# The whole and the sum of its parts : formation of blocs revisited


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# The Whole and the Sum of Its Parts: Formation of Blocs Revisited

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ABSTRACT

For any simple voting game (SVG), we raise the question as to whether forming a given bloc is advantageous. We consider this question from two different points of view as to what voting power means. We also distinguish between blocs imposed by annexation and those formed voluntarily. We illustrate our theoretical findings with examples using both toy SVGs and the Qualified Majority Voting rule of the Council of Ministers of the European Community.

# The Whole and the Sum of Its Parts: Formation of Blocs Revisited

## 1 Introduction

We present here a sort of preview and summary of work to be published in Felsenthal and Machover (2002)—briefly F&M (2002)—where full proofs and detailed examples can be found. For various general concepts and results concerning voting power, we shall refer to our book Felsenthal and Machover (1998), briefly F&M (1998).

We are concerned with the formation of a bloc in a simple voting game (SVG) $\mathcal{W}$. Note the difference between *bloc* and *coalition*. The latter term, borrowed from current usage in cooperative game theory (cf. Myerson 1991, p. 418), simply means an arbitrary subset $S$ of $\mathcal{W}$'s voters; this term is not meant to imply that the members of $S$ always vote in the same way; indeed, they may never do so.

But when a coalition $S$ of $\mathcal{W}$ fuses into a *bloc*, denoted by '$\&_S$' say, then this bloc is a new single voter. The SVG $\mathcal{W}$ ceases to exist: it is transformed into a new SVG $\mathcal{W}|\&_S$ whose voters are all those voters of $\mathcal{W}$ who do not belong to $S$, as well as the new voter $\&_S$, who inherits, so to speak, the voting mandates of all the members of $S$; but the members of $S$ themselves are no longer voters of $\mathcal{W}|\&_S$. For a rigorous definition of $\mathcal{W}|\&_S$ see Section 2.

The question we address in this paper is: *When is it advantageous to form a bloc?* Before this question can be answered correctly, it must however be made more precise; several clarifications are called for.

For a start, we must make it clear that the 'advantage' we are referring to is to be reckoned in terms of *voting power*. But now one might wonder: Advantage *to whom*? The point here is that—leaving aside external intervention—there are two ways in which a bloc can be thought to arise: annexation or voluntary consent.

Annexation occurs when one voter takes over the voting mandates of other voters, in order to use them in his or her own interest—as when a shareholder buys up the voting shares of other shareholders. In this case, it is only the annexer's advantage that counts; and common sense suggests that annexing non-dummy voters must *always* increase the voting power of the annexer and is advantageous in this respect (while annexing a dummy obviously makes no difference). In our opinion, this common-sense view is sound. However, more needs to be said about this issue, and we shall return to it in Section 2.

But if the members of a coalition $S$ combine into a bloc voluntarily, by mutual consent, then clearly they must all derive some voting-power advantage from this. It turns out that in order to analyse this issue correctly, we must be more precise about what we mean by 'voting power'.

First, we must state that the voting power we are concerned with in this paper is *a priori*—rather than *a posteriori* or *actual*—voting power. This is the power that each voter derives from the structure of the decision rule, the SVG itself. In taking this stance, we go 'behind a veil of ignorance' regarding each voter's likes and dislikes, affinities and disaffinities among voters, and the nature of the bills to be voted on. (Cf. F&M 1998, Com. 2.2.3 and references cited there; see also Felsenthal and Machover 2001, Holler and Widgrén 1999 and Lane and Berg 1999.) Of course, in reality when deciding whether to form a bloc, voters will take into account the kind of information that we ignore here, in so far as it is available. But such information may not be reliably available; and even if it is, the a priori theory developed here can serve as a benchmark, against which considerations using this information may better be appraised.

Second, it turns out that in assessing the voting-power advantages of forming a bloc, the distinction between two underlying—intuitive and pre-formal—notions of what voting power is all about makes a major difference. We are referring here to the distinction between *I-power* and *P-power*. This distinction is explained in detail in F&M (1998) and summarized in Felsenthal and Machover (2001), so we need not repeat those explanations here.

In Section 2, after addressing (from both viewpoints, P-power and I-power) the easy problem of a bloc formed by annexation, we turn to the problem of a voluntary bloc, from the viewpoint of P-power. This also turns out to be easy (assuming, of course, that we have a reasonable index of P-power): forming a bloc can be of advantage to all the prospective partners, iff the bloc's expected share in the fixed prize is greater than the sum of the shares that the partners expect to obtain when acting as separate individuals.

