overline{e}} + a_{\overline{e}}}{B}\end{pmatrix} a^t + N_e^t Q N - \alpha\sigma^2 h\Gamma)e^{-rh} N v_{\ell-1}. \tag{B34}$$

Combining equation (B33) with equation (B34), and noting that

$$a^t N v_{\ell-1} = a^t E_\ell v_\ell = E_\ell a^t v_\ell = E_\ell x_{\ell+1},$$

$$\hat{n}^t N_e^t = \hat{n}^t - (g a_s - a_{\overline{e}})(1+g)(-1,1,0),$$

and

$$\hat{n}^t \Gamma N v_{\ell-1} = -(1,1,0) N v_{\ell-1} = -e_\ell,$$

we get

$$\frac{1-e^{-rh}}{h}\left(p_\ell - \frac{1}{B}x_\ell\right) - d_\ell - \frac{1-e^{-rh}}{h}\left(g\frac{A_s - a_s}{B} + \frac{A_{\overline{e}} + a_{\overline{e}}}{B}\right)E_\ell x_{\ell+1}e^{-rh}$$
$$+\left(\hat{n}^t - (g a_s - a_{\overline{e}})(1+g)(-1,1,0)\right)Q N^2 v_{\ell-1}e^{-rh} + \alpha\sigma^2 h e_\ell e^{-rh} = 0. \tag{B35}$$

Noting that $E_\ell v_\ell = N v_{\ell-1}$, we can write the expectation in period ℓ, of equation (B33) in period $\ell+1$, as

$$\frac{1-e^{-rh}}{h}\left(E_\ell p_{\ell+1} - \frac{1}{B}E_\ell x_{\ell+1}\right) - d_\ell + \hat{n}^t Q N^2 v_{\ell-1} = 0. \tag{B36}$$

We multiply equation (B36) by e^{-rh} and subtract it from equation (B35). To simplify the resulting equation, we use two facts. The first fact is

$$\frac{1}{B} - g\frac{A_s - a_s}{B} - \frac{A_{\overline{e}} + a_{\overline{e}}}{B} = (ga_s - a_{\overline{e}})(1+g)\frac{A_s - a_s}{B}\frac{1}{1 - a_s(1+g)}.$$

To derive this fact, we multiply the optimality conditions (A11), (A12), and (A13) by -1, $-g$ and $1 + ga_s - a_{\overline{e}}$, respectively, and add them up. We then use equations (B21) and (B23). The second fact is

$$(-1,1,0)QNv_{\ell-1} = (-\frac{1-e^{-rh}}{h}\frac{A_s - a_s}{B}E_\ell x_{\ell+1} + (1 - a_s(1+g))(-1,1,0)QN^2 v_{\ell-1})e^{-rh}. \tag{B37}$$

To derive this fact, we multiply equation (B26) by $Nv_{\ell-1}$. Using these two facts, we get

$$\frac{1-e^{-rh}}{h}\left(p_\ell - \frac{1}{B}x_\ell - d_\ell h - E_\ell p_{\ell+1}e^{-rh}\right) + \alpha\sigma^2 h e_\ell e^{-rh}$$
$$-\frac{(ga_s - a_{\overline{e}})(1+g)}{1 - a_s(1+g)}(-1,1,0)QNv_{\ell-1} = 0. \tag{B38}$$

Combining equations (B37) and (B38), and noting that $E_\ell v_\ell = Nv_{\ell-1}$, we get equation (27).

Step 3.2: The Three Optimality Conditions (A30)

We will show that the three optimality conditions (A30) are equivalent to equations (21), (22), and (25), which are satisfied. To show the equivalence, we first show that

$$G(1 - Ne^{-rh})v_{\ell-1} = (\alpha e_\ell - \overline{\alpha}\overline{e}_\ell)\sigma^2 h e^{-rh} + \frac{1-e^{-rh}}{h}\left(\overline{E}_\ell p_{\ell+1} - E_\ell p_{\ell+1}\right)e^{-rh}$$
$$-\frac{1-e^{-rh}}{h}\frac{1}{B}x_\ell - \frac{(ga_s - a_{\overline{e}})(1+g)}{1 - a_s(1+g)}(-1,1,0)QNv_{\ell-1}, \tag{B39}$$

where G is the row vector formed by the LHS of the three optimality conditions (A30), $\overline{E}_\ell$ is the expectation w.r.t. the market makers' information, and E_ℓ is the expectation w.r.t. the large trader's information. In Section B we wrote the LHS of the first-order condition (B12) as $Gv_{\ell-1} + \mathcal{G}\Delta x_\ell$. Therefore, the LHS of equation (B33) is equal to $Gv_{\ell-1}$. Moreover, since $E_\ell v_\ell = Nv_{\ell-1}$, the LHS of equation (B36) is equal to $GNv_{\ell-1}$, and the LHS of equation (B38) is equal to the LHS of equation (B39). The LHS of equation (B38) is also equal to the RHS of equation (B39). This follows by substituting the price p_ℓ from equation (26) into the LHS of equation (B38).

