Theoretical Economics Discussion Paper

# Gradualism in Dynamic Agenda Formation

Gilat Levy and Ronnie Razin

September 2009

LSE STICERD Research Paper No. TE/2009/543
This paper can be downloaded without charge from:
http://sticerd.lse.ac.uk/dps/te/te543.pdf

# Gradualism in Dynamic Agenda Formation

**Gilat Levy and Ronny Razin**

**Discussion Paper No. TE/2009/543**
September 2009
The Suntory-Toyota International Centres for
Economics and Related Disciplines
London School of Economics and Political Science
Houghton Street
London WC2A 2AE
Tel.: 020-7955 6674

## Abstract

We analyze a dynamic model of agenda formation in which players compete in each period to put their ideal policies on the agenda. In each period, with some probability, a decision maker is called upon to take an action from the agenda. We show that in any Markov equilibrium of this game, players with extreme ideal policies will always compete to be in the agenda. On the other hand, there is a positive probability that in each round a more moderate policy will arise on the agenda. Therefore, agenda formation is a gradual process which evolves to include better policies for the decision maker but at a relatively slow pace.

# 1 Introduction

The process of group decision making involves two main (possibly intertwined) subprocesses: that of the formation of a set of alternatives to choose from (which below we refer to as the *agenda*) and that of choosing an option from this set. In some cases, the agenda might be exogenously given. Often however, decision makers are not even aware of the feasible options at hand. A newly elected President who has to tackle a major issue such as a Health reform, is usually not an expert on the subject and must be, somehow, introduced to the feasible policies. In such a case, the agenda formation process plays an important role in bringing to his or her attention the set of feasible options. Even in decision processes that are more formally grounded, for example when voters elect this new President, the set of feasible candidates has to form first. Candidates have to fulfil some formal and official steps as well as to bring themselves to the public attention via a lengthy campaign.

Securing a place on the agenda may indeed be costly, as in such political campaigns. Time or attention constraints of the decision maker (be it politicians or voters) imply that at each point in time, the agenda can evolve to potentially include more policies but not as many as there are feasible.[1] This gives rise to a competition to be on the agenda, as we often observe in intense media and advertising campaigns. Since the process of agenda formation is dynamic in practice (for example, when there is no explicit deadline for action), these competing players would also need to decide when to compete: Should they be first on the agenda or are they better off waiting for others to place their alternatives? In this paper we explore such questions by studying the dynamic process of agenda formation when interested parties compete to place policies on the agenda.

To fix ideas let us think about the public debate about global warming. The public might be initially unaware about the important parameters affecting climate change and the possible policies one could implement to alleviate the problem. Interested parties will use costly media campaigns to bring their favorite policies to the public's attention. Bio fuels or advanced technological industries may push for subsidies for producing alternative fuels or for low carbon technologies while others may call for taxes on emissions; countries who are worse affected may push to decrease global consumption altogether; polluting industries may push to do nothing. These

[1]See Cox (2006) for an argument about the importance of time and attention constraints in legislatures.

different alternatives form the de facto agenda. As there is no clear deadline as to when the parliament or international bodies will act on climate change, we can ask: Will the alternatives that are better for the public emerge on the agenda? How long will it take? who are the players who "speak" first? are these the ones with the better policies for the public (the "moderates") or the worst (the "extremists")?

We propose an infinite horizon model in which a decision maker has to choose an alternative from an evolving agenda. The decision maker is assumed to have single-peaked preferences on a one-dimensional policy space. The timing of the decision is stochastic: in each period, with a probability $\rho$, the decision maker will choose an alternative from the agenda (specifically, the alternative closest to his ideal policy) and with probability $1-\rho$ the game continues to the next period. The parameter $\rho$ captures the (stochastic) length of the decision making process.

A finite number of interested players (each with single-peaked preferences on the one-dimensional policy space) try to influence the agenda: In each period, players play an influence game whose winner adds his ideal policy to the agenda. In the influence game, players simultaneously take a costly action, and the probability that a player wins is some exogenous function of the vector of the costly actions. This formulation includes many all-pay mechanisms that have been used in the literature (such as all-pay auctions or the Tullock influence functions).[2] We analyze Markov perfect equilibria of this game where the state is defined as the policy that is closest to the decision maker's ideal policy in the current agenda.

Our first set of results brings to the fore the main tension that arises in the model, between extreme and moderate players. Extreme players, due to negative externalities, are willing to compete harder than others. Moderates on the other hand, represent better policies for the decision maker, and will thus have a bigger impact when on the agenda. We show that in short decision making processes ($\rho \to 1$), negative externalities are more important, resulting in a polarized agenda. That is, extreme players will always be active and win with a strictly positive probability. On the other hand, long processes (low $\rho$) benefit moderates. Specifically, the player who

[2] Becker (1983) in his seminal paper laid down the framework of influence functions. The literature has entertained many functional forms, including the Tullock family of influence functions. Skaperdas (1996) provides a survey of the different "contest success functions" used in the literature and provides an axiomatic representation of the generalized Tullock family of functions. Hillman and Riley (1989) have analyzed influence games with an all-pay-auction mechanism, leading to a large literature analyzing both static and dynamic influence games. See also the survey in Konrad (2009).

represents the best policy for the decision maker has to win only one stage in order to insure that the decision maker chooses his ideal policy while other players need to win all stages in order to crowd him out. We show that as $\rho \to 0$, the player who represents the best policy wins with probability converging to one, and with payments approaching zero.

Our main result focuses on the positive features of the process. In particular, we are interested in the tendency of extremists to be active participants in the initial stages of the process. This is clearly the case for high values of $\rho$. When $\rho$ is very low, we know from the above that the best player will eventually win, but will extremists stay out of the process and give up quickly? We show that this is not the case. In particular, for any $\rho$, including arbitrarily small $\rho$, at any stage of the process there is a probability, strictly positive and bounded away from zero, that a player different from the most favorable will win the stage. We also show that at any stage there is a substantial probability that a new and more favorable policy will be added to the agenda. Thus the agenda evolves forward with a positive probability, but in a gradual, or relatively slow, manner.

It is easy to see the intuition for why the agenda will not stagnate as long as more and better policies for the decision maker have not been placed on it yet; if no such new policies are added to the agenda in equilibrium, players have nothing to fight for, and will not place strictly positive bids. But then players whose (better) policy is not on the agenda yet will find it optimal to participate.

The intuition for the result about the gradual evolution of the agenda rests on the balancing between the short run and long run motivations for putting a policy on the agenda. In the short run, players are motivated (often by negative externalities) to win in order to take advantage of the possibility (perhaps small) that the decision maker will take a decision in the present. On the other hand, players know that their action today will affect the game in the future. If the process is not gradual, the most favorable player will be on the agenda fairly quickly. This implies that players expect that their influence on the future is rather small, and as a result they would tend to act today mainly on the basis of the short run considerations. But this gives precedence to negative externalities and to more extreme players bidding aggressively. But then the most favorable player cannot be on the agenda too quickly; thus, any agenda formation process must be gradual.

In other words, as the future entails less and less polarization in the policies that will be brought forward (as these become increasingly appealing to the decision maker), players are less worried about long term considerations. This implies a focus on short term considerations and thus a relatively high degree of polarization. Extreme players are therefore always active.

The gradualism result is quite general and therefore does not allow us to tie the degree of gradualism to relevant parameters such as the distribution of the preferences of players or available policies in society. To this end we apply the model to a more specific environment with three players and an all-pay-auction contest. We analyze in this environment how the distribution of available policies affects the timing decisions of players' proposals. We distinguish between two-sided and one-sided influence games where in the former players' ideal policies are on both sides of the decision maker, and in the latter, they are all concentrated on one side of the decision maker.

We show that two-sided influence games involve more polarization and more gradualism than one-sided influence games. In other words, in two-sided games, it takes longer for the favorable policy for the decision maker to be placed on the agenda. The player who represents this policy waits and does not participate in the early stage of the agenda formation. We also show that in both types of games, a more symmetric distribution of the ideal policies is more conducive to a quicker convergence to this favorite policy.

The rest of the paper is organized as follows. In the next section we review the related literature. In Section 3 we present the model. In Section 4 we present our main result of gradualism. Section 5 applies the model to study one and two-sided influence games with three players. In Section 6 we further illustrate the role of negative externalities by considering a model in which players are only motivated by winning, where we show that there is less gradualism and in particular, the player representing the favorite policy of the decision maker is active in any stage of the agenda formation process. We discuss the robustness of the model to various assumptions, as well as possible extensions, in Section 7. All proofs that are not in the text are in an appendix.

# 2 Related Literature

Our model, in which players exert influence in order to place their policies on the agenda, combines two strands in the political economy literature, the one on endogenous agenda formation, and the one on influence games.

The literature on endogenous agenda formation, arising from "chaos" results, has analyzed an agenda formation process in which agents propose policies to be placed on the agenda according to some protocol. Given the agenda, some voting mechanism determines the final outcome.[3] Early contributions include Austen-Smith (1989) and Baron and Ferejohn (1989) who consider random proposers. More recently, Duggan (2006) provides a general existence result for games of endogenous agenda formation in which the order of the proposers and the number of proposals is known while Penn (2008) and Dutta et al. (2004) consider "protocol-free" games and focus on agenda processes which end endogenously. None of these papers considers environments with negative externalities.

More closely related are papers by Barbera and Coelho (2009) and Copic and Katz (2007). Barbera and Coelho (2009) analyze equilibrium existence in voting games in a committee which has to choose $k$ candidates from a longer list, following which a decision maker will choose his favorite candidate. Copic and Katz (2007) similarly analyze existence in a game in which $N$ players can propose different amendments to a policy where an agenda setter chooses thereafter which $T < N$ proposals to include in the final voting stage. The assumption that $T < N$ represents the scarcity of time or attention of legislatures, as in our model. By choosing different policies, the players therefore attempt to increase the probability that their amendment is placed on the agenda.

In our framework, the method by which the agenda is formed is different: we use contests to allow players to place their policies on the agenda. The cost of placing a policy on the agenda is therefore endogenous and depends on the evolution of the agenda. This allows us to focus on the dynamics of the agenda formation. By allowing cost to be endogenous, and by analyzing a dynamic process, we also differ from models such as town meetings (Osborne et all (2004)) or the citizen candidate models (Osbrone and Slivinski (1996), Besley and Coate (1997)), in which politicians have to pay some fixed cost in order for their policy to be considered.

The literature on influence games (Becker (1983), Grossman and Helpman (1994)) have mostly used contest functions or auctions (such as the generalized Tullock, all pay auctions, or menu auctions) to model how players can directly affect political outcomes. For example, Grossman and Helpman (2001) assume that money buys

[3] Typically in the voting game that follows two consecutive policies on the agenda compete, and then the winning policy competes against the next one on the agenda and so on.

votes, that is, that some voters are affected by advertising. Thus, the more a candidate spends, the more she is likely to gain these votes. Our approach differs from the above as we assume that players exert influence only in order to be able to be considered by the decision maker and cannot be guaranteed to be her choice.[4] The most related paper in this literature is Polborn and Klumpp (2006). In their paper about primary elections, two candidates compete, via a contest function, to win different districts, where the one who gains a majority of the districts wins the overall competition. Winning districts in their paper can be analogous to winning slots of attention in our model. In contrast, our model considers many players, and thus the effect of negative externalities on the dynamic incentives to win these slots.

