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Dan S. Felsenthal and Moshé Machover

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Myths and Meanings of Voting Power: Comments on a Symposium

Dan S Felsenthal
University of Haifa

Moshé Machover
King's College, London

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Please address all correspondence to:

Moshé Machover
Department of Philosophy
King's College
Strand
London WC2R 2LS
England

Phone: +44 20 8969 5356 (home)
Fax: +44 20 7848 2270 (office)
E-mail: moshe.machover@kcl.ac.uk

ABSTRACT

These are comments on the Symposium *Power Indices and the European Union* in the July 1999 issue of this *Journal*. We point out several common inter-connected confusions and errors concerning the meaning of voting power. We stress the vital distinction between two different intuitive notions of voting power. We emphasize the need for a unified approach to the study of a priori and actual voting power. We show that the family of 'strategic' measures proposed by some of the participants in the Symposium are a natural generalization of the Banzhaf measure.

KEY WORDS • Actual and a priori voting power • European Union • I-power • P-power • strategic measures

Myths and Meanings of Voting Power: Comments on a Symposium

1. Introduction

The following familiar statements are often repeated in the voting-power literature.

- Shapley and Shubik (1954) proposed the first index of voting power—hereafter referred to as the 'S-S index'.

- The justification of the S-S index rests on a permutation model: coalition formation occurs in a random order.

- Eleven years after the S-S index was proposed, Banzhaf (1965) invented an additional measure.[1]

- The Banzhaf (Bz) index is a variant of the S-S index, except that it ignores the order in which a coalition is formed.

- Both indices are essentially constructs of cooperative game theory.

- They have similar meaning and behaviour: they measure the same kind of thing in slightly different ways.

Yet our contention is that all these widely believed statements are mere myths. And in our view their wide acceptance has had the unfortunate effect of distorting and confusing many discussions of the theoretical foundations of the measurement of voting power.

A case in point is the Symposium on Power Indices and the European Union in the July 1999 issue of the *Journal of Theoretical Politics*, consisting of five papers: Garrett and Tsebelis (1999) and (1999a); Lane and Berg (1999); Holler and Widgrén (1999); and Steuneneberg, Schmidtchen and Kobaldt (1999). For the sake of brevity, we shall refer to these papers as 'G&T', 'G&T(a)', 'L&B', 'H&W' and 'SS&K', respectively.

The main bone of contention in the Symposium is whether voting-power indices are at all applicable to the measurement of voting power in the decision-making bodies of the European Union (EU). On this issue, the first-mentioned authors take an adamantly negative position, while L&B and H&W defend the use of power indices, at least for some purposes and within certain limits. SS&K take an intermediate position: they are critical of the existing widely-used measures, primarily the S-S index and the Bz measure; but they propose a new 'strategic' measure of voting power—or, more precisely, a whole family of such measures—which in their view accommodates the objections of G&T.

In our view, the arguments of all participants in the Symposium are, to varying degrees, flawed by the common errors mentioned above. In Section 2 we explain what is wrong with the commonly held views cited at the beginning of the present Introduction. In Section 3 we point out some of the errors on these matters made by the Symposium's participants.

In Section 4 we turn to the main issue of the Symposium and take up a position in favour of using power indices. In Section 5 we discuss the 'strategic' measures proposed by SS&K and argue that they are a natural (and promising) generalization of the Bz measure. The main conclusions appear in Section 6.

Several of the general points we make below are presented in greater detail in our recent (1998) book *The Measurement of Voting Power* (hereafter referred to as 'MVP'). In what follows we provide ample references to the relevant sections of this book.

2. Two Notions of Voting Power

The so-called Bz measure of voting power was in fact proposed by Lionel Penrose (1946). To be precise, the measure invented by Penrose is $\beta'/2$, where β' is the Bz measure. So Penrose's paper predated Banzhaf's (1965) by nineteen years, and Shapley and Shubik's (1954) by eight. However, his pioneering contribution was almost totally unnoticed by mainstream writers on voting power, who date the inauguration of the subject to the last-mentioned paper.[2]

Normally, questions of priority in scientific ideas and inventions are of little substantive importance, though they may be of interest to historians of science. But in the present case the historiographic mistake has had a detrimental effect upon the understanding of the foundations of the subject itself. The point is that Penrose was a mathematical statistician, and his paper makes no reference to game theory (cooperative or of any other kind); it is couched entirely in probabilistic terms, which are sufficient to justify his measure. Shapley, on the other hand, is a celebrated game theorist, who in his (1953) paper proposed his famous value for cooperative games, of which the S-S index is but a special case.