We next evaluate the RHS of equation (B39) for three values of $v_{\ell-1}$, the column vectors of the matrix

$$\hat{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{a_{\overline{e}}}{1-e^{-rh}} \\ 0 & 0 & \frac{a_s}{1-e^{-rh}} \end{pmatrix}.$$

We divide the result by $\sigma^2 h e^{-rh}/(1-e^{-rh})$, and denote it by $\hat{G}_e$ for the first column vector, G_s for the second, and $G_{\overline{e}}$ for the third. We have

$$G(1 - Ne^{-rh})\hat{N} = \frac{\sigma^2 h e^{-rh}}{1 - e^{-rh}}(\hat{G}_e, G_s, G_{\overline{e}}).$$

Since $1 - a_e(1+g)$ and $1 - a_s - a_{\overline{e}} \in (0,1)$, the matrices $1 - Ne^{-rh}$ and $\hat{N}$ are invertible. Therefore, the three optimality conditions (A30), i.e. $G = 0$, are equivalent to $\hat{G}_e = G_s = G_{\overline{e}} = 0$. We will show that $G_s = 0$ and $G_{\overline{e}} = 0$ are equations (22) and (25), respectively. Moreover, we will show that $\hat{G}_e = k_e G_e + k_s G_s + k_{\overline{e}} G_{\overline{e}}$, where G_e is the LHS of equation (21), and $k_e \neq 0$. Therefore, the three optimality conditions (A30) will be equivalent to equations (21), (22), and (25).

We first compute $G_{\overline{e}}$. Equation (A19) implies that $x_\ell = 0$. Equation (A21) implies that

$$\begin{pmatrix} e_\ell - s_\ell \\ s_\ell \\ \overline{e}_\ell \end{pmatrix} = N v_{\ell-1} = \begin{pmatrix} 0 \\ \frac{a_{\overline{e}}}{1-e^{-rh}} \\ \frac{a_s}{1-e^{-rh}} \end{pmatrix}.$$

Equations (A20) and (B32) imply that

$$\overline{E}_\ell p_{\ell+1} - E_\ell p_{\ell+1} = \frac{1}{B} a_e (e_\ell - s_\ell) = 0. \tag{B40}$$

Plugging into equation (B38), we find that $G_{\overline{e}}$ is the LHS of equation (25).

We next compute G_s. Equation (A19) implies that $x_\ell = a_s$. Equation (A21) implies that $(e_\ell - s_\ell, s_\ell, \overline{e}_\ell) = (0, 1 - a_s, a_s)$. Equation (B40) implies that $\overline{E}_\ell p_{\ell+1} - E_\ell p_{\ell+1} = 0$. Equations (B22), (B23), (B28), and (B29), imply that

$$(1 - a_s)(Q_{2,2} - Q_{1,2}) + a_s(Q_{2,3} - Q_{1,3}) = -(1 - a_s(1+g)) \frac{a_s^2(1 - a_s - a_{\overline{e}})\overline{\alpha}\sigma^2 h e^{-3rh}}{(1 - e^{-rh}) D_1 D_3}.$$

Plugging into equation (B38), we get

$$G_s = (\alpha(1 - a_s) - \overline{\alpha} a_s)(1 - e^{-rh}) - \frac{a_s D_2 \overline{\alpha}}{D_1} + (g a_s - a_{\overline{e}})(1+g) \frac{a_s^2(1 - a_s - a_{\overline{e}})\overline{\alpha} e^{-2rh}}{D_1 D_3}.$$

It is easy to check that G_s is in fact the LHS of equation (22).

We finally compute $\hat{G}_e$. Equation (A19) implies that $x_\ell = a_e$. Equation (A21) implies that $(e_\ell - s_\ell, s_\ell, \overline{e}_\ell) = (1-a_e(1+g), ga_e, a_e)$. Equation (B40) implies that $\overline{E}_\ell p_{\ell+1} - E_\ell p_{\ell+1} = a_e(1-a_e(1+g))/B$. Plugging into equation (B38), we get

$$\hat{G}_e = (\alpha(1-a_e) - \overline{\alpha}a_e)(1-e^{-rh}) + \frac{a_e(1-a_e(1+g))D_2\overline{\alpha}e^{-rh}}{D_1} - \frac{a_e D_2 \overline{\alpha}}{D_1}$$

$$+(ga_s - a_{\overline{e}})(1+g)a_s a_e \frac{(1-a_e(1+g))D_3 + ga_s - a_{\overline{e}}}{D_1 D_3 D_4}\overline{\alpha}e^{-2rh}.$$

Setting

$$k_e = -\frac{D_2 D_6}{D_1 D_4 (1-a_s-a_{\overline{e}})},$$

$$k_s = -\frac{1}{D_1 D_4}\left[(ga_s - a_{\overline{e}})(1+g)a_e e^{-rh} + \frac{\left(\frac{\overline{\alpha}}{\alpha} - g\right) D_3 a_e (1-(1-a_e(1+g))e^{-rh})}{1-a_s-a_{\overline{e}}}\right],$$

$$k_{\overline{e}} = \frac{1}{D_1 D_4}\left[a_e(1+g)(1-e^{-rh})D_2 + \frac{a_e(1-(1-a_e(1+g))e^{-rh})}{1-a_s-a_{\overline{e}}}\hat{k}_{\overline{e}}\right],$$