We deal with those easy matters mainly for the sake of contrast with the main issue of this paper, to which we turn in Section 3: the problem of a voluntary bloc from the viewpoint of I-power. The solution to this problem is far less obvious than those of Section 2, because—unlike the payoffs considered under the notion of P-power, rooted in cooperative game theory—influence is not an additive quantity. Forming a bloc can be of advantage to all the prospective partners iff each of them can obtain via the bloc greater indirect influence than s/he has directly, when they all act as separate individuals. But it is a fallacy to suppose that the influence of the bloc is in general equal to the sum of these indirect influences that the partners obtain via the bloc. In fact, these indirect influences depend crucially on the mechanism fixed by the partners for determining the voting behaviour of the bloc. This leads us to the notions of *alliance* and *expedient bloc*. We state some general theoretical results—proved in F&M (2002)—concerning these notions.

In Section 4 we summarize examples—which are presented in detail in F&M (2002)—showing that, from the viewpoint of I-power, a voluntary bloc may be expedient even if its voting power is smaller than the sum of the original powers of its members; and it may not be feasible even if its voting power is greater than that sum. Some of these are toy examples, but we also consider examples relating to the Council of Ministers of the European Community (CMEC).

We do not provide definitions of concepts that we assume to be familiar to most participants of this meeting; a reader who is in doubt about these is advised to consult F&M (1998).

# 2 Preliminaries

Here and in the next section, $\mathcal{W}$ is some arbitrary simple voting game (SVG); $N$ is the assembly (set of all voters) of $\mathcal{W}$; and $S$ is a coalition of $\mathcal{W}$ (in other words, $S \subseteq N$). To avoid trivialities, we assume that $S$ has at least two members.

We let $\mathcal{W}|\&_S$ be the SVG that results from $\mathcal{W}$ when $S$ fuses into a bloc. Informally, this means that the members of $S$ henceforth vote as one person. Formally, the assembly of $\mathcal{W}|\&_S$ is $(N - S) \cup \{\&_S\}$, obtained from $N$ by removing all the members of $S$ and adding a new voter, $\&_S$, the *bloc of $S$*. We denote this assembly by '$N|\&_S$'. The winning coalitions of $\mathcal{W}|\&_S$ are all those $X \subseteq N - S$ such that $X$ is winning in $\mathcal{W}$, as well as all $X \cup \{\&_S\}$ such that $X \subseteq N - S$ and $X \cup S$ is winning in $\mathcal{W}$. (Cf. F&M 1998, Def. 2.3.23.)

If $\mathcal{W}$ is a weighted voting game (WVG), then so is $\mathcal{W}|\&_S$: take the weight of $\&_S$ to be the sum of the weights that the members of $S$ had in $\mathcal{W}$, while the weights of all other voters as well as the quota are kept the same as in $\mathcal{W}$.

If $\xi$ is a measure of voting power, we denote by '$\xi_a[\mathcal{W}]$' the value that $\xi$ assigns to voter $a$ in $\mathcal{W}$. Following F&M (1998), we reserve the term *index of voting power* for a measure whose values for all voters of any SVG always add up to 1: so $\sum_{x \in N} \xi_x[\mathcal{W}] = 1$ for any $\mathcal{W}$. An index is thus a measure of *relative* voting power.

Note that all measures of P-power are in fact indices. This is because, by definition, the P-power of a voter is the expected *share* of that voter in a fixed prize, whose total value can always be fixed, by convention, as 1.

In contrast, when it comes to a priori I-power, the primary notion is the *absolute* amount of influence that a voter can exert on the outcome of a division; and the sum of these, for all voters of an SVG, cannot be taken as a fixed quantity, independent of the SVG. Indeed, the Bz measure $\beta'$—the only serious contender as a measure of a priori absolute I-power—is not an index in the present strict sense.