Finally, other papers have analyzed gradualism in different contexts, in bargaining games, public good games, and patent races, albeit stemming from different reasons than the one analyzed in our model. Compte and Jehiel (2004) analyze a bargaining game in which the outside option of the players at each stage is some compromise of the most generous offers made by the players. This implies that a player cannot make a generous offer too quickly, as this will induce the other player to reject it and rip the benefit from his outside option instead. Admati and Perry (1991) show gradualism in contribution games with sunk investments. Agents holds back their contribution in any stage to insure that the other agent contributes his share as well. Finally, in a multistage patent race game among two players, Konrad and Kovenock (2009) show that an agent who is losing in the patent race still does not give up in the stage game, as long as he can win some strictly positive instantaneous prize in that stage.

# 3 The Model

There are $N$ players, who are trying to influence a final policy $y \in [-1, 1]$. The players have ideal policies and the utility of player $i$ from his ideal policy $x_i$ and the final policy decision is $-|x_i - y|$. The final decision is a policy in $\{x_1, .., x_i, .., x_n\}$.[5]

[4] Austen-Smith (1995) assumes that lobbies need to pay a fixed exogenous "access" cost in order to be heard by a politician.

[5] These simple utility functions are not necessary for our results in Section 4 and can be easily generalized to other forms of utilities which exhibit negative externalities, i.e., situations in which players care about the final policy or about the identity of the winner. The results in Section 5 and 6 do depend on the specification of these utilities.

For concreteness, let $x_1 = 0$, and $|x_i| < |x_{i+1}|$ for all $i$.

We now describe the dynamic game that determines how the final decision is selected. At any stage $t$ in the game, the players engage in an all-pay competition whose details we specify below. The winner of the competition at stage $t$ places his ideal policy $x^t$ on the agenda. The agenda at time $t$, $A^t \subseteq \{x_1, .., x_i, .., x_n\}$, evolves in the following way: $A^0 = x_n$, $A^t = A^{t-1} \cup x^t$.[6] After any stage $t$, with probability $\rho \in (0, 1)$, the game terminates. At the termination node, a decision maker chooses the policy in $A^t$ that is closest to zero. With probability $1 - \rho$ the game continues to stage $t + 1$.

We now describe the all-pay-competition that the players play at each stage. In this competition each player $i$ simultaneously places a bid $b_i \geq 0$ which he must pay regardless of the outcome. The probability with which player $i$ wins the competition at stage $t$ is determined according to a function $H_i^t(b)$, where $b$ is the vector of bids. We assume that the function $H_i^t(b)$ satisfies the following properties:[7]

H1. $\sum_{i \in N} H_i(b) = 1$

H2. For any $K > 0$, there exists a $K' > 0$ such that if $b_i = \max_j b_j$ and $\frac{b_i}{b_j} > K'$ then $\frac{H_i(b)}{H_j(b)} > K$.

H3. Monotonicity: $H_i(b)$ (weakly) increases in $b_i$ and (weakly) decreases in $b_j$ for $j \neq i$.

Assumption H1 is made for expositional purposes. Assumption H2 is a weaker version of a requirement that if one player bids in relative terms infinitely more than another player, then he must win with a probability that is relatively large to the other player. H3 is a standard monotonicity requirement implying that it is costly to influence decisions. In particular, the above set of assumptions are general enough that they include many of the functional forms used in the literature, including the generalized Tullock contests, and the all-pay-auction mechanism. In the generalized

[6] The results remain the same if $A^0 = \phi$.

[7] We ignore the supercript $t$ in the description of $H$ from now on.

Tullock contest,[8]

$$H_i(b_1,..,b_i,...,b_m) = \begin{cases} \frac{f(b_i)}{\sum_j f(b_j)} \text{ If } \exists b_j > 0 \\ \frac{1}{n} \text{ otherwise} \end{cases}$$

for some strictly increasing continuous function $f(x)$ satisfying $f(0) = 0$, and the all-pay-auction satisfies:

$$H_i(b_1,..,b_i,...,b_m) = \begin{cases} \frac{1}{|\arg\max b_j|} \text{ if } i \in \arg\max b_j \\ 0 \text{ if } b_i < \max b_j \end{cases}$$

We therefore focus on competitions in which players must pay their bids, but remain very general about how the winner of this competition is selected. The all-pay feature is a relevant one in political economy, where agents do invest efforts and resources to gain the attention of (or access to) a decision maker. In these circumstances, explicit contracts cannot be legally written or enforced and so these efforts must be taken upfront. As it is not transparent how the decision maker assigns access or attention, we adopt a more general approach to our main results although we solve some more specific examples in Sections 5 and 6.

Let $J^t \in \{1, ..., n\}$ be the index of the player with the ideal policy that the decision maker would choose from the agenda $A^{t-1}$ (the "most moderate" policy in $A^{t-1}$). We will focus our analysis on Markov Perfect Equilibria in which players condition their strategies (their bids), on and off the equilibrium path, only on the state variable $J^t$ and ignore both the time index $t$ and past histories.[9] Finally, we say that a player is *active* in state $J$ (in some equilibrium) if the measure of non-zero bids in his support is strictly positive in this equilibrium.

Our main interest in this paper is to consider the dynamics of the process of agenda formation. We wish to highlight the trade-off or tension between "moderates", those with policies close to the decision maker, and "extremists", who, due to negative externalities, may be willing to pay more in order to win. The advantage of the

[8]See Skaperdas (1996) for an axiomatic approach that imposes more conditions on the general $H$ and yields the generalized Tullock contest. Skaperdas (1996)'s axiomatization uses an independence axiom. Clark and Riis (1998) also use independence and homogeneity and lose anonymity to get non anonymous Tullock .

[9]It is left to show that an MPE always exist. We are currently constructing a proof for continuous $H$ functions; the proof for the all-pay-auction case can be done by construction. Sections 5 and 6 illustrate the existence of MPE for the all-pay-auction mechanisms.

moderates arises when the game is long - once they place their policy on the agenda, they are more likely to win. Hence, they need to win fewer competitions as opposed to extremists who need to repeatedly win competitions in order to crowd out better policies. On the other hand, when the game is short, the extremists are more eager to win as the advantage described above disappears.

Our focus in the paper is on the dynamic process in which $\rho < 1$, but first let us consider the benchmark in which $\rho = 1$. This benchmark highlights the importance of negative externalities; these imply that some degree of polarization will always exist. For simplicity we focus here on distributions of preferences that include players with the most extreme ideal policies 1 and -1.

**Proposition 1:** *There exists an $\bar{\varepsilon} > 0$, such that in any equilibrium, for any interval $I \subset [-1, 1]$ of size $\varepsilon < \bar{\varepsilon}$, the probability that the winning policy is in $I$ is strictly smaller than one.*

**Proof of Proposition 1:** Suppose by way of contradiction that for all $\bar{\varepsilon} > 0$ there exists a distribution of ideal policies, an equilibrium and an interval $I$ of size $\varepsilon < \bar{\varepsilon}$ such that the probability that a policy from within $I$ wins is one. Note that the willingness to pay of players within $I$ is bounded by $\varepsilon$. From H1 and H3 all others must bid zero. Choose the player who is furthest from the interval $I$. Assume without loss of generality that this player is a player with ideal policy at 1. The willingness to pay of this player is at least $1-\varepsilon$. By submitting a bid $k\varepsilon$, so that $k \to \infty$ and $k\varepsilon \to 0$, by H2, this player wins with a probability converging to one and his bid converges to zero. This implies an expected utility close to zero. Alternatively, in equilibrium his expected payoff is at most $-(1-\varepsilon)$ and hence he has a profitable deviation.■

For example, in an all-pay auction $\bar{\varepsilon} = 1$. This is the case as for any interval of smaller size there will exist a player outside this interval whose expected distance from the policy is larger than the length of the interval. In an all-pay auction, the highest bidder wins with probability one, so such a player will have a higher willingness to pay than any of the active players. This implies that this player must be active. In the simple Tullock influence model where $f(b_i) = b_i$, we show that $\bar{\varepsilon} = 2$ and that the probability that the winning policy in any smaller interval is at most a half (see Appendix).

# 4 Gradualism in dynamic agenda formation

We now turn our attention to the dynamic game. First, we focus on very long games in which $\rho$ is small. Intuitively, in such games, the moderate player's advantage is highlighted as all he needs to win the game is to win one stage. On the other hand, other more extreme players, need to win multiple stages in order to crowd out more moderate players. We therefore have,

**Proposition 2:** *For any $\varepsilon > 0$, there exists $\rho^* > 0$, such that for any $\rho < \rho^*$, the expected utility of player 1 from the game is larger than $-\varepsilon$.*

**Proof of Proposition 2:** Suppose by contradiction that the statement above is false. This implies, that there exists an $\varepsilon > 0$, such that there is a sequence of equilibria with $\rho \to 0$, such that player 1's expected utility is smaller than $-\varepsilon$.

First note that there are no "absorbing" states besides $J = 1$, i.e., states at which the equilibrium probability that we remain in this state converges to one. If players payments are positive at this state then their utility becomes unboundedly low. Thus their payments must converge to zero and player 1 can deviate and instead of getting a negative utility from an absorbing state, pay a bid close to 0 and by H2 win this state and gain utility close to zero, which implies that the state is either not absorbing or that it is not an equilibrium.

Thus the game must unravel to $J = 1$ in finite expected time. This implies that players payments must converge to zero at a rate $\rho$. Thus player 1 can pay in the initial state $\varepsilon/2$ and win for sure; this provides him with a utility of $-\varepsilon/2 > -\varepsilon$, a contradiction.$\square$

The proposition implies that player 1 must win with probability converging to one and that his payments converge to zero. By H3 and H1, the payments of other players must go to zero as well. But do the extreme players give up from the start? The median will win, the future is set, the present is not important, so why should players participate in the bidding?

Our main result below is concerned with the positive properties of the dynamics of agenda formation, and in particular, how active are extremists. We show that even when the game is long -and that the advantage of player 1 is stark as shown above- extremists do not stay out of the competition:

Theorem 1 *There exists an $\varepsilon > 0$ such that in any Markov Perfect Equilibrium,* *for any state $J > 0$, for any $\rho$, (i) Player 1 wins with a probability lower than $1 - \varepsilon$. (ii) Some player $i < J$ wins with a probability larger than $\varepsilon$.*

The intuition behind this result could be understood through the decomposition of players' incentives into short run and long run considerations. In particular, we show that players' willingness to pay at each stage is what determines their decision of whether to be active or not. A typical willingness to pay of a player $i$ at state $J$ in the game takes the form,

$$w_i^J = \rho \tilde{X}_i^J + (1 - \rho)(V_i^{\min\{i,J\}} - \tilde{V}_i^J)$$

where $\tilde{X}_i^J$ is player $i's$ utility difference between winning and being inactive *in the current stage*. This expression is multiplied by $\rho$, the probability that the policy is chosen today. As a result, this represents the short term incentive to be active and its magnitude is of order $\rho$.