$$\hat{k}_{\overline{e}} = \frac{\overline{\alpha}}{\alpha}D_5 + (1-e^{-rh})\left(\frac{\overline{\alpha}}{\alpha} - g\right) D_3 - (1-a_s-a_{\overline{e}})(1-e^{-rh})(1+g),$$

$$D_5 = 1-(1-a_s(1+g)(2-a_s-a_{\overline{e}}))e^{-rh},$$

and

$$D_6 = 1-(1-a_e(1+g))(1-a_s-a_{\overline{e}})e^{-rh},$$

we have the identity $\hat{G}_e = k_e G_e + k_s G_s + k_{\overline{e}} G_{\overline{e}}$. The proof of this identity is omitted, and is available from the author upon request. For the solution of (s), we have $1-a_e(1+g)$, $1-a_s(1+g)$, and $1-a_s-a_{\overline{e}} \in (0,1)$. Therefore, $k_e = -D_2 D_6 / D_1 D_4 (1-a_s-a_{\overline{e}}) < 0$.

Step 3.3: Equation (A31)

We use the vector b defined by equation (A34). We also define the vector $\hat{b}$ by

$$\hat{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{B41}$$

We can write the first of the three optimality conditions (A30) as

$$-\frac{1-e^{-rh}}{h}\frac{2a_e}{B} + \hat{n}^t Q(a_e \hat{n} + b) = 0.$$

Therefore, equation (A31) is equivalent to $\hat{n}^t Q b > 0$, or, since Q is symmetric, $b^t Q \hat{n} > 0$. Using the envelope conditions (B25), we get

$$b^t Q \hat{n} = b^t(\hat{Q}' - \alpha\sigma^2 h \Gamma)e^{-rh}\hat{n} = (b^t Q N \hat{n} + \alpha\sigma^2 h)e^{-rh}$$

$$= (b^t Q((1 - a_e(1+g))\hat{n} + (a_{\overline{e}} - g a_s)\hat{b}) + \alpha\sigma^2 h)e^{-rh}.$$

Therefore,

$$b^t Q\hat{n} = \frac{\left((a_{\overline{e}} - g a_s) b^t Q\hat{b} + \alpha\sigma^2 h\right) e^{-rh}}{1 - (1 - a_e(1+g))e^{-rh}}.$$

Using the envelope conditions again, we get

$$b^t Q\hat{b} = ((1 - a_s - a_{\overline{e}}) b^t Q\hat{b} - \alpha\sigma^2 h)e^{-rh}.$$

Therefore, $b^t Q\hat{b} = -\alpha\sigma^2 h / D_1$, and

$$b^t Q\hat{n} = \frac{D_2 \alpha\sigma^2 h}{D_1(1 - (1 - a_e(1+g))e^{-rh})} > 0.$$

The Solution of (s)

We proceed in three steps. First, we use equations (23) and (24) to solve for g and $\overline{\Sigma}_e^2$, as functions of $a_e \in (0,1)$. Second, we use equation (22) to solve for a_s as a function of g. Finally, we plug a_s and g into equation (21), obtain an equation only in a_e, and show that this equation has a solution $a_e \in (0,1)$.

Step 1: Determination of g and $\overline{\Sigma}_e^2$

We define the function $f(a_e)$ by

$$f(a_e) = \frac{2\overline{\sigma}_u^2 - a_e\overline{\sigma}_u^2 + a_e}{2(1 - a_e)},$$

and the function $F(x, a_e)$ by

$$F(x, a_e) = x^2 + 2f(a_e)x - \overline{\sigma}_u^2.$$

Since $a_e \in (0,1)$, we have $f(a_e) > 0$ and $f'(a_e) > 0$. Moreover, equation $F(x, a_e) = 0$ has a unique positive solution, that we denote by $x(a_e)$.

We can write equation (24) as

$$F\left(\frac{a_e\overline{\Sigma}_e^2}{(1 - a_e)h}, a_e\right) = 0 \Rightarrow \overline{\Sigma}_e^2 = \frac{(1 - a_e)h x(a_e)}{a_e}.$$

Dividing equation (23) by equation (24), we get

$$g = \frac{a_e\overline{\Sigma}_e^2}{(1 - a_e)h\overline{\sigma}_u^2} \Rightarrow g = \frac{x(a_e)}{\overline{\sigma}_u^2}.$$

We will show that $0 > dg/da_e > -(1+g)/a_e$, a fact that we will use in step 3. Differentiating equation $F(x(a_e), a_e) = 0$, we get

$$\frac{dg}{da_e} = \frac{1}{\overline{\sigma}_u^2}\frac{dx(a_e)}{da_e} = -\frac{f'(a_e)x(a_e)}{\overline{\sigma}_u^2(x(a_e) + f(a_e))} = -\frac{f'(a_e)g}{x(a_e) + f(a_e)} < 0.$$

To show that $dg/da_e > -(1+g)/a_e$ we will show that

$$\frac{1+g}{a_e} + \frac{dg}{da_e} = \frac{1}{a_e} + g\left(\frac{1}{a_e} - \frac{f'(a_e)}{x(a_e)+f(a_e)}\right) > 0.$$

Since $F(x(a_e), a_e) = 0$, we have $x(a_e) < \overline{\sigma}_u^2/2f(a_e)$, i.e. $g < 1/2f(a_e)$ Therefore, it suffices to show that

$$\frac{1}{a_e} + \frac{1}{2f(a_e)}\left(\frac{1}{a_e} - \frac{f'(a_e)}{f(a_e)}\right) > 0.$$

Noting that $f'(a_e)/f(a_e) < 1/a_e(1-a_e)$, it is easy to show this result.

