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# Assessing the costs of protection in a context of switching stochastic regimes

Pauline Barrieu\*, Nadine Bellamy & Jean-Michel Sahut

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## Abstract

We consider the problem of costs assessment in the context of switching stochastic regimes. The dynamics of a given asset include a background noise, described by a Brownian motion and a random shock, the impact of which is characterised by changes in the coefficient diffusions. A particular economic agent that is directly exposed to variations in the underlying asset price, incurs some costs,  $F(L)$ , when the underlying asset price reaches a certain threshold,  $L$ . Ideally, the agent would make advance provision, or hedge, for these costs at time 0. We evaluate the amount of provision, or the hedging premium,  $\Pi(L)$ , for these costs in the disrupted environment, with changes in the regime for a given time horizon, and analyse the sensitivity of this amount to possible model misspecifications.

*Keywords:* Provision, hedging, costs assessment, stochastic regimes.

*Mathematical Subject Classification (2010):* 60 G 99; 60 K30; 90 B05 ; 91 B70.

*JEL Classification:* C 65; D 80; D 92; E 22; L71.

## 1 Introduction

In this paper, we are interested in the question of provision assessment in a framework of regime switching. More precisely, we consider an agent whose economic activities depend on the price evolution of a given asset, for instance a commodity, the dynamics of which are subject to important modifications, inducing some changes to the underlying regime. Different factors may affect the dynamics of the asset over time, and, we assume, that the modifications stem from two main sources: some ordinary factors, represented by a Brownian motion, and an extraordinary factor, which seldom occurs within the considered time frame. The impact of this extraordinary factor on the asset is represented by a sudden switch in the diffusion coefficients at the random time of occurrence. The agent incurs some costs when the price of the underlying asset reaches a certain level and would like to hedge these costs, or to provision for them. The purpose of this

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paper is precisely to evaluate the hedging, or provisioning, of the costs incurred by the agent in this disrupted environment.

The framework that has been described can be related to many different situations, including, for instance, those encountered by oil or gas providers that are subject to a high degree of competition. Indeed, an energy provider cannot directly pass an increase in production costs to their customers, and any protective measures undertaken against "extraordinary factors" will help the company to smooth their costs, whether the price increase is due to ordinary fluctuations or the occurrence of extraordinary factors. A perfect hedging strategy would be extremely costly, hence, the structure of the contract studied here is designed to cover the agent's main risk exposure. Note that the method presented here is not limited to this particular application and can also be used to evaluate, among other things, particular insurance contracts that are based upon an index and not directly upon the incurred losses, as it is often the case in agricultural microinsurance (see, for instance, the report from Munich Re Foundation (2008)). In this case, the payment of an indemnity to the insured depends upon the evolution of a relevant underlying asset or index, such as a weather-related index or the price of a given commodity (such as corn or wheat), the dynamics of which may also be subject to drastic changes that are due to extraordinary factors.

However, one of the obvious applications of this study is, arguably, to particular situations concerning the decision making process for non-conventional oil-field exploitation, which we refer to as the main illustrative study of this paper. When considering such situations, differing phenomena can be observed: on the one hand, because of the strategic nature of oil, its prices are affected by various factors. Even if some pure market factors do exist, most driving forces, which impact the crude oil spot price, are related to the fundamentals of production capacity, supply and demand. The ability of an oil producer to respond to demand depends upon various types of factors. More precisely, the ordinary factors include the working costs (not including accidents or social crises) and the equipment costs for extraction, but also the variations in customer demand according to the seasons. Changes in the working regulations and site conditions are also typically referred to as ordinary factors. Common to all these factors is the fact that the incurred additional costs can be estimated with an upper bound. On the other hand, it is impossible to assess the costs related to extraordinary factors, such as political crises, or the speculation on crude oil prices. The highly publicised explosion of the "Deepwater Horizon" off shore oil rig, in April 2010, and the disaster of the subsequent oil slick constitute an example of what we call "extraordinary factors" . These extraordinary factors are, by nature, very different from ordinary factors as they affect the price dynamics in a global way.

The owner of the non-conventional field is directly affected by these various factors. His decision to exploit his fields depends directly upon the crude oil price. Indeed, since the extraction of oil from non-conventional fields is more costly than it is from traditional fields, it will only be

meaningful to do so, from an economic point of view, if the price of oil is sufficiently high. When it becomes interesting for the field owner to start exploiting his fields, he will incur some initial costs. Provisioning in advance, or hedging for these costs, is, therefore, a natural question. The aim of this paper is to evaluate at time 0 the amount of provision, or the hedging premium, for these costs in the disrupted environment previously described.

The paper is organised as follows: Section 2 presents the modelling framework, the main assumptions, and describes the contract we want to evaluate. The main results of the valuation, and some numerical illustrations, are given in Section 3. Section 4 concludes.

## 2 Modelling framework

In this first section, we introduce the framework of the paper, detailing, in particular, the underlying asset and the various factors - ordinary and extraordinary - affecting its dynamics. We then describe the situation of the economic agent, exposed to some particular costs when the asset reaches a certain level, and describe his problem of evaluating, at time 0, the amount of provisions he will have to put aside to hedge his potential future costs. This initial amount can either be seen as a technical provision, or as the premium of an insurance contract he is buying at time 0 to cover his potential costs.

### 2.1 Underlying asset price dynamics and impact of random factors

In this paper, the stochastic framework is described by a standard probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\mathbb{P}$  is a reference probability measure. In particular  $\mathbb{P}$  can stand for the historical or statistical probability measure, but also a probability measure representing the beliefs of the economic agent we consider. The price of the underlying asset, denoted by  $S$ , has the following dynamics:

$$dS_t = S_t [a(t) dt + \sigma(t) dW_t] ; S_0 = s_0, \quad (1)$$

where  $(W_t; t \geq 0)$  is a standard  $\mathbb{P}$ -Brownian motion.

