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Portfolio credit risk of default and spread widening
Research paper

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Abstract
This paper introduces a new model for portfolio credit risk incorporating default and spread widening in a simple and consistent framework. Credit spreads are modelled by geometric Brownian motions with a dependence structure powered by a $t$-copula. Their joint evolution drives the spreads widening and triggers defaults, and then the loss can be calculated accordingly. It is a heterogeneous model that takes account of different credit ratings and term structures for each underlying spread. This model is applicable to portfolio credit risk management, stress test, or to fit into regulatory capital requirements. The procedures of parameter calibration and scenario simulation are provided. A detailed example is also given to see how this proposed model can be implemented in practice.

Keywords: Portfolio credit risk, Stress test, Economic capital, Default risk, Spread widening risk, Copula, Basel III

1. Introduction
Credit risk and market are intrinsically related to each other and, more importantly, they are not separable (Jarrow and Turnbull 2000). During the recent financial crisis, especially after the collapse of Lehman Brothers in September 2008, the interaction of credit risk and market becomes more evident. It has exposed some severe problems in the original systems of risk assessment, management and supervision, due to inadequate recognition of this interaction. This is also one of the main issues that have pushed G20 to propose a major agreement of the Basel III for a global banking reform. It is a new global regulatory standard on bank capital adequacy and liquidity, and will impose new capital requirements on the world’s banking system, in an effort to strengthen the ability of absorbing shocks from economic stress and to avoid future financial meltdowns.

Modelling the credit risk itself has already been a challenge, as more credit products are more frequently bought and sold, instead of being held to maturity. Nowadays, researchers, practitioners and regulators start to explore the complexity of the interactive behavior of credit risk and market, and attempt to develop more sophisticated models to capture the credit risk and
other forms of risk in a consistent methodological framework. Simply calculating the market risk and credit risk separately and then summing them up could either underestimate or overestimate the overall risk involved (BCBS, 2009). The evidences were provided by Breuer, Jandačka, Rheinberger and Summer (2010), and Alessandri and Drehmann (2010). Gupton, Finger and Bhatia (1997) developed a mark-to-market credit migration model, CreditMetrics, widely used in practice for portfolio credit risk management and economic capital calculation. Kupiec (2007) extended the CreditMetrics model to additionally incorporate the valuation effects of market risk on non-default credits. Altman, Brady, Resti and Sironi (2005) and Bruche and González-Aguado (2010) investigated the negatively correlated interaction between default and recovery rate. Alessandri and Drehmann (2010) implemented the stress test by integrating the default and interest rate risk. Tang and Yan (2010) analysed the impact of the interaction between market and default on corporate credit spreads based on macroeconomic factors, such as the GDP growth rate and its volatility, and consumer confidence. In particular, inspiring discussion and comprehensive summary on this issue can also be found from the conference on “Interaction of Market and Credit Risk” (IMCR), held by Deutsche Bundesbank and the working group of Research Task Force established by Basel Committee on Banking Supervision (BCBS) in Berlin in 2007 (Hartmann, 2010; BCBS, 2008).

In this paper, we attempt to model the interaction of default risk and spread widening risk, which has been often neglected before the recent financial crisis. Credit spread risk as defined by BCBS (2008) is the risk of potential loss due to a change in an instrument’s credit spread (defined as the instrument’s yield relative to that of a comparable-duration default-free instrument) that is not attributable to defaults or credit migrations (e.g. a change in liquidity premia). During the recent credit market turmoil, banks were hit hard by these structured credit instruments they held in their trading books, with significant losses coming from spread widening (Madigan 2010). Calculating this risk has been required by new financial capital regulation, in particular, the incremental risk charge (IRC). The IRC expands the scope of the capital charge to capture not only price changes due to defaults but also other sources of price risk, such as significant moves of credit spreads (BCBS, 2008). Banks must conduct stress tests that include widening credit spreads in recessionary scenarios, as also proposed by Basel III.

This paper introduces a new portfolio credit risk model incorporating default and spread widening in one consistent framework over a single horizon. It is applicable to credit risk management, stress test, or to fit into the new regulatory requirements being raised from the recent financial crisis, in particular, the IRC. This model is similar to the seminal Merton (1974)’s model, a structural model based on an unobservable underlying firm’s value. The main difference is that we assume the underlying processes are the firms’ credit spreads rather than the firms’ values, and they have a dependence structure powered by a copula. Their joint evolution drives the spreads widening and triggers defaults simultaneously, and then the loss can be calculated accordingly. The credit spreads are observable from credit market and can be calibrated from historical data. They are also closely linked to the underlying default risk by implying the default probability under risk-neutral measure (Hull, Predescu and White, 2005). This is a simple heterogeneous model that can take account of different credit ratings and term structures for each underlying spread, and also can integrate different marginal distributions for credit spreads with a flexible dependence structure via a copula function. The setting of heterogeneity is similar to the CreditMetrics model by Gupton, Finger and Bhatia (1997). The parameters can be calibrated from the historical daily time series of corporate bond spreads, and our model then can be imple-
mented by Monte Carlo simulation for a bond portfolio in practice.