The second expression represents the long run effect of being active. The expression $V_i^{\min\{i,J\}} - \tilde{V}_i^J$ represents the effect of today's action on the future continuation values; $V_i^{\min\{i,J\}}$ is the continuation value following player $i$ winning this period's contest while $\tilde{V}_i^J$ represents the expected continuation value if player $i$ remains inactive.

To understand the magnitude of the long term consideration, $V_i^{\min\{i,J\}} - \tilde{V}_i^J$, we prove in the appendix, that in state $J$, some player $i < J$ always wins with a probability bounded away from zero. This implies that the game will endogenously end (i.e., reach $J = 1$) in finite time in expectations and that the difference in continuation values $V_i^{\min\{i,J\}} - \tilde{V}_i^J$ is also of order $\rho$. To derive this technically -in Lemma 1 in the appendix- we use an induction on $J$ which relies both on the MPE structure, and on the structure of the game which renders it is impossible to move from some state $J$ to any state $J' > J$.

Thus, both the short run and the long run considerations composing the willingness to win of players are comparable and of order $\rho$. But then player 1 cannot win any stage with probability converging to one. The proof of the Theorem uses the following arguments: If player 1 wins with probability converging to 1, other players, by H1 and H3, will place infinitesimally small bids. But then, by H2, player 1's optimal behaviour is to scale the maximum bid of others by some large $K'$ and still win with a probability converging to one while paying a bid converging to zero. This however cannot constitute an equilibrium as then another player can deviate and place an infinitely higher bid than 1's which still converges to zero, but allows him to win with probability converging to one. This deviation is guaranteed to be profitable as the deviating player's willingness to win is of the same order as that of player 1 as explained above.

More generally, in equilibrium, the magnitude of the two effects- the short run and the long run-must be balanced. If it is expected that the most moderate player will be on the agenda in the very near future, this will imply that $V_i^{\min\{i,J\}}$ is fairly close to $\tilde{V}_i^J$ and so the short run incentives will play an important role. But this will imply that other more extreme players will decide to be active and so that our initial supposition cannot be sustained. As a result, the equilibrium will tend to balance out the two effects. Thus, equilibria will always involve some short run considerations, and relatively extreme players will always be active.

# 5 One-sided and two-sided influence games

Our previous result showed that in the general setting extremists never give up, leading to gradualism in dynamic agenda setting. In this section we focus on a particular specification of the model to investigate the positive attributes of this process. In particular, we are interested in timing decisions and the effect of the distribution of preferences on the dynamic unravelling of the agenda.

To fix ideas we will focus in the remainder of the paper on all-pay-auctions. In all pay auctions, the highest bidder at each stage wins that stage. If several players place the highest bid, each of them has some strictly positive probability of winning it. Formally:

$$H_i(b_1, .., b_i, ..., b_m) = \begin{cases} \frac{1}{|\arg\max b_j|} \text{ if } i \in \arg\max b_j \\ 0 \text{ if } b_i < \max b_j \end{cases}$$

General results on all-pay-auctions without negative externalities can be extended to our environment.[10] These results imply that in any equilibrium, in any stage, the player with the highest willingness to win that stage (which is endogenous in our model and depends on the equilibriums strategies of others in that stage and in the future) extracts some rent in equilibrium. On the other hand, all other players are squeezed out of their rent. The way that the above features are obtained in equilibrium is that all players use mixed strategies with the zero bid in their support, and that all players besides the one with the highest willingness to win place an atom on zero; this

[10] See Baye et al (1996). We show in the appendix how their results can be extended.

implies that a close to zero bid of the player with the highest willingness to win still results in some probability of winning for this player, at almost no cost (see Lemma A1 in the appendix). Finally, in equilibrium, the willingness to win of players who do not place positive bids is lower than the minimum willingness to win of those who do place bids.

We assume that there are three players with $x_1 = 0$, $x_2 > 0$, and $|x_3| > x_2$. In particular, we distinguish between one-sided and two-sided influence games. In one-sided influence environments, the influencing parties, although hetrogeneous, all support policies on one side of the decision maker, i.e., $x_3 > 0$. For example, imagine a politician who decides on the regulation of the financial sector where only financial firms can propose policies.

Alternatively, two-sided environments are situations in which the influencing parties support policies on both sides of the decision maker, i.e., $x_3 < 0$. For example, think about a politician who decides on the regulation of the financial sector, where financial firms, as well as consumer groups, are organized in order to propose policies.

## 5.1 Two-sided influence

In this subsection we assume that $x_3 < 0$, so that players 2 and 3 are on different sides of the decision maker. This configuration implies that player 1, while always the most favorable position in terms of the decision maker, will often have a relatively lower intensity to win.

Our results below establish two main themes. The first theme is that this configuration of preferences implies polarization. In particular, often in equilibrium (and for some environments in all equilibria) the most moderate player will not participate in bidding, while the extreme players will, in the initial states of the agenda setting process. The second theme, is the importance of the distribution of preferences. Polarization is easier to sustain when the distribution of preferences is asymmetric around the most moderate player.

The next result considers equilibria in which player 1 is inactive in the initial stages of the game, i.e., when the agenda includes only the most extreme policy $x_3$.

**Proposition 3** *In the initial stages of the game: (i) There is no equilibrium in which player 1 is active, if $|x_3|$ is relatively close to $x_2$ and $\rho$ is sufficiently high; (ii) There exists an equilibrium in which player 1 is not active for all $\rho$ if and only if*

$3x_2 \leq |x_3|$.

The intuition for part (i) above follows from negative externalities. Consider $|x_3| \approx |x_2|$ and $\rho = 1$. But then, when only two players are active, it must be players 2 and 3. To see this, note that if only player 1 and another player bid, their willingness to pay is in the order of $|x_2|$. But the player who is not bidding, expects that each of the other two players will win with positive probability and so will have, by negative externalities, a willingness to pay higher than $|x_2|$. This implies that he will want to deviate and place a strictly positive bid. Therefore the only equilibrium with two players bidding must consist of players 2 and 3 as in this case the willingness to pay of player 1 is strictly smaller than theirs. The proof in the appendix shows that there cannot be equilibria with three active players and generalizes the argument above for a range of $\rho < 1$.

Part (ii) focuses on all values of $\rho$. We therefore need to consider more carefully how the game will fold in the future. When $J = 1$, the game has practically ended; no player will be willing to expend any amount as the decision maker will choose $x_1$ in any eventuality. Given the continuation payoffs for $J = 1$, suppose now that the state is $J = 2$. Note that player 1 must be active in equilibrium by Theorem 1. Player 3 on the other hand has nothing to fight for: he can only spoil the probability that player 1 will win and maintain player 2 as a winner, which is the worst outcome for him. Thus, an equilibrium must consist of players 1 and 2 bidding and such an equilibrium will continue to be played as long as player 1 does not win. In equilibrium player 2 has a lower willingness to win at any stage game, as in order to win the game he would have to win every period, a stark contrast to player 1 who needs to win just one stage game. This implies that player 2 is squeezed out of his rent so his continuation value ($V_i^J$) is $V_2^2 = -x_2$ (the utility from losing to player 1). We compute in the appendix $V_1^2 = \rho x_2$ and $V_3^2 = \frac{-z|x_3|-(1-z)(\rho(|x|_3+x_2)}{1-(1-z)(1-\rho)}$ where $z = \frac{3-2\rho}{2(2-\rho)}$ is the probability that player 1 wins the stage game. Note that in equilibrium, player 1 wins in finite time in expectations (although the probability he wins any stage is bounded away from 1, by Theorem 1).

We can now consider the existence of equilibria in the initial stages of the agenda setting, in which player 1 is not active. We can solve for the two-player equilibrium in which only players 2 and 3 are active by computing the willingness to win of these two players (and solving for these simultaneously with the continuation values of player $i$

in state 3):

$$\begin{aligned} w_2^3(\rho) &= \rho(|x_3| + x_2) + (1-\rho)(V_2^2(\rho) - V_2^3(\rho)); \\ w_3^3(\rho) &= \rho(|x_3| + x_2) + (1-\rho)(V_3^3(\rho) - V_3^2(\rho)); \end{aligned}$$

where $w_i^J$ is the willingness to win of player $i$ in state $J$.

Given the solution we have to make sure that player 1 does not want to deviate and bid as well. By Lemma A1 in the Appendix, we need to check that $w_1^3 < w_3^3$, where

$$w_1^3(\rho) = \rho(z'x_2 + (1-z')|x_3|) + (1-\rho)(-z'V_1^2(\rho) - (1-z')V_1^3(\rho))$$

where $z'$ is the probability that 2 wins the stage game at $J = 3$. As in Theorem 1, it is more interesting to consider low values of $\rho$, for which:

$$w_1^3(\rho) < w_3^3(\rho) \Leftrightarrow_{\lim \rho \to 0} 3x_2 < |x_3|$$

In fact, the equilibrium holds for all $\rho$ if $3x_2 < |x_3|$. The conditions for existence are weaker when $\rho$ is higher as such values lower the advantage of player 1.

To see how the condition for $\rho \to 0$ arises note that the temporary gain of player 3 from winning vs. losing to 2 is $\rho(|x_3| + x_2)$. The future gain of 3 winning today vs. losing to 2 is actually a loss, as he is just delaying the convergence to 1 winning and incurring a loss of $\rho x_2$- losing to player 2 in the next period. This balanced with the contemporary gain leaves us with a total gain of $\rho|x_3|$. From the point of view of player 1, the willingness to win is in the order of $\rho(2x_2 + \frac{x_3}{3})$ : in the war between 2 and 3, player 2 wins with probability converging to one in finite time. Thus, winning over player 2 vs. losing to him, implies a benefit of $\rho x_2$ in the present *and* in the future. On the other hand, in equilibrium, there is a probability which is strictly lower than one that player 3 would win; winning now vs. staying out does not incur a great utility benefit vis a vis player 3. The requirement that $\rho x_3 > \rho(2x_2 + \frac{x_3}{3})$ yields the condition above.

We next focus on the most extreme player, player 3. Proposition 4 establishes that he will always be an active player.

**Proposition 4** *In the initial stages of the game player 3 is active in any equilibrium.*

Suppose to the contrary that there is an equilibrium in which only players 1 and 2 are active. As on equilibrium path the continuations would only involve players 1 and 2 as well, the equilibrium in the initial stages will be exactly as the equilibrium when $x_2$ is already on the agenda. Thus, the willingness to win of player 2 in the initial stages is, as later on, $\rho x_2$ : player 2 can only achieve the contemporary gain, as his future utility from being on the agenda is the utility from losing to player 1. Player 3 on the other hand, has a larger motivation to win in this equilibrium. His future loss from delaying the time in which player 1 wins the game is outweighed by the contemporary gain from winning today, implying that his willingness to win is at least $\rho x_3$, larger than that of player 2. Player 3 will therefore deviate and place a positive bid, a contradiction.

Note that in the equilibrium analysis above, *polarization in the present builds on less polarization in the future*: as the future unfolds, players do not affect the outcomes much as polarization decreases, which implies that they focus on the present or on the static consideration. In the present, negative externalities are important and imply large polarization.