The historiographic error of dating the beginning of the mathematical study of measures of voting power to Shapley and Shubik's (1954) paper fostered the firm impression that *the whole* subject is a branch of cooperative game theory. In particular, it came to be believed that the Bz measure is qualitatively the same kind of thing as the S-S index.[3]

This impression was further reinforced by the fact that the well known permutation model, in terms of which the S-S index is often justified, looks very similar to the model that serves for defining and justifying the Bz measure; and indeed the resulting mathematical formulas are also quite similar.

But it seems to us that the justifications of the S-S index in terms of the permutation model or in terms of other, more elaborate, representations are in fact all spurious. These representations are no more than that—mere representations, which cannot be taken seriously as convincing models of coalition formation. We believe that the only plausible way of justifying the S-S index is in terms of the persuasive power of postulates characterizing the Shapley value, such as those of Shapley (1953) or Young (1985).[4] On the other hand, while it is possible to characterize the Bz measure axiomatically, as in Dubey and Shapley's (1979), such characterizations do not lend any additional justification to this measure: it is amply justified directly in terms of a probabilistic model of division of votes.[5]

More importantly, the Bz measure and the S-S index are based on totally different notions as to what a priori voting power is all about.

In our view, there are in fact two quite distinct underlying (intuitive and pre-formal) notions of a priori voting power, which the various measures of voting power attempt to explicate and formalize.

The first notion is that of power as *influence*: a voter's ability to affect the outcome of a division of a voting body—whether the bill in question will be passed or defeated. We have called this notion of power 'I-power'.[6]

The second notion is that of power as a voter's expected *relative share in some prize*, which a winning coalition can put its hands on by the very act of winning. We have called this notion of power ‘P-power’.

In his critique of the S-S index, Coleman (1971) drew attention to this distinction (without of course using our terminology), but his insight was largely ignored by most writers on voting power, who tended to conflate the two notions and were thereby led to all sorts of error.

In fact, I-power and P-power are fundamentally different. Their explication and formalization lead in rather different directions, and there is no reason to expect, in general, that a property or piece of behaviour that is essential for an acceptable index of P-power should apply also to a valid measure of I-power; or vice versa. We have taken up and amplified Coleman’s insight elsewhere.[7] Here we shall merely outline a few important points.

The notion of I-power has essentially nothing to do with cooperative game theory or, for that matter, with game theory generally, as it is normally understood.[8] According to this notion, voting behaviour is motivated by *policy seeking*. The action of a given voter does not depend on what other voters may be expected to do, let alone on bargaining and concluding binding agreements with them. *In fact, it applies equally well to decision-making bodies in which voting is secret, so that such bargaining among the voters is pointless if not impossible.* Each voter simply votes for or against a given bill on what s/he considers to be the merit of this bill; and the way s/he votes is independent of the decision rule. The passage or failure of a bill is here best regarded as a public good (or public bad), which affects all voters, irrespective of how they have voted on that bill.

Since no binding pacts are assumed in connection with I-power, it is misleading to talk here about the ‘formation’ of a coalition in any conscious sense. Even the very term ‘coalition’, as referring to an arbitrary set of voters, is perhaps somewhat misleading, as it seems to imply conscious coordination.

But as far as I-power is concerned no such coordination is envisaged. In a division of the voters on a given bill, a set of voters happen to find themselves voting on the same side; this is all. Unfortunately, the use of terminology borrowed from game theory has contributed to the widespread confusion by creating the false impression that voting power is necessarily a game-theoretic notion.

(We shall conform with this conventional game-theoretic terminology here, even when discussing I-power. In particular, we shall refer to a binary decision rule as a *simple voting game (SVG).*)

Another way of seeing that the notion of I-power is not fundamentally game-theoretic is the observation that, under this notion, the voting *power* of a voter has nothing whatsoever to do with payoffs. Rather, a voter's I-power depends only on the structure of the SVG itself, which contains no information about any payoffs. (So from the viewpoint of I-power, it is not really a game in the true game-theoretic sense, which requires some information about payoffs to be specified.) Of course, one may assume that payoffs do affect voting *behaviour*: they enter the calculations of voters when making up their minds how to vote on a specific given bill. A rational voter will vote for or against a bill by comparing the expected payoff of the passage of the given bill with the expected payoff of its defeat. But the point is that these payoffs are individually determined: they can vary from voter to voter and from bill to bill; and they are completely exogenous to the structure of the SVG itself.

Note also that in the case of I-power one can talk meaningfully not only about relative voting power but also about voting power in an *absolute* sense. In fact, absolute I-power is the primary notion, whereas relative I-power is derived from it by normalization.