**Remark 2.1** *This modelling is consistent with the examples mentioned in the introduction: for example, the dynamics for oil prices have been widely studied in the literature and various models have been suggested to capture the specificities of this commodity. In the seminal papers of Brennan and Schwartz (1985) and McDonald and Siegel (1985), the oil prices are represented by a geometric Brownian motion. Even if such a model may appear to be over-simplistic, various empirical tests show some mixed results as the relevance of a model depends on the length of the study period. In particular, as noticed by Picchetti and Postali (2007): "We conclude that the average half-life of oil price (between four and eight years depending on the model chosen) is long enough to allow a good approximation as a geometric Brownian motion".*

Note that Equation (1) is very general, and allows different situations to be taken into account, depending on whether or not the market price dynamics have been effected by the extraordinary shock. In the remainder of the paper, we assume that the modifications in the price stem from two main sources: ordinary factors that are represented by the Brownian motion  $W$ , and one extraordinary factor, which seldom occurs within the considered time frame. The impact of the extraordinary factor upon the price is represented by a sudden switch in the dynamics at the random time  $\tau$  of the occurrence of such an extraordinary shock. We also assume that the extraordinary factor is independent from the ordinary ones and observable by the agents in the market. In many situations, the independence assumption is not particularly strong. Indeed, the occurrence of a catastrophe or of an accident may be seen as uncorrelated from the natural evolution and fluctuations of the market prices. As a consequence, the random variable  $\tau$  is supposed to be independent from the Brownian motion; it is assumed to be distributed according to an exponential law with parameter  $\lambda$ . The available information structure is characterised by the filtration  $(\mathcal{F}_t; t \geq 0)$ . This information includes the observations of the prices and the occurrence of the random shock  $\tau$ :

$$\mathcal{F}_t = \sigma(S_s, 0 \leq s \leq t) \vee \sigma(N_s, 0 \leq s \leq t),$$

where  $N_s = \mathbf{1}_{\{\tau \leq s\}}$ .

In this paper, we are interested in the switch of the stochastic regime consecutive to the occurrence of a shock. The impact of the shock is, therefore, characterised by a change to both the drift and the volatility; thus, Equation (1) becomes:

$$dS_t = S_t [(a_1 \times \mathbf{1}_{t < \tau} + a_2 \times \mathbf{1}_{t \geq \tau}) dt + (\sigma_1 \times \mathbf{1}_{t < \tau} + \sigma_2 \times \mathbf{1}_{t \geq \tau}) dW_t],$$

with  $a_2$ ,  $a_1$ ,  $\sigma_2$  and  $\sigma_1$  some positive constants.

Note that the changes in volatility are used to model the adjustment of economic anticipations held by the various market participants, consecutive to the shock. Not only the anticipated future price trend, but the level of uncertainty is also modified.

## 2.2 The economic problem

We now consider a particular economic agent who is directly exposed to the variations in the underlying asset price. Let  $L$  be a given threshold such that  $L \geq s_0$ . We assume that the agent will face some costs when the underlying asset price reaches  $L$ . The amount of the costs the agent is facing depends mainly upon the threshold  $L$ . Some other parameters may also have an impact on the costs, but to a lesser extent, as  $L$  acts as the threshold for the payment of these costs. Therefore, we simply denote the costs by  $F(L)$ .

Therefore, for the sake of generality, these parameters are not explicitly specified and our purpose will be to study how the threshold and the cost impact the economic problem, rather than to determine their values.

More precisely, our intention is to evaluate, at time 0, the amount of provision, or the hedging premium  $\Pi(L)$ , for these costs in the disrupted environment previously described, when the time horizon is a fixed time  $T > 0$ .

Coming back to our example of non-conventional oil fields, the agent can, for instance, be the owner of such a non-conventional oil field. His decision to exploit his field depends directly upon the price level of crude oil; and, therefore, when it becomes economically interesting to start the exploitation, he will cover some initial exploitation costs. The threshold  $L$  corresponds to the price level above which it becomes profitable to exploit the fields. Above such a threshold, the profits generated from the sale of oil are sufficient to cover the potential costs related to the extraction and clean-up processes. There are some initial costs  $F(L)$  related to the exploitation of such oil fields.

The problem for the oil producer is, therefore, to assess the amount of provision that needs to be put aside to cover the future and potential costs of exploitation of these non-conventional fields over a certain time period  $[0, T]$ . In such a framework, we evaluate, at time 0, the amount of provisions  $\Pi(L)$ , given the various types of factors that may affect the market price of oil.

More precisely, let  $\tau_L$  be the first time threshold  $L$  is met by the price process:

$$\tau_L = \inf \{t; S_t \geq L\}.$$

This  $\mathcal{F}_t$ -stopping time is the trigger time for the payment of the fixed costs  $F(L)$  by the agent. The premium at time 0, that can be viewed either as a technical provision or as the premium of an insurance contract the agent is buying at time 0 to cover his potential costs, and is therefore given as:

$$\Pi(L) = \mathbb{E} \left[ e^{-\mu\tau_L} F(L) \times \mathbf{1}_{\tau_L < T} \right],$$

where  $\mu$  denotes a (constant) instantaneous discount rate, which translates the preference of the agent at the present time. It may be related to the instantaneous risk free rate, but this is not necessarily the case and it may be more general. It may be totally subjective or even imposed by regulation.

The premium is expressed as an expected value under the reference probability measure  $\mathbb{P}$ . Note that this probability measure is chosen by the agent. The framework of the paper is very general as this probability measure can be the prior probability measure, calibrated by historical or statistical data, but also a subjective probability measure taking into account the agent's beliefs and anticipations. It can also correspond to the more classical framework of the equivalent martingale measure (with the discount rate being the risk-free rate in this case).

Moreover, we assume that the costs are independent of the various factors affecting the underlying price. In other words, the premium may be rewritten as:

$$\Pi(L) = \mathbb{E} [F(L)] \mathbb{E} \left[ e^{-\mu\tau_L} \times \mathbf{1}_{\tau_L < T} \right].$$

They are generally represented as a deterministic function of the threshold  $L$ . Without any loss in generality, and for the sake of simplicity, we assume the cost function to be deterministic, and so

$$\Pi(L) = F(L)\mathbb{E}\left[e^{-\mu\tau_L} \times \mathbf{1}_{\tau_L < T}\right].$$

### 3 Evaluation of the premium

In this section, the contract premium at time 0 is obtained, in the modelling framework previously described, where the occurrence of an extraordinary factor impacts upon the drift and volatility, and, therefore, changes the stochastic regime of the underlying price dynamics. We start by introducing some preliminary notation and results, before presenting the main results and some numerical tests of the sensitivity of the premium with respect to various parameters.