The paper is organised as follows. Section 2 gives a mathematical description of our framework, including the marginal distribution for each credit spread in Section 2.1, and a dependence structure for all spreads in a bond portfolio in Section 2.2. Based on this common framework, the credit risk of spread widening and default can be modelled respectively in Section 3.1 and Section 3.2, and an integrated model of overall credit risk is given in Section 3.3. We use the database of historical corporate bond spreads as given in Section 4 and calibrate parameters in Section 5. In Section 6 we provide a detailed example to demonstrate how this proposed model can be implemented in industrial practice. Section 7 concludes this paper and gives some suggestions for future research.

2. Model Framework

The whole framework for modelling the joint evolution of credit spreads consists of two parts:

1. a marginal distribution for the spread evolution of each bond in the portfolio (in Section 2.1);
2. a copula function for the dependence structure of these spreads (in Section 2.2).

We have investigated this framework by using different types of marginal distributions and copulas. To illustrate the modelling idea, in this paper, we only use log-normal marginal distributions with a t-copula dependency as an example.

2.1. Marginal Distribution for the Individual Spread

There are $N$ bonds in the portfolio, and we assume $\{S_{t,j}^{<j}\}_{t \geq 0}$, the spread of bond $j \in \{1, 2, ..., N\}$, follows a geometric Brownian motion,

$$\frac{dS_{t,j}^{<j}}{S_{t,j}^{<j}} = \mu_d dt + \sigma_d dW_{t,j}^{<j},$$

where the drift $\mu_d$, volatility $\sigma_d$ are positive constants, and $W_{t,j}^{<j}$ is a Brownian motion. It is well known that the analytic solution is given by

$$S_{t,j}^{<j} = S_{0,j}^{<j} \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma_d dW_{t,j}^{<j}\right),$$

and $S_{t,j}^{<j}$ follows the log-normal distribution,

$$\ln S_{t,j}^{<j} \sim N\left(\ln S_{0,j}^{<j} + \left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right).$$

Since $\mu_d - \frac{1}{2}\sigma_d^2$ is very small$^1$, we simply assume $\mu_d - \frac{1}{2}\sigma_d^2 \approx 0$, then,

$$\ln S_{t,j}^{<j} \sim N\left(\ln S_{0,j}^{<j}, \sigma^2 t\right).$$

---

$^1$This can be verified by the observation from the spread database in Section 4.
namely,
\[ S_t^{<j_p} = S_0^{<j_p} e^{\sigma_j t} \sqrt{Z_j} \]  
(1)

where \( Z_j \sim N(0, 1) \).

To build a heterogeneous model that can take account of different credit ratings and term structures for each bond in the portfolio, we assume the parameter volatility is a function of credit rating \( R \) and maturity bucket \( T \), i.e.
\[ \sigma_j = \sigma(R, T) \]

where \( R \in \{AAA, AA, A, BBB\} \), and \( T \in \{0Y - 05Y, 05Y - 10Y, 10Y - 15Y, 15Y - 20Y, 20Y+\} \) grouped according to the bond’s time to maturity.

2.2. Dependence Structure for All Spreads

We adopt the copula approach for modelling the dependence structure because of its flexibility of incorporating different marginal distributions and the ease of simulation (Li, 2000; Nelsen, 2006; Kole, Koedijk and Verbeek, 2007). There are a variety of other techniques for dependency modelling in the literature, such as the methodology of credit contagion introduced by Jarrow and Yu (2001), Errais, Giesecke and Goldberg (2009), and more recently Dassios and Zhao (2011).

The dependence structure of all spreads \( \vec{S}_t = (S_t^{<1}, ..., S_t^{<N})' \) given by (1) is built via a vector of dependent standard normal distributed random variables \( \vec{Z} = (Z_1, ..., Z_N)' \), with a dependence structure following a \( t \)-copula parameterised by a \( N \times N \) correlation matrix

\[
\Sigma = \begin{bmatrix}
1 & \rho_{1,2} & \cdots & \rho_{1,N} \\
\rho_{2,1} & 1 & \cdots & \rho_{2,N} \\
\vdots & \ddots & \ddots & \vdots \\
\rho_{N-1,1} & \rho_{N-1,2} & \cdots & 1 \\
\rho_{N,1} & \rho_{N,2} & \cdots & 1
\end{bmatrix}
\]  
(2)

and degree of freedom \( \nu \). \( \rho_{i,j} \) is the correlation coefficient for bond \( i \) and \( j \) (\( i \neq j \)) in the portfolio.

We adopt the procedure by Romano (2002) to construct the vector \( \vec{Z} \) as follows:

1. generate a vector of independent random variables \( \vec{X} = (X_1, ..., X_N)' \) where \( X_j \sim N(0, 1) \);
2. obtain a vector of dependent random variables \( \vec{Y} = (Y_1, ..., Y_N)' \) with a joint distribution \( \Phi_\Sigma(\vec{Y}) \) where
\[
\vec{Y} = \mathcal{A} \vec{X},
\]
and matrix \( \mathcal{A} \) is the Cholesky decomposition of \( \Sigma \), i.e.
\[ \Sigma = \mathcal{A} \mathcal{A}^T; \]

2We assume each bond is in the investment grade.
obtain a vector of dependent random variables \( \tilde{t} = (t_1, ..., t_N)' \) by

\[
\tilde{t} = \frac{Y}{\sqrt{c}}
\]

where \( c \) is a random variable following a chi-squared distribution \( \chi^2_\nu \), and independent of \( \tilde{X} \);

4. transform vector \( \tilde{t} \) to a unit space \([0, 1]^N\) by

\[
(U_1, ..., U_N) = \left( t_1(t_1), ..., t_N(t_N) \right).
\]

and then vector \( \tilde{U} =: (U_1, ..., U_N)' \) has a \( t \)-copula dependence structure;

5. \( \tilde{Z} \) is constructed via

\[
(Z_1, ..., Z_N) = \left( \Phi^{-1}(U_1), ..., \Phi^{-1}(U_N) \right)
\]

where \( \Phi^{-1}(\cdot) \) is the inverse function of the accumulative standard normal distribution.