All serious attempts to explicate and formalize the notion of I-power have led—and in our view must lead—in one direction: to the Banzhaf measure, or to some generalization of it. This has happened several times to people who [re]-invented essentially the same measure independently of one another. They include Lionel Penrose (1946), who as far as we know was the original inventor; Banzhaf (1965); Rae (1969), whose measure is the Banzhaf measure in thin disguise; Coleman (1971), whose two measures are slightly more sophisticated variations on the same theme; and Barry (1980).[9] We shall show below that the same applies also to the strategic index proposed by SS&K.

The Banzhaf measure has a clear probabilistic meaning: *the power of a given voter in a given voting game is the a priori probability of that voter being decisive, tipping the balance between passage and failure of the bill in question.*

P-power, on the other hand, is a thoroughly game-theoretic notion. It presupposes *office-seeking* voting behaviour aimed at winning, for the sake of obtaining part of the prize, which is available *only* to the winners and therefore cannot be a public good in the true sense. It also assumes bargaining and binding agreements. For this reason it makes no sense where voting is secret, because that excludes meaningful bargaining and binding agreement. In order to know to what share of the prize you are entitled, if any, we have to know how you and others have voted.

Also, P-power is an *essentially relative* notion. Absolute P-power makes no coherent sense.

The S-S index, as well as the index proposed by Deegan and Packel (1978), are clearly attempts to explicate and formalize this pre-formal notion of P-power. The Johnston (1978) index in our view is a hybrid, based on a confused and misconceived attempt to graft a P-power 'correction' on the

Banzhaf index. In this way it transformed a good index of I-power into a bad index of P-power.[10] Holler's (1982) 'Public Good' index seems to us, conversely, to be an incoherent result of grafting an I-power modification on the Deegan–Packel index.

3. Errors in the Symposium

In this section we shall point out a few examples of how the errors discussed in Section 2 are reflected in the arguments of all participants in the Symposium. These examples are by no means exhaustive.

Let us start with one of the main arguments used by G&T against power indices: their non-additivity. Section 2 of their paper (pp. 296–98) is almost wholly devoted to this argument. Criticizing Lane and Mæland (1995), they say:

> According to Lane and Mæland, pooling the Mediterranean governments' votes would lead to a reduction in their combined power. If each voted separately in a 15-member Council [of Ministers of the EU], their combined power (using the Banzhaf normalized index) would be 0.112 + 0.092 + 0.059 + 0.059 = 0.332 [*sic*] ... . Voting as a bloc, however, their index would be reduced to 0.247 ... . One should immediately ask the question: Why would these governments ever choose to vote as a bloc if in so doing they lose power? (p. 296)

What G&T are discussing here is of course an instance of a well known phenomenon, the so-called *paradox of (large) size*: if the members of a coalition S of an SVG merge and form a bloc $\&_S$, which henceforth acts as a single voter—thereby giving rise to a new SVG—then the voting power of the new bloc-voter $\&_S$ may be smaller than the sum of the powers of the members of S in the original SVG. This phenomenon is displayed by each of the commonly used voting-power indices, and indeed by any conceivable half-way reasonable measure of voting power.[11]

Now, if it were a question of P-power, then G&T's argument would be quite reasonable. Clearly, if the share of the fixed prize that the bloc $\&_S$ expects to obtain is less than the sum of the shares that the members of S expect to obtain when acting as separate individuals, then the bloc will not be formed voluntarily. (Although in a decision-making body such as a shareholders' meeting it could still be formed by annexation.)

But the same argument does not apply to I-power, and in particular to the Bz measure. As we explained in Section 2, the Bz powers of the members of S have nothing to do with payoffs; they are *probabilities.* Moreover, one should be very cautious in interpreting the sum of these probabilities, because they are probabilities of events that are in general not mutually disjoint. So the sum of the Bz powers of the members of S cannot be interpreted as the total influence of the coalition S when its members act as individuals.[12] Since the calculations of Lane and Mæland (1995) are concerned with I-power, G&T's argument is a *non sequitur.*

Their fallacy is further compounded by the fact that the figures they quote from Lane and Mæland are those for the [normalized] Bz *index.* Those figures indeed show that the *relative* influence of a Mediterranean bloc, if it were formed, would be smaller than the sum of the original *relative* influences of the Mediterranean members acting separately; but it does not follow that the *absolute* influence of the bloc would also be smaller than the sum of the *absolute* individual influences—even if such a comparison were meaningful.

Indeed, if the Mediterranean bloc were formed, its absolute I-power, according to the Bz measure, would be 0.355 whereas the sum of the absolute I-powers of the four individual Mediterranean countries is only 0.325.

Clearly, when considering the formation of a bloc, the prospective partners are primarily interested in the absolute amount of influence it would wield. If this is sufficiently large, why should they mind if by forming the bloc they would also cause an increase in the absolute influence of other voters?