#### 3.1 Some preliminary notation and results

Let us first introduce a series of simplifying notation which are used throughout the paper, and allow us to give a relatively simple formula for the premium.

i) For  $x$  and  $\alpha$  in  $\mathbb{R}$ , and  $t$  in  $\mathbb{R}^+$ , let

$$g(x, \alpha, t) = \mathcal{N}(d_1(x, \alpha, t)) + \exp(2\alpha x) \mathcal{N}(d_2(x, \alpha, t)) \quad (2)$$

and

$$G(x, \alpha, \lambda, t) = \int_0^t \lambda e^{-\lambda u} g(x, \alpha, u) du, \quad (3)$$

where  $\mathcal{N}$  is the cumulative distribution of the Gaussian distribution,  $\lambda$  is the parameter of the random time  $\tau$  of the regime switch and where

$$d_1(x, \alpha, t) = \frac{-x + \alpha t}{\sqrt{t}} \quad ; \quad d_2(x, \alpha, t) = \frac{-x - \alpha t}{\sqrt{t}}. \quad (4)$$

ii) For  $x$ ,  $k$  and  $h$  in  $\mathbb{R}$ , and  $u$  in  $[0, T]$ , let

$$H(x, u, k, h) = \exp\left(-kx + \frac{1}{2}k^2u\right) \left[1 - \exp\left(-\frac{2h(h - x + ku)}{u}\right)\right]. \quad (5)$$

Moreover, the following lemma is a key result in order to obtain an explicit formula for the premium, as we will see in the next subsection.

**Lemma 3.1** *Let  $\theta_1$  is the unique positive solution of the equation  $\frac{1}{2}\theta^2 + \theta\left(\frac{a_1}{\sigma_1} - \frac{1}{2}\sigma_1\right) - \mu = 0$  and  $\alpha_0 = \frac{1}{\sigma_1}a_1 - \frac{1}{2}\sigma_1 + \theta_1$ . Then*

$$\mathbb{E}\left[e^{-\mu\tau_L} \mathbf{1}_{\tau_L < \tau} \times \mathbf{1}_{\tau_L < T}\right] = \left(\frac{s_0}{L}\right)^{\frac{\theta_1}{\sigma_1}} \left[G\left(\frac{1}{\sigma_1} \ln \frac{L}{s_0}, \alpha_0, \lambda, T\right) + e^{-\lambda T} \times g\left(\frac{1}{\sigma_1} \ln \frac{L}{s_0}, \alpha_0, T\right)\right],$$

where  $g$  and  $G$  are defined in Equations (2) and (3).

**Proof.** We first remark that on  $\{\tau_L < \tau\}$  we have:

$$W_{\tau_L} = \frac{1}{\sigma_1} \ln \left( \frac{L}{s_0} \right) + \left( \frac{1}{2} \sigma_1 - \frac{a_1}{\sigma_1} \right) \tau_L,$$

Let  $\mathbb{P}^{\theta_1}$  be the probability measure equivalent to  $\mathbb{P}$  defined by  $\frac{d\mathbb{P}^{\theta_1}}{d\mathbb{P}} |_{\mathcal{F}_t} = \exp \left( \theta_1 W_t - \frac{1}{2} \theta_1^2 t \right)$ , and  $W^{\theta_1}$  is the  $\mathbb{P}^{\theta_1}$ -Brownian motion:  $W_t^{\theta_1} = W_t - \theta_1 t$ . Therefore:

$$\begin{aligned} \mathbb{E} \left[ e^{-\mu \tau_L} \times \mathbf{1}_{\tau_L < \tau} \times \mathbf{1}_{\tau_L < T} \right] &= \mathbb{E}^{\theta_1} \left[ e^{-\theta_1 W_{\tau_L} + (\frac{1}{2}(\theta_1)^2 - \mu) \tau_L} \times \mathbf{1}_{\tau_L < \tau} \times \mathbf{1}_{\tau_L < T} \right] \\ &= \left( \frac{s_0}{L} \right)^{\frac{\theta_1}{\sigma_1}} \mathbb{E}^{\theta_1} \left[ \mathbf{1}_{\tau_L < \tau} \times \mathbf{1}_{\tau_L < T} \right] \\ &= \left( \frac{s_0}{L} \right)^{\frac{\theta_1}{\sigma_1}} \mathbb{E}^{\theta_1} \left[ \mathbf{1}_{\tau_L < \tau} \times \mathbf{1}_{\tau < T} \right] + \left( \frac{s_0}{L} \right)^{\frac{\theta_1}{\sigma_1}} \mathbb{E}^{\theta_1} \left[ \mathbf{1}_{\tau_L < T} \times \mathbf{1}_{\tau \geq T} \right]. \end{aligned}$$

As a consequence on  $\{t < \tau\}$ :

$$\mathbb{P}^{\theta_1} (\tau_L < t) = \mathbb{P}^{\theta_1} \left( \sup_{0 \leq s \leq t} \left[ \left( a_1 - \frac{1}{2} \sigma_1^2 + \sigma_1 \theta_1 \right) s + \sigma_1 W_s^{\theta_1} \right] > \ln \left( \frac{L}{s_0} \right) \right).$$

From the law of the supremum of a Brownian motion (see Chapter 3, Section 3.1 in Jeanblanc *et al* (2009) for instance), we get

$$\begin{aligned} \mathbb{P}^{\theta_1} (\tau_L < t) &= \mathcal{N} \left( d_1 \left( \frac{1}{\sigma_1} \ln \left( \frac{L}{s_0} \right), \alpha_0, t \right) \right) + \left( \frac{L}{s_0} \right)^{\frac{2\alpha_0}{\sigma_1}} \mathcal{N} \left( d_2 \left( \frac{1}{\sigma_1} \ln \left( \frac{L}{s_0} \right), \alpha_0, t \right) \right) \\ &= g \left( \frac{1}{\sigma_1} \ln \frac{L}{s_0}, \alpha_0, t \right), \end{aligned}$$

with  $\alpha_0$  defined in the above lemma,  $g$  defined in Equation (2) and  $d_1, d_2$  defined in Equation (4).

Now

$$\mathbb{E} \left[ e^{-\mu \tau_L} \times \mathbf{1}_{\tau_L < \tau} \times \mathbf{1}_{\tau_L < T} \right] = \left( \frac{s_0}{L} \right)^{\frac{\theta_1}{\sigma_1}} \left[ G \left( \frac{1}{\sigma_1} \ln \frac{L}{s_0}, \alpha_0, \lambda, T \right) + e^{-\lambda T} g \left( \frac{1}{\sigma_1} \ln \frac{L}{s_0}, \alpha_0, T \right) \right].$$

Hence the result. □

### 3.2 The main result

We are now able to state the main results of this paper. More precisely, in the framework of regime switching previously described, the value of the premium, which can also be interpreted as the amount of provisions to be put aside to cover for future costs, can be explicitly computed as follows:

**Proposition 3.2** *At time 0, the premium of the contract is given by:*

$$\Pi(a_1, a_2, L, \sigma_1, \sigma_2) = F(L) (\Pi_1(a_1, L, \sigma_1) + \Pi_2(a_1, a_2, L, \sigma_1, \sigma_2)), \quad (6)$$



with  $\Pi_1(a_1, L, \sigma_1) = \mathbb{E}[e^{-\mu\tau_L} \mathbf{1}_{\tau_L < \tau} \times \mathbf{1}_{\tau_L < T}]$  is given in Lemma 3.1 and

$$\Pi_2(a_1, a_2, L, \sigma_1, \sigma_2) = \left(\frac{s_0}{L}\right)^{\frac{\theta_2}{\sigma_2}} \int_0^T \lambda e^{(\alpha-\lambda)u+\beta x} \times \int_{-\infty}^{\gamma(u)} A(x, u) \frac{1}{\sqrt{2\pi}u} \exp\left(-\frac{x^2}{2u}\right) dx du, \quad (7)$$

where  $\theta_2$  is the unique positive solution of the equation  $\frac{1}{2}\theta^2 + \theta\left(\frac{a_2}{\sigma_2} - \frac{1}{2}\sigma_2\right) - \mu = 0$ ,

$$\beta = -\theta_2 \left(\frac{\sigma_2 - \sigma_1}{\sigma_2}\right) \text{ and } \alpha = -\theta_2 \left(\frac{a_2 - a_1}{\sigma_2} + \frac{\sigma_1^2 - \sigma_2^2}{2\sigma_2} + \theta_2 \left(\frac{\sigma_2 - \sigma_1}{\sigma_2}\right)\right);$$

$$A(x, u) = g(h_2 + k_2u - x, -k_2, T - u) \times H(x, u, k_1, h_1); \quad (8)$$

$$\gamma(u) = \frac{1}{\sigma_1} \ln \frac{L}{s_0} + \left(\frac{1}{2}\sigma_1 - \frac{1}{\sigma_1}a_1 - \theta_2\right)u; \quad (9)$$

and

$$k_1 = -\frac{1}{\sigma_1} \left(a_1 - \frac{1}{2}\sigma_1^2 + \theta_2\sigma_1\right) \quad ; \quad h_1 = \frac{1}{\sigma_1} \ln \frac{L}{s_0}$$

$$k_2 = -\frac{1}{\sigma_2} \left(a_2 - \frac{1}{2}\sigma_2^2 + \theta_2\sigma_2\right) \quad ; \quad h_2 = \frac{1}{\sigma_2} \ln \frac{L}{s_0} - \left(\frac{a_1 - a_2}{\sigma_2} - \frac{\sigma_1^2 - \sigma_2^2}{2\sigma_2} + \frac{\theta_2(\sigma_1 - \sigma_2)}{\sigma_2}\right)u - \frac{\sigma_1 - \sigma_2}{\sigma_2}x.$$

**Proof.** The situation where  $\tau_L < \tau$  is solved in Lemma 3.1. So we simply have to focus on the situation where  $\tau \leq \tau_L < T$  and evaluate  $\Pi_2 \equiv \mathbb{E}(e^{-\mu\tau_L} \times \mathbf{1}_{\tau \leq \tau_L < T})$

We proceed in several steps:

1<sup>st</sup> step: From

$$S_t = s_0 \exp\left(\left((a_1 - a_2) - \frac{1}{2}(\sigma_1^2 - \sigma_2^2)\right)\tau + (\sigma_1 - \sigma_2)W_\tau + \left(a_2 - \frac{1}{2}\sigma_2^2\right)t + \sigma_2 W_t\right) \quad \text{on } t \geq \tau,$$

we get:

$$W_{\tau_L} = \frac{1}{\sigma_2} \ln \left(\frac{L}{s_0}\right) + \left(\frac{a_2 - a_1}{\sigma_2} - \frac{\sigma_2^2 - \sigma_1^2}{2\sigma_2}\right)\tau + \left(1 - \frac{1}{\sigma_2} \times \sigma_1\right)W_\tau - \left(\frac{a_2}{\sigma_2} - \frac{1}{2}\sigma_2\right)\tau_L,$$

and then:

$$\begin{aligned} \mathbb{E}(e^{-\mu\tau_L} \times \mathbf{1}_{\tau \leq \tau_L < T}) &= \mathbb{E}^{\theta_2} \left[ e^{-\theta_2 W_{\tau_L} + \left(\frac{1}{2}\theta_2^2 - \mu\right)\tau_L} \times \mathbf{1}_{\tau \leq \tau_L < T} \right] \\ &= \left(\frac{s_0}{L}\right)^{\frac{\theta_2}{\sigma_2}} \mathbb{E}^\theta \left[ e^{-\theta_2 \left(\frac{a_2 - a_1}{\sigma_2} + \frac{\sigma_1^2 - \sigma_2^2}{2\sigma_2} + \theta_2 \left(1 - \frac{\sigma_1}{\sigma_2}\right)\right)\tau - \theta_2 \left(1 - \frac{\sigma_1}{\sigma_2}\right)W_\tau^{\theta_2}} \times \mathbf{1}_{\tau \leq \tau_L < T} \right], \end{aligned}$$

where  $\mathbb{P}^{\theta_2}$  and  $W^{\theta_2}$  are defined in a similar way as in Lemma 3.1.

Using the simplifying notation of  $\alpha$  and  $\beta$  introduced in the above proposition, we get:

$$\mathbb{E}(e^{-\mu\tau_L} \times \mathbf{1}_{\tau \leq \tau_L < T}) = \left(\frac{s_0}{L}\right)^{\frac{\theta_2}{\sigma_2}} \mathbb{E}^{\theta_2} \left[ e^{\alpha\tau + \beta W_\tau^{\theta_2}} \times \mathbf{1}_{\tau \leq \tau_L < T} \right].$$

$$\begin{aligned}
\mathbb{E} (e^{-\mu\tau_L} \times \mathbf{1}_{\tau \leq \tau_L < T}) &= \\
&= \left(\frac{s_0}{L}\right)^{\frac{\theta_2}{\sigma_2}} \mathbb{E}^{\theta_2} \left[ e^{\alpha\tau + \beta W_\tau^{\theta_2}} \times \mathbf{1}_{\tau \leq \tau_L < T} \right] \\
&= \left(\frac{s_0}{L}\right)^{\frac{\theta_2}{\sigma_2}} \int_0^T \lambda e^{-\lambda u} \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{\gamma(u)} \exp\left(-\frac{x^2}{2u}\right) \mathbb{E}^{\theta_2} \left[ e^{\alpha\tau + \beta W_\tau^{\theta_2}} \times \mathbf{1}_{\tau \leq \tau_L < T} | \tau = u, W_\tau^{\theta_2} = x \right] dx du,
\end{aligned}$$

where  $\gamma$  is defined by the fact that, on  $\{\tau = u\}$  the condition  $\tau \leq \tau_L$  is equivalent to:

$$W_u^{\theta_2} < \gamma(u).$$

Therefore, we find

$$\gamma(u) = \frac{1}{\sigma_1} \ln \frac{L}{s_0} + \left( \frac{1}{2} \sigma_1 - \frac{1}{\sigma_1} a_1 - \theta_2 \right) u.$$

And so we can write

$$\mathbb{E} (e^{-\mu\tau_L} \times \mathbf{1}_{\tau \leq \tau_L < T}) = \left(\frac{s_0}{L}\right)^{\frac{\theta_2}{\sigma_2}} \int_0^T \lambda e^{-\lambda u} \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{\gamma(u)} \exp\left(-\frac{x^2}{2u}\right) e^{\alpha u + \beta x} A(x, u) dx du,$$

with  $A(x, u) = \mathbb{E}^{\theta_2} [\mathbf{1}_{\tau \leq \tau_L < T} | \tau = u, W_\tau^{\theta_2} = x]$ .