We assume the correlation coefficient for bond \( i \) and \( j \) \((i \neq j)\) is a function of their credit ratings, i.e.

\[
\rho_{i,j} = \rho(R_1, R_2) \quad (i \neq j)
\]

where \( R_1, R_2 \in \{AAA, AA, A, BBB\} \), and \( \rho_{i,j} = 100\% \) for \( i = j \). This assumption is based on the observation of the joint evolution of spreads from, for instance, the historical time series of benchmark spreads of \( \text{iBoxx Index 01/03/2006–27/06/2008} \) for rating-maturity buckets as given by Figure 1. Their dependence pattern of joint movement is particularly evident during the period of credit crunch.

Remark 2.1. We use a \( t \)-copula rather than Gaussian copula, since the dependence structure in a \( t \)-copula is controlled by not only a \( N \times N \) correlation matrix \( \Sigma \) given by (2) (which is the same as Gaussian copula) but also an extra parameter – degree of freedom \( \nu \). It provides higher
flexibility to capture joint extreme events, which are the center concern of the risk management (Embrechts, McNeil and Straumann, 1999). Also, the Gaussian copula can also be recovered by setting \( \nu \) to infinity.

In this paper, we initially assume degree of freedom \( \nu = 3 \) for a \( t \)-copula for instance, then compare the results by choosing different \( \nu \). Based on the assumptions given above, the parameters in our model needed to be calibrated are volatility \( \sigma_j = \sigma(R, T) \) for each bond and correlation \( \rho_{ij} = \rho(R_1, R_2) \) \((i \neq j)\) for each pair of bonds. The detail for this calibration is provided in Section 5.

3. Portfolio Credit Risk of Spread Widening and Default

Based on the framework for the joint evolution of credit spreads given by Section 2, now we can model the portfolio credit risk of spread widening and default consistently. In particular, we provide two types of models:

1. stand-alone models for the spread widening risk (in Section 3.1) and the default risk (in Section 3.2);
2. an integrated model for overall risk of spread widening and default (in Section 3.3).

We assume there is no default (loss) in the stand-alone model for spread widening risk, and there is no spread widening loss for a bond given default in the integrated model.

3.1. Credit Risk of Spread Widening

In the short term, the major contribution of credit risk for a portfolio is from the spread widening, rather than default. The credit spread widens, the value of the bond decreases, by mark-to-market valuation. This risk is usually measured daily, monthly, or yearly for risk management.

An approximate formula for calculating the value loss for any bond \( j \) due to spread widening is given by

\[
L^W_j = PV_j(1 + \Delta_j)^{-D_j} - PV_j
\]  

(5)

where \( PV_j \) is the present value, \( \Delta_j \) is the change of spread, \( S_j^{<j,p} - S_0^{<j,p} \), and \( D_j \) is the duration. By (1), the changes of spreads can be expressed as \( \bar{\Delta} = (\Delta_1, ..., \Delta_N)' \) where

\[
\Delta_j = S_0^{<j,p} \left( e^{\sigma_j \sqrt{t} Z_j} - 1 \right).
\]  

(6)

Therefore, based on the assumption of no default (loss) in this stand-alone model, the portfolio loss \( L^W \) due to spread widening is given by

\[
L^W = \sum_{j=1}^{N} L^W_j,
\]  

(7)

or, the portfolio loss in percentage

\[
L^W% = \frac{\sum_{j=1}^{N} L^W_j}{\sum_{j=1}^{N} PV_j}.
\]  

(8)
3.2. Credit Risk of Default

In the long term, the main portfolio credit risk is from default. Default model can be consistently incorporated into the spread widening model above. The approach used here is similar to the credit barrier models, developed by Iscoe, Kreinin and Rosen (1999), Hull and White (2001), and later extended by Albanese and Chen (2006).

We assume the default probabilities \( \vec{p} = (p_1, ..., p_N)' \) are given, for instance, by Moodys (2009), and depend on the credit rating and time to maturity, i.e.

\[ p_j = p(\mathcal{R}, T) \]

where \( \mathcal{R} \in \{\text{AAA, AA, A, BBB}\} \) and \( T \) is time to maturity.

Based on the inverse transform sampling method from probability theory, there are three types of equivalent boundaries for triggering the default of bond \( j \) in the portfolio:

1. if the uniform distributed random variable \( U_j \) overshoots the boundary \( 1 - p_j \) (i.e. the survival probability);
2. if the standard normal distributed random variable \( Z_j \) overshoots the boundary \( b_j \) where

\[ b_j = \Phi^{-1}(1 - p_j) \]

3. if the change of spread \( \Delta_j \) exceeds the boundary \( c_j \) where

\[ c_j = S_{0}^{-\epsilon\mathcal{P}} \left( e^{\sigma_{j} \sqrt{t} b_{j}} - 1 \right) \]

\( U_j, Z_j \) and \( \Delta_j \) are all latent random variables with a common dependence structure powered the \( t \)-copula constructed in Section 2.2. \( 1 - p_j, b_j \) and \( c_j \) are called the corresponding implied default boundaries.