On top of all this, G&T commit the simple error of speaking interchangeably about forming a *bloc* and forming a *coalition*, as if they were the same thing. But in the theory of voting power they have very different meanings. A bloc is a fusion of several voters into one new stable entity, which henceforth will act as a single voter, not just in one division but so long as it remains in existence. This gives rise to a new SVG in place of the original one in which the voters in question acted as separate individuals.[13] The formation of a coalition is quite another matter. As we have pointed out in Section 2, this concept makes sense under the notion of P-power, but not of I-power. It is borrowed from cooperative game theory, and refers to a binding agreement of several voters to act together *in one play of the game* and to re-distribute their payoffs in a certain way. A coalition may form even when it is not in the interest of its members to merge into a bloc.

(This is not their only simple error. For example, on pp. 293–94 they imply that according to the Bz index only *minimal winning* coalitions have positive (and equal) probabilities. In the case they consider, there are altogether 128 possible coalitions, of which 21 are minimal winning. They say that each of these minimal winning coalitions has probability $\frac{1}{21}$.)

The distinction between I-power and P-power is also of great importance in assessing G&T's insistence (in connection with actual voting power) that only *connected* coalitions are admissible.

Before we go any further, we wish to note that, technically speaking, the correct requirement should be not connectedness but *convexity*, in the following sense: if a coalition S contains k voters whose most preferred states (represented as points in some Euclidean space) are $X_1, \ldots, X_k$ then any other voter whose most preferred state is a convex combination of these k positions (that is, $\sum_{i=1}^{k} a^i X_i$, where the a^i are non-negative reals that add up to 1) must also be in S.[14]

Now note that G&T impose this condition of admissibility only on a *positive* coalition (consisting of the 'yes' voters) but not on a *negative* coalition (consisting of those who vote 'no'). However, if this asymmetry between 'yes' and 'no' can be justified at all, the justification applies only to P-power. This is because the notion of P-power itself is asymmetric: the concept of *formation* of a coalition, based on a binding agreement, applies only to the positive coalition, which hopes to lay its hands on the spoils by winning. The negative coalition is not 'formed' in this sense, but consists simply of all those voters who are left out.

But for I-power this asymmetry is quite unjustified. Here each voter votes 'yes' or 'no' according as his or her most preferred position is nearer to the proposed bill or to the status quo. The two options are treated symmetrically. Thus, if the convexity condition is to be imposed at all, it must apply to both the positive and the negative coalition in each division. This is equivalent to the existence of a hyperplane such that all voters whose preferred positions are on one side of it vote 'yes', and all those on the other side vote 'no'.

To return to the main point: one of G&T's chief arguments against the use of power indices, as well as their asymmetric admissibility condition, are based on the fundamental misconception that *all* power indices are constructs of cooperative game theory. But the truth is that this does not apply to the Bz measure and Bz index, which—for very good reason—is the index that most researchers apply to the European Union.[15]

Unfortunately, this mistaken belief is shared by other participants in the Symposium. Thus, L&B—with whose conclusion on the main issue of the Symposium we broadly agree—undermine their own case by admitting at the outset that the power-index method 'constitutes an approach within n-person game theory . . . ' (p. 309). Their paper devotes much space to defending the thesis that the assumptions of cooperative game theory are realistic in the case of the EU. We find this thesis both unconvincing and irrelevant, because the use of the Bz measure does not depend on it in any way.

As for H&W, they weaken their position—with which we have much in common—by the mistaken thesis (p. 322) that the S-S index takes into account the voters' preferences, albeit in a randomized way. Here they take seriously the well known permutation model as a justification for the Shapley value and the S-S index, despite the explicit warning of Shapley and Shubik (1954, p. 790) 'that the scheme we have been using (arranging the individuals in all possible orders, etc.) is just a convenient conceptual device'.[16]

4. Why Voting-Power Indices?

Let us now address the main issue of the Symposium. We shall be quite brief, as there is no need to repeat at length those points put forward by L&B and H&W with which we agree.

G&T have two main arguments against the application of power indices to the EU. The first is that power indices are incapable of taking into account the intricate procedures of decision making in the EU: the complex play between the Commission, the Council of Ministers (CM) and the European Parliament.

This argument, even if it were correct, cannot be used against the application of power indices to the analysis of a priori power distribution within the CM in isolation, or the Parliament in isolation. Surely, this question of a priori power distribution within each of these bodies, when it operates under a given decision rule, is of considerable interest.[17]

But the argument is incorrect even as regards the EU decision making as a whole. The notion of *composite* voting 'game' allows the construction of extremely complex voting 'games' from simpler ones, thus providing models for highly intricate interactions among these simpler components.[18]

In this connection we must sound a caveat. In the CM, each voter has just two options: voting 'yes' or 'no'. Abstention is not a real *tertium quid*: under the unanimity rule it counts as a 'yes', and under the qualified majority rule it counts as a 'no'. But in the Parliament abstention is a distinct third option. Such voting rules should be modelled as *ternary voting games*.[19]

G&T's second argument against the use of power indices is that these indices do not take into account the structure of preferences of the voters.