It is also worth pointing out that it is the *relative values of* $x_2$ *and* $x_3$ that are important in characterizing the equilibria above. This prediction is different from models in which the payments in equilibria are exogenously fixed (for example, the citizen-candidate or town meeting models). In such models, the absolute values of ideal policies matter, whereas in our analysis, it is the relative values of these policies.

Finally, we show that when the distribution of preferences is more symmetric, player 1 will become active. Thus, polarization is more likely to be sustained when the distribution of preferences is asymmetric.

**Proposition 5** *For low values of* $\rho$*, if* $|x_3| < 2x_2$ *player 1 is active in any equilibrium in any stage of the game.*

## 5.2 One-sided influence

We now assume that $x_3 > 0$. This configuration implies that player 1 will often have a relatively high willingness to win as on top of being the most moderate, he is also an "extremist".

Our results below show that relatively to the case of two-sided influence, one-sided influence environments will have less polarization. In particular, there is always an equilibrium in which player 1 is active and the environments in which he is not active are less prevalent than in the two-sided influence configuration. In addition, our last result focuses on a new strategic behaviour that emerges in one-sided interactions; players who are less moderate than the current state may become active, trying to crowd-out other policies from showing up on the agenda in order to "defend" the existing policies.

Our first result, as before, focuses on player 1.

**Proposition 6** *In the initial stage of the agenda formation process, (i) There is always an equilibrium in which player 1 is active; (ii) There exists an equilibrium in which player 1 is not active for all values of* $\rho$ *if and only if* $4x_2 < |x_3|$.

To see why there always exists an equilibrium in which player 1 is active, consider an equilibrium in which only him and player 3 bid (in the initial stages). If player 2 were to deviate and win, then the continuation game is such that he will have to fight against player 1: in this equilibrium, as before, player 2 is "squeezed-out" of all rents and thus has no willingness to pay for it. Moreover, as player 2's ideal policy is between 1 and 3's, they are bidding too aggressively for him to be able to rip any short run benefits. As a result, 2 will not join in and such an equilibrium holds for all parameters.

To see whether there exists an equilibrium in which player 1 is not active, we need, as in the previous section, to check whether player 1's willingness to win is lower than that of player 3 when 3 and 2 are the only active players. The willingness to win of player 3 in this equilibrium is, as in the two-sided case, roughly $\rho x_3$ when $\rho \to 0$. Given the equilibrium, the willingness to win of player 1 is calculated in a similar manner to the one in the two-sided configuration. The only difference is that player 3 wins with a higher probability in the one-sided game: he is better off in this environment (as his worst utility is $-|x_3|$) and thus bids more aggressively against player 2 in the initial stages. This increases the willingness to win of player 1, which implies that this equilibrium becomes *harder to sustain* in this environment. Again, the necessary condition for this equilibrium to hold is sufficient for other values of $\rho$.

We now focus on player 3. In the two-sided case, player 3 was always active (Proposition 4), whereas in the one sided configuration we have,

**Proposition 7** *For all* $\rho$, *if* $x_3 < \frac{4}{3}x_2$, *there exists an equilibrium in which player 3 is not active.*

Note that for low values of $\rho$, the equilibrium constructed in the proof relies on a new strategic behaviour that is unique to the one-sided configuration (for three players). In particular, the equilibrium relies on player 3 bidding in $J = 2$ against player 1 in order to "defend" 2. Alternatively, if the continuation game is such that 1 and 2 fight in $J = 2$, then an equilibrium with players 1 and 2 fighting in state $x_3$ holds only if $x_3 < x_2(1+\rho\frac{2-\rho}{3-\rho^2})$. This condition becomes more restrictive as $\rho$ shrinks and is impossible to sustain when $\rho = 0$. The continuation game in which in $J = 2$ player 3 fights, increases the motivation of player 2 to fight in $J = 3$ as he expects player 3 to carry the burden of maintaining his position the agenda on his behalf.

REMARK 1: Defending other players raises the following question. What if there are many agents similar to $x_2$ that can all share the fight against $x_1$? Could many players sharing the position of $x_2$ overcome the advantage of player 1? One way that this can possibly be done, is that at each stage, another player fights against player 1. Assume that there are infinitely many of them, and so each player in the $x_2$ position knows that he needs to win only one stage to sustain the equilibrium and that there is always another player that will fight later on. However, in this case we still find that the advantage of player 1 is preserved:

**Proposition 8** *Suppose that there are infinitely many players with ideal policy $x_2$ and one single player with ideal policy $x_1$. Then there exists an equilibrium in which in each period a new player at $x_2$ fights against player 1. In this equilibrium, the probability that player 1 wins each stage is bounded away from zero.*

Again, the intuition is that player 1 needs just one slot to win-and practically terminate-the game, whereas when a player at $x_2$ wins a stage, the game is not over. Thus, player 1 still has an advantage.

# 6 Policy motivation vs. winning motivation

In many set ups, it is reasonable to consider the utility of politicians or interest groups to be also about winning and not only about the final political outcome. Note by the way that all equilibria in the section above are robust to some small office or winning motivation, as does our main theorem.

In this section we consider therefore the opposite extreme case in which agents are not policy motivated and hence negative externalities play no role. Instead, each cares only about winning (and does not care about the identity of the winner in case he loses). We maintain, as in the previous section, the assumption that the mechanism is an all-pay-auction.[11]

Suppose then that a player receives 0 when he loses and some $v$ when he wins. This implies that the advantage of the extremists wears off, as all players have the same utility from winning vs. losing. Hence, the only element that plays a role here is the advantage of the moderate players.

**Proposition 9** *For any $\rho < 1$, player 1 bids at any stage with a positive probability. Moreover, at every stage, player 1 has the highest probability of winning.*

In the absence of negative externalities, the game above becomes a dynamic version of the $N$ players static all-pay-auction analyzed in Baye et al (1996).[12] In the proof we show how we can apply their results. In particular, we show that player 1 has the highest willingness to win at any subgame and hence must participate in every stage. Moreover, all others players are squeezed out of their rent.

The result above illustrates the role of negative externalities. These give advantage to extremists, and allow for a higher level of gradualism that is exhibited when such externalities do not exist. In particular, the best player does not "wait" here as he might do in the negative externalities case.

To illustrate the Proposition, consider again the two-sided configuration of preferences, but now with players only benefiting $v$ from winning and 0 otherwise. As before, when $J = 1$ then the game is over, whereas when $J = 2$, only players 1 and 2 are active, with player 2 being squeezed out of rent, and hence: $V_1^1 = v$, $V_1^2 = (1-\rho)v$, and $V_i^1 = 0$, $V_i^2 = 0$ for $i \in \{2, 3\}$. To solve for the equilibrium when $J = 3$, note that given the continuation, the willingness to win of player 1 is higher than all the others, and the willingness to win of all the others is equal (as their contemporary gain as well as their continuation values are equal). This implies a continuum of equilibria in which: (i) Player 1 mixes uniformly on $[0, \rho v]$, (ii) Player 2(3) mixes uniformly on $[0, \rho v]$ with an atom on 0; (iii) Player 3(2) mixes uniformly on some $[b', \rho v]$ and places an atom on 0, for some $b' \in [0, \rho v]$.[13]

[11] Cases-Arces and Martinez-Jerez (2007) and Konrad and Kovenock (2009) also consider dynamic all pay-auctions (without negative externalities) albeit in different contexts.

[12] For a two-player version, see Hillman and Riley (1989).

[13] Among these, the equilibrium in which player 1 wins with the lowest probability is when players 2 and 3 fight symmetrically i.e., $b' = 0$. The equilibrium distribution functions over bids in this case are $F_2(b) = F_3(b) = \sqrt{\frac{\rho v(1-\rho)+b}{\rho v(2-\rho)}}$, $F_1(b) = \frac{b}{\rho v} \frac{1}{\sqrt{\frac{\rho v(1-\rho)+b}{\rho v(2-\rho)}}}$. Thus, in the first stage, player 1 will win with a probability of at least $\frac{(1-\rho)}{(2-\rho)} + \int_0^{\rho v} (\frac{1}{\rho v} (\sqrt{\frac{\rho v(1-\rho)+b}{\rho v(2-\rho)}}) - \frac{b}{\rho v(2-\rho)\rho v} \frac{1}{\sqrt{\frac{\rho v(1-\rho)+b}{\rho v(2-\rho)}}}) db$.

Finally, note that if one allows for some perturbation of the preferences where players have (small) policy concerns as they do in the main model, then although Proposition 9 is still maintained, there is some equilibrium selection, and not all moderate players can be active. For example, in the example above, the symmetry between players 2 and 3 breaks down and we will not be able to sustain an equilibrium in which only players 1 and 2 fight in the first stage of the game, even for infinitesimally small policy concerns or more generally, negative externalities.

# 7 Discussion

We now consider several of our main assumptions. One assumption we make is that players can only propose their ideal policies on the agenda and strategically cannot choose to offer other policies. Allowing for such an option will in fact not change our main results but will just add another dimension of complexity to the model. Even if players can strategically choose which policies to offer from some finite set of policies, convergence to the best policy from the point of view of the decision maker will be gradual and relatively slow.

Another assumption in our model is that the set of policies or players in our model is finite. We believe it is possible to construct an analogous version of Theorem 1 and our other results to the case of a continuum of players. The assumption of a finite set simplifies our exposition considerably.

In our game we assume that in each stage only one player can propose a policy. This is not crucial, and instead one can analyze a game in which several, lets say $k$, policies can be chosen to be presented on the agenda. This type of game is complicated by the fact that players may bid for several winning positions and it thus becomes a simultaneous multi-prize competition.[14]

The utility function that players have in our model is linear in the distance of their ideal policy from the outcome. As noted in Section 2, our main result will generalize for other types of negative externalities and we have focused on a natural and simple form of such externalities.

Finally, another simplifying assumption that we make is that players have complete information about other players' ideal policies or more generally about the

[14] Siegel (2009) analyzes general static all pay competitions with several prizes in which the players with the highest bids are the ones that win the prizes.

available set of policies.[15] A possible extension of our model is to the case of private information which is left for future analysis.

[15] In Levy and Razin (2009) we analyze an endogenous agenda formation model with two period, two players, and private information on ideal policies.