Then, the default probability for bond \( j \) is equivalently given by

\[ p_j = \mathbb{P}(U_j > 1 - p_j) = \mathbb{P}(Z_j > b_j) = \mathbb{P}(\Delta_j > c_j) \]

and the default indicator for bond \( j \) is given by

\[ I_j = \mathbb{I}(U_j > 1 - p_j) = \mathbb{I}(Z_j > b_j) = \mathbb{I}(\Delta_j > c_j) \]

where \( \mathbb{I} \) is an indicator function. Hence, we have the portfolio default scenario \( \vec{I} = (I_1, ..., I_N)' \), and the loss due to default for bond \( j \) can be calculated by

\[ L^D_j = -I_j PV_j (1 - \delta_j) \]

where \( \delta_j \) is the recovery rate from the present value of bond \( j \). The whole portfolio loss due to default is given by

\[ L^D = \sum_{j=1}^{N} L^D_j \]

or, the portfolio loss in percentage

\[ L^D\% = \frac{\sum_{j=1}^{N} L^D_j}{\sum_{j=1}^{N} PV_j} \]
3.3. Overall Credit Risk

The stand-alone credit risk assessment for spread widening and default risk separately are given above in Section 3.1 and Section 3.2. Now we consider the integrated model for overall credit risk involved.

To avoid double-counting two types of risk, in the integrated model, we assume there will be no any further loss of spread widening for a bond given default. This is the major difference between the stand-alone models and the integrated risk model. Hence, the aggregate portfolio loss of default and spread widening is given by

\[ L = \sum_{j=1}^{N} L_j \]

where

\[ L_j = (1 - I_j)L_j^W + I_jL_j^D, \]  

or, the portfolio loss in percentage

\[ L\% = \frac{\sum_{j=1}^{N} L_j}{\sum_{j=1}^{N} PV_j}. \]  

**Remark 3.1.** The loss of spread widening in this integrated model is capped by the loss due to default. The worst scenario, or the maximum total loss in percentage, is given by

\[ \lim_{S^j \to \infty} L\% = 1 - \bar{\delta} \]

where \( \bar{\delta} = \frac{\sum_{j=1}^{N} \delta_j}{N} \) is the average recovery rate and \( S^j \to \infty \) means all spreads in the portfolio simultaneously widen to infinity, i.e. \( S^j \to \infty \) for all \( j \), and trigger joint default of all bonds.

The stand-alone default loss \( L^D\% \) in Section 3.2 is actually the loss of default without spread interaction, and the overall loss \( L\% \) can be decomposed by

\[ L\% = L^D\% + L^{I}\%, \]

where \( L^{I}\% \) is the extra loss due to spread interaction, upon the stand-alone default loss \( L^D\% \).

4. Data Description

We use two types of data:

- **Spread Data:** the historical daily time series of bond spreads over Gilts in the *iBoxx Sterling Universe 01/01/1999–13/05/2008* (including a period of credit crunch), provided by *Deutsche Bank*.

- **Default Data (Table 1):** the 2008’s annual issuer-weighted corporate default rates \( p(R, T = 1\text{ year}) \) from Moody’s (2009).
In particular, the spread data includes two separate csv-files:

- **SpreadData.csv**: a \((2443 \text{ dates} \times 1159 \text{ issuers})\) matrix of the daily spreads (in basis point) over Gilts for all \(GBP \text{ iBoxx}\) bonds from 01/01/1999 to 13/05/2008 including all business days.

- **Description.csv**: a \((1159 \text{ issuers} \times 7 \text{ descriptions})\) matrix provides information of ISIN, Maturity Date, Maturity Bucket, Type, Sector, Rating and Issuer.

A specified sample of time series of spreads in SpreadData.csv and the associated information in Description.csv is given by Figure 2 and Table 2, respectively. Note that, there is no data at the beginning of the plot in Figure 2, as this bond had not yet been issued.

**Remark 4.1.** We assume the two rating systems by Moody’s and Standard & Poor’s have the equivalence: \(Aaa \equiv AAA, Aa \equiv AA, A \equiv A, Baa \equiv BBB\).

**Remark 4.2.** There are two types of bonds in the database: Corp and Govt, here we only select Corp-type bonds for calibration, as we assume there is no credit risk for the bonds issued by governments.

**5. Parameter Calibration**

We calibrate the parameters (by programming in R) from the spread database by using a procedure similar to Blamont, Hauviller and Prieul (2007). The calibration is based on the daily
time series of credit spreads excluding the dates before the bond offering.

To be consistent with our bucketing assumption given in Section 2, we group the bonds in the database by (4 ratings × 5 maturity buckets) 20 rating-maturity buckets. The mean of the spreads is calculated based on the actual spreads\(^3\). The volatility and correlation are calibrated on the daily changes of log-spreads (or continuously compounded rate).

The calibration procedure and the corresponding results are given as below:

- **Mean Calibration** *(Table 3)*:
  1. calculate the average of daily spreads for each bond;
  2. average by each rating-maturity bucket.