2<sup>nd</sup> step: We now need to derive  $A(x, u)$ . To do so, we rewrite it as:

$$A(x, u) = \mathbb{P}^{\theta_2} \left[ \left\{ \sup_{u \leq t \leq T} S_t \geq L \right\} \cap \left\{ \sup_{0 \leq t \leq u} S_t < L \right\} | \tau = u, W_\tau^{\theta_2} = x \right].$$

Let  $h_1, k_1, k_2$  and  $h_2$  be the parameters defined in Proposition 3.2.

• Now we can notice that

$$\left[ \left\{ \sup_{u \leq t \leq T} S_t \geq L \right\} | \tau = u, W_u^{\theta_2} = x \right] = \left[ \left\{ \sup_{u \leq t \leq T} (W_t^{\theta_2} - k_2 t) \geq h_2 \right\} | W_u^{\theta_2} = x \right].$$

• So we get:

$$\begin{aligned}
A(x, u) &= \mathbb{P}^{\theta_2} \left[ \left\{ \sup_{u \leq t \leq T} S_t \geq L \right\} \cap \left\{ \sup_{0 \leq t \leq u} S_t < L \right\} | \tau = u, W_u^{\theta_2} = x \right] \\
&= \mathbb{P}^{\theta_2} \left[ \left\{ \sup_{u \leq t \leq T} (W_t^{\theta_2} - k_2 t) \geq h_2 \right\} \cap \left\{ \sup_{0 \leq t \leq u} (W_t^{\theta_2} - k_1 t) < h_1 \right\} | \tau = u, W_u^{\theta_2} = x \right] \\
&= \mathbb{P}^{\theta_2} \left[ \left\{ \sup_{t \in [0, T-u]} (\widetilde{W}_t^{\theta_2} - k_2 t) \geq h_2 + k_2 u - W_u^{\theta_2} \right\} \cap \left\{ \sup_{0 \leq t \leq u} (W_t^{\theta_2} - k_1 t) < h_1 \right\} | \tau = u, W_u^{\theta_2} = x \right]
\end{aligned}$$

where  $\widetilde{W}^{\theta_2}$  is defined as  $\widetilde{W}_t^{\theta_2} = W_{u+t}^{\theta_2} - W_u^{\theta_2}$ , and is independent of  $W^{\theta_2}$ . Therefore, using the independence of both Brownian motions, we can write  $A(x, u)$  as:

$$A(x, u) = B(x, u) \times C(x, u),$$

with :

$$\begin{aligned}
B(x, u) &= \mathbb{P}^{\theta_2} \left[ \left\{ \sup_{t \in [0, T-u]} \left( \widetilde{W}_t^{\theta_2} - k_2 t \right) \geq h_2 + k_2 u - x \right\} \mid \tau = u, W_u^{\theta_2} = x \right] \\
&= g(h_2 + k_2 u - x, -k_2, T - u),
\end{aligned}$$

and

$$C(x, u) = \mathbb{P}^{\theta_2} \left[ \left\{ \sup_{0 \leq t \leq u} \left( W_t^{\theta_2} - k_1 t \right) < h_1 \right\} \mid \tau = u, W_u^{\theta_2} = x \right].$$

- The determination of  $C(x, u)$  requires a bit of work.

Let  $W^{k_1}$  be defined as:  $W_t^{k_1} \equiv W_t^{\theta_2} - k_1 t$ . Then  $W^{k_1}$  is a  $\mathbb{P}^{k_1}$ -Brownian motion, where  $\mathbb{P}^{k_1}$  is the probability measure defined as  $\frac{d\mathbb{P}^{k_1}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp\left(-k_1 W_t^{\theta_2} + \frac{1}{2} k_1^2 t\right)$ . Therefore,

$$\begin{aligned}
C(x, u) &= \mathbb{P}^{\theta_2} \left[ \left\{ \sup_{0 \leq t \leq u} \left( W_t^{\theta_2} - k_1 t \right) < h_1 \right\} \mid W_u^{\theta_2} = x, \tau = u \right] \\
&= \mathbb{E}^{k_1} \left[ \exp\left(-k_1 W_u^{\theta_2} + \frac{1}{2} k_1^2 u\right) \mathbf{1}_{\left\{ \sup_{0 \leq t \leq u} W_t^{k_1} < h_1 \right\}} \mid W_u^{k_1} = x - k_1 u, \tau = u \right] \\
&= \exp\left(-k_1 x + \frac{1}{2} k_1^2 u\right) \mathbb{P}^{k_1} \left[ \left\{ \sup_{0 \leq t \leq u} W_t^{k_1} < h_1 \right\} \mid W_u^{k_1} = x - k_1 u, \tau = u \right].
\end{aligned}$$

From the formula for the supremum of a Brownian bridge (see Theorem 1 in Boukai (1988) for example), we can write:

$$\mathbb{P}^{k_1} \left[ \sup_{0 \leq t \leq u} W_t^{k_1} < h_1 \mid W_u^{k_1} = x - k_1 u, \tau = u \right] = 1 - \exp\left[-\frac{2h_1(h_1 - x + k_1 u)}{u}\right],$$

and therefore we finally get

$$\begin{aligned}
C(x, u) &= \exp\left(-k_1 x + \frac{1}{2} k_1^2 u\right) \left[ 1 - \exp\left(-\frac{2h_1(h_1 - x + k_1 u)}{u}\right) \right] \\
&= H(x, u, k_1, h_1),
\end{aligned}$$

with  $h_1$  and  $k_1$  as defined in the above proposition and where the function  $H$  is given by Equation (5).  $\square$

Any particular economic agent who is directly exposed to the variations in the underlying asset price, and will face some fixed costs  $F(L)$  when the underlying asset price reaches  $L$ , can compute, explicitly, the amount of provision to be set aside at time 0 or the hedging premium at time 0,  $\Pi(L)$ , in the disrupted environment with a random regime switch. Coming back to our example of non-conventional fields, Proposition 3.2 gives an estimation of the amount of provision that an oil producer has to put aside, should he want to exploit some unconventional fields over a period of time  $[0, T]$ , in a setting where a change in the regime for the oil price affects the drift and the volatility.