Step 2: Determination of a_s

We can write equation (22) as

$$G_s(a_s, g) \equiv -a_s\frac{D_5}{D_3}\overline{\alpha} + (\alpha(1-a_s) - \overline{\alpha}a_s)(1-e^{-rh}) = 0, \tag{B42}$$

where D_3 and D_5 were defined in Section C.2, and $a_{\overline{e}} = a_s\overline{\alpha}/\alpha$. For $a_s \in (0, \alpha/(\alpha+\overline{\alpha}))$, $1 - a_s - a_{\overline{e}} \in (0,1)$ and $D_3 > 0$. The function $G_s(a_s, g)D_3$ is a third-order polynomial in a_s, which is strictly positive for $a_s = 0$, strictly negative for $a_s = \alpha/(\alpha+\overline{\alpha})$, and goes to ∞ when a_s goes to ∞. Therefore, equation (B42) has a solution $a_s \in (0, \alpha/(\alpha+\overline{\alpha}))$, which is unique in $(0, \alpha/(\alpha+\overline{\alpha}))$. Moreover, at the solution we have $\partial G_s(a_s, g)/\partial a_s < 0$.

We will show that $da_s/dg < 0$, a fact that we will use in step 3. Noting that

$$G_s(a_s, g) = -a_s\left(1 + \frac{a_s\left(g - \frac{\overline{\alpha}}{\alpha}\right)e^{-rh}}{D_3}\right)\overline{\alpha} + (\alpha(1-a_s) - \overline{\alpha}a_s)(1-e^{-rh}), \tag{B43}$$

we get

$$\frac{\partial G_s(a_s, g)}{\partial g} = -a_s^2\frac{1-(1-a_s\left(1+\frac{\overline{\alpha}}{\alpha}\right))^2e^{-rh}}{(D_3)^2}\overline{\alpha}e^{-rh} < 0. \tag{B44}$$

Differentiating equation $G_s(a_s, g) = 0$, we get $da_s/dg < 0$.

We will also show that for $g > \overline{\alpha}/\alpha$ and $1 - a_s(1+g) \in [0,1)$, we have $da_s/dg > -a_s/(1+g)$. Using equation (B43) and noting that

$$\frac{\partial}{\partial a_s}\frac{a_s^2}{D_3} = \frac{a_s}{(D_3)^2}(2-(2-a_s\left(2+g+\frac{\overline{\alpha}}{\alpha}\right))e^{-rh}) > 0,$$

we get $\partial G_s(a_s, g)/\partial a_s < -\overline{\alpha}$. Using equation (B44) and noting that

$$D_3 > 1-(1-a_s\left(1+\frac{\overline{\alpha}}{\alpha}\right))^2e^{-rh} > 0,$$

and $D_3 > a_s(1+g)e^{-rh}$, we get $\partial G_s(a_s, g)/\partial g > -a_s\overline{\alpha}/(1+g)$. Differentiating equation $G_s(a_s, g) = 0$, we get $da_s/dg > -a_s/(1+g)$.

Step 3: Determination of a_e

We denote by $G_e(a_e, a_s, g)$ the LHS of equation (21), and by $G_e(a_e)$ the LHS when we plug g and a_s as functions of a_e. The function $G_e(a_e)$ is continuous in $a_e \in (0,1)$. When a_e goes to 0, g goes to a limit $g(0) > 0$, and a_s goes to a limit $a_s(0) \in (0, \alpha/(\alpha + \overline{\alpha}))$. The function $G_e(a_e)$ goes to

$$-(1 - a_s(0)\left(1 + \frac{\overline{\alpha}}{\alpha}\right))(1 - e^{-rh})\alpha < 0.$$

When a_e goes to 1, g goes to 0, and a_s goes to a limit $a_s(1) \in (0, \alpha/(\alpha + \overline{\alpha}))$. The function $G_e(a_e)$ goes to

$$(1 - a_s(1))\overline{\alpha} > 0.$$

Therefore, equation (21) has a solution $a_e \in (0,1)$. Since $a_s \in (0, \alpha/(\alpha + \overline{\alpha}))$, we have $1 - a_s - a_{\overline{e}} \in (0,1)$. Equation (23) implies that $g < (1 - a_e)/a_e$. Therefore, $1 - a_e(1 + g) \in (0,1)$. Finally, equation (21) implies that $1 - a_s(1+g) > 0$. Therefore, $1 - a_s(1+g) \in (0,1)$.