- **Volatility Calibration** *(Table 4)*:
  1. calculate the volatility of daily log-spread changes for each bond;
  2. average by each rating-maturity bucket, i.e. \(\sigma(R, T)\) where \(R \in \{AAA, AA, A, BBB\}\) and \(T \in \{00Y - 05Y, 05Y - 10Y, 10Y - 15Y, 15Y - 20Y, 20Y^*\}\);
  3. convert the daily volatility \(\sigma(R, T)\) to the yearly volatility \(\sigma(R, T) \times \sqrt{t}\) where time \(t = 260\) days\(^4\).

- **Correlation Calibration** *(Table 5)*:
  1. calculate the correlation for each pair of daily log-spread changes, and obtain a full (1159 issuers × 1159 issuers) correlation matrix;
  2. average this full matrix by each rating-rating bucket, and obtain a concentrated (4 ratings × 4 ratings) correlation matrix \(\rho(R_1, R_2)\) where \(R_1, R_2 \in \{AAA, AA, A, BBB\}\).

### Table 3: The Mean of Spreads (bp) by Rating-maturity Bucket

<table>
<thead>
<tr>
<th></th>
<th>00Y-05Y</th>
<th>05Y-10Y</th>
<th>10Y-15Y</th>
<th>15Y-20Y</th>
<th>20Y*</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>49.36</td>
<td>51.76</td>
<td>67.18</td>
<td>70.46</td>
<td>54.76</td>
</tr>
<tr>
<td>AA</td>
<td>92.65</td>
<td>114.84</td>
<td>118.49</td>
<td>109.30</td>
<td>91.67</td>
</tr>
<tr>
<td>A</td>
<td>118.13</td>
<td>150.81</td>
<td>139.28</td>
<td>126.76</td>
<td>126.3</td>
</tr>
<tr>
<td>BBB</td>
<td>148.32</td>
<td>151.78</td>
<td>170.30</td>
<td>186.83</td>
<td>138.15</td>
</tr>
</tbody>
</table>

### 6. Model Implementation

In this section, we use a specified example portfolio to demonstrate how this heterogeneous model can be implemented to capture the portfolio credit risk of default and spread widening in a single consistent framework, by using the mathematical framework in Section 2, 3 and the calibrated parameters in Section 5.

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\(^3\)The mean calibration is only used to investigate the average level of spreads for each bucket and provide a board picture of the database used for calibration, but it will not be implemented in our model.

\(^4\)We assume there are 260 business days in one year.
Table 4: The Yearly Volatility $\sqrt{\frac{260}{\nu}} \sigma(R, T)$ of Log-spread Changes by Rating-maturity Bucket

<table>
<thead>
<tr>
<th></th>
<th>00Y-05Y</th>
<th>05Y-10Y</th>
<th>10Y-15Y</th>
<th>15Y-20Y</th>
<th>20Y+</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>85.81%</td>
<td>59.69%</td>
<td>71.86%</td>
<td>50.50%</td>
<td>68.42%</td>
</tr>
<tr>
<td>AA</td>
<td>42.98%</td>
<td>34.94%</td>
<td>31.48%</td>
<td>39.01%</td>
<td>56.30%</td>
</tr>
<tr>
<td>A</td>
<td>45.35%</td>
<td>34.33%</td>
<td>27.99%</td>
<td>33.13%</td>
<td>32.80%</td>
</tr>
<tr>
<td>BBB</td>
<td>42.48%</td>
<td>30.11%</td>
<td>38.57%</td>
<td>33.24%</td>
<td>50.40%</td>
</tr>
</tbody>
</table>

Table 5: The Correlation Matrix $\rho(R_1, R_2)$ of Log-spread Changes by Rating Bucket

<table>
<thead>
<tr>
<th></th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>18.26%</td>
<td>15.84%</td>
<td>8.50%</td>
<td>13.47%</td>
</tr>
<tr>
<td>AA</td>
<td>15.84%</td>
<td>14.92%</td>
<td>9.10%</td>
<td>11.48%</td>
</tr>
<tr>
<td>A</td>
<td>8.50%</td>
<td>9.10%</td>
<td>10.43%</td>
<td>7.15%</td>
</tr>
<tr>
<td>BBB</td>
<td>13.47%</td>
<td>11.48%</td>
<td>7.15%</td>
<td>10.93%</td>
</tr>
</tbody>
</table>

One year’s period of time well combines the short and long term perspectives, and it is also a convention for risk management and regulatory requirements. Hence, we assess the credit risk over one-year capital horizon for an example portfolio composed of 20 corporate bonds with the detail specified by Figure 3.

We model the volatility and correlation for each bond in the example portfolio by matching its rating and maturity to the corresponding bucket of calibrated volatility and correlation provided by Table 3 and Table 4, respectively. Hence, the results of yearly volatilities $\{\sqrt{\frac{260}{\nu}} \sigma_j\}_{j=1,2,...,20}$ and the $20 \times 20$ correlation matrix $\Sigma$ for this example portfolio are specified by Figure 4.

6.1. Credit Risk of Spread Widening

According to Section 3.1, now we have all information needed for calculating the portfolio percentage loss $L^W\%$ due to one year’s spread widening:

- the calibrated parameters for spreads: the yearly volatility $\sqrt{\frac{260}{\nu}} \sigma_j$ as given by the first column of the table in Figure 4;
- the calibrated parameters for $t_{\nu,3}$-copula: the $20 \times 20$ full correlation matrix $\Sigma$ for the whole portfolio as given by Figure 4;
- the information for each bond in the portfolio: present value $PV_j$, duration $D_j$, and current spread $S_j^{<}$ as given by Figure 3.