### 3.3 Sensitivity analysis and numerical applications

We are now interested in the analysis of the sensitivity of the premium with respect to some key parameters. This is an important step in the study of the robustness of the modelling approach as it gives some quantification of the impact that a model misspecification could have on the premium valuation. Furthermore, from the definition of the premium itself

$$\Pi(L) = F(L)\mathbb{E}\left[e^{-\mu\tau_L} \times \mathbf{1}_{\tau_L < T}\right],$$

this sensitivity analysis will mainly depend on the term in expected value, the costs  $F(L)$  acting as a size factor. Note also that the analysis can be made by studying the impact on the both parts of the normalised premium, depending on whether threshold  $L$  is met at a time before or after the exogenous shock; and the premium can be written as:

$$\begin{aligned}\Pi &= F(L) \times \mathbb{E}\left[e^{-\mu\tau_L} \times \mathbf{1}_{\tau_L < T}\right] \\ &= F(L) \times [\Pi_1 + \Pi_2],\end{aligned}$$

with  $\Pi_1 = \mathbb{E}\left[e^{-\mu\tau_L} \mathbf{1}_{\tau_L < \tau} \times \mathbf{1}_{\tau_L < T}\right]$  and  $\Pi_2 = \mathbb{E}\left(e^{-\mu\tau_L} \times \mathbf{1}_{\tau \leq \tau_L < T}\right)$ .

- *Sensitivity analysis of the premium with respect to the threshold  $L$* : First of all, we can note that the threshold  $L$  plays a specific role among all the different parameters; and knowing the premium variations with respect to this parameter is of a major importance. It can easily be seen that  $\mathbb{E}\left[e^{-\mu\tau_L} \times \mathbf{1}_{\tau_L < T}\right]$  is decreasing with respect to the parameter  $L$ , and is valued in  $(0, 1)$ .

Indeed, the more the level  $L$  increases, the more the probability that  $S$  reaches the threshold is small and tends to zero. However, it does not mean that the premium vanishes, since the value of  $F(L)$  has to be considered. In the framework we consider, it appears usual to take  $F$  as an increasing function of the threshold. As a consequence, nothing more can be said about the premium behaviour without any specification of the cost function choice.

*Costs function impact*: We first analyse the variations of according to the choice of the costs function  $F(L)$ . We consider the particular case  $F(L) = C + L^\kappa$ , where  $\kappa > 0$  and  $C$  is a constant standing for the fixed costs. The other term  $L^\kappa$  is consistent with some desirable economic properties of the cost function (such as the monotonicity with respect to  $L$ ), but also can be seen as a decomposition basis for many other cost functions. Moreover, as we are interested in the impact of  $L$  on the premium value, the numerical studies are made for  $C = 0$ , with no loss of generality. We obtain the following results, summarised in the table below for the premium value with respect to different values of  $\kappa$  and different values of the threshold  $L$ . Note that for all these cases, the parameters of the model are considered as follows:

$S_0$	$T$	$\mu$	$\lambda$	$a_1$	$a_2$	$\sigma_1$	$\sigma_2$
100	1	5%	1	7%	20%	30%	70%

Premium values with respect to  $L$  and  $\kappa$  when  $F(L) = L^\kappa$

$L \setminus \kappa$	0.3	1.5	2.5	2.6	3	3.5	4
100	3.981	1000.000	99999.997	158489.315	999999.9737	9999999.737	99999997.37
120	2.604	813.928	97671.301	157646.777	1069935.494	11720556.1	128392259.3
140	1.678	631.174	88364.310	144840.393	1045540.618	12371003.43	146375686.5
160	1.084	478.398	76543.699	127151.487	968209.7146	12246991.8	154913554.3
180	0.713	362.406	65233.055	109646.550	875193.2781	11741949.98	157534790.1
190	0.584	316.641	60161.852	101670.874	829273.9016	11430751.89	157562041.3
	decreasing	decreasing	decreasing	bump	bump	bump	increasing

- We previously put to the fore that, in the context we deal with, the premium  $\Pi$  is the product of two monotonic functions with respect to the threshold, since:

$$\Pi(L) = F(L)\mathbb{E}[e^{-\mu\tau_L} \times \mathbf{1}_{\tau_L < T}],$$

and the map  $L \rightarrow G(L) = E[e^{-\mu\tau_L} \times \mathbf{1}_{\tau_L < T}]$  is decreasing. As a consequence we expect that for some "small" values of  $\kappa$ , the decreasing property of  $G$  wins, whereas for "bigger" values of  $\kappa$  the increasing property of  $F$  prevails. We can be precise about this fact and assert that there exist two limit values  $\kappa_1$  and  $\kappa_2$  such that  $0 < \kappa_1 < \kappa_2$  and

- For  $0 < \kappa < \kappa_1$  the decreasing impact of  $L \rightarrow E[e^{-\mu\tau_L} \times \mathbf{1}_{\tau_L < T}]$  prevails and the premium  $\Pi$  is a decreasing function of the threshold.

- For  $\kappa_2 \leq \kappa$  the increasing impact of  $L \rightarrow F(L)$  prevails and the premium  $\Pi$  is an increasing function of the threshold.

This fact is consistent with the numerical tests; moreover, in the case where  $\kappa_1 < \kappa < \kappa_2$ , then  $\Pi$  is not monotonic with respect to  $L$  and we can observe that there is a value of the threshold  $L(\kappa)$  for which we get:

$$\begin{aligned} s_0 \leq L \leq L(\kappa) &\implies \Pi \text{ is increasing with respect to } L \\ L(\kappa) \leq L &\implies \Pi \text{ is decreasing with respect to } L. \end{aligned}$$

Note that, in order to make his hedging decision, the agent will consider the threshold  $L$  and the cost function  $F(L)$  in addition to the price dynamics. Therefore, estimating the various parameters is an essential step in the decision making process. From now on, we assume, in the following numerical applications, that  $F(L) \equiv 1$ . A careful sensitivity analysis will show how

robust the pricing formula is with respect to a model misspecification and the amplitude of its impact. The parameters of the shock and of the price dynamics after the shock are certainly the most difficult to assess given the lack of calibration data. In this sense, the understanding of the sensitivity of the premium with respect to  $a_2$ ,  $\sigma_2$  and  $\lambda$  are particularly relevant.