We will show that at the solution a_e, we have $dG_e(a_e)/da_e > 0$. This fact implies that the solution is unique and, as we show in Section C.3, allows us to use the implicit function theorem. We have

$$\frac{dG_e(a_e)}{da_e} = \frac{\partial G_e(a_e, a_s, g)}{\partial a_e} + \left(\frac{\partial G_e(a_e, a_s, g)}{\partial g} + \frac{\partial G_e(a_e, a_s, g)}{\partial a_s}\frac{da_s}{dg}\right)\frac{dg}{da_e}.$$

Since $\partial G_e(a_e, a_s, g)/\partial a_e > 0$ and $0 > dg/da_e > -(1+g)/a_e$, we have $dG(a_e)/da_e > 0$ if

$$\frac{\partial G_e(a_e, a_s, g)}{\partial a_e} - \left(\frac{\partial G_e(a_e, a_s, g)}{\partial g} + \frac{\partial G_e(a_e, a_s, g)}{\partial a_s}\frac{da_s}{dg}\right)\frac{1+g}{a_e} > 0.$$

We have

$$\frac{\partial G_e(a_e, a_s, g)}{\partial a_e} - \frac{\partial G_e(a_e, a_s, g)}{\partial g}\frac{1+g}{a_e} = (1 - (1 - a_e(1+g))e^{-rh})\overline{\alpha},$$

and

$$\frac{\partial G_e(a_e, a_s, g)}{\partial a_s} = -a_e(1 - (1 - a_e(1+g))e^{-rh})(1+g)\overline{\alpha} + (1 - a_e(1+g))\left(1 + \frac{\overline{\alpha}}{\alpha}\right)(1 - e^{-rh})\alpha.$$

Using equation (21), we can write $\partial G_e(a_e, a_s, g)/\partial a_s$ as

$$-\frac{a_e(1 - (1 - a_e(1+g))e^{-rh})\overline{\alpha}}{1 - a_s\left(1 + \frac{\overline{\alpha}}{\alpha}\right)}\left(g - \frac{\overline{\alpha}}{\alpha}\right).$$

Therefore, we have $dG_e(a_e)/da_e > 0$ if

$$1 + \frac{(1+g)\left(g - \frac{\overline{\alpha}}{\alpha}\right)}{1 - a_s\left(1 + \frac{\overline{\alpha}}{\alpha}\right)}\frac{da_s}{dg} > 0.$$

If $g \leq \overline{\alpha}/\alpha$, this condition is satisfied, since $da_s/dg < 0$. If $g > \overline{\alpha}/\alpha$, this condition is satisfied since $da_s/dg > -a_s/(1+g)$.

C.3. The Solution for Small σ_e^2

We first prove Lemma 1. To state the lemma, we use the following notation. Consider a function $F(x,y)$, where x is a $1\times N$ vector, y a $1\times M$ vector, and F a $K\times 1$ vector. We denote by $J_xF(x,y)$ the $K\times N$ matrix of partial derivatives of $F(x,y)$ w.r.t. x, i.e. the Jacobian matrix of $F(x,y)$ w.r.t. x.

Lemma 1 *Consider a function*

$$F(x,y)=\begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix},$$

where x is a $1\times N$ vector, y a $1\times M$ vector, $F_1(x,y)$ a $N\times 1$ vector, and $F_2(x,y)$ a $M\times 1$ vector. Suppose that (i) there exists a function $y(x)$ such that $F_2(x,y(x))=0$, (ii) $J_yF_2(x,y)$ is invertible, and (iii) $J_xF_1(x,y(x))$ is invertible. Then $J_{x,y}F(x,y)$ is invertible for $y=y(x)$.

In words, Lemma 1 says that the Jacobian matrix of $F(x,y)$ w.r.t. (x,y) is invertible if (i) we can solve equation $F_2(x,y)=0$ for y, and (ii) the Jacobian matrix of the function $F_1(x,y(x))$, that we obtain by plugging $y(x)$ in the function $F_1(x,y)$, is invertible. Lemma 1 allows us to "eliminate" the function $F_2(x,y)$ and consider a smaller Jacobian matrix.

Proof: We have

$$J_{x,y}F(x,y)=\begin{pmatrix} J_xF_1(x,y) & J_yF_1(x,y) \\ J_xF_2(x,y) & J_yF_2(x,y) \end{pmatrix}.$$

The matrix $J_{x,y}F(x,y)$ is invertible if the matrix obtained by multiplying the last M columns by $J_yF_2(x,y)^{-1}J_xF_2(x,y)$ and subtracting them from the first N, is invertible. This matrix is

$$\begin{pmatrix} J_xF_1(x,y)-J_yF_1(x,y)J_yF_2(x,y)^{-1}J_xF_2(x,y) & J_yF_1(x,y) \\ 0 & J_yF_2(x,y) \end{pmatrix},$$

and is invertible if the matrix

$$J_xF_1(x,y)-J_yF_1(x,y)J_yF_2(x,y)^{-1}J_xF_2(x,y)$$

is invertible. We will show that for $y=y(x)$, this matrix is $J_xF_1(x,y(x))$. Differentiating $F_2(x,y(x))=0$, we get

$$J_xF_2(x,y(x))+J_yF_2(x,y(x))J_xy(x)=0\Rightarrow J_xy(x)=-J_yF_2(x,y(x))^{-1}J_xF_2(x,y(x)).$$

Therefore,

$$J_x F_1(x, y(x)) = J_x F_1(x, y(x)) + J_y F_1(x, y(x)) J_x y(x)$$

$$= J_x F_1(x, y(x)) - J_y F_1(x, y(x)) J_y F_2(x, y(x))^{-1} J_x F_2(x, y(x)).$$

Q.E.D.