Due to the heterogeneity of this model, there is no analytic solution for the whole loss of the portfolio, hence we implement Monte Carlo simulation for $\vec{Z}$ with 500,000 sample paths. The percentiles of loss percentage $L^W\%$ are given by Table 6.

To investigate the underlying scenarios, for each percentile of $L^W\%$ in Table 6, we present a sample of realised underlying spread changes in Figure 5 from the simulation. Take the percentile 100th% for instance, we can observe from the last row of Figure 5 that, it is mainly the significant spread increase of these specific bonds in the portfolio that contributes to a total loss of 39.15%.
Table of Sample Portfolio Detail on 13/05/2008

<table>
<thead>
<tr>
<th>Index</th>
<th>Rating</th>
<th>PV</th>
<th>Duration</th>
<th>Maturity Bucket</th>
<th>Current Spread</th>
<th>Nominal Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>AA</td>
<td>953,557</td>
<td>5.96</td>
<td>05Y-10Y</td>
<td>258.39</td>
<td>1,069,375</td>
</tr>
<tr>
<td>2</td>
<td>A</td>
<td>279,189</td>
<td>6.49</td>
<td>05Y-10Y</td>
<td>427.07</td>
<td>331,875</td>
</tr>
<tr>
<td>3</td>
<td>BBB</td>
<td>606,531</td>
<td>3.81</td>
<td>05Y-05Y</td>
<td>84.34</td>
<td>590,000</td>
</tr>
<tr>
<td>4</td>
<td>AA</td>
<td>310,798</td>
<td>5.67</td>
<td>05Y-10Y</td>
<td>9.67</td>
<td>295,000</td>
</tr>
<tr>
<td>5</td>
<td>A</td>
<td>1,954,029</td>
<td>7.20</td>
<td>05Y-10Y</td>
<td>209.93</td>
<td>1,750,000</td>
</tr>
<tr>
<td>6</td>
<td>A</td>
<td>1,619,433</td>
<td>7.80</td>
<td>10Y-15Y</td>
<td>113.46</td>
<td>1,585,625</td>
</tr>
<tr>
<td>7</td>
<td>A</td>
<td>2,149,645</td>
<td>8.68</td>
<td>10Y-15Y</td>
<td>210.25</td>
<td>2,000,000</td>
</tr>
<tr>
<td>8</td>
<td>AAA</td>
<td>3,449,815</td>
<td>12.35</td>
<td>15Y-20Y</td>
<td>54.93</td>
<td>3,262,000</td>
</tr>
<tr>
<td>9</td>
<td>A</td>
<td>63,994</td>
<td>6.62</td>
<td>05Y-10Y</td>
<td>351.17</td>
<td>60,000</td>
</tr>
<tr>
<td>10</td>
<td>A</td>
<td>2,070,411</td>
<td>2.78</td>
<td>05Y-05Y</td>
<td>285.22</td>
<td>2,101,875</td>
</tr>
<tr>
<td>11</td>
<td>AA</td>
<td>1,907,433</td>
<td>2.88</td>
<td>05Y-05Y</td>
<td>152.42</td>
<td>2,000,000</td>
</tr>
<tr>
<td>12</td>
<td>AA</td>
<td>672,450</td>
<td>6.45</td>
<td>05Y-10Y</td>
<td>354.08</td>
<td>700,000</td>
</tr>
<tr>
<td>13</td>
<td>BBB</td>
<td>1,676,103</td>
<td>4.97</td>
<td>05Y-10Y</td>
<td>133.29</td>
<td>1,735,125</td>
</tr>
<tr>
<td>14</td>
<td>A</td>
<td>1,920,156</td>
<td>6.82</td>
<td>05Y-05Y</td>
<td>113.68</td>
<td>1,900,000</td>
</tr>
<tr>
<td>15</td>
<td>BBB</td>
<td>668,915</td>
<td>9.79</td>
<td>10Y-15Y</td>
<td>138.46</td>
<td>663,750</td>
</tr>
<tr>
<td>16</td>
<td>AA</td>
<td>9,321,789</td>
<td>6.90</td>
<td>05Y-10Y</td>
<td>446.81</td>
<td>10,000,000</td>
</tr>
<tr>
<td>17</td>
<td>A</td>
<td>6,086,188</td>
<td>13.93</td>
<td>20Y+</td>
<td>180.26</td>
<td>6,100,000</td>
</tr>
<tr>
<td>18</td>
<td>A</td>
<td>4,889,691</td>
<td>8.15</td>
<td>10Y-15Y</td>
<td>346.26</td>
<td>5,000,000</td>
</tr>
<tr>
<td>19</td>
<td>AA</td>
<td>349,339</td>
<td>10.94</td>
<td>20Y+</td>
<td>146.81</td>
<td>290,000</td>
</tr>
<tr>
<td>20</td>
<td>BBB</td>
<td>7,357,536</td>
<td>10.79</td>
<td>10Y-15Y</td>
<td>154.80</td>
<td>3,000,000</td>
</tr>
</tbody>
</table>

Figure 3: The Detail of the Example Portfolio

Table 6: The Percentiles of Loss Percentage $L^\%$ Due to One Year’s Spread Widening