- *Sensitivity analysis of the premium with respect to the exogenous shock intensity  $\lambda$* : From  $\mathbb{P}(T \geq \tau) = 1 - e^{-\lambda T}$  we deduce, in a heuristic way, that the more the parameter  $\lambda$  increases, the bigger the impact of the exogenous shock on the dynamics. Recalling that in our setting, the impact outcome is an increase in both the drift and the volatility, and consequently an increase in the dynamics values, we intuitively deduce that an increase in the shock intensity will result in an increase in the premium. This is confirmed by the numerical study.

*Premium values with respect to  $L$  and  $\lambda$  when  $F(L) = 1$  and*

$S_0$	$T$	$\mu$	$a_1$	$a_2$	$\sigma_1$	$\sigma_2$
100	1	5%	7%	20%	30%	70%

$L \setminus \lambda$	0.1	1	2	3	5
101	0.975	0.976	0.977	0.977	0.976
120	0.569	0.619	0.646	0.659	0.669
140	0.294	0.381	0.427	0.449	0.465
160	0.144	0.236	0.285	0.308	0.325
180	0.070	0.150	0.193	0.214	0.229
190	0.049	0.121	0.160	0.179	0.194
199	0.036	0.100	0.135	0.153	0.167

- *Sensitivity analysis of the premium with respect to the drift parameters  $a_1$  and  $a_2$* : using similar arguments, since an increase in the drift coefficients implies an increase in the dynamics values, we can intuitively write

$$a_i < a'_i \implies S_t^{a_i} \leq S_t^{a'_i}, \text{ for } t \text{ a.s.} \implies \tau_L^{a_i} \leq \tau_L^{a'_i} \implies \Pi^{a_i}(L) \leq \Pi^{a'_i}(L) \text{ for } i = 1, 2.$$

Hence, the behaviours of  $\Pi$  with respect to  $a_1$  and  $a_2$  are very similar and the premium increases with these two parameters, as can be seen in the following tables:

*Premium values with respect to  $L$  and  $a_1$  when  $F(L) = 1$  and*

$S_0$	$T$	$\mu$	$\lambda$	$a_2$	$\sigma_1$	$\sigma_2$
100	1	5%	1	20%	30%	70%

$L \backslash a_1$	0.02	0.07	0.12
101	0.974	0.976	0.979
120	0.587	0.619	0.651
140	0.345	0.381	0.420
160	0.208	0.236	0.270
180	0.130	0.150	0.176
190	0.104	0.121	0.143
199	0.086	0.100	0.119

Premium values with respect to  $L$  and  $a_2$  when  $F(L) = 1$  and

$S_0$	$T$	$\mu$	$a_1$	$\lambda$	$\sigma_1$	$\sigma_2$
100	1	5%	7%	1	30%	70%

$L \backslash a_2$	0.08	0.15	0.2	0.3	0.5
101	0.975	0.976	0.978	0.981	0.974
120	0.605	0.619	0.644	0.678	0.585
140	0.363	0.381	0.413	0.463	0.339
160	0.220	0.236	0.269	0.322	0.197
180	0.136	0.150	0.179	0.230	0.116
190	0.108	0.121	0.148	0.197	0.091
199	0.088	0.100	0.125	0.172	0.073

- *Sensitivity analysis of the premium with respect to the volatility parameter  $\sigma_1$  and  $\sigma_2$* : It is worth noticing that the contributions of these parameters are not similar. Indeed, as shown in the tables below, the premium is increasing with  $\sigma_1$  whereas it is not monotonous with respect to  $\sigma_2$ . This dissymmetry can be explained since we have:

$$\begin{aligned}
 S_t &= s_0 \exp\left(\left(a_1 - \frac{1}{2}\sigma_1^2\right)t + \sigma_1 W_t\right) && \text{on } t < \tau \\
 S_t &= s_0 \exp\left(\left((a_1 - a_2) - \frac{1}{2}(\sigma_1^2 - \sigma_2^2)\right)\tau + (\sigma_1 - \sigma_2)W_\tau + \left(a_2 - \frac{1}{2}\sigma_2^2\right)t + \sigma_2 W_t\right) && \text{on } t \geq \tau.
 \end{aligned}$$

The contributions of the parameters  $\sigma_1$  and  $\sigma_2$  in the specification of the asset at time  $t$  are very different: the first one plays a part before time  $\tau$  whereas the contribution of the second one only occurs after the random shock.

Premium values with respect to  $L$  and  $\sigma_1$  when  $F(L) = 1$  and

$S_0$	$T$	$\mu$	$\lambda$	$a_1$	$a_2$	$\sigma_2$
100	1	5%	1	7%	20%	70%

$L \setminus \sigma_1$	0.1	0.3	0.4	0.6
101	0.955	0.976	0.980	0.983
120	0.388	0.619	0.669	0.723
140	0.153	0.381	0.449	0.532
160	0.067	0.236	0.304	0.396
180	0.032	0.150	0.208	0.298
190	0.022	0.121	0.173	0.260
199	0.016	0.100	0.147	0.230

Premium values with respect to  $L$  and  $\sigma_2$  when  $F(L) = 1$  and

$S_0$	$T$	$\mu$	$a_1$	$a_2$	$\sigma_1$	$\lambda$
100	1	5%	7%	20%	30%	1

$L \setminus \sigma_2$	0.4	0.5	0.6	0.7	0.8
101	0.977	0.977	0.977	0.976	0.976
120	0.615	0.623	0.624	0.619	0.610
140	0.360	0.378	0.384	0.381	0.371
160	0.203	0.226	0.237	0.236	0.228
180	0.112	0.135	0.148	0.150	0.145
190	0.083	0.105	0.118	0.121	0.117
199	0.064	0.084	0.097	0.100	0.097

- *Sensitivity analysis of the premium with respect to the maturity date  $T$* : The maturity date is chosen by the agent, and not imposed by the inherent characteristics of the project. For practical purposes, the maturity date for the hedging strategy is obviously bounded. Moreover, it can easily be seen that  $\mathbb{E}[e^{-\mu\tau_L} \times \mathbf{1}_{\tau_L < T}]$  is increasing in  $T$ , as so is the premium  $\Pi$ . The limit values for the premium  $\Pi$  are 0 and  $F(L) \mathbb{E}[e^{-\mu\tau_L}]$ , which corresponds to the limit situation when the maturity tends to infinity.