# 8 Appendix

## 8.1 Proofs for Section 4

### 8.1.1 Proposition 1: An application to the Tullock function:

**Claim 1** *Assume the simple Tullock influence function, $H_i(b_1, .., b_i, ..., b_m) = \frac{b_i}{\sum b_j}$. In any equilibrium (i) at most two individuals submit strictly positive bids. (ii) The unique equilibrium is that the two most extreme players with positions at $-1$ and $1$ are the only ones who are active.*

**Proof of Claim 1** : (i) Let $M$ be the set of active players. Rename the set of active players with player 1 being the player with the minimum $x_i$ and player $m$ being the player with the highest $x_i$ among the active players. Each $i \in M$ will satisfy the f.o.c:

$$\begin{aligned} b_1(x_i - x_1) + ... + b_{i-1}(x_i - x_{i-1}) + b_{i+1}(x_{i+1} - x_i) + ... + b_m(x_m - x_i) &= (\sum_{j=1}^{m} b_j)^2 \text{ if } i \neq 1, m \\ b_2(x_2 - x_1) + .... + b_m(x_m - x_1) &= (\sum_{j=1}^{m} b_j)^2 \text{ if } i = 1 \\ b_1(x_m - x_1) + ... + b_{m-1}(x_m - x_{m-1}) &= (\sum_{j=1}^{m} b_j)^2 \text{ if } i = m \end{aligned}$$

Take the difference between the f.o.c of two consecutive individuals in $M$, $i$ and $j$ and assume w.l.o.g that $i < j$:

$$b_1(x_i - x_j) + ... + b_{i-1}(x_i - x_j) + b_i(x_i - x_j) + b_j(x_j - x_i) + ... + b_m(x_j - x_i) = 0$$

This implies that

$$\sum_{l=1}^{i} b_l = \sum_{l=j}^{m} b_l$$

But if we take $i = 1$ and $j = 2$ and then $i' = 2$ and $j' = 3$ we get:

$$\begin{aligned} b_1 &= b_2 + \sum_{l=3}^{m} b_l \text{ and} \\ b_1 + b_2 &= \sum_{l=3}^{m} b_l \end{aligned}$$

implying that $b_2 = 0$, a contradiction.

(ii) Suppose two players $i$ and $j$ are active and is not the case that they are the players at 1 and $-1$. Suppose without loss of generality that $x_i, x_j \neq 1$. Note that in equilibrium they both submit a symmetric bid $b^*$. Let $\varepsilon$ be the distance between the policies of $i$ and $j$. The solution to the first order conditions implies that,

$$\begin{aligned} b\varepsilon &= (2b)^2 \Leftrightarrow \\ b &= \frac{\varepsilon}{4} \end{aligned}$$

Now write down the first order condition for player at $x_i = 1$ at zero,

$$\frac{\varepsilon}{4}(1 - x_i) + \frac{\varepsilon}{4}(1 - x_j) - \frac{\varepsilon^2}{4}$$

but note that as $(1 - x_i) > \varepsilon$ or $(1 - x_j) > \varepsilon$ this implies that the expression above is positive, and hence this cannot be an equilibrium.

Finally suppose that only the players with ideal policies 1 and $-1$ are active, and they each bid $\frac{1}{2}$. For any other player, the first order condition evaluated at zero is,

$$\frac{1}{2}(x_i + 1) + \frac{1}{2}(1 - x_i) - 1 = 0$$

and so this corresponds to an equilibrium. $\square$

### 8.1.2 Proof of Theorem 1:

Let $u_{ij}^l$ be the utility of player $i$ when player $j$ wins at state $l$, abstracting from the possible payments made by player $i$ at state $l$. That is, $u_{ij}^l = -\rho|x_i - x_k| + (1 - \rho)V_i^k$ where $k = \min\{l, j\}$. Let $w_{ij}^l = u_{ii}^l - u_{ij}^l$ denote the willingness to win of player $i$ against player $j$ in state $l$. We first prove the following Lemma.

**Lemma 1:** *(i) There exists an $\bar{\varepsilon} > 0$, such that for all $\rho$, for all states $l$, the probability that some $i < l$ wins is larger than $\bar{\varepsilon}$. (ii) For any state $l$ and for any $j \neq 1$, $w_{ji}^l$ is of order $\rho$ or lower, and $w_{21}^l$ is strictly positive and of order $\rho$.*

**Proof of Lemma 1:** First note that $w_{1i}^l$ for all $i$ is of an order $\rho$ or higher for any state, as $w_{1i}^l = \rho|x_{\min\{i,l\}}| + (1 - \rho)(-V_1^{\min\{i,l\}})$ where $V_j^J \leq 0$ for all $j, J$.

We will prove the Lemma by induction.

I. Consider $J = 2$.

I(i): We will first show that the probability that player 1 wins is bounded away from zero. Suppose to the contrary, that there exists a sequence of equilibria in which player 1 wins with probability $\varepsilon$ converging to zero.

In what follows we assume mixed strategies for the players denoted by $f_i$ and suppress the notation for the element of the sequence and for state 2.

As $\Pr(1\text{ wins}) = \int_{b_1} f_1 \Pr(1\text{ wins}|b_1) db_1 < \varepsilon$, then $\Pr(1\text{ wins}|b_1) < k\varepsilon$ for a measure of at least $1 - \frac{1}{k}$ of bids in the support of player 1 for all $k > 2$. Choosing a sequence of $k \to \infty$ and $k\varepsilon \to 0$ this implies that for almost any bid $b_1^*$ in the support of player 1, $\Pr(1\text{ wins}|b_1) < k\varepsilon$.

We now compare the utility of each bid in the support of player 1 with a bid of zero. Given the strategies of all other players, player 1 is better off using $b_1^*$ rather than zero only if:

$$\sum_{i \neq 1} (\Pr(i\text{ wins}|b_1 = 0) - \Pr(i\text{ wins}|b_1^*))(w_{12}) > b_1^*$$

For almost all $b_1^*$, by H1 and H3, $\Pr(i\text{ wins}|b_1 = 0) - \Pr(i\text{ wins}|b_1^*) < k\varepsilon$, implying that $b_1^* < k\varepsilon w_{12}$.

Consider other active players. A possible strategy for each such player $j$ is to bid a sequence of $b_j = \gamma b_1^*$ where $\gamma \to \infty$ and $\gamma k\varepsilon \to 0$ so that $b_j \to 0$. By H2, such bid guarantees winning (and thus maintaining state 2, as in the equilibrium) with probability converging to 1 and a bid converging to zero. Thus the equilibrium strategy of all other active players must involve bids $b_j^*$ with $\frac{b_j^*}{w_{12}} \to 0$.

Now we reach a contradiction. Player 1 can deviate from his equilibrium strategy and place a bid $b_1'$ such that $\frac{b_1'}{b_j^*} \to \infty$ and $\frac{b_1'}{w_{12}} \to 0$. His (relative) gain is $w_{12}$ while his (relative) cost is at most infinitely smaller than $w_{12}$, yielding is strictly positive benefit.

I(ii). Given I(i), let the probability that 1 wins in $J = 2$ denoted by $z$, which is bounded from zero. As $V_j^2 = \frac{-z|x_2| - b_j^2}{1-(1-\rho)(1-z)}$ and $V_1^2 = \frac{-\rho(1-z)|x_2| - b_1^2}{1-(1-\rho)(1-z)}$, we have:

$$w_{j1}^2 = \rho|x_2| + (1-\rho)(V_j^2 + |x_2|) = \frac{\rho|x_2| - (1-\rho)b_j^2}{1-(1-\rho)(1-z)}$$

Where, with some abuse of notation, $b_j^J$ refer to the expected payments of player $j$ in state $J$. By H1 and H3, $0 \leq b_j^2 < z w_{j1}^2$, implying that $\frac{\rho|x_2| - (1-\rho)z w_{j1}^2}{1-(1-\rho)(1-z)} < w_{j1}^2 < \frac{\rho|x_2|}{1-(1-\rho)(1-z)}$ and is therefore of order $\rho$.

Note that for all $j > 2$, $w_{j2}^2 = 0$. Note that for $j$ who "wish" 1 would win, we have

$$\begin{aligned}
w_{j1}^2 &= -\rho|x_2| + (1-\rho)(V_j^2 + |x_2|) \\
V_j^2 &= -\rho(z|x_j - x_1| + (1-z)|x_j - x_2|) - b_j^2 + (1-\rho)(z(-|x_j - x_1|) + (1-z)V_j^2) \\
V_j^2 &= \frac{-\rho(z|x_j - x_1| + (1-z)|x_j - x_2|) - b_j^2 + (1-\rho)z(-|x_j - x_1|)}{1-(1-\rho)(1-z)} \\
&= \frac{\rho|x_2| - (1-\rho)b_j^2}{1-(1-\rho)(1-z)}
\end{aligned}$$

I(iii): follows from (ii).

II. Assume that the Lemma is true for all states $J \le l-1$.

III. Consider state $l$.

III(i). Let $\varepsilon$ be the probability that a player with $j < l$ wins. We will show that it cannot be that $\varepsilon$ converges to zero. Suppose it does. By arguments similar to I(i), almost all bids must be infinitely smaller than $\max_j \max_i w_{ji}$ for $j < l$ and $i \le l$.

Now consider player 1. His utility is at most $(1-\varepsilon)u_{1l} + \varepsilon\tilde{u}_{1i}$ where $\tilde{u}_{1i}$ is the expectations over the utility from players $i < l$ winning. On the other hand, there exists some sequence of bids $b_1'$ with $\frac{b_1'}{\max_j \max_i w_{ji}} \to 0$ that guarantees winning with probability almost 1, and $b_1' \to 0$. Thus from such a deviation his utility is $u_{11} - b_1'$ so his gain is $(1-\varepsilon)w_{1l} + \varepsilon\tilde{w}_{1i} - b_1'$. We will now show that $\frac{\max_j \max_i w_{ji}}{w_{1l}}$ is bounded from above and thus the gain is strictly positive - a contradiction to the equilibrium hypothesis.

Note that $w_{1l}$ is of order $\rho$ or higher. If $\max_j \max_i w_{ji} = w_{j'i}$ for some $i < l$, then by the induction $\frac{\max_j \max_i w_{ji}}{w_{1l}}$ is bounded. Assume therefore that $\max_j \max_i w_{ji} = w_{j'l}$ for some $j' < l$, $j \neq 1$. Then:

$$\begin{aligned}
\frac{w_{j'l}}{w_{1l}} &= \frac{\rho|x_{j'} - x_l| + (1-\rho)\frac{\rho V_{j'}^{j'} + \rho(|x_{j'} - x_l|(1-\varepsilon) + \sum_{i<l} p_i^l |x_{j'} - x_i|) + (1-\rho)\sum_{i<l} p_i^l (V_{j'}^{j'} - V_{j'}^i) + b_{j'}^l}{1-(1-\rho)(1-z)}}{\rho|x_l| + (1-\rho)\frac{\rho(|x_l|(1-\varepsilon) + \sum_{i<l} p_i^l |x_i|) + (1-\rho)\sum_{i<l} p_i^l(-V_1^i) + b_1^l}{1-(1-\rho)(1-\varepsilon)}} \\
&\approx \frac{\rho|x_{j'} - x_l| + (1-\rho)\frac{\rho V_{j'}^{j'} + \rho|x_{j'} - x_l|(1-\varepsilon) + \sum_{i<l} p_i^l w_{j'i}}{1-(1-\rho)(1-\varepsilon)}}{\rho|x_l| + (1-\rho)\frac{\rho|x_l|(1-\varepsilon) + \sum_{i<l} p_i^l w_{1i}}{1-(1-\rho)(1-\varepsilon)}}
\end{aligned}$$

as $b_i^l < \varepsilon w_{il}$ for $i \in \{1, j'\}$ so that $b_i^l$ is negligible in the above ratio.[16] By the induction,

$$\frac{\rho V_{j'}^{j'} + \rho|x_{j'} - x_l|(1-z) + \sum_{i<l} p_i^l w_{j'i}}{\rho|x_l|(1-z) + \sum_{i<l} p_i^l w_{1i}}$$

[16] By $\approx$ we mean that the ratio of the two expressions goes to 1.

is bounded and thus $\frac{w_{j'l}}{w_{1l}}$ is bounded from above.