To extend the solution of (S') for small σ_e^2, we use the implicit function theorem. We denote by z the 1×23 vector of unknowns a_e, a_s, $a_{\overline{e}}$, g, $\overline{\Sigma}_e^2$, $A_{\overline{e}}/B$, A_s/B, $1/B$, $\overline{Q}$, and Q. We denote by $K(z, \sigma_e^2) = 0$ the 23×1 vector of optimality conditions of the large trader's problem, equations (23) and (24) of the recursive filtering problem, equations of the market makers' problem, and envelope conditions of the large trader's problem. Finally, we denote by z_0 the solution of (S') for $\sigma_e^2 = 0$. The function $K(z, \sigma_e^2)$ is C^1 at $(z_0, 0)$, and $K(z_0, 0) = 0$. The implicit function theorem applies if the matrix $J_z K(z, 0)$ is invertible for $z = z_0$.

To show that $J_z K(z, 0)$ is invertible, we use Lemma 1. We set $(x, y) = z$ and $F(x, y) = K(z, 0)$, denote by y the 1×15 vector of unknowns $A_{\overline{e}}/B$, A_s/B, $1/B$, $\overline{Q}$, and Q, and by $F_2(x, y) = 0$ the 15×1 vector of equations of the market makers' problem and envelope conditions of the large trader's problem. In Section C.2 we solved equation $F_2(x, y) = 0$ for y. Since $F_2(x, y)$ is linear in y, and since we could solve for y, the matrix $J_y F_2(x, y)$ is invertible. Lemma 1 implies that $J_z K(z, 0)$ is invertible if $J_x F_1(x, y(x))$ is invertible.

In Section C.2 we showed that the optimality conditions of the large trader's problem are connected to equations (21), (22), and (25), though an invertible linear transformation. Therefore, we can assume that $F_1(x, y(x)) = 0$ consists of equations (21), (22), (25), (23), and (24).

To show that $J_x F_1(x, y(x))$ is invertible, we use Lemma 1 for the function $F_1(x, y(x))$. The "new" (x, y) is the "old" x, the new $F(x, y)$ is $F_1(x, y(x))$, the new y is the 1×4 vector of unknowns a_s, $a_{\overline{e}}$, g, and $\overline{\Sigma}_e^2$, and the new $F_2(x, y) = 0$ is the 4×1 vector of equations (22), (25), (23), and (24). In Section C.2 we solved $F_2(x, y) = 0$ for y. Using the results of this section, it is easy to check that the matrix $J_y F_2(x, y)$ is invertible. Lemma 1 implies that $J_{x,y} F(x, y)$ is invertible if $J_x F_1(x, y(x))$ is invertible. Using the notation of Section (25), $J_x F_1(x, y(x)) = dG_e(a_e)/da_e > 0$. Q.E.D.

D. Proofs of Propositions 3, 5, and 6

Proof of Proposition 3: We denote by a_e^* and a_s^*, the a_e and a_s that solve (S). We will show that $a_e^* > a_s^*$ for $\sigma_e^2 = 0$, and conclude by continuity. We proceed by contradiction and assume that $a_e^* \leq a_s^*$. As in Section C.2, we fix $a_e \in (0,1)$ and define g and $\overline{\Sigma}_e^2$ from equations (23) and (24), and a_s from equation (22). Since $a_s < \alpha/(\alpha+\overline{\alpha}) < 1$, there exists $a_e \geq a_e^*$ such that $a_e = a_s$. For this a_e we have $G_e(a_e) \geq 0$, since equation $G_e(a_e) = 0$ has a unique solution $a_e^* \in (0,1)$, and the function $G_e(a_e)$ goes to a strictly positive limit when a_e goes to 1.

Noting that $a_s = a_e$ and $a_{\overline{e}} = a_s\overline{\alpha}/\alpha$, we have

$$G_e(a_e) = (1 - a_e(1+g))\left(a_e(1-(1-a_e(1+g))e^{-rh})\overline{\alpha} - (\alpha(1-a_e) - \overline{\alpha}a_e)(1-e^{-rh})\right).$$

Using equation (22), we get

$$G_e(a_e) = (1 - a_e(1+g))a_e\overline{\alpha}F,$$

where

$$\begin{aligned} F &= 1 - (1-a_e(1+g))e^{-rh} - \frac{1-(1-a_e(1+g)(2-a_e\left(1+\frac{\overline{\alpha}}{\alpha}\right)))e^{-rh}}{1-(1-a_e(1+g))(1-a_e\left(1+\frac{\overline{\alpha}}{\alpha}\right))e^{-rh}}. \\ &= -(1-a_e\left(1+\frac{\overline{\alpha}}{\alpha}\right))e^{-rh}\frac{1-(1-a_e(1+g))^2e^{-rh}}{1-(1-a_e(1+g))(1-a_e\left(1+\frac{\overline{\alpha}}{\alpha}\right))e^{-rh}}. \end{aligned}$$

Since $1-a_e(1+\overline{\alpha}/\alpha) = 1-a_s(1+\overline{\alpha}/\alpha) \in (0,1)$ and $1-a_e(1+g) \in (0,1)$, we have $G_e(a_e) < 0$, a contradiction. Q.E.D.