## 4 Concluding comments and possible extensions

In this paper, we have considered the problem of costs assessment in the context of switching stochastic regimes. The dynamics of a given asset involve a background noise, described by a Brownian motion and a random shock, the impact of which is characterised by changes in the coefficient diffusions. A particular economic agent, who is directly exposed to the variations in the underlying asset price, will incur some costs  $F(L)$  when the underlying asset price reaches



a certain threshold  $L$ . He would like to make an advance provision, or hedge for these costs, at time 0. We evaluate the amount of provision or the hedging premium  $\Pi(L)$  for these costs in the disrupted environment with changes in the regime, when the time horizon is a fixed time  $T > 0$ , and study the sensitivity of the hedging premium with respect to the various parameters involved in the modelling framework. Note that the hedging strategy has been considered from time  $t = 0$ . However, the results can easily be extended for any  $t$ , such that the exogenous factors have not yet appeared.

The regime switch problem we consider in the paper can be extended to the situation where many independent shocks can affect the process dynamics. However, the formulae become rapidly very heavy. Let us consider, for instance, a two regime-switch model, involving two random shocks  $\tau_i$ ,  $i = 1, 2$ , exponentially distributed with parameters  $\lambda_i$ . The filtration  $(\mathcal{F}_t; t \geq 0)$  is:

$$\mathcal{F}_t = \sigma(S_s, 0 \leq s \leq t) \vee \sigma(N_s^1, 0 \leq s \leq t) \vee \sigma(N_s^2, 0 \leq s \leq t),$$

where  $N_s^i \equiv \mathbf{1}_{\{\tau_i \leq s\}}$ ,  $i = 1, 2$ , and the dynamics of  $S$  is given by:

$$\begin{aligned} dS_t &= S_t \left( a_1 \times \mathbf{1}_{t < (\tau_1 \wedge \tau_2)} + a_2 \times \mathbf{1}_{(\tau_1 \wedge \tau_2) \leq t < (\tau_1 \vee \tau_2)} + a_3 \mathbf{1}_{t \geq (\tau_1 \vee \tau_2)} \right) dt \\ &+ S_t \left( \sigma_1 \times \mathbf{1}_{t < (\tau_1 \wedge \tau_2)} + \sigma_2 \times \mathbf{1}_{(\tau_1 \wedge \tau_2) \leq t < (\tau_1 \vee \tau_2)} + \sigma_3 \mathbf{1}_{t \geq (\tau_1 \vee \tau_2)} \right) dW_t, \end{aligned}$$

where  $a_j, \sigma_j$  are some positive constants for  $j = 1, 2, 3$ .

The economic problem

$$\Pi(L) = F(L) \mathbb{E} \left[ e^{-\mu\tau_L} \times \mathbf{1}_{\tau_L < T} \right] \quad \text{with } \tau_L = \inf \{t; S_t \geq L\}$$

reduces to

$$\Pi(L) = F(L) \times [\Pi_1 + \Pi_2 + \Pi_3],$$

where:

$$\begin{aligned} \Pi_1 &= \mathbb{E} \left[ e^{-\mu\tau_L} \mathbf{1}_{\tau_L < (\tau_1 \wedge \tau_2)} \times \mathbf{1}_{\tau_L < T} \right] \\ \Pi_2 &= \mathbb{E} \left( e^{-\mu\tau_L} \times \mathbf{1}_{(\tau_1 \wedge \tau_2) \leq \tau_L < (\tau_1 \vee \tau_2)} \times \mathbf{1}_{\tau_L < T} \right) \\ \Pi_3 &= \mathbb{E} \left( e^{-\mu\tau_L} \times \mathbf{1}_{(\tau_1 \vee \tau_2) \leq \tau_L < T} \right). \end{aligned} \tag{10}$$

Assuming that the processes  $N^1$ ,  $N^2$  and  $W$  are pairwise independent, the characterisation of (10) is a generalization of the one regime-switch result.

In this study the cost function  $F(L)$  is assumed to be independent of the various characteristics of the underlying price process. The results obtained can not be directly generalized to any cost function without an indication of the type of dependency. However, once a dependency structure is introduced, the model can be extended in many cases. One may think for instance of the following interesting and straightforward extension where some dependency of the cost function  $F$  is introduced through the hitting time  $\tau_L$ . More precisely, a dissymmetry between reaching the threshold  $L$  before or after the occurrence of extraordinary shocks is introduced.

This would allow the agent to hedge differently according to the timing. To do so, two different cost functions  $F_1$  and  $F_2$  may be considered and the premium becomes:

$$\Pi = F_1(L) \mathbb{E} [e^{-\mu\tau_L} \mathbf{1}_{\tau_L < \tau} \times \mathbf{1}_{\tau_L < T}] + F_2(L) \mathbb{E} [e^{-\mu\tau_L} \times \mathbf{1}_{\tau \leq \tau_L < T}],$$

with  $0 \leq F_1(L) \leq F_2(L)$  whenever the main objective is to hedge against exogenous shocks. The extension of our results in this framework is straightforward and does not introduce any additional difficulty. Another interesting extension could be the introduction of a cap on the value of some parameters in the premium. For instance, the premium could be an increasing function of the volatility  $\sigma_2$  up to a certain level and then remain constant if the parameter goes beyond this level. This would enable the introduction of greater flexibility into the contract and the associated hedging.

Another interesting direction of study could be regarding the decision of the agent to hedge. More precisely, we can consider a slightly different framework where the agent does not necessarily consider a full hedge, but bases his decision on the probability of the risk occurring. In this perspective, the agent has a risk aversion level  $\alpha$  and wants to control the probability  $\mathbb{P}(\tau_L \geq T)$ . Considering the set

$$\mathcal{A}(\alpha) = \{L / \mathbb{P}(\tau_L \geq T) \leq \alpha\}$$

and using the fact that  $\mathbb{P}(\tau_L > T)$  is increasing in the level  $L$ , there is a threshold  $L^{\max}(\alpha)$ , such that  $\mathcal{A}(\alpha) = (s_0, L^{\max}(\alpha))$ . In other words:

$$\begin{aligned} s_0 \leq L \leq L^{\max}(\alpha) &\implies \mathbb{P}(\tau_L \geq T) \leq \alpha \quad \text{and} \\ L^{\max}(\alpha) \leq L &\implies \mathbb{P}(\tau_L \geq T) \geq \alpha. \end{aligned}$$

The strategy of the agent will then be based upon how the economic level  $L$  compares with  $L^{\max}(\alpha)$ . This question can be explicitly solved using the results of the paper after having noticed that the meeting of probability  $P(\tau_L < T)$  corresponds to  $\Pi(L)$  in the particular situation of normalised costs  $F(L) \equiv 1$  and of no discount rate  $\mu = 0$ .

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