It is quite true that the Bz measure of I-power and the S-S index (as well as other indices of P-power) do not take into account the actual preferences of the voters or their mutual affinities and disaffinities; nor are they designed to do so. This is because these are a priori measures, which address the distribution of power under a given decision rule as such, which is regarded as an empty shell, ignoring the actual personalities of the voters. As both L&B and H&W put it (using Rawls's apt expression) the a priori indices 'go behind a veil of ignorance'; and they *must* do so in order to provide a *constitutional* normative analysis. When designing a constitution, it would be very wrong to tailor it to a particular structure of preferences of the voters, their affinities and disaffinities, because these are highly volatile and transient.[20] This is especially the case in a body such as the CM, whose voters represent governments whose political colour, policies and alliances keep changing. A relatively stable one-dimensional (say 'left-to-right') space, which is often used to model preferences in a national decision-making body, is generally inapplicable here.

The application of power indices to the EU can only be ruled out if one dismisses such aprioristic constitutional considerations as having no objective importance—a position that we regard as quite arbitrary and untenable.

Of course, no-one is compelled to be interested in such constitutional normative analysis. It is a matter of personal taste. One may prefer to study actual, a posteriori voting power. In this highly important and useful enterprise, available information regarding the protagonists' preferences, affinities and disaffinities must be factored in.

But here the following question arises. Should the study of actual voting power proceed by an altogether separate method, totally unconnected to that of the a priori power indices, as G&T apparently suggest; or should the aprioristic method of power indices be adapted to the aposterioristic study by enriching the structure of SVGs so that the needed information can be factored in?

We believe that the latter, unified method is preferable by far. This is because actual voting power is in fact a *superposition*: the specific power that a voter derives from the actual conjuncture of preferences and affinities is superimposed upon the a priori power the voter derives from the bare decision rule. The a priori voting power serves as the benchmark, to which actual voting power ought to be compared. Such a comparison is possible only under a unified method.

As a matter of fact, structures that are elaborations of SVGs, in which mutual affinities and disaffinities between voters can be modelled, have been proposed in the past.[21] And as we shall argue in the next section, the structures proposed by SS&K provide another interesting instance of a unified method, in which information about both affinities and preference structures can be modelled.

Note that the EU institutions are called upon to make decisions on a great variety of issues, relating to various spheres of policy and economics. The geometrical distribution of preferences is likely to be quite different on different kinds of issues, even within a short time-span (and even more so if we consider a time span of a few years, during which preferences may change quite radically). On fishing, for example, the positions of Britain and Spain may be at opposite ends of the spectrum, but on another issue they may be quite close together.

So a valid approach to actual (a posteriori) voting power must not assume a constant structure of preferences (as G&T seem to do), but allow a great deal of latitude in the amount of information concerning preferences that can be incorporated in it. Again, as we shall see in the next section, the structures proposed by SS&K allow such flexibility.

5. Strategic Measures

In their Abstract and Introduction, SS&K display some of the usual errors about power indices, stemming from the lack of distinction between I-power and P-power.

Thus they claim in their Abstract (p. 339) that 'all previous indices are based on cooperative game theory'; and this claim is repeated in expanded form later (p. 340).

SS&K agree with G&T that ' ... voting power indices do not represent the distribution of power between players in the EU in a satisfactory and meaningful way ... ' (p. 343). To meet G&T's objections, they propose 'a new method' (p. 339) or 'new approach' (p. 346), yielding measures of 'strategic power' (p. 342).

Let us restate the definition of these new measures, in a somewhat more general form, using our own terminology and notation.

We start with an arbitrary SVG $\mathcal{W}$, which models the decision rule of the decision-making body in question. Without loss of generality, we may assume that the assembly (set of voters) of $\mathcal{W}$ is $N = \{1, 2, \ldots, n\}$.

Whereas according to the conventional measures of a priori voting power the power of each voter depends only on $\mathcal{W}$, the strategic power of a voter depends also on an additional structure that supplements $\mathcal{W}$. This additional structure is defined as follows.

First, we fix a space $\mathbf{S}$ of *states*. Each member of $\mathbf{S}$ represents a possible state of the world that may be the actual state (the status quo), or the outcome of a vote division (that is, a decision arrived at according to $\mathcal{W}$). The state space $\mathbf{S}$ must be embedded in some Euclidean space of dimension $m \geq 1$. Typically, $\mathbf{S}$ will be either a finite set of points or a bounded continuous set such as an arc, a surface or a region of the Euclidean space; but other, more general choices are also possible.