Proof of Proposition 5: We first show that if $ga_s - a_{\overline{e}} \geq 0$, the large trader's stock holdings decrease over time. Since $a_e > a_s$, we have

$$a_e(1+g) - a_s - a_{\overline{e}} > ga_s - a_{\overline{e}} \geq 0.$$

The coefficients of $(1-a_s-a_{\overline{e}})^{\ell'-\ell}$ and $(1-a_e(1+g))^{\ell'-\ell}$ in equation (30) are thus positive, and stock holdings decrease over time.

We next show that if $ga_s - a_{\overline{e}} < 0$, stock holdings decrease and then increase over time. Stock holdings are equal to 1 for $\ell' = \ell$, and go to $a_{\overline{e}}/(a_s + a_{\overline{e}}) < 1$ when ℓ' goes to ∞. Their derivative w.r.t. ℓ' changes sign at most once. Therefore, stock holdings decrease and then increase over time if they increase for large ℓ'. We distinguish two cases. If $a_e(1+g) - a_s - a_{\overline{e}} > 0$, then $1 - a_s - a_{\overline{e}} > 1 - a_e(1+g)$. Stock holdings are approximately

$$\frac{a_{\overline{e}}}{a_s + a_{\overline{e}}} + \frac{a_e(ga_s - a_{\overline{e}})}{(a_e(1+g) - a_s - a_{\overline{e}})(a_s + a_{\overline{e}})}(1-a_s-a_{\overline{e}})^{\ell'-\ell}$$

for large ℓ', and they increase. If $a_e(1+g) - a_s - a_{\overline{e}} < 0$, stock holdings are approximately

$$\frac{a_{\overline{e}}}{a_s + a_{\overline{e}}} + \frac{a_e - a_s}{a_e(1+g) - a_s - a_{\overline{e}}}(1 - a_e(1+g))^{\ell'-\ell},$$

and they also increase.

We next show that if $ga_s - a_{\overline{e}} \geq 0$, the trading rate decreases over time. The trading rate is

$$\frac{x_{\ell'}}{\sum_{\ell'' \geq \ell'} x_{\ell''}} = \frac{x_{\ell'}}{e_{\ell'-1} - e_\infty}.$$

Equation (A37) implies that

$$x_{\ell'} = a_e(e_{\ell'-1} - s_{\ell'-1}) + a_s s_{\ell'-1} - a_{\overline{e}}\overline{e}_{\ell'-1}$$

$$= \frac{a_e(ga_s - a_{\overline{e}})}{a_e(1+g) - a_s - a_{\overline{e}}}(1 - a_s - a_{\overline{e}})^{\ell'-\ell} + \frac{a_e(1+g)(a_e - a_s)}{a_e(1+g) - a_s - a_{\overline{e}}}(1 - a_e(1+g))^{\ell'-\ell}. \quad \text{(B45)}$$

Using equations (30) and (B45), we can write the trading rate as

$$\frac{ga_s - a_{\overline{e}} + (1+g)(a_e - a_s)f(\ell')}{\frac{ga_s - a_{\overline{e}}}{a_s + a_{\overline{e}}} + \frac{a_e - a_s}{a_e} f(\ell')},$$

where

$$f(\ell') = \frac{(1 - a_e(1+g))^{\ell'-\ell}}{(1 - a_s - a_{\overline{e}})^{\ell'-\ell}}.$$

Using the inequalities $a_e > a_s$ and $a_e(1+g) - a_s - a_{\overline{e}} > 0$, it is easy to check that the trading rate increases in $f(\ell')$, and that $f(\ell')$ decreases in ℓ'. Therefore, the trading rate decreases over time.

We finally show that if $ga_s - a_{\overline{e}} \geq 0$, the price impact decreases over time. We can write the expected price change, $p_{\ell'-1} - p_{\ell'}$, successively as

$$\left(\frac{h}{1-e^{-rh}}\overline{Q}_{1,2} + \frac{A_{\overline{e}}}{B}\right)\overline{e}_{\ell'-1} + \left(\frac{h}{1-e^{-rh}}\overline{Q}_{1,3} + \frac{A_s}{B}\right)s_{\ell'-1} + \frac{a_e}{B}(e_{\ell'-1} - s_{\ell'-1})$$

$$= \frac{h}{1-e^{-rh}}(\overline{Q}_{1,3} - \overline{Q}_{1,2})(a_s s_{\ell'-1} - a_{\overline{e}}\overline{e}_{\ell'-1}) + \frac{a_e}{B}(e_{\ell'-1} - s_{\ell'-1})$$

$$= \frac{h}{1-e^{-rh}}(\overline{Q}_{1,3} - \overline{Q}_{1,2})\frac{a_e(ga_s - a_{\overline{e}})}{a_e(1+g) - a_s - a_{\overline{e}}}(1 - a_s - a_{\overline{e}})^{\ell'-\ell}$$

$$+ \left(-\frac{h}{1-e^{-rh}}(\overline{Q}_{1,3} - \overline{Q}_{1,2})\frac{a_e(ga_s - a_{\overline{e}})}{a_e(1+g) - a_s - a_{\overline{e}}} + \frac{a_e}{B}\right)(1 - a_e(1+g))^{\ell'-\ell}. \quad \text{(B46)}$$