The value ${\beta'}_a[\mathcal{W}]$—the [absolute] Bz power of voter $a$ in $\mathcal{W}$—can be characterized probabilistically as follows. Suppose voters act independently of one another, each voting 'yes' or 'no' with equal probability of $\frac{1}{2}$. Then ${\beta'}_a[\mathcal{W}]$ is equal to the probability that the voters other than $a$ are so divided that $a$ is in a position to decide the outcome: by joining the 'yes' voters $a$ will give rise to a winning coalition, so that the proposed bill will be passed; but if $a$ joins the 'no' voters the bill will be defeated. (Cf. F&M 1998, Thm. 3.2.4. For an explicit formula for ${\beta'}_a[\mathcal{W}]$, see F&M 1998, Def. 3.2.2.) The Bz *index* $\beta$, which can be used to measure *relative* a priori I-power, is obtained from $\beta'$ by *normalization*:

$$\beta_a[\mathcal{W}] = \frac{{\beta'}_a[\mathcal{W}]}{\sum_{x \in N} {\beta'}_x[\mathcal{W}]}.$$

Now let us suppose that the bloc $\&_S$ arises by annexation: a particular voter $a \in S$ takes over the voting mandates of all other members of $S$. Under what circumstances will this be of advantage to the annexer $a$?

First, let us approach this question from the viewpoint of P-power. (Here we are obviously assuming that the notion of P-power is coherent.) The annexation is a priori advantageous to $a$ iff it gives $a$ a greater expected share in the prize than $a$ had originally. To formalize this condition is very easy—once we decide on a reasonable index of P-power, which may be quite a controversial matter. *Assuming* that $\xi$ is such an index, the condition is expressed by the inequality

$$\xi_{\&_S}[\mathcal{W}|\&_S] > \xi_a[\mathcal{W}]. \tag{1}$$

Similarly, the condition that the annexation is a priori disadvantageous to $a$ is expressed by the reverse inequality:

$$\xi_{\&_S}[\mathcal{W}|\&_S] < \xi_a[\mathcal{W}]. \tag{2}$$

If $\xi$ satisfies (2) for some $S$ and $a \in S$, then $\xi$ is said to display thereby the *bloc paradox*.

Common sense suggests that annexing the voting mandates of other voters cannot possibly worsen the bargaining position of the annexer and will therefore never be disadvantageous. Anyone who, like us, finds this common-sense view compelling must consequently reject as invalid any purported index $\xi$ of P-power that displays the bloc paradox.

Of all indices of a priori P-power known to us, the only one that does not suffer from the bloc paradox is the S-S index. In fact, this index always satisfies (1), provided $S$ has at least one member, other than $a$, who is not a dummy. For a proof of this fact, and for instances in which the Deegan–Packel and Johnston indices display the bloc paradox see F&M (1988, pp. 256–7). Anyone who, for some reason, prefers one of those other indices for measuring P-power must be prepared to live with the bloc paradox and accept the counter-intuitive consequence that in some cases annexation will reduce the a priori expected payoff of the annexer.

Now let us consider the same question from the viewpoint of I-power. Here again common sense suggests that annexing the voting mandates of other voters cannot diminish the a priori influence of the annexer; and if at least one of those other voters is not a dummy, this influence must actually increase. This common-sense view is vindicated by the behaviour of the Bz measure, which is in our view the only serious contender for measuring absolute a priori I-power. To see this, it is enough to consider the case where $S$ has just two members, $a$ and one other voter, say $b$. In fact, we have

$$\beta'_{\&_{\{a,b\}}}[\mathcal{W}|\&_{\{a,b\}}] = \beta'_a[\mathcal{W}] + \beta'_b[\mathcal{W}]. \tag{3}$$

So in any case $\beta'_{\&_{\{a,b\}}}[\mathcal{W}|\&_{\{a,b\}}] \geq \beta'_a[\mathcal{W}]$; and if $b$ is not a dummy in $\mathcal{W}$ then $\beta'_{\&_{\{a,b\}}}[\mathcal{W}|\&_{\{a,b\}}] > \beta'_a[\mathcal{W}]$. (For a proof, see F&M 1998, pp. 47–48.) Note that this strictly additive property of Bz power does not extend to three or more voters. This is because the Bz power of a third voter, say $c$, in $\mathcal{W}|\&_{\{a,b\}}$ need not be the same as in $\mathcal{W}$; for example, $c$ may be a non-dummy in $\mathcal{W}$ and become a dummy in $\mathcal{W}|\&_{\{a,b\}}$. Still, if $a$ annexes the voting mandates of several voters successively, one at a time, starting with one who is initially not a dummy, then in the first step the annexer's Bz power increases, and in subsequent steps it never decreases.