Next, we fix $n+2$ random variables $\mathsf{X}_1, \ldots, \mathsf{X}_n$, Y and Z. All these random variables take their values in the space $\mathbf{S}$, and accordingly we shall call them *the state variables.* The variable X_i represents the state preferred by voter i; Y represents the state that will result if a proposed bill will be passed by $\mathcal{W}$; and Z represents the status quo, which will continue to prevail if the bill is defeated. The reason for taking these as random variables rather than as $n+2$ definite points of $\mathbf{S}$ is that we are considering a hypothetical situation, about which we may not have complete information. By 'fixing' the state variables we mean that a particular joint probability distribution of these variables is fixed. This distribution can be quite arbitrary,[22] except that, for an obvious reason, the equality $\mathsf{Y} = \mathsf{Z}$ (which would mean that the bill is vacuous) must have probability 0.[23]

The strategic measure corresponding to the given $\mathcal{W}$, $\mathbf{S}$ and the distribution of state variables is defined as follows.

First, consider an $(n+2)$-tuple of values, $X_1, \ldots, X_n$, Y and Z taken respectively by $\mathsf{X}_1, \ldots, \mathsf{X}_n$, Y and Z. This $(n+2)$-tuple of values represents a particular division of the assembly of voters. Voter i will vote 'yes' if the distance $\|X_i - Y\|$ between i's preferred state and the proposed state is smaller than the distance $\|X_i - Z\|$ between i's preferred state and the status quo; in the opposite case, i will vote 'no'; and if the two distances happen to be equal, i will flip a true coin and will vote 'yes' or 'no' with equal probability. (These distances are measured in the Euclidean space in which $\mathbf{S}$ is embedded.)

The voters having cast their votes in this way, the SVG $\mathcal{W}$ now determines the outcome: whether the bill is passed or defeated. Let U be the resulting state: thus, $U = Y$ or $U = Z$, according as the bill is passed or defeated.

Now let D_i be the distance $\|X_i - U\|$ between i's preferred state and the outcome of the division. This distance, as just defined, is a function of

$X_1, \ldots, X_n$, Y and Z; say

$$D_i = f_i(X_1, \ldots, X_n, Y, Z).$$

Thus D_i can be regarded as a value of a random variable D_i, where

$$\mathsf{D}_i = f_i(\mathsf{X}_1, \ldots, \mathsf{X}_n, \mathsf{Y}, \mathsf{Z}).$$

The distribution of D_i is completely determined by $\mathcal{W}$ and the joint distribution of the state variables.

Let $\Delta_i[\mathcal{W}] = \mathrm{E}\mathsf{D}_i$ be the expected (or mean) value of D_i. The intuition behind the strategic measure is that the smaller this mean distance, the greater the power of voter i.

In order to standardize the measurement based on $\Delta_i[\mathcal{W}]$, we compare it to $\Delta_d[\mathcal{W}]$, where d is a dummy voter. (We can assume that $\mathcal{W}$ has such a voter; otherwise, a dummy can be added.)[24]

Following SS&K, we now define the *strategic power* $\Psi_i[\mathcal{W}]$ of voter i by putting

$$\Psi_i[\mathcal{W}] := \frac{\Delta_d[\mathcal{W}] - \Delta_i[\mathcal{W}]}{\Delta_d[\mathcal{W}]}.$$

Note that what we have here is not a single measure but a very large family of measures. This is because $\Psi_i[\mathcal{W}]$ depends not only on $\mathcal{W}$ but also on the choice of the state space and the joint distribution of the state variables. This choice gives us an enormous latitude for building into the model all kinds of information concerning the actual state of the world, the kinds of bill to be put to the vote, and affinities or disaffinities between voters.

From our discussion in Section 2 it should be obvious that Ψ is a measure of I-power rather than P-power. It bears all the unmistakable hallmarks of I-power: it assumes no binding agreements and is thus applicable to secret voting; it does not necessarily have anything to do with payoffs (although information about payoffs can, if desired, be built into the distribution of the state variables); and it is an *absolute* measure from which an index (in the present sense) can be obtained by normalization.

But it is not necessarily an a priori measure, because, as we have just pointed out, much information can be built into the state space and the distribution of the state variables.

In order to obtain an a priori strategic measure, we must go behind a veil of ignorance: we must minimize the information built into the state space and the distribution of the state variables. SS&K seem to believe that to do this it is sufficient to assume that the state variables are mutually independent and uniformly distributed on the state space. In what follows, we too shall make this assumption.[25] But it is not sufficient, because the geometric structure of the state space itself also carries some information. In particular, any asymmetry of this space implies a bias in favour of some states and against others. We shall therefore assume now that $\mathbf{S}$ is perfectly symmetric. In the discrete case, this means that $\mathbf{S}$ is the set of vertices of a regular polygon (including the simplest case, where it consists of just two points), or of a regular polyhedron of higher dimension. In the continuous case this means that $\mathbf{S}$ is a circle, or the surface of a sphere of some higher dimension.