For the first equality we use equation (8) for $p_{\ell'}$ and equation (B30) for $p_{\ell'-1}$, for the second equality we use the optimality conditions (A11) and (A12), and for the third equality we use equation (A37). Using equations (B45) and (B46), we can write the price impact as

$$\frac{\frac{h}{1-e^{-rh}}(\overline{Q}_{1,3} - \overline{Q}_{1,2})(ga_s - a_{\overline{e}}) + \left(-\frac{h}{1-e^{-rh}}(\overline{Q}_{1,3} - \overline{Q}_{1,2})(ga_s - a_{\overline{e}}) + \frac{a_e(1+g) - a_s - a_{\overline{e}}}{B}\right)f(\ell')}{ga_s - a_{\overline{e}} + (1+g)(a_e - a_s)f(\ell')}.$$

The function $f(\ell')$ decreases in ℓ'. Therefore, the price impact decreases over time if it increases in $f(\ell')$. It is easy to check that the price impact increases in $f(\ell')$ if and only if

$$\frac{1}{B} - \frac{h}{1-e^{-rh}}(\overline{Q}_{1,3} - \overline{Q}_{1,2}) > 0. \tag{B47}$$

Substituting $1/B$ from the optimality condition (A13), we can write inequality (B47) as

$$\frac{h}{1-e^{-rh}}(1+g)\overline{Q}_{1,3} < 0.$$

This inequality holds as long as $\overline{Q}_{1,3} < 0$. Equations (B21) and (B22) imply that for $\sigma_e^2 = 0$, $\overline{Q}_{1,3} < 0$. Therefore, for small σ_e^2, $\overline{Q}_{1,3} < 0$. Our numerical solutions confirm that $\overline{Q}_{1,3} < 0$ for large σ_e^2. Q.E.D.

Proof of Proposition 6: We replace (S') by an equivalent system (S'_c), that we obtain as follows. We set $a_e = \phi_e\sqrt{h}$, $a_s = \phi_s h$, $a_{\overline{e}} = \phi_{\overline{e}} h$, and $\overline{\Sigma}_e^2 = \phi_\Sigma\sqrt{h}$, and replace the unknowns a_e, a_s, $a_{\overline{e}}$, and $\overline{\Sigma}_e^2$, by ϕ_e, ϕ_s, $\phi_{\overline{e}}$, and ϕ_Σ. We divide the Bellman conditions of the market makers' problem by h, the envelope conditions of the large trader's problem by $\sqrt{h}$ if they correspond to $Q_{1,1}$, $Q_{2,1}$, and $Q_{3,1}$, and by h otherwise, and equation (24) of the recursive filtering problem by h. Finally, we multiply the optimality conditions of the large trader's problem by the invertible matrix

$$(1-Ne^{-rh})\hat{N}\frac{1-e^{-rh}}{\sigma^2 h^2 e^{-rh}}\begin{pmatrix} k_e & 0 & 0 \\ k_s & 1 & 0 \\ k_{\overline{e}} & 0 & 1 \end{pmatrix}^{-1}.$$

For $\sigma_e^2 = 0$, we get equations $G_e/h = G_s/h = G_{\overline{e}}/h = 0$. We denote by z the vector of unknowns of (S'_c) and by $K(z, \sigma_e^2, h) = 0$ the vector of equations.

We will use the implicit function theorem for $\sigma_e^2 = h = 0$. It is easy to check the following. First, the function $K(z, \sigma_e^2, h)$ can be extended by continuity for $h = 0$, and is C^1. (We use the fact that the matrices $\overline{R}'$, R'_u, and R', are "of order" h.) Second, for $\sigma_e^2 = h = 0$, the optimality conditions of the market makers' problem, the equations obtained from the Bellman conditions of the market makers' problem, and the equations obtained from the envelope conditions of the large trader's problem, are linear in $A_{\overline{e}}/B$, A_s/B, $1/B$, $\overline{Q}$, and Q, and can be solved in these unknowns. Third, for $\sigma_e^2 = h = 0$, the equations obtained from the optimality conditions of the large trader's problem, equation (23), and the equation obtained from equation (24), become

$$(\phi_e)^2(1+g)\overline{\alpha} - r\alpha = 0,$$

$$-\phi_s \frac{r + 2\phi_s(1+g)}{r + \phi_s(2+g) + \phi_{\overline{e}}}\overline{\alpha} + r\alpha = 0,$$

$$\phi_{\overline{e}}\alpha - \phi_s\overline{\alpha} = 0,$$

$$g - \frac{\phi_e\phi_\Sigma}{\overline{\sigma}_u^2} = 0,$$

and

$$(\phi_e\phi_\Sigma)^2 + 2\overline{\sigma}_u^2\phi_e\phi_\Sigma - \overline{\sigma}_u^2 = 0,$$

respectively. Fourth, the ϕ_e, ϕ_s, $\phi_{\overline{e}}$, and g_0 of the Proposition solve these equations. Therefore, for $\sigma_e^2 = h = 0$, we have a solution, z_0, to $K(z, \sigma_e^2, h) = 0$. To show that the matrix $J_zK(z, 0, 0)$ is invertible for $z = z_0$, we proceed as in Section C.3. Q.E.D.