<table>
<thead>
<tr>
<th>Percentile</th>
<th>50th%</th>
<th>90th%</th>
<th>95th%</th>
<th>97.5th%</th>
<th>99th%</th>
<th>99.5th%</th>
<th>100th%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loss in Percentage (L%)</td>
<td>0.52%</td>
<td>5.39%</td>
<td>7.27%</td>
<td>9.17%</td>
<td>11.75%</td>
<td>13.77%</td>
<td>39.15%</td>
</tr>
</tbody>
</table>

Figure 4: The Yearly Volatilities ($\sqrt{256 \sigma_{ij}^2}$) and The Full Correlation Matrix $\Sigma$ for the Portfolio
Remark 6.1. Note that, the underlying scenarios corresponding to each loss percentile are not necessary unique, particularly for these lower percentiles. We only provide one of the samples in Figure 5 from the simulation.

6.2. Credit Risk of Default

According to Section 3.2, we can model the default rate \( p_j \) for each bond, by matching its rating to one year’s default rate in Table 1, and use latent variables against the implied boundaries for triggering defaults. For any bond \( j \), by (9), we can provide three types of equivalent implied boundaries, \( 1 - p_j, b_j \) and \( c_j \), under distributions of uniform \( U_j \) (as given by (3)), normal \( Z_j \) (as given by (4)) and change of spread \( \Delta_j \) (as given by (6)), respectively. The results are given by Figure 6.

We can simply compare the change of spread \( \Delta_j \) already simulated in Section 6.1 with the corresponding implied boundary \( c_j \), and determine whether bond \( j \) has defaulted or not. By using the information of the present value for each defaulted bond as given by Figure 3 and recovery rate \( \delta = 40\% \), the percentiles of loss percentage \( L_j \% \) due to one year’s bond defaults are given.
For each percentile in Table 7, we provide one sample of the underlying scenarios $I$ in Figure 7. In each scenario, the default is indicated by $I$ and the no default by $0$. Each $I$ or $0$ is determined by comparing the underlying realised spread change $\Delta_j$ with the corresponding implied boundary $c_j$ given by Figure 6. If $\Delta_j$ overshoots $c_j$, then it generates a number of $I$; otherwise, it gives $0$. Take the worst scenario 100th% for instance, we can identify that it is the default of these specific bonds indicated by $I$ in the last row of Figure 7 that contributes to the total loss as much as 53.36%.

### 6.3. Overall Credit Risk

According to Section 3.3, now we assess the overall portfolio credit risk of spread widening and default. The distribution and percentiles 5 of integrated portfolio loss $L\%$ are given by Figure 8 and Table 8, respectively.

In Figure 9, similarly, if the spread increment $\Delta_j$ has exceeded the corresponding implied boundary $c_j$ (given by Figure 6), then the bond is assumed to have defaulted. Each defaulted

<table>
<thead>
<tr>
<th>Percentile</th>
<th>50th%</th>
<th>90th%</th>
<th>95th%</th>
<th>97.5th%</th>
<th>99th%</th>
<th>99.5th%</th>
<th>100th%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loss in Percentage ($L%$)</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>1.99%</td>
<td>9.99%</td>
<td>13.36%</td>
<td>53.36%</td>
</tr>
</tbody>
</table>

**Table 7: The Percentiles of Loss Percentage $L\%$ Due to One-Year Defaults**

**Table 8: The Percentiles of Integrated Loss Percentage $L\%$**
Figure 8: The Distribution of Integrated Loss $L\%$ of the Whole Portfolio

Figure 9: Sample: The Simulated One Year’s Spread Changes $\Delta$ (bp) and Defaults for Corresponding Percentile of $L\%$ in Table 8

Figure 10: Sample: The Loss Decomposition for the Percentiles of the Integrated Loss $L\%$ (Table 8)
In Figure 10, the decomposition of the underlying spread widening loss and default loss from the integrated loss $L\%$ is not stable and not necessarily unique. This is because one single value of integrated loss $L\%$ could be decomposed into two types of loss with various possible combinations.

The difficulty of risk decomposition raised here in Figure 10 is in line with the conclusion from the IMCR conference summarised by Hartmann (2010): Credit risk and other types of risk could be defined distinctly, but their differences should not be overstated. Drawing a clear distinction between them in practical risk measurement and management is very difficult, and we need to account for their joint influence. Hence, for calculating the economic capital, we suggest use the results of integrated loss as given Table 8.

Alternatively, we can investigate the underlying risk contributions of spread widening and default by calculating their loss percentiles separately in this integrated model and the results are given by Figure 11. We can observe that the relative contribution of default increases more significantly than the one from spread widening when approaching towards the (negative) left-tail of the loss distribution.

In Figure 11, the spread widening loss from the integrated model is less than the spread widening loss $L^W\%$ (in Table 6) from the stand-alone model (in Section 6.1), because for some scenarios in the integrated model, the spread widening loss could be capped by default, under the additional assumption of no spread widening loss for a bond given default (formally as given by (11)).