Under these assumptions we shall now show that Ψ is essentially the Bz measure of a priori voting power.

To this end, first note that from the symmetry of $\mathbf{S}$ and the assumption that the X_i are independent and uniformly distributed on $\mathbf{S}$ it follows that the preferred state of each voter is equally likely to be nearer to Y than to Z as the other way around. Therefore each voter will vote 'yes' or 'no' with probability $\frac{1}{2}$; and they will do so independently of each other—just as in the Bernoulli model underlying the Bz measure.

Now fix i and let R and r be the greater and smaller, respectively, of the two distances $\|\mathsf{X}_i - \mathsf{Y}\|$ and $\|\mathsf{X}_i - \mathsf{Z}\|$. Then, by the definition of D_i we have

$$\mathsf{D}_i = (1-p)\mathsf{R} + p\mathsf{r},$$

where p is the probability that the outcome of the division agrees with the way i voted (that is, that i votes 'yes' and the bill is passed, or votes 'no' and the bill is defeated).

Now, according to Penrose's Theorem,[26] we have

$$p = \frac{1+\beta'_i[\mathcal{W}]}{2}.$$

Feeding this into the previous equality, we get:

$$\mathsf{D}_i = \frac{1-\beta'_i[\mathcal{W}]}{2}\mathsf{R} + \frac{1+\beta'_i[\mathcal{W}]}{2}\mathsf{r}.$$

To get the mean value $\Delta_i[\mathcal{W}]$ of D_i, we must take the average over all values of Y, Z and X_i. We obtain:

$$\Delta_i[\mathcal{W}] = \frac{1-\beta'_i[\mathcal{W}]}{2}R + \frac{1+\beta'_i[\mathcal{W}]}{2}r,$$

where R and r are defined as follows. Choose at random and independently two distinct points Y and Z in $\mathbf{S}$, and then, independently of them, a point X. Then R and r are respectively the expected values of the greater and lesser of the distances $\|\mathsf{X} - \mathsf{Y}\|$ and $\|\mathsf{X} - \mathsf{Z}\|$.

For the dummy, whose Bz power is 0, we get, in particular,

$$\Delta_d[\mathcal{W}] = \frac{R+r}{2}.$$

Feeding these values for $\Delta_i[\mathcal{W}]$ and $\Delta_d[\mathcal{W}]$ into the definition of $\Psi_i[\mathcal{W}]$, we obtain

$$\Psi_i[\mathcal{W}] = \frac{R-r}{R+r}\beta'_i[\mathcal{W}].$$

Thus $\Psi_i[\mathcal{W}]$ is simply the Bz power of i multiplied by a constant that depends on the shape of $\mathbf{S}$. Note, in particular, that in the simplest possible case, where $\mathbf{S}$ consists of just two points, r is clearly 0, so in this case $\Psi_i[\mathcal{W}] = \beta'_i[\mathcal{W}]$ exactly.

In our view, this result vindicates the Bz measure: not for the first time, a new approach to the measurement of a priori I-power has, yet again, led to β'.

It also suggests that the strategic measure proposed by SS&K is a natural generalization of a priori I-power, which allows the incorporation of additional information, and thus the study of a posteriori voting power.

On the other hand, it must be pointed out that the practicability of these strategic measures may be problematic. This is so for two reasons. First, the choice of the state space and joint distribution of the state variables—on which the numerical values yielded by the measure depend in a crucial way—is by no means an easy matter. Second, as SS&K themselves point out, the strategic measures 'are difficult to manipulate analytically' (p. 350). The perfectly symmetric case, which yields the Bz measure, seems to be a fortunate exception.

6. Conclusions

In conclusion, we would first like to stress once more the vital distinction between the two different pre-formal notions of voting power. While the notion of P-power is a construct of cooperative game theory, the older alternative notion of I-power is independent of game theory.

Failure to recognize this distinction has led to many errors in appraising the various measures of voting power, and has also confused the arguments regarding the need for an a priori measure in the European context.

As the n-person bargaining problem has not been solved conclusively, there is no completely satisfactory and universally agreed index of a priori P-power. On the other hand, the Bz measure and resulting normalized index are satisfactory and intuitively justified from the viewpoint of a priori I-power. Indeed, for this reason the Bz measure has been reinvented several times.