We must point out that, unlike the Bz measure, the Bz index does display the bloc paradox: for instances of this see F&M (1998, pp. 256–7). However, this does not mean that annexation can reduce the annexer's I-power. The Bz index does not measure voters' absolute I-powers but their respective shares in the total I-power, which varies from one SVG to another. Cases in which the Bz index displays the bloc paradox occur where annexation, while increasing the influence of the annexer, also causes, as a by-product, a sufficiently great increase in the influence of other voters. That this can indeed happen may seem paradoxical; but is a fact all the same. (And this is one of the reasons why the Bz index cannot be used also to measure voters' a priori P-power.)

Note that the index proposed by Holler (1982) can also easily be shown to display the paradox. However, unlike the Bz index, Holler's index is not derived by normalization from some (other) putative measure of absolute I-power. Thus the only way it gives us for comparing the powers of a given voter in two different SVGs is to compare the two values of the index itself in these SVGs. So anyone who, for some reason, prefers this index for measuring I-power must be prepared to live with the bloc paradox and accept the counter-intuitive consequence that in some cases annexation will reduce the a priori influence of the annexer.

Now let us consider a bloc $\&_S$ formed voluntarily, by consent of all the members of $S$. Leaving the viewpoint of I-power to the next section, we adopt here the viewpoint of P-power. As is commonly done in cooperative game theory—in which the notion of P-power is rooted—we must assume that the payoffs received by voters who carry a vote-division to a successful outcome consist of quantities of transferable utility that behave in an additive way.

Since all members of $S$ must consent to forming the bloc, we must now ask under what condition the bloc may be advantageous to all of them. Clearly, the answer is: iff the expected share of the bloc in the [fixed] prize is greater than the sum of the expected shares that the members of $S$ receive when acting as separate individuals. Presumably, when forming the bloc the partners will agree to divide its payoff in such a way as to leave each of them better off than before. In the reverse case, where the expected share of the bloc is smaller than that sum, the bloc must be disadvantageous to at least one of the voters in $S$, and will therefore not be formed. (In the remaining case, when the two quantities happen to be equal, the bloc can at best leave all the prospective partners in the same position as before.) Again, formalizing these conditions is easy—leaving aside the controversial issue of selecting a reasonable index of P-power. If $\xi$ is such an index, then the bloc is a priori advantageous iff

$$\xi_{\&_S}[\mathcal{W}|\&_S] > \sum_{x \in S} \xi_x[\mathcal{W}], \tag{4}$$

and a priori disadvantageous iff

$$\xi_{\&_S}[\mathcal{W}|\&_S] < \sum_{x \in S} \xi_x[\mathcal{W}]. \tag{5}$$

Cases where (5) holds are displayed by any half-way reasonable measure $\xi$ of voting power. They have been dubbed *the paradox of large size*. But there is nothing genuinely paradoxical about them. It stands to reason that in some cases voters can achieve more, and in some cases less, by acting as one body than they can achieve in total by acting separately. (For a more detailed discussion, see F&M 1998, § 7.2.)

## 3 Alliances and expedient blocs

Having dismissed some easy problems of bloc formation in Section 2, let us now turn to the meatier problem of voluntary blocs, considered from the viewpoint of I-power.

Clearly, from this viewpoint the bloc $\&_S$ will be advantageous to all the members of $S$ iff after forming $\&_S$ every one of them will be able to exercise more influence over the outcome of a division than s/he was able to exercise originally, in $\mathcal{W}$. So in formalizing the present problem we should use the Bz measure $\beta'$ rather than the Bz index $\beta$. Using the latter makes little sense in the present context because, as we saw in Section 2, this index only measures a voter's *relative* I-power, which may increase even when the voter's absolute I-power decreases. We may use $\beta$ for comparing the I-powers of different voters in one given SVG; but we cannot use it for comparing the voting powers of a voter in two different SVGs.

At first sight, it seems as though, in analogy with (4), the condition for the bloc being advantageous from the I-power viewpoint should be formalized as

$$\beta'_{\&_S}[\mathcal{W}|\&_S] > \sum_{x \in S} \beta'_x[\mathcal{W}]. \tag{6}$$

But this does not stand up to closer examination. For one thing, whereas the left-hand side of (4) represents the bloc's expected share of transferable utility, which can be directly portioned out among the partners, just as coffee can be poured out from a jug into several cups, the left-hand side of (6) represents the influence of the bloc, quantified as probability. How is influence-as-probability to be portioned out?