Thus $\varepsilon > \bar{\varepsilon} > 0$.

III(ii). Note that $w^l_{ji} = w^{\max\{j,i\}}_{ji}$ for $j, i < l$ and that $w^l_{ji} = 0$ for $j, i > l$. We now show that for all other cases, $w^l_{ji}$ is of order $\rho$ or lower.

Suppose that in equilibrium the state remains $l$ with probability $1 - z$ and that $p^l_i$ is the probability that some player $i < l$ wins in state $l$.

Case 1: $w^l_{j1}$ for $j \geq l$ : After some manipulation:

$$w^l_{j1} = \rho|x_l| + (1-\rho)\frac{\rho|x_l|(1-z) + \sum_{i<l} p^l_i w^l_{ji} - b^l_j}{1-(1-\rho)(1-z)}$$

This expression is of order $\rho$ or lower by the induction hypothesis.

Case 2: $w^l_{ji}$ for $j \geq l > i$. We have:

$$\begin{aligned} w^l_{ji} &= \rho(|x_j - x_i| - |x_j - x_l|) + (1-\rho)(V^l_j - V^i_j) \\ &= w^l_{j1} - w^i_{j1} \end{aligned}$$

both of order $\rho$ or lower by the induction and case 1 and hence $w^l_{ji}$ is of order $\rho$ or lower.

Case 3: $w^l_{ji}$ for $j < l < i$ : After some manipulation:

$$\begin{aligned} w^l_{ji} &= w_{jl} \\ &= \rho(|x_j - x_l|) + (1-\rho)\tfrac{\rho V^j_j + \rho(|x_j - x_l|(1-z) + \sum_{i<l} p^l_i w^l_{ji} + b^l_j}{1-(1-\rho)(1-z)} \end{aligned}$$

Note that $b^l_j < \max_i w^l_{ji}$; if $\max_i w^l_{ji} = w_{jk}$ for some $k < l$ then we know that the above is of order $\rho$ or lower. Suppose then that $\max_i w^l_{ji} = w_{jl}$. Plugging this maximal value we get that $w_{jl}$ must be of the same order as $\rho(|x_j - x_l|) + (1 - \rho)\frac{\rho V^j_j + \rho(|x_j - x_l|(1-z) + \sum_{i<l} p^l_i w^l_{ji}}{1-(1-\rho)(1-z)}$ which is of order $\rho$ or lower by the induction hypothesis.

III(iii). Note that $w^l_{21} = w^2_{21}$.

This completes the proof of Lemma 1.$\square$

We can now prove the Theorem. Suppose that at some state $l$ player 1 wins with probability $1 - \varepsilon$ converging to 1. Similar arguments as in I(i) in Lemma 1 imply that for all $j \neq 1$, for almost all bids in the support of $j$, $\frac{b^*_j}{\max_{i \leq l} w_{ji}} \to 0$ and thus player 1's bid satisfies $\frac{b^*_1}{\max_j \max_{i \leq l} w_{ji}} \to 0$ almost surely. Now consider player 2 for whom $w^l_{21} > 0$ and is of order $\rho$ from Lemma 1. Player 2 can deviate to some bid $b'_2$ with $\frac{b'_2}{\max_j \max_{i \leq l} w_{ji}} \to 0$ and $\frac{b'_2}{b^*_1} \to \infty$ which will guarantee winning with probability converging to 1, and therefore, relative to his equilibrium strategy, a gain of at least $(1-\varepsilon)w_{21} + \varepsilon\tilde{w}_{2i} - b'_2$ which is strictly positive, a contradiction.$\blacksquare$

## 8.2 Proofs for Section 5

### 8.2.1 A useful Lemma

We start with a useful result about equilibria in which only two players bid strictly positive bids.

**Lemma A1:** *Suppose that in equilibrium, at some stage $t$, only two players, $i$ and $j$, place strictly positive bids with strictly positive probability. Then: (i) Strategies are continuous, integrable and differentiable $F_i, F_j$, there are no gaps in the support and atoms can be placed only on zero, and both players have same support.(ii) Let $w_j(F_i, \rho)$ be the willingness to win of player $j$ given $F_i$ and define analogously $w_i(F_j, \rho)$. Without loss of generality, let $w_j(F_i, \rho) \geq w_i(F_j, \rho)$. Then $w_i(F_j, \rho) > 0$ and $F_i$ and $F_j$ are:*

$$F_j(b) = \frac{b}{w_i(F_j, \rho)}; \; F_i(b) = \frac{w_j(F_i, \rho) - w_i(F_j, \rho) + b}{w_j(F_i, \rho)} \text{ for all } b \in [0, w_i(F_j)]$$

*(iii) The expected utility of player $i$ from this game is the utility of losing to $j$, whereas the expected utility of player $j$ is, with probability $\frac{w_i(F_j,\rho)}{w_j(F_i,\rho)}$ the utility from losing to $i$ and with probability $1 - \frac{w_i(F_j,\rho)}{w_j(F_i,\rho)}$ the utility from winning. Player $j$ wins with probability $1 - \frac{w_i(F_j,\rho)}{w_j(F_i,\rho)} + \frac{w_i(F_j,\rho)}{2w_j(F_i,\rho)}$. (iv) For any other player $k$, let $w_{ki}$ be the willingness to win of player $k$ against particular player $i$. If $w_{ki}(\rho) + w_{kj}(\rho) > 0$, then $w_k(F_i, F_j, \rho) \leq w_i(F_j, \rho)$.*

**Proof of Lemma A1**: (i) Follows from standard analysis in the literature; see Hillman and Riley (1989), Baye et al (1996), and Levy and Razin (2008). (ii) Consider the first order condition for player $i$ :

$$f_j(b) w_i(F_j, \rho) = 1$$

and similarly the one for player $j$. This implies the form of the distribution function above, with an atom on zero for $F_i$. (iii) These are computed at a bid infinitesimally close to zero which is in the support of the players. (iv) For some player $k$, any utility maximizing bid must satisfy the first order condition $f_i F_j w_{ki}(\rho) + f_j F_i w_{kj}(\rho) - 1 = 0$ but second order condition, using (i), is $f_i f_j (w_{ki}(\rho) + w_{kj}(\rho)) > 0$. Hence utility maximizing bids are either 0 or the maximum bid which is $w_i(F_j, \rho)$. So for player $k$ not to enter, we must have that his utility from a bid of zero is higher than the utility from the maximum bid, which implies that $w_k(F_i, F_j, \rho) \leq w_i(F_j, \rho)$.■

### 8.2.2 Computations for Two-sided influence games:

We now analyze, by backward induction, the MPE for all states $J$.

1. Equilibria for $J = 2$ :

Only players 1 and 2 can be active, as explain in the text. The analysis follows Lemma A1: We conjecture that in equilibrium player 2 has a lower willingness to win[17], and is therefore the one who places an atom on zero of size $1 - \frac{w_2^2}{w_1^2}$, where[18]

$$\begin{aligned} w_2^2 &= \rho x_2 + (1-\rho)(V_2^2 - V_2^1) \\ w_1^2 &= \rho x_2 + (1-\rho)(V_1^2 - V_1^1) \end{aligned}$$

(recall that subscript indicates the player, and superscript indicates the state). By (ii) in Lemma A1,

$$V_2^1 = -x_2;\ V_1^2 = \frac{w_2^2}{w_2^1}(\rho x_2 + (1-\rho)V_1^2)$$

and plugging for these values, we can solve for the ratio $\frac{w_2^2}{w_2^1} = \frac{1}{2-\rho}$, implying that the atom is of size $\frac{1-\rho}{2-\rho}$. Thus, $V_1^2 = -\rho x_2$, $V_2^2 = -x_2$, and $V_3^2 = \frac{-3|x_3| - x_2\rho + |x_3|\rho}{3-\rho}$.[19]

2. Equilibria for $J = 3$ :

Note that the two above states are strategically equivalent, and we will therefore assume that the same MPE is played in both.

We first consider the equilibrium in which players 2 and 3 only are active. The willingness to win of each player is:

$$\begin{aligned} w_2^3 &= \rho(|x_3| + x_2) + (1-\rho)(V_2^2 - V_2^3); \\ w_3^3 &= \rho(|x_3| + x_2) + (1-\rho)(V_3^3 - V_3^2); \end{aligned}$$

Conjecture that the atom on zero is on player 3 (the opposite cannot arise). Let the size of the atom be $\delta$. Then:

$$\begin{aligned} V_2^3 &= \delta(1-\rho)(-x_2) + (1-\delta)(-\rho(|x_3| + x_2) + (1-\rho)V_2^3 \\ V_2^3 &= \frac{\delta(1-\rho)(-x_2) - (1-\delta)\rho(|x_3| + x_2)}{1-(1-\delta)(1-\rho)} \\ V_2^2 - V_2^3 &= \frac{\rho(|x_3|(1-\delta) - \delta x_2)}{\delta(1-\rho) + \rho} \\ V_3^3 - V_3^2 &= -\rho(|x_3| + x_2 + V_3^2) = \frac{x_2(3-2\rho)}{3-\rho} \end{aligned}$$

[17] Indeed, conjecturing the opposite leads to a contradiction.

[18] For brevity, we have dropped the index $\rho$ and the index of the distribution functions.

[19] Given these, it is easy to check the informal argument stated in the text that player 3 will not enter (by condition (iii) in Lemma A1).

We can solve for $\delta = 1 - \frac{w_3^3}{w_2^3}$ to find:

$$\delta(\rho) = \frac{3|x_3| + 3x_2\rho - 4|x_3|\rho - 5x_2\rho^2 + 2x_2\rho^3 + |x_3|\rho^2}{6|x_3| + 7x_2\rho - 5|x_3|\rho - 7x_2\rho^2 + 2x_2\rho^3 + |x_3|\rho^2}$$

Note that $\delta(\rho) \geq 0$ for all $\rho$. This allows to compute

$$\begin{aligned} w_1^3 &= \rho(\frac{1+\delta(\rho)}{2}x_2 + (\frac{1-\delta(\rho)}{2})x_3) + \\ &\quad (1-\rho)(-\frac{1+\delta(\rho)}{2}V_1^2 - (\frac{1-\delta(\rho)}{2})V_1^3) \end{aligned}$$

where

$$V_1^3 = -\frac{\frac{1+\delta}{2}(\rho x_2 + (1-\rho)\rho x_2) + \frac{1-\delta}{2}\rho x_3}{1-(1-\rho)\frac{1-\delta}{2}}.$$

To check that $w_1^3 - w_3^3 < 0$ we note that the $lhs$ is maximal for $\rho \to 0$. We therefore compute $\lim_{\rho\to 0}[w_1^3(\rho) - w_3^3(\rho)] = 3x_2 - x_3$ to get the required condition.