Note that, calculating the percentile of the integrated loss by just simply summing up the spread widening loss and default loss for the corresponding percentile in Figure 11 is not appropriate, as this summation involves double-counting of two types of risk and does not agree with
Table 9: Risk Contribution of Spread Interaction

<table>
<thead>
<tr>
<th>Percentile</th>
<th>97.5th%</th>
<th>99th%</th>
<th>99.5th%</th>
<th>100th%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integrated Loss</td>
<td>L%</td>
<td>9.88%</td>
<td>14.03%</td>
<td>18.36%</td>
</tr>
<tr>
<td>Loss of Default</td>
<td>D%</td>
<td>1.99%</td>
<td>9.59%</td>
<td>13.36%</td>
</tr>
<tr>
<td>Extra Loss</td>
<td>I%</td>
<td>7.89%</td>
<td>4.45%</td>
<td>5.00%</td>
</tr>
<tr>
<td>Multiplier</td>
<td>L%/D%</td>
<td>3.97</td>
<td>0.46</td>
<td>0.37</td>
</tr>
</tbody>
</table>

the corresponding percentile of overall loss for the integrated model. This would overestimate the whole credit risk for higher percentiles, and this is similar to the finding by Alessandri and Drehmann (2010).

To see the risk contribution from spread interaction, we decompose the integrated credit risk by

\[ L\% = L^{D}\% + L^{I}\% \]

where the percentiles of \( L\% \) and \( L^{D}\% \) are already given by Table 8 and Table 7 (see also Figure 11), respectively, and the higher percentiles of \( L^{I}\% \) is provided in Table 9. In this table, the spread-interaction multiplier \( L^{I}\%/L^{D}\% \), is a simple measure for the relative risk contribution from spread interaction upon the original default risk, which is often neglected in credit risk management, particularly before the 2007-2008’s credit crunch. We observe that the value of the multiplier decreases when the percentile increases. This output is consistent with the reality: in a short period of one year, for a relatively normal economic condition, the major risk comes from spread widening whereas for a more severe environment the main risk originates from default.

6.4. Overall Credit Risk by Alternative Dependence Structures

The issue of dependence structure is of central importance in all portfolio credit risk modelling methodologies (Jarrow and Turnbull 2000), especially that concerns the high-percentile loss in the tail. We provide some alternative settings for the dependency in our model for assessing the overall credit risk.

6.4.1. Degree of Freedom \( \nu \)

The results above are all based on a \( t \)-copula with degree of freedom \( \nu = 3 \). In Figure 12, we calculate the overall loss \( L\% \) by choosing different \( \nu \) for a comparison. We can observe from this sensitivity study that, when the dependency becomes weaker (\( \nu \) larger), the loss \( L\% \) increases for lower percentiles but decreases for higher percentiles. The critical point for loss \( L\% \) as an increasing or decreasing function of \( \nu \) is approximately 95th%. By probability theory, when \( \nu \) goes to infinity, it will eventually converge to a Gaussian copula model which is widely used in practice.

6.4.2. Correlation \( \Sigma \)

Calibrating the correlation remains a challenge for researchers and practitioners. In our model we calibrate correlation parameters for driving spreads widening and defaults based on the spread database, because credit spread provides us an instantaneous market view of the risk premium for the underlying credit, with the majority \(^7\) implying the level of default risk (Hull, Predescu

\(^7\)There are other types of risk implied from credit spreads, such as liquidity.
There are other approaches for correlation calibration. Düllmann, Küll and Kunisch (2010) calibrated the correlations from the time series of the underlying equity prices and default rates, and also had a comparison between them. The Internal Ratings-Based (IRB) approach for Basel II by BCBS (2001) assumed a (homogeneous) average around 20% for correlations (comparing with an average of 18.45% in our model given by Figure 4).

To be more convenient to compare our model with these models based on different calibration methods for correlation, we provide a reasonable approximation by setting one homogeneous correlation $\bar{\rho}$ to all pairs of bonds in $\Sigma$ in our model, $\{\rho_{ij}\}_{i,j} \equiv \bar{\rho}$ to replace the calibrated heterogeneous correlations in Table 5. All other parameters remain the same. We can see how these percentiles of the loss $L\%$ migrate accordingly when increasing $\bar{\rho}$ from 0% to almost 100% as given by Figure 13.

**Remark 6.2.** As we assume the recovery rate $\delta = 40\%$ for all bonds, by (13) it implies that the upper limit of total portfolio loss percentage is 60%. It is approximately the worst scenario (59.58%) simulated by setting the homogeneous correlation $\rho = 99.99\%$ for percentile 100th% as given by Figure 13.
7. Conclusions

The model developed in this paper provides a simple and fast way to assess the credit risk of a portfolio on spread widening and default under a heterogeneous and consistent framework. It is also able to capture the behavior of each underlying bond by distinguishing its credit rating and maturity. The procedure of calibration and implementation are simple and easy to be extended. Alternative marginal distributions and copula functions can be easily replaced under different assumptions tailored to meet different purposes for portfolio risk management.

For future research, instead of using a simple log-normal process to model the evolution of credit spread, we could use other non-negative processes, such as the mean-reverting process. For the dependency modelling, we could consider other copula functions, such as Gaussian and family of Archimedean copulas. For more general portfolio risk management, this framework powered by a copula could possibly be generalised to consistently integrate other types of risk, such as liquidity risk, or to incorporate the negatively related behavior between default and recovery rate. Moreover, rather than fixing a one-year horizon, we could set different time horizons in this model, and then would be able to see the evolution of the portfolio credit risk of default and spread widening over short, median and long terms. We could also extend our approach for pricing credit derivatives under an appropriate probability measure, or a dynamic model to capture the credit risk by calibrating the parameters from instantaneously daily updating database.
References