Second, we must register our agreement with L&B and H&W on the need for a priori measures of voting power. These are of great importance for constitutional analysis. Moreover, the problem of measuring *actual* voting power—which has so far not been solved in a *practical* way—should in our view be tackled by a unified method: a method that takes the models used in the aprioristic theory and supplements them by additional structure that allows the incorporation of additional information.

We believe that the method proposed by SS&K is promising in this respect. The fact that—apparently without the authors' intention—when one goes behind a veil of ignorance their method yields the Bz measure is in our view a positive aspect of their approach, as well as an additional vindication of the Bz measure.

Notes

1. In this paper, we reserve the term *index* to a measure that is normalized such that the sum of its values for all voters is 1. Thus by *Bz index* we mean what is often referred to as the 'relative Bz index' and denoted by 'β', whereas by *Bz measure* we mean what Straffin (1982) and others call the 'absolute Bz index' and denote by 'β''.

2. For further details see MVP, *passim*, especially Section 1.2.

3. This view was actively advocated by Shapley and his collaborators, who managed to persuade Banzhaf himself, despite his early criticism of the S-S index and of the game-theoretic approach. On these matters see MVP, Section 4.2.

4. For detailed arguments, see MVP, Sections 6.2 and 6.3, especially Comments 6.2.8, 6.3.9 and 6.3.10.

5. See MVP, Sections 3.1 and 3.2, especially Comment 3.2.5.

6. See MVP, Section 3.1.

7. See MVP, *passim*, especially Comment 2.2.2 and Sections 3.1, 6.1 and 7.10.

8. Of course, it is possible to view game theory in a very broad sense, in

which it subsumes the whole of Social Choice, and in particular the whole of the topic of voting power.

9. On Rae's measure, see MVP, Remarks 3.2.17. On the Coleman measures, see MVP, Definition 3.2.20 and Remarks 3.2.21. The case of Barry is particularly ironic, because he vehemently rejects the Banzhaf index, which he mistakenly regards as a game-theoretic measure of P-power; and being in any case hostile to the very idea of P-power he dismissed this index as a 'gimmick'. But he fails to notice that his own measure of 'decisiveness' is exactly the same as Penrose's original proposal, which equals $\beta'/2$. See Barry (1980, pp. 191, 338) and MVP, Comment 3.2.5.

10. For further details on these matters see MVP, Chapter 6.

11. For a detailed discussion, analysis and explanation of this phenomenon, see MVP, Section 7.2.

12. A warning against this fallacy was sounded already by Dubey and Shapley (1979, p. 103).

13. For a precise definition, see MVP, Definition 2.3.23.

14. In one-dimensional space, convexity is the same as connectedness; but in higher dimensions connectedness would make little sense in the

present context.

15. In this connection, see MVP, pp. 160–61.

16. Further on this matter, see MVP, Comment 6.3.9.

17. For such an analysis of the CM, when operating under the so-called qualified majority rule, see for example MVP, Chapter 5 and various papers cited there; also various papers cited in the Symposium.

18. For the definition of composite SVGs, see MVP, Definition 2.3.12.

19. Regarding such structures and the appropriate adaptations of the Bz measure and S-S index, see MVP, Chapter 8.

20. Further on this, see MVP, Comment 2.2.3.

21. See, for example, references cited in MVP, Comment 2.2.3.

22. SS&K assume that the state variables are mutually independent and all the X_i have the same distribution; but for the time being we need not impose this severe restriction.

23. We regard this as a minor but necessary amendment of SS&K's assumptions. It is however automatically satisfied if the distribution is continuous.

24. See MVP, Remark 2.3.24(iii). As for the state variable X_d of the added dummy, a natural choice is to assume that, for each state, the probability (or, in the continuous case, probability density) of X_d is the arithmetical mean of those of Y and Z.

25. In the case of a discrete space, this implies that the variables Y and Z take all pairs of *distinct* values with equal probability.

26. See MVP, Theorem 3.2.16.

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DAN FELSENTHAL is Professor of Political Science at the University of Haifa. He is the author of *Topics in Social Choice: Sophisticated Voting, Efficacy and Proportional Representation* (Praeger, 1990), and co-author of *The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes* (Edward Elgar, 1998). He has written extensively on the theory of voting and on voting behaviour. ADDRESS: Department of Political Science, University of Haifa, Haifa 31905, Israel. [email: msdanfl@mscc.huji.ac.il]

MOSHÉ MACHOVER is Professor Emeritus at King's College, University of London. He is a mathematician, author or co-author of books on mathematical logic, mathematical economics and voting power. His papers range over various topics in foundations of mathematics, mathematical logic and social choice. ADDRESS: Department of Philosophy, King's College, Strand, London WC2R 2LS, England. [email: moshe.machover@kcl.ac.uk]