We now consider the equilibrium in which 1 and 3 are active. The willingness to win of each player is:

$$\begin{aligned} w_1^3 &= \rho(|x_3|) + (1-\rho)(V_1^1 - V_1^3); \\ w_3^3 &= \rho(|x_3|) + (1-\rho)(V_3^3 - V_3^1); \end{aligned}$$

Conjecture that the atom on zero is on player 3 (the opposite cannot arise). Let the size of the atom be $\delta$. Then:

$$\begin{aligned} V_1^3 &= (1-\delta)(-\rho(|x_3|)) + (1-\rho)(1-\delta)V_1^3 \\ V_1^3 &= -\frac{(1-\delta)\rho|x_3|}{1-(1-\delta)(1-\rho)} \\ V_1^1 - V_1^3 &= \frac{(1-\delta)\rho|x_3|}{1-(1-\delta)(1-\rho)} \\ V_3^3 - V_3^1 &= -|x_3| + |x_3| = 0 \end{aligned}$$

We can solve for $\delta = 1 - \frac{w_3^3}{w_1^3}$ to find:

$$\delta(\rho) = \frac{1}{2-\rho} \geq 0.$$

This allows to compute

$$\begin{aligned} w_2^3 &= \rho(\frac{1+\delta(\rho)}{2}x_2 + (\frac{1-\delta(\rho)}{2})(x_3 + x_2)) \\ &\quad +(1-\rho)(\frac{1-\delta(\rho)}{2})(-x_2 - V_2^3) \end{aligned}$$

where

$$V_2^3 = \frac{\rho(\frac{1+\delta(\rho)}{2}(-x_2) + (\frac{1-\delta(\rho)}{2})(-x_3 - x_2)) + (1-\rho)(\frac{1+\delta(\rho)}{2}(-x_2))}{1-(1-\rho)(\frac{1-\delta(\rho)}{2})}.$$

To check that $w_2^3 - w_3^3 < 0$ we note that the $lhs$ is maximal for $\rho \to 0$. We therefore compute $\lim_{\rho \to 0}[w_2^3(\rho) - w_3^3(\rho)] < 0 \Leftrightarrow 2x_2 < |x_3|$ to get the required condition for equilibrium existence.

Finally, we consider an equilibrium in which 1 and 2 are active. This equilibrium must be identical to the one in $J = 2$, so that $w_2^3(\rho) = \rho x_2$. The probability that 2 wins in this equilibrium is $\frac{1}{2(2-\rho)}$. The willingness to win of player 3 is therefore (noting that $V_3^2 = V_3^3$):

$$w_3^3 = \rho(|x_3| + \frac{1}{2(2-\rho)}x_2) + (1-\rho)(1 - \frac{1}{2(2-\rho)})(V_3^2 + |x_3|)$$

where $V_3^2 = \frac{-3|x_3| - x_2\rho + |x_3|\rho}{3-\rho}$. Thus,

$$\begin{aligned} w_3^3 &= \rho(|x_3| + \frac{1}{2(2-\rho)}x_2) + (1-\rho)(1 - \frac{1}{2(2-\rho)})(-\frac{\rho x_2}{3-\rho}) = \\ & \rho|x_3| + \rho x_2(\frac{\rho}{3-\rho}) \\ &> \rho x_2 \end{aligned}$$

so 3 would deviate and this equilibrium cannot be sustained.

From the above note that when $|x_3|$ is close enough to $x_2$, the only equilibrium that can arise is that all three players are active.

### 8.2.3 Proofs of Propositions 3,4,5:

**Proof of Proposition 3(i):** Assume that $\rho = 1$. In the text we explain why, in an equilibrium with only two active players, the such two are players 2 and 3 when $|x_3|$ is relatively close to $x_2$. By continuity, the analysis above would hold for sufficiently high $\rho$ as well.

We now show that there doesn't exist an equilibrium with three active players when $\rho = 1$ and $|x_3| = x_2$. It is easy to show that the bid 0 must be in the support of all players. Denote the distribution function over bids in their support by $F_i(b)$ for $i \in \{1, 2, 3\}$.

Suppose that all players reach the same maximum $\bar{b}$. But then the utility of all players at 0 must be the same (as their utility from $\bar{b}$ is $-\bar{b}$ and the utility of all actions in the support must be the same). However, this is impossible: the utility of player 1 will always be at least $-|x_2|$ and that of at least one other player will be at most $-|x_2|$.

Note that at least two players must reach the maximum bid $\bar{b}$. We will now show that it cannot be that one player will bid at most $\tilde{b} < \bar{b}$, which will therefore imply that no equilibrium with active three players exist.

Suppose first that player 2 or 3 reaches a lower maximum $\tilde{b}$. Suppose wlog that it is player 2. But then for player 1, from equal utility at $\bar{b}$ and $\tilde{b}$, we have that $\bar{b} - \tilde{b} = (1 - F_3(\tilde{b}))|x_2|$, whereas the utility difference for player 2 from $\bar{b}$ and $\tilde{b}$ is at least $(1 - F_3(\tilde{b}))|x_2|$; thus if 1 finds it worthwhile to pay the highest bid, so will 2.

Suppose now that player 1 reaches a lower maximum $\tilde{b}$. Note that players 2 and 3 must reach the same maximum $\bar{b}$ and must behave symmetrically on $[\tilde{b}, \bar{b}]$ implying that $F_2(\tilde{b}) = F_3(\tilde{b})$. As the difference in utilities for player 1 from $\tilde{b}$ and a bid close to zero has to be zero, we have that $\tilde{b} = \int_0^{\tilde{b}} (f_2 F_3 + f_3 F_2)|x_2| db = F_3(\tilde{b})^2 |x_2|$. On the other hand, the same calculation for player 2 implies that $\tilde{b} = \int_0^{\tilde{b}} (f_3 F_1 + f_1 F_3)|x_2| db + \int_0^{\tilde{b}} f_3 F_1 |x_2| db > F_3(\tilde{b})|x_2| > F_3(\tilde{b})^2 |x_2|$, a contradiction to the above.

Thus, with continuity, when $\rho$ is sufficiently high and $|x_3|$ is sufficiently close to $x_2$, we cannot sustain an equilibrium with three active players.■

**Proof of Proposition 3(ii):** Follows from the computations of the equilibria.■

**Proof of Proposition 4:** Follows from the computations of the equilibria.■

**Proof of Proposition 5:** Follows from the computations of the equilibria.■

### 8.2.4 Computation for one-sided influence games:

We now compute the MPE for all $J$.

1. Equilibria for $J = 2$.

We first consider an equilibrium in which only players 1 and 2 are active. This is exactly the same equilibrium as calculated in the two-sided game. The only difference is that $V_3^2 = \frac{-\frac{3-\rho}{2(2-\rho)} x_3 - \frac{1}{2}\frac{1-\rho}{2-\rho}\rho(x_3 - x_2)}{1 - \frac{1}{2}\frac{1-\rho}{2-\rho}(1-\rho)}$ is now higher.

Consider now an equilibrium in which only players 1 and 3 are active. Note that the role of player 3 here is to defend the position of player 2 and compete in his stead.

The equilibrium actions will be exactly as in the case above where 1 and 2 are active. The continuation values are $V_3^2 = -x_3$, $V_1^2 = -\rho x_2$, and $V_2^2 = \frac{3x_2 - 2\rho x_2}{\rho - 3}$.

It is therefore clear that there exists a continuum of equilibria, in which players 2 and 3 "share" the burden of being active against player 1 (as in Baye et al (1994)), and all three players are therefore active. We focus here on a symmetric equilibrium (in which players 2 and 3 have the same strategy) which is sufficient for our results, as it is the equilibrium in which player 1 wins with the lowest probability (whereas in the equilibria described above player 1 wins with the highest possible probability). Let players 2 and 3 use the density function $g(b)$ and let player 1 use $f(b)$. The first order conditions are given by:

$$\begin{aligned} 2g(b)G(b)w_{12}^2 &= 1 \\ f(b)G(b)w_{21}^2 &= 1 \\ f(b)G(b)w_{31}^2 &= 1 \end{aligned}$$

where $w_{21}^2 = w_{31}^2$ in a symmetric equilibrium. This implies that $(GG)' = \frac{1}{w_{12}}$ i.e.,

$$G(b) = \sqrt{\frac{b+c}{w_{12}}}; F(b) = \frac{2w_{12}}{w_{21}}\sqrt{\frac{b+c}{w_{12}}} + c'$$

for some constants $c$ and $c'$. As $F(0) = 0$,

$$c' = -\frac{2w_{12}}{w_{21}}\sqrt{\frac{c}{w_{12}}}$$

Also, as $F(b^{\max}) = 1$ and $G(b^{\max}) = 1$, we have:

$$c = w_{12}(1 - \frac{w_{21}}{2w_{12}})^2$$

where the atom that player 1 is facing is $G(0)^2 = (1 - \frac{w_{21}}{2w_{12}})^2$

In equilibrium player 1 wins with probability:

$$\alpha = \int_0^{b^{\max}} f(x)G^2(x)dx = \frac{2w_{12}}{3w_{21}} - \frac{2w_{12}}{3w_{21}}((1 - \frac{w_{21}}{2w_{12}})^3$$

and:

$$\frac{w_{21}^2}{w_{12}^2} = \frac{\rho + \sqrt{-2\rho + \rho^2 + 2} - 2}{\rho - 1}$$

Hence, $G(0)^2 \to 0.5$, and the probability that 1 wins in equilibrium converges to 0.7357. Finally, note that $V_2^2 = \frac{-\alpha' x_2}{1-(1-\rho)(1-\alpha')}$ where $\alpha' = \frac{1}{a}(1 - (1 - \frac{a}{2})^2)$.

2. Equilibria for $J = 3$.

Consider first an equilibrium in which only players 1 and 3 are active. Suppose that off the equilibrium path, if 2 deviates and win, then in $J = 2$, only 1 and 2 are active.

Assume that the atom is on player 3; simple computations as above imply that the atom is of size $\delta(\rho) = \frac{1-\rho}{2-\rho}$, and the willingness to win of player 3 is $\rho x_3$. The willingness to win of player 2 is $\rho x_2 + \rho(x_3 - x_2)\frac{\frac{1}{2}(1-\delta(\rho))}{1-\frac{1}{2}(1-\delta(\rho))} < \rho x_3$. Thus, the equilibrium holds for all parameters.

Consider now an equilibrium when only players 2 and 3 are active. Suppose first that when 2 wins, then at $J = 2$, either only 1 and 2 are active or only 1 and 3 are active.

We solve for the equilibrium as above, by conjecturing that the atom on zero, $\delta(\rho)$, is played by player 3. We then find that

$$\delta(\rho) = \frac{6x_2 - 3x_3 - 6\rho^2 x_2 + \rho^2 x_3 + 3\rho^3 x_2 - \rho^3 x_3 - 3\rho x_2 + 3\rho x_3}{6x_2 - 6x_3 - 8\rho^2 x_2 + 2\rho^2 x_3 + 3\rho^3 x_2 - \rho^3 x_3 + \rho x_2 + 3\rho x_3}$$

with $\lim_{\rho\to 1} \delta(\rho) = 0$ and $\lim_{\rho\to 0} \delta(\rho) = \frac{x_3-2x_2}{2(x_3-x_2)} > 0$ if $x_3 - 2x_2 > 0$. We also find that $\delta'(\rho) < 0$. Thus, a condition for this equilibrium to arise is that $x_3 - 2x_2 > 0$. Otherwise, we can conjecture that the atom is on player 2, but then we find that his willingness to win is negative, a contradiction.