And whereas the right-hand side of (4) can be regarded as the total expected share of $S$ in the prize in $\mathcal{W}$, when its members act as separate individuals, the right-hand side of (6) does not have an analogous meaning. The terms of this sum are probabilities, and in general they are probabilities of events that are not disjoint from one another. So the sum does not represent anything like the 'total influence of $S$ in $\mathcal{W}$'—a concept that is in fact rather meaningless. The apparent analogy between the problems of forming a bloc from the viewpoint of P-power and that of I-power is a false one.

Nevertheless, it is intuitively clear that, from the latter viewpoint, if the bloc $\&_S$ is to be advantageous to all the partners, then each of them, each $x \in S$, must expect to obtain via $\&_S$ greater influence over the outcome of a division than s/he had individually. But how can $x$ obtain influence 'via $\&_S$' over the outcome? *Surely, the only way is for $x$ to influence the way $\&_S$ votes in a division of the assembly $N|\&_S$ of $\mathcal{W}|\&_S$.*

We reach a similar conclusion by approaching the problem from a somewhat different direction. The notion of I-power presupposes policy-seeking voting behaviour, whereby each voter votes on any given bill according to his or her own interests, which are totally exogenous to the SVG. But what might the 'interests' of the voluntarily formed bloc $\&_S$ be? What meaning can be ascribed to this concept? Let us take an example that is at least potentially realistic. Lane and Mæland (1995) consider a scenario in which four members of the Council of Ministers of the European Union (CMEU)—Italy, Spain, Greece and Portugal—form a *Mediterranean bloc*. (Here 'Mediterranean' is obviously used as a geo-political rather than strictly geographical term: Portugal does not have a Mediterranean shore, whereas France does.) This scenario implies that the four representatives of the *Medbloc* (as we may dub it) will always vote in the same way. They may even delegate their mandates to a single Medbloc representative. But Lane and Mæland hardly mean to suggest that, even under this hypothetical scenario, the four member-states would merge completely and cease to exist as separate countries, or that the interests of Italy on every single issue that may ever come before the CMEU will always coincide with those of Portugal. Surely, when forming the bloc the prospective partners must come to some binding agreement as to how to instruct the Medbloc delegate(s) to vote on any given bill.

These considerations suggest that in order to analyse the problem of bloc formation from the present viewpoint, that of I-power, we ought to postulate that when a bloc $\&_S$ is formed, the partners also fix a particular SVG $\mathcal{W}_S$, whose assembly is $S$. The job of this *internal* SVG is to decide, for each bill that comes before the 'top' SVG $\mathcal{W}|\&_S$, how the bloc $\&_S$ (or its delegate) will vote in $\mathcal{W}|\&_S$.

We shall call such a structure—a bloc $\&_S$ together with an internal SVG $\mathcal{W}_S$—an *alliance*.

(Note, by the way, that from the rival viewpoint, that of P-power, an internal SVG $\mathcal{W}_S$ was not needed. The notion of P-power presupposes office-seeking voting behaviour, whereby each voter bargains with other voters, striving to reach an agreement that will maximize his or her payoff. In forming the bloc $\&_S$, the partners must agree how to split its payoff between them; but they need not set up a special mechanism for instructing the delegate of $\&_S$ how to vote. This delegate will turn for instructions not to the partners, but to experts in the theory of bargaining and cooperative game theory, hoping to receive from them advice—for what it's worth—as to what optimal bargaining and voting strategy s/he ought to use.)

When the members of $S$ form an alliance whose internal SVG is $\mathcal{W}_S$, this gives rise to a new *composite* SVG, which we shall denote by '$\mathcal{W}\|\mathcal{W}_S$'. This is in fact a special case of the general operation of composition of SVGs, as defined in F&M (1998, Def. 2.3.12). In the Appendix of F&M (2002) we present a rigorous definition of $\mathcal{W}\|\mathcal{W}_S$ in that format. Here we shall just define $\mathcal{W}\|\mathcal{W}_S$ directly, in its own terms.

The assembly of $\mathcal{W}\|\mathcal{W}_S$ is $N$, the same as that of $\mathcal{W}$. The winning coalitions of $\mathcal{W}\|\mathcal{W}_S$ are all sets of the form $X \cup Y$, with $X \subseteq S$ and $Y \subseteq N - S$, satisfying at least one of the following two conditions:

- $Y$ is a winning coalition of $\mathcal{W}$;
- $X$ is a winning coalition of $\mathcal{W}_S$ and $S \cup Y$ is a winning coalition of $\mathcal{W}$.

Informally speaking, $\mathcal{W}\|\mathcal{W}_S$ works as follows. When a bill is proposed, the members of $S$ decide about it using $\mathcal{W}_S$, the internal SVG of their alliance. Then, when the bill is brought before the plenary, the assembly of $\mathcal{W}$, all the members of $S$ vote as a bloc, in accordance with their internal decision; so that now the final outcome is the same as it would have been in $\mathcal{W}|\&_S$ with the bloc voter $\&_S$ voting according to the internal decision.

Note that each member of $S$ now has *direct* I-power in the internal SVG $\mathcal{W}_S$, as well as *indirect* I-power in $\mathcal{W}\|\mathcal{W}_S$, which s/he exercises via the bloc $\&_S$.

Clearly, when the members of $S$ consider forming an alliance, they are well advised to compare their prospective indirect I-powers with the I-powers they have in the original SVG $\mathcal{W}$. We shall therefore say that an alliance with internal SVG $\mathcal{W}_S$ is *feasible* [*relative to a given SVG $\mathcal{W}$*] if

$$\beta'_x[\mathcal{W}\|\mathcal{W}_S] \geq \beta'_x[\mathcal{W}] \quad \text{for all } x \in S; \tag{7}$$

and we shall say that the alliance is *expedient* [*relative to a given SVG $\mathcal{W}$*] if

$$\beta'_x[\mathcal{W}\|\mathcal{W}_S] > \beta'_x[\mathcal{W}] \quad \text{for all } x \in S. \tag{8}$$

Moreover, we shall say that a bloc is *feasible* or *expedient* [*relative to a given SVG $\mathcal{W}$*] if there exists some internal SVG such that the resulting alliance is feasible or expedient, respectively.

We state here three general theorems, whose proofs are given in the Appendix of F&M (2002).

**3.1 Theorem** *For every $x \in S$*

$$\beta'_x[\mathcal{W}\|\mathcal{W}_S] = \beta'_x[\mathcal{W}_S] \cdot \beta'_{\&_S}[\mathcal{W}|\&_S].$$

∎

Thus, to obtain the indirect Bz power of $x$ in $\mathcal{W}\|\mathcal{W}_S$, multiply the direct Bz power of $x$ in $\mathcal{W}_S$ by the Bz power of the bloc $\&_S$ in $\mathcal{W}|\&_S$.

As for the Bz powers of voters $y \in N - S$ in $\mathcal{W}\|\mathcal{W}_S$, it is tempting to jump to the conclusion that they are the same as in $\mathcal{W}|\&_S$. But this is not generally true. The reason for this is that in the probabilistic characterization of $\beta'_y[\mathcal{W}|\&_S]$ it is assumed a priori that the bloc voter $\&_S$ votes 'yes' or 'no' with equal probability of $\frac{1}{2}$ (see Section 2). But in $\mathcal{W}\|\mathcal{W}_S$ the members of $S$, although they vote 'as a bloc', do not in general vote 'yes' or 'no' with equal a priori probability of $\frac{1}{2}$. They do so only in the special case where the number of winning coalitions of the internal SVG $\mathcal{W}_S$ is $2^{|S|-1}$, exactly half of the number of all coalitions. In this special case the Bz powers of voters $y \in N - S$ in $\mathcal{W}\|\mathcal{W}_S$ are indeed the same as in $\mathcal{W}|\&_S$.

**3.2 Theorem** *A bloc made up of two voters is never expedient. It is feasible iff originally the two voters have equal Bz powers, or at least one of them is a dummy.* ∎

**3.3 Theorem** *Let $a, b$ and $c$ be distinct voters of $\mathcal{W}$ such that $\beta'_a[\mathcal{W}] = \beta'_b[\mathcal{W}] \geq \beta'_c[\mathcal{W}]$. Then the bloc $\&_{\{a,b,c\}}$ is feasible. This bloc is expedient iff $c$ is not a dummy in $\mathcal{W}|\&_{\{a,b\}}$.* ∎

# 4 Summary of examples

In §§4 and 5 of F&M (2002) we present several examples of expedient and infeasible blocs. Some of these are toy examples, but some are hypothetical blocs that could be formed in the current CMEU under its weighted voting rule, known in Eurospeak as 'qualified majority voting' (QMV).

We shall summarize these examples here; for full details the reader is referred to F&M (2002).

The first type of example is rather unsurprising: these are cases in which inequality (6) of Section 3 holds and the bloc $\&_S$ turns out to be expedient. An example of this type is provided by the Medbloc mentioned in Section 3, consisting of Italy, Spain, Portugal and Greece under the current QMV in the CMEU. In fact, there are two different proper SVGs that these four members can use as an internal decision rule to create an expedient alliance. (For details, see F&M 2002, Example 5.1.)

As we shall explain in Section 5, it was the example of this particular bloc, discussed by Lane and Mæland (1995) and Garrett and Tsebelis (1999), which first aroused our interest in the topic of this paper. Other examples of potential blocs in the CMEU discussed by Lane and Mæland (1995) also belong to this first type.

A second type of case, which is also not surprising, is one in which the inequality opposite to (6), namely

$$\beta'_{\&_S}[\mathcal{W}|\&_S] < \sum_{x \in S} \beta'_x[\mathcal{W}], \tag{9}$$

holds, and the bloc $\&_S$ turns out to be infeasible. No example of this type is given in F&M (2002), so let us give one here. Let $\mathcal{W}$ be the WVG with assembly $\{a, b, c, d\}$ such that

$$\mathcal{W} \cong [5; 3, 2, 1, 1],$$

in alphabetical order. (Thus the voters, in alphabetic order, are assigned weights $3, 2, 1, 1$ and the quota is set at 5.) The Bz powers of the voters are $\frac{5}{8}, \frac{3}{8}, \frac{1}{8}$ and $\frac{1}{8}$ respectively. If $a, b$ and $c$ form a bloc, then in $\mathcal{W}|\&_{\{a,b,c\}}$ the bloc voter $\&_{\{a,b,c\}}$ is a dictator, and has Bz power 1—which is smaller than $\frac{9}{8}$, the sum of the original powers of $a, b$ and $c$. Also, there is no possible choice of $\mathcal{W}_{\{a,b,c\}}$ which would satisfy inequality (7) of Section 3 with $S = \{a, b, c\}$.

The third type of example is quite surprising to anyone who assumes that voting power, as quantified by the Bz measure, behaves like a transferable utility. Here inequality (9) holds, but the bloc $\&_S$ is nevertheless expedient. A simple toy example of this type is the following: Let $\mathcal{W}$ be the majority WVG with assembly $\{a, b, c, d, e, f\}$; thus

$$\mathcal{W} \cong [4; 1, 1, 1, 1, 1, 1].$$

That is, each voter has weight 1, and the quota is 4. Here the Bz power of each voter is $\frac{5}{16}$.

Now suppose that the first three voters form a bloc $\&_{\{a,b,c\}}$. We get a new WVG,

$$\mathcal{W}|\&_{\{a,b,c\}} \cong [4; 3, 1, 1, 1].$$

Here the bloc voter has Bz power $\frac{7}{8}$ and each of the remaining ones has $\frac{1}{8}$. Note that the Bz power of the bloc is *smaller* than the sum of the original

Bz powers of the three partners. Nevertheless, according to Theorem 3.3 the bloc is expedient. For details see F&M (2002, Example 4.1).

The fourth type of example exhibits a phenomenon that is the exact opposite of the previous one. Here inequality (6) of Section 3 holds, but the bloc $\&_S$ nevertheless turns out to be infeasible. Here is a toy example of this type. Let $\mathcal{W}$ be the WVG with assembly $\{a, b, c, d, e, f, g\}$ such that

$$\mathcal{W} \cong [11; 6, 5, 1, 1, 1, 1, 1],$$

in alphabetical order. Here the heaviest voter, $a$, has Bz power $\frac{33}{64}$ and each of the voters with weight 1 has Bz power $\frac{1}{64}$. Now let $a$ form a bloc with $c$ and $d$. Then

$$\mathcal{W}|\&_{\{a,c,d\}} \cong [11; 8, 5, 1, 1, 1],$$

in which the new bloc voter $\&_{\{a,c,d\}}$ has Bz power $\frac{9}{16} = \frac{36}{64}$. This is greater than $\frac{35}{64}$, the sum of the original Bz powers of the partners. Nevertheless, the bloc is infeasible: any internal SVG will either make $c$ and $d$ dummies, or give $a$ direct Bz power $\leq \frac{3}{4}$, hence indirect Bz power $\leq \frac{27}{64}$. (This is Example 4.4 in F&M 2002.)

The same type of behaviour is provided by an example based on the CMEU under its current QMV rule. If Germany, The Netherlands and Luxembourg were to form a bloc, then its Bz power would be greater than the sum of the present Bz powers of these three member states; but there does not exist any internal SVG that would make this bloc feasible. For details see F&M (2002, Example 5.3).

# 5 Discussion

Our interest in the topic of this paper was aroused by a critical comment of Garrett and Tsebelis (1999) on Lane and Mæland (1995). Lane and Mæland use the Bz index to investigate, among other things, the power distribution in the CMEU, with its QMV decision rule, under various scenarios of bloc formation.

The first scenario they consider is the formation of a Medbloc, consisting of Italy, Spain, Portugal and Greece. They do not raise the question as to whether the formation of this bloc (or indeed any of the other blocs they consider) would be advantageous. Their critics—who are vehemently opposed in general to the application of power indices to the European Union—rebuke them for this.

> According to Lane and Mæland, pooling the Mediterranean governments' votes would lead to a reduction in their combined power. If each voted separately in a 15-member Council, their combined power (using the Banzhaf normalized index) would be $0.112 + 0.092 + 0.059 + 0.059 = 0.332$ [*sic*] ... . Voting as a bloc, however, their index would be reduced to 0.247 ... . One should immediately ask the question: Why would these governments ever choose to vote as a bloc if in so doing they lose power? (Garrett and Tsebelis 1999, p. 296.)

Apart from the slight arithmetical or typographical error—0.332 instead of 0.322—this critique contains two fallacies.

First, the figures quoted—as all the figures in Lane and Mæland (1995)—are those for the [relative] Bz index rather than the [absolute] Bz measure. (This may be justified, since the issue studied there is potential changes in the *distribution* of voting power.) However, as we argued in the beginning of Section 3, it makes no sense to use the Bz index when enquiring whether a bloc is advantageous. Surely, when considering the formation of a Medbloc, the four potential partners are primarily interested in the consequent changes in their *absolute* I-power.

Second, while the figures given by Lane and Mæland (1995) are those for [relative] I-power, measured by the Bz index, the criticism in Garrett and Tsebelis (1999) treats them as though they were values of an index of P-power, and applies the criterion of our inequality (5) in Section 2 to imply that the Medbloc would be disadvantageous. This is another fallacy against which we warned in the beginning of Section 3.

Actually, as we noted in Section 4, the [absolute] Bz power of the Medbloc would be greater than the sum of the present Bz powers of the four partners. But, as illustrated by examples of the fourth type outlined in Section 4, this in itself does not guarantee that the Medbloc is expedient or even feasible: it all depends on whether the four partners can find a suitable internal SVG. In this particular case it turns out that they can; in fact, there are two such proper SVGs. On the other hand, as shown by examples of the third type, a bloc can be expedient even if its Bz power is smaller than the sum of the original Bz powers of the partners.

To conclude, let us note that the theory developed in this paper raises an interesting issue in the theory of cooperative games with *non-transferable* utility.

Starting from a given SVG $\mathcal{W}$, consider the following game of *alliance formation*. A play of this game consists of forming one or more alliances among the voters of $\mathcal{W}$. (Of course, if two or more alliances are formed simultaneously, then their members must be disjoint: no voter of $\mathcal{W}$ is allowed to belong to more than one alliance.) This gives rise to a new composite SVG, which can be defined along the same lines as $\mathcal{W}\|\mathcal{W}_S$; the latter being the special case in which just one alliance is formed. The Bz power of each voter in this composite SVG can be regarded as a payoff; but it is a non-transferable quantity, so no side payments are admitted.

One can then ask various questions regarding this game: for example, what system of alliances can be regarded as stable, and what conditions must $\mathcal{W}$ satisfy in order that the alliance-formation game associated with it should have a stable solution.

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