We now have to make sure that player 1 indeed prefers not to be active. Given the solution for $\delta(\rho)$ we check the condition $w_1^3(\rho) - w_3^3(\rho) < 0$. Again the $lhs$ decreases in $\rho$ and so we check $\lim_{\rho\to 0}[w_1^3(\rho) - w_3^3(\rho)] = 4x_2 - x_3$ to get the condition in the text.

Consider now the same equilibrium at $J = 3$ (i.e., only players 2 and 3 are active) but assume that all players are active at $J = 2$.

Assume that the atom, $\delta(\rho)$, is on player 3. Computing the equilibrium, we find that the atom is as above, converging to $\frac{x_3-2x_2}{2x_3-2x_2}$. To compute whether player 1 wants to enter, we find that when $\rho \to 0$, $\frac{w_1^3}{w_3^3} \approx \frac{x_3+(\frac{x_3-2x_2}{2x_3-2x_2}+\frac{1}{2}(1-\frac{x_3-2x_2}{2x_3-2x_2}))2x_2}{(1-\frac{1}{2}(1-\frac{x_3-2x_2}{2x_3-2x_2}))x_3} > 1$ for all $x_2, x_3$ and hence the equilibrium does not hold.

We now consider an equilibrium when $J = 3$ and only players 1 and 2 are active. Suppose first that at $J = 2$, only players 1 and 2 are active as well. Thus, at $J = 3$, the equilibrium is exactly as in $J = 2$ so that $w_2^3 = \rho x_2$. Consider player 3. His willingness to win is $\rho(x_3 - zx_2) + (1-\rho)(V_3^3 - zV_3^2 - (1-z)(-x_3))$ where $z$ is the probability that player 2 wins in the game between 1 and 2, which is $\frac{1}{2(2-\rho)}$. Note also that $V_3^2 = V_3^3$. We therefore have:

$$\begin{aligned}
w_3^3(\rho) &= \rho(x_3 - zx_2) + (1-\rho)(V_3^2 + x_3)(1 - \frac{1}{2(2-\rho)}) \\
&= \rho(x_3 + x_2\rho\frac{2-\rho}{3-\rho^2})
\end{aligned}$$

and thus $w_3^3(\rho) < w_2^3(\rho)$ iff $x_3 < x_2(1 + \rho\frac{2-\rho}{3-\rho^2})$.

Suppose now that at $J = 2$, only players 1 and 3 are active and consider the game at $J = 3$ when players 1 and 2 are active.

Suppose that the atom on zero is on player 2. It is easy to see that

$$\begin{aligned}
w_1^3 &= \rho x_2 + (1-\rho)(0 - V_1^2) = \rho x_2(2-\rho) \\
w_2^3 &= \rho x_2 + (1-\rho)(V_2^2 + x_2) = \rho x_2(\frac{2(2-\rho)}{3-\rho})
\end{aligned}$$

and that player 2 places an atom of $1 - \frac{2}{3-\rho}$.

Will player 3 enter? $w_3^3(\rho) = \rho(x_3 - (1-z)x_2) + (1-\rho)(V_3^3 + x_3)$ for $z = \frac{\rho-2}{\rho-3}$ (the probability that 1 wins in equilibrium), and $V_3^3 = \rho(-x_3 + (1-z)x_2) + (1-\rho)(-x_3)$. Plugging this in $w_3^3(\rho)$, player 3 does not enter as long as $\rho(x_3 - \frac{\rho x_2}{3-\rho}) < \rho x_2(\frac{2(2-\rho)}{3-\rho})$, or when $x_3 < x_2\frac{4-\rho}{3-\rho}$.

Note that the condition for this equilibrium to hold is more lenient than in the case in which the continuation game in $J = 2$ is that players 1 and 2 are active, as for all $\rho$, $\frac{4-\rho}{3-\rho} > (1 + \rho\frac{2-\rho}{3-\rho^2})$. Moreover, this would represent the most lenient condition among the set of feasible equilibria in the stage $J = 2$.

### 8.2.5 Proofs of propositions 6,7,8:

**Proof of Proposition 6:** Follows from the above computations.■

**Proof of Proposition 7:** Follows from the above computations.■

**Proof of Proposition 8:** We need to only consider this equilibrium at $J = 2$. Suppose that player 1 places an atom or that there is no atom. Then: $w_1 = \rho x_2 + (1-\rho)(-V_1^2) = x_2$ as player 1 loses. For the 2-player who bids: $w_i = \rho x_2 + (1-\rho)(V_i^2 + x_2)$; $V_i^2 = \frac{z(-x_2)}{1-(1-z)(1-\rho)}$ as he does not bid later on, where $z$ is the probability that player 1 wins any stage when $J = 2$. Thus, $w_i = \rho x_2 + (1-\rho)(\rho x_2\frac{1-z}{z+\rho-z\rho}) = \rho(\frac{x_2}{z+\rho-z\rho})$. Obviously the willingness to win of player 1 is higher unless $z = 0$, a contradiction to 1 placing an atom or no atom.

Thus the 2-players must place an atom. We have:

$$\begin{aligned}
w_1 &= \rho x_2 + (1-\rho)(-V_1^2);\ V_1^2 = \frac{-(1-\delta)\rho x_2}{1-(1-\delta)(1-\rho)} \\
w_1 &= \rho x_2 [1 + \frac{(1-\rho)(1-\delta)}{1-(1-\delta)(1-\rho)}] \\
V_i^2 &= \frac{z(-x_2)}{1-(1-z)(1-\rho)};\ z = \delta + \frac{1-\delta}{2} \\
w_i &= \rho x_2 + (1-\rho)(-\rho x_2 \frac{z-1}{z+\rho-z\rho}) = \rho x_2 \frac{2}{\delta+\rho-\delta\rho+1} \\
\delta &= -\frac{1}{\rho-1}\left(\sqrt{-2\rho+\rho^2+2}-1\right)
\end{aligned}$$

The probability that 1 wins is roughly between $(0.5, 0.7)$.■

## 8.3 Proofs for Section 6

**Proof of Proposition 9:** First note that players' continuation values are at least 0 at any stage game. Second, consider $J = 2$ and note that the only players that may potentially submit strictly positive bids are 1 and 2. The atom must be on 2 and the solution is the same as in the standard model, and we have that $V_1^2 = v(1-\rho)$ and $w_1^2 = \rho v(2-\rho)$. Suppose we are now at some state $J$ and that the Proposition is true for all states that are more moderate than $J$. Note that players who are more extreme than $J$ do not participate. We then have that $w_i^J = \rho v + (1-\rho)(V_i^I - \sum_{j\neq i} V_i^j)$ but by the induction, $V_i^i = 0$, and $V_i^j = 0$ for all $j$ that participate, thus $w^J = \rho v$ for all $x_i$ that are weakly more moderate than $J$. On the other hand, $w_1^J = \rho v + (1-\rho)(v - \sum_{j\neq 1} V_1^j) > \rho v$ as $V_1^j < v$ by the induction hypothesis. We can therefore apply Baye et al (1996) for each stage to find that, $\Pi_i F_i(0) = \frac{1-\rho}{2-\rho}$ for all $i \neq 1$ that participates, and so $F_i(0) > 0$ for any such $i$, that player 1 must participate in every stage and that he wins with a higher probability than any other. Also, there is a continuum of equilibria as in Baye et al (1996).■

# References


[1] Admati, A. and M. Perry (1991), "Joint projects without commitment", *Review of Economic studies*, 58, 259-276.

[2] Austen-Smith, D. (1995), "Campaign contribution and access", American Political Science Review, 89(3), 566-581.

[3] Austen-Smith, D. (1989), "Sophisticated sincerity: voting over endogenous agendas" *American Political science review* 81(4) 1323-1330.

[4] Barbera, S. and D. Coelho (2009), "On the rule of k names", working paper.

[5] Baron, D. and J. Ferejon (1989), "Bargaining in legislatures", *American political science review* 83(4), 1181-1206.

[6] Baye, M. R., D. Kovenock, C. G. de Vries (1996), "The all-pay auction with complete information", Economic Theory Volume 8(2), 291-305.

[7] Becker, G. (1983), "A Theory of Competition among Pressure Groups for Political Influence", The Quarterly Journal of Economics 98: 371-400.

[8] Besley T. and S. Coate (1997), "An Economic Model of Representative Democracy," *Quarterly Journal of Economics* 108, 85-114.

[9] Casas-Arce, F. and F. Asis Martinez-Jerez (2007), "The all pay auctions with Handicaps", working paper.

[10] Clark D. and C. Riis (1998), "Contest success function: an extension", *Econ Theory* 11:201–204.

[11] Compte, O. and P. Jeheil (2004), "Gradualism in Bargaining and contribution games", Review of economic studies, 71(4), 975-1000.

[12] Copic, J, and J. N. Katz (2007), "Auctioning off the agenda: bargaining in legislatures with endogenous schedualing", working paper.

[13] Cox, G. (2006), The organization of democratic legislatures, in B. Weingast and D. Wittman (eds.), The oxford handbook of political economy, Oxford: Oxford University Press.

[14] Duggan, J. (2006), "Endogenous voting agendas", *Social Choice and Welfare* 27(3), 495-540.

[15] Dutta, B. Jackson, M, and Lebreton, M. (2004) "Equilibrium agenda formation", *Social choice and welfare* 23(1), 21-57.

[16] Grossman, G. and E. Helpman (1994), "Protection for Sale", *American Economic Review*, vol. 84, no.4.

[17] Grossman, G. and E. Helpman. (2001), Special Interest Groups, MIT Press.

[18] Hillman, A. and J. Riley (1989), "Politically contestable rents and transfers", Economics and Politics1, 17–39.

[19] Konrad, K. and D. Kovenock, (2009), "Multibattle contests", *Games and economic behaviour.*

[20] Konrad, K. A. (2009), Strategy and Dynamics in Contests, Oxford University Press, 2009.

[21] Levy, G. and R. Razin (2009), "Bidding for attention and competing for votes in political debates", in The proceedings of The Political Economy of Democracy conference.

[22] Levy, G. and R. Razin (2008), "Attention for Sale: Endogenous Agenda Formation in Sequential vs. Simultaneous Debates", working paper, LSE.

[23] Osborne, M., J. Rosenthal, and M. Turner (2004), "Meeting with Costly Participation", *American Economic Review* 90(4), 927-943.

[24] Osborne M. and A. Slivinski (1996), "A Model of Political Competition with Citizen-Candidates," *Quarterly Journal of Economics* 111, 65-96.

[25] Penn, E. (2008), "A Distributive N-Amendment Game with Endogenous Agenda Formation", mimeo, Harvard.

[26] Polborn, M.K. and T. Klumpp, "Primaries and the new hampshire effect", *Journal of Public Economics* 90, 1073-1114.

[27] Skaperdas, S. (1996), "Contest success functions.", *Econ. Theory* 7, 283–290.

[28] Siegel, R. (2009)), "All pay contests", *Econometrica*, 77(1), 